# ONE-DIMENSIONAL q-STATE MODIFIED POTTS MODEL AND ITS THERMODYNAMIC FUNCTIONS

HASAN AKIN

DEPARTMENT OF MATHEMATICS, FACULTY OF ARTS AND SCIENCES, HARRAN UNIVERSITY, SANLIURFA TR63050, TURKEY, E-MAIL: akinhasan@harran.edu.tr; akinhasan25@gmail.com

ABSTRACT. Since its introduction, the Potts model has gained widespread popularity across various fields due to its diverse applications. Even minor advancements in this model continue to captivate scientists worldwide, and small modifications often intrigue researchers from different disciplines. This paper investigates a one-dimensional q-state modified Potts model influenced by an external magnetic field. By leveraging the transfer matrix method, exact expressions are derived for key thermodynamic quantities, including free energy, entropy, magnetization, susceptibility, and specific heat capacity. Numerical analyses explore how these thermodynamic functions vary with relevant parameters, offering insights into the system's behavior. Additionally, the asymptotic properties of these quantities are examined in the limiting cases  $T \to 0$  and  $T \to \infty$ . The findings contribute to a deeper understanding of the model's thermodynamic characteristics and highlight its potential applications across various disciplines. **Keywords**: Modified Potts model, thermodynamic functions, free energy, susceptibility, magnetization.

1. INTRODUCTION

Lattice spin systems encompass a wide range of models within statistical mechanics [1]. While some directly correspond to physical phenomena, others act as simplified analogs of more intricate systems [2]. The lattice structure plays a fundamental role in studying spin systems. Since Potts introduced the model [3], systems involving three or more spins on various grids have been extensively investigated across numerous disciplines (statistical mechanics, mathematics, material science, biology [4], quantum entanglement and topological order [5]) under different names to represent complex systems [6]. The Potts model, developed as a generalization of the Ising model [7], revolves around interacting spins, which in the Ising model are limited to parallel or antiparallel states [3, 8]. In Ref. [9], the authors studied the one-dimensional q-state Potts model (1D q-SPM) with ferromagnetic pair interactions that decay with the distance r according to a power-law function.

To investigate various fundamental properties of lattice systems on specific grids, it is essential to analyze the model's partition functions. Numerous methods exist for constructing partition functions. In the 1D case, these can be expressed using recurrence relations, but they are more generally derived using the transition matrix method. This method was first introduced by Kramers and Wannier in 1941 [10, 11]. This approach enables the analysis of thermodynamic properties of the lattice model via the partition function [12]. In this study, we concentrate on determining the largest eigenvalue of the transition matrix associated with a q-state modified Potts model in the 1D setting (shortly, 1D q-SMPM). Calculating the partition function for a chain of two-state elements with nearest-neighbor interactions,

Date: April 25, 2025.

known as the 1D Ising model, is a standard exercise in many textbooks [13]. This computation is typically carried out using the transfer matrix method, which sums the exponential terms derived from the Ising model Hamiltonian. In the same way, the partition function for other 1D lattice models, such as the Ising, Potts, or Hard-core models, is calculated using the same method. The transfer matrix approach simplifies the summation of exponential terms according to the specific Hamiltonian of each model [14, 15, 16, 17]. In Ref. [18], Creswick and Kim utilized the microcanonical transfer matrix to investigate the zeros of the partition function of the Q-SPM. Potts models, studied under various names, have been the focus of recent research. In Ref. [19], Kryzhanovsky analyzed a modified version of the Potts model featuring binarized synaptic coefficients.

By analyzing the eigenvalues of the transfer matrix, one can directly obtain the partition function. The key distinction between the combinatorial method (used by Ising) and the transfer matrix method lies in how the Boltzmann weights are handled. Recently, the transfer matrix method has yielded successful results in studying the thermodynamic functions of 1D mixed-type and mixed-spin lattice models [20, 21, 22, 23, 24, 25].

In [26], we examined various thermodynamic properties of the q-SPM on the Cayley tree using the cavity method. In Reference [27], the phase transition problem was investigated by the cavity method on the Cayley tree for the Potts model with finite number of spins. Similarly, in [28], Yang employed a modified recursion technique to derive the exact partition function for a 1D Potts model with free or periodic boundary conditions. This approach was then compared to the transfer matrix method and the technique introduced by Marchi and Vila. Furthermore, Chang and Shrock, in [29], proposed measures of spin ordering for the q-state ferromagnetic Potts model under a generalized external magnetic field.

In this present paper, we introduce a 1D q-SMPM with an external field. The article will be approached using two distinct methods. First, the iterative equation system will be derived using the cavity method, and the solutions of these equations identify the Gibbs measure for the specified model. Second, the transition matrix will be constructed based on the Markov chain rule, and the largest eigenvalue of this matrix will be used to compute the corresponding partition function. Using this partition function, we will determine the model's free energy function. By taking partial derivatives of the free energy function with respect to the relevant parameters, we will derive exact formulas for thermodynamic functions such as entropy, magnetization, susceptibility, and specific heat capacity. Finally, numerical calculations will be performed to plot the graphs of these functions, allowing for comparisons with previous results.

The remainder of this study is organized as follows: Section 2 introduces the Hamiltonian that defines the 1D q-SMPM and constructs the transition matrix associated with the linear equation system used to compute partial partition functions for the 1D case. Sections 3 and 4 focus on deriving the model's basic thermodynamic functions using the transition matrix for the 1D q-SMPM. These functions are analyzed numerically, particularly in relation to the specified parameters, and their peak points at critical temperatures are explored. Finally, the concluding section 5 provides a comprehensive evaluation, highlights key findings, and offers predictions to guide future research.

### 2. Construction of the partial partition functions and transition matrix

In this section, we introduce the essential concepts and results that form the foundation of our analysis. To calculate the free energy for the 1D q-SMPM with an external field, we use the following Hamiltonian:

(2.1) 
$$H(\sigma) = -J \sum_{\langle x,y \rangle} \cos(\pi \delta_{\sigma(x),\sigma(y)}) - h \sum_{\langle x,y \rangle} \cos(\pi \delta_{\sigma(x),\sigma(y)}),$$

where  $\langle x, y \rangle$  denotes nearest-neighbor pairs, and  $\delta_{\sigma(x),\sigma(y)}$  represents the Kronecker delta.

Let us go through a more specialized approach that considers the discrete nature of the spin states. For spins  $\sigma(x), \sigma(y) \in \Phi := \{1, 2, \dots, q\}$ , the term  $\cos(\pi \delta_{\sigma(x), \sigma(y)})$  introduces different interactions depending on whether the neighboring spins are equal or not:

- If  $\sigma(x) = \sigma(y)$ ,  $\delta_{\sigma(x),\sigma(y)} = 1$ , so  $\cos(\pi \cdot 1) = -1$ .
- If  $\sigma(x) \neq \sigma(y)$ ,  $\delta_{\sigma(x),\sigma(y)} = 0$ , so  $\cos(\pi \cdot 0) = 1$ .

Thus, the interaction between neighbors x and y can be summarized as:

$$J\cos(\pi\delta_{\sigma(x),\sigma(y)}) = \begin{cases} -J, & \text{if } \sigma(x) = \sigma(y), \\ J, & \text{if } \sigma(x) \neq \sigma(y). \end{cases}$$

On 1D lattice  $\mathbb{N}$ , the interaction energy of a given spin configuration  $\{\sigma_i\} \in \Phi^N$  is expressed as

(2.2) 
$$E^{(N)}(\{\sigma_i\}) = -J \sum_{i=1}^{N} \cos(\pi \delta_{\sigma_i,\sigma_{i+1}}) - \frac{h}{\beta} \sum_{i=1}^{N} \cos(\pi \delta_{\sigma_i,\sigma_{i+1}}).$$

As is typical, the partition function of the 1D-qSMPM is the sum over all possible spin configurations:

(2.3) 
$$Z^{(N)}(\beta, h, J) = \sum_{\{\sigma_i\}\in\Phi^N} \exp\left(-\beta E^{(N)}\left(\{\sigma_i\}\right)\right),$$

where  $\beta = \frac{1}{k_B T}$  is the inverse temperature,  $k_B$  is the Boltzmann constant, and T represents the temperature.

Let  $Z_{\sigma(x)}^{(N)}(\beta, h, J)$  represent the partial partition function of a subtree rooted at vertex x, where the spin at x is fixed to  $\sigma(x)$ , expressed as a function of the parameters  $\beta$ , h, and J. At the boundary vertices (the last level of the tree), their contribution to the partial partition function depends on the spin state, as they have no neighboring vertices. For any initial vertex x, the value of  $Z_{\sigma(x)}^{(N)}(\beta, h, J)$  is calculated by summing over all possible spin configurations of its consecutive vertices. When  $\sigma(x) \in \Phi$ , the contribution to  $Z_{\sigma(x)}^{(N)}(\beta, h, J)$  is determined by whether the spins of the consecutive vertices with or deviate from  $\sigma(x)$ . The recursion for  $Z_{\sigma(x)}^{(N)}(\beta, h, J)$  is expressed through the following partial partition sums:

(2.4) 
$$Z_{\sigma(x)}^{(N)}(\beta,h,J) = \sum_{\sigma(y)\in\Phi} e^{\beta J\cos(\pi\delta_{\sigma(x),\sigma(y)}) + h\cos(\pi\delta_{\sigma(x),\sigma(y)})} Z_{\sigma(y)}^{(N-1)}(\beta,h,J),$$

where y represents the consecutive vertex of x on the lattice  $\mathbb{N}$ .

The total partition function is given by:

(2.5) 
$$Z^{(N)}(\beta, h, J) = \sum_{\{states\}} e^{\beta J \sum_{i=1}^{N} \cos(\pi \delta_{\sigma_i \sigma_{i+1}}) + h \sum_{i=1}^{N} \cos(\pi \delta_{\sigma_i \sigma_{i+1}})}$$
$$= \sum_{\sigma(x) \in \Phi} Z^{(n)}_{\sigma(x)}(\beta, h, J),$$

and the sum:

$$\sum_{\{states\}}(\dots) = \prod_{i=1}^N \sum_{\sigma_i \in \Phi}(\dots)$$

indicates summation over the spin states at all sites. The total partition function can be rewritten as a product of terms, each depending on two neighboring spins, with periodic boundary conditions  $\sigma_{N+1} = \sigma_1$ :

$$Z^{(N)}(\beta, h, J) = \sum_{\{states\}} M(\sigma_1, \sigma_2) M(\sigma_2, \sigma_3) \dots M(\sigma_{N-1}, \sigma_N) M(\sigma_N, \sigma_1)$$

where  $M(\sigma, \sigma') = e^{(\beta J + h) \cos(\pi \delta_{\sigma, \sigma'})}$  for  $\sigma, \sigma' \in \Phi$ .

From Equation (2.4), recursive relations can be written for  $Z_{\sigma(x)}^{(N)}(\beta, h, J)$ , and it is convenient to express them in matrix form as follows:

$$\begin{pmatrix} Z_1^{(N)} \\ Z_2^{(N)} \\ \vdots \\ Z_q^{(N)} \end{pmatrix} = \mathbf{M}^{(q)} \begin{pmatrix} Z_1^{(N-1)} \\ Z_2^{(N-1)} \\ \vdots \\ Z_q^{(N-1)} \end{pmatrix}$$

where the transfer matrix  $\mathbf{M}^{(q)}$  is given by

(2.6) 
$$\mathbf{M}^{(q)} = \begin{pmatrix} e^{-h-J\beta} & e^{h+J\beta} & e^{h+J\beta} & \cdots & e^{h+J\beta} & e^{h+J\beta} \\ e^{h+J\beta} & e^{-h-J\beta} & e^{h+J\beta} & \cdots & e^{h+J\beta} & e^{h+J\beta} \\ e^{h+J\beta} & e^{h+J\beta} & e^{-h-J\beta} & \cdots & e^{h+J\beta} & e^{h+J\beta} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ e^{h+J\beta} & e^{h+J\beta} & e^{h+J\beta} & \cdots & e^{-h-J\beta} & e^{h+J\beta} \\ e^{h+J\beta} & e^{h+J\beta} & e^{h+J\beta} & \cdots & e^{h+J\beta} & e^{-h-J\beta} \end{pmatrix}$$

In the upcoming sections, we will perform a numerical analysis of these functions by computing the thermodynamic quantities associated with the model, using the largest eigenvalue of the matrix  $\mathbf{M}^{(q)}$ .

### 3. Free energy

The free energy of a lattice model is essential for understanding various thermodynamic and physical characteristics of the system [5]. The free energy function can give many clues about the stability, phase transition and equilibrium state of a system [12]. In particular, the free energy governs aspects such as thermodynamic stability, phase transitions, entropy, specific heat capacity, and equilibrium properties. For instance, first-order phase transitions are characterized by a discontinuity in the first derivative of the free energy (such as entropy or magnetization), while second-order phase transitions correspond to discontinuities in the second derivative of the free energy (like specific heat capacity or susceptibility) [30].

It is well-established that the partition functions associated with lattice models can be computed using various approaches. On the Bethe lattice, methods such as the cavity method [31, 32] or the Kolmogorov consistency theorem are applicable. For one-dimensional lattices, the transition matrix method is the most commonly utilized technique. The partition function can be expressed as a linear combination of the N-th powers of the eigenvalues of the transfer matrix [12]. Thus, in the thermodynamic limit, using the total partition function  $Z^{(N)}(\beta, h, J)$  from (2.5), we determine the free energy per site for the 1D q-SMPM. As  $N \to \infty$ , only the largest eigenvalue dominates, resulting in the following expression:

(3.1) 
$$f(\beta, h, J) = -k_B T \lim_{N \to \infty} \frac{1}{N} \ln Z^{(N)}(\beta, h, J),$$

where N is the number of vertices. For the sake of simplicity, we will set the constant  $k_B$  equal to 1 throughout the paper.

By transfer matrix method, the total partition function  $Z^{(N)}(\beta, h, J)$  is determined by evaluating the trace of the matrix product as follows:

$$Z^{(N)}(\beta, h, J) = \operatorname{Tr}\left(\left(\mathbf{M}^{(q)}\right)^{N}\right) = \sum_{j=1}^{q} (\lambda_{j})^{N},$$

where  $\lambda_j$  are the eigenvalues of the transfer matrix  $\mathbf{M}^{(q)}$  given in (2.6) for an arbitrary spin q.

From Equation (2.5), this can be rewritten as:

(3.2) 
$$Z^{(N)}(\beta, h, J) = \sum_{j=1}^{q} (\lambda_j)^N = (\lambda_{\max})^N \left[ 1 + \sum_{j=1}^{q-1} \left( \frac{\lambda_j}{\lambda_{\max}} \right)^N \right],$$

where  $\lambda_j$  are the eigenvalues of the transfer matrix  $\mathbf{M}^{(q)}$ , and  $\lambda_{\max} = \max\{\lambda_1, \lambda_2, \dots, \lambda_q\}$ .

In the thermodynamic limit  $N \to \infty$ , the partition function simplifies to:

(3.3) 
$$Z^{(N)}(\beta, h, J) = (\lambda_{\max})^N$$

as the contributions from smaller eigenvalues become negligible. It is well established that the critical behavior of the model is governed by this largest eigenvalue in the thermodynamic limit as  $N \to \infty$  [15, 17, 20, 21]. From Equation (3.3), the bulk free energy is governed by the largest eigenvalue of the transfer matrix, and is given by:

$$f(\beta, h, J) = -\frac{1}{\beta} \lim_{N \to \infty} \frac{1}{N} \ln Z^{(N)}(\beta, h, J) = -\frac{1}{\beta} \ln \lambda_{\max}.$$

The eigenvalues of  $\mathbf{M}^{(q)}$  are derived based on the structure of the matrix, taking into account its symmetry and the constant diagonal values. By means of the characteristic equation det $(\mathbf{M}^{(q)} - \lambda I) = 0$ , the eigenvalues of the symmetric matrix  $\mathbf{M}^{(q)}$  are obtained as

(3.4) 
$$\lambda_{i} = \begin{cases} e^{-h - J\beta} \left( 1 - e^{2h + 2J\beta} \right), & \text{for } 1 \le i \le q - 1, \\ e^{-h - J\beta} \left( 1 + (q - 1)e^{2h + 2J\beta} \right), & \text{for } i = q. \end{cases}$$

From Equation (3.4), it is evident that  $\lim_{N\to\infty} (\lambda_i/\lambda_q)^N = 0$  since  $\lambda_i/\lambda_q < 1$  for  $1 \le i \le q-1$ . Consequently, considering the largest eigenvalue  $\lambda_q = \lambda_{\max} = e^{-h-J\beta} \left(1 + (q-1)e^{2h+2J\beta}\right)$  as given in (3.4), the free energy function for the 1D-qSPM, expressed in terms of J, h,  $\beta$ , and q, is determined as:

(3.5) 
$$f(\beta, h, J, q) = -\frac{1}{\beta} \ln\left(\frac{1 + (q - 1)e^{2h + 2J\beta}}{e^{h + J\beta}}\right).$$

This function is also referred to as the Gibbs free energy per site [29]. For the disordered phase, we expect each spin configuration to contribute equally, leading to a higher entropy and potentially higher free energy, depending on  $\beta$ , h, J and q, respectively. This recursive approach offers a way to approximate f numerically and analyze phase behavior on the model with the given Hamiltonian (2.1) and spin set  $\Phi$ .



FIGURE 1. (Color online) 3D plots of the free energy function (3.5) for the range  $\beta \in [0.001, 30]$  and  $h \in [-3, 3]$ : (a) for J = -12, q = 16; (b) for J = 0.95, q = 16.

Figures 1 display 3D representations of the free energy function within the ranges  $\beta \in [0.001, 30]$  and  $h \in [-3, 3]$ . These graphs illustrate that the key determinant is the *J* parameter. For antiferromagnetic cases (J < 0), peaks are visible on the surface Figure 1(a), whereas for ferromagnetic cases (J > 0), the surface gradually increases with  $\beta$  and exhibits a smooth appearance Figure 1(b).



FIGURE 2. (Color online) Graphs of free energy given in (3.5) (left) for h = -3, J = 5.15 (right) for h = 4, J = -3.

In Fig. 2, the free energy functions associated with the 1D q-SMPM are plotted as a function of  $\beta$  for the model with q = 3, 4, 5, 6, 7, 8, 9, 21. The graphs are generated for selected values of the coupling constants J and h. It is observed that the free energy function decreases with increasing q. Specifically, within certain intervals of  $\beta$ , the following inequality holds:  $f(\beta, h, J, 21) < f(\beta, h, J, 9) < f(\beta, h, J, 8) < f(\beta, h, J, 7) < f(\beta, h, J, 6) < f(\beta, h, J, 5) < f(\beta, h, J, 4) < f(\beta, h, J, 3)$ . In [26], the authors studied the classical q-SPM on the Cayley tree, considering competing NN interactions and prolonged NNN interactions. They also explored the behavior of various thermodynamic functions associated with the

model. In the current work, we observe that the graphs presented in Figure 2 for the 1D q-SMPM display similar patterns to those reported in [26]. Notably, for J = 5.15, h = -3, the free energy functions show peaks in specific regions, whereas for J = -3, h = 4, they exhibit a smooth and continuous behavior. Note that the figures were drawn with the help of Mathematica [34].

*Remark* 3.1. It is worth mentioning that the free energy function for the model can also be determined using the total partition function derived in equation (2.5).

## 4. Thermodynamic Functions

In this section, we extend our analysis of the 1D q-SMPM by deriving additional thermodynamic properties, such as entropy, magnetization, susceptibility, and specific heat capacity, using the free energy function defined in (3.5). Furthermore, we investigate how these properties depend on the parameters  $\beta$ , h, J, and q, and we perform a numerical analysis to study the behavior of their corresponding graphs.

4.1. The entropy. It is important to note that the derivatives of the free energy  $F(\beta, h, J, q)$  with respect to different parameters provide insights into the thermodynamic variables. Specifically, the derivative of the free energy function  $f(\beta, h, J, q)$  with respect to temperature T is related to the system's entropy, as outlined by Georgii [33] (also see [26, 27] for details). So, the entropy of the 1D q-SMPM is given by:

(4.1) 
$$S(\beta, h, J, q) = -\frac{\partial F(\beta, h, J, q)}{\partial T} = \beta^2 \frac{\partial F(\beta, h, J, q)}{\partial \beta}$$

From the expressions (3.5) and (4.1), the entropy  $S(\beta, h, q, J)$  is obtained as:

(4.2)

$$S(\beta, h, J, q) = \frac{J\left(1 - (q - 1)e^{2(h + J\beta)}\right)\beta + \left(1 + (q - 1)e^{2(h + J\beta)}\right)\ln\left[e^{-h - J\beta}\left(1 + (q - 1)e^{2(h + J\beta)}\right)\right]}{1 + (q - 1)e^{2(h + J\beta)}}$$
$$= \frac{J\left(1 - e^{2\left(h + \frac{J}{T}\right)}(q - 1)\right)T^{-1} + \left(1 + e^{2\left(h + \frac{J}{T}\right)}(q - 1)\right)\ln\left[e^{-\frac{J + hT}{T}}\left(1 + e^{2\left(h + \frac{J}{T}\right)}(q - 1)\right)\right]}{1 + e^{2\left(h + \frac{J}{T}\right)}(q - 1)\right)}.$$



FIGURE 3. (Color online) Three-dimensional (3D) plots of the entropy function (4.2) within the range  $T \in [0.001, 2]$  and  $h \in [-3, 18]$ : (a) for J = -2, q = 17 (b) for J = 8, q = 22 within the range  $T \in [0.001, 22]$  and  $h \in [-3, 22]$ 

The formula (4.2) expresses the entropy in terms of the parameters h and  $\beta$ , incorporating the behavior of the system based on these variables. So, one examines the effects of J (interaction strength) and q (number of states) on the system's thermodynamic behavior.

In figures 3, we have plotted two 3D graphs based on the formula in (4.2), with  $\beta = 1/T$  as a function of h and T. From these plots, we observed that J has a greater influence than the q-spin value. In the antiferromagnetic region (J < 0) 3(a), the entropy function shows significant variation, with some areas exhibiting negative values. In contrast, in the ferromagnetic region (J > 0) 3(b), the entropy function appears more plane-like. In these graphs, we have also included the 0 plane in the three-dimensional coordinate system to illustrate the regions where the entropy is positive. This allows us to identify the areas where the entropy value is greater than zero.



FIGURE 4. (Color online) Entropy function plots (4.2) as a function of temperature T: (left) for h = -3, J = 5.3, q = 7; (right) for h = 2, J = -5.3, q = 17, 22.

Figure 4 shows the graphs of the entropy function as a function of temperature T for specific values of h, J, and q. A notable observation is that the entropy value is negative at low temperatures.

4.2. The magnetization per spin. The magnetization  $m(\beta, h, J, q)$  is derived within the framework of the canonical ensemble using equation (3.2). To determine the magnetization for arbitrary values of the external magnetic field h, we will utilize Onsager's exact solution for zero-field magnetization as a reference. Following the methodology commonly used in the literature [14, 15, 20], the magnetization per spin is determined by taking the partial derivative of the free energy function with respect to the external magnetic field h.

Using the largest eigenvalue  $(\lambda_{\max})$  of the transition matrix  $\mathbf{M}^{(q)}$ , we can compute the magnetization per spin as

(4.3) 
$$m(\beta, h, J, q) = -\frac{\partial f(\beta, h, q, J)}{\partial h} = \frac{-1 + e^{2(h+J\beta)}(q-1)}{(1 + e^{2(h+J\beta)}(q-1))\beta}.$$

In Equation (4.3), the free energy attains its extreme value with respect to h when  $h = -\frac{1}{2} \ln \left(e^{2\beta J}(q-1)\right)$ . In Fig. 5, we present three-dimensional plots of the magnetization function (4.3) for the following parameter ranges: (a) J = -2, q = 17 with  $T \in [0.001, 30]$  and  $h \in [-3, 3]$ ; (b) J = 8, q = 22 with  $T \in [0.001, 40]$  and  $h \in [-8, 8]$ . From the graphs, it can be observed that the magnetization function increases as h increases, exhibiting both negative and positive values. Additionally, as the temperature approaches zero, the magnetization becomes entirely negative (see Fig. 6).



FIGURE 5. (Color online) Three-dimensional plots of the magnetization function (4.3) are presented for the following parameter ranges: (a) J = -2, q = 17 with  $T \in [0.001, 30]$  and  $h \in [-3, 3]$ ; (b) J = 8, q = 22 with  $T \in [0.001, 40]$  and  $h \in [-8, 8]$ .



FIGURE 6. (Color online) Three-dimensional plot of the magnetization function (4.3) is presented for the following parameter ranges: J = -2, q = 17 with  $T \in [0.001, 1]$  and  $h \in [-3, 3]$ .

4.3. Susceptibility. The susceptibility of lattice models has been the subject of extensive research, with a particular focus on its relationship with other thermodynamic functions, especially the Curie temperature. In Ref. [35], Melnikov, Paradezhenko, and Reser applied dynamic spin fluctuation theory to calculate the magnetic susceptibility of ferromagnetic metals above the Curie temperature  $T_c$ , and they made a thorough comparison with experimental data. The susceptibility is often essential for understanding the magnetic properties of materials [35]. By taking the first derivative of the free energy, as presented in (3.5), calculated using the partition function for the lattice model, with respect to the external magnetic field h, the magnetization per site is obtained. The partial derivative of the function in (3.5) with respect to h defines the model's susceptibility [29]. This function helps identify the critical temperatures  $T_c$  and determine the regions where phase transitions occur in the system [14, 15]. Typically, sharp peaks marking the transition from a disordered to an ordered state provide valuable insight into the presence of a phase transition.

Magnetic susceptibility  $\chi(\beta, h, J, q)$  measures the response of magnetization  $m(\beta, h, q, J)$  to an external magnetic field h, and is given by:

$$\chi\left(\beta,h,J,q\right) = \left(\frac{\partial m\left(\beta,h,q,J\right)}{\partial h}\right)_{T}$$

In systems like the Curie-Weiss model, susceptibility near the Curie temperature  $T_c$  follows:

$$\chi(T) = \frac{C}{T - T_c}$$

where C is the Curie constant.

Consequently, by taking the second partial derivative of the free energy function with respect to the field variable h, we arrive at the following expression:

(4.4) 
$$\chi(h,\beta,J,q) = -\frac{\partial^2 f(h,\beta,J,q)}{\partial h^2} = \frac{4e^{2(h+J\beta)}(-1+q)}{\left(1+e^{2(h+J\beta)}(-1+q)\right)^2\beta}.$$



FIGURE 7. (Color online) Graphs of the magnetic susceptibility  $\chi(h, \beta, J, q)$  from equation (4.4) are shown as a function of the magnetic field h for the following parameter sets:  $\beta = 0.01, J = 1.3, q = 17; \beta = 0.1, J = 1.3, q = 17; \beta = 0.9, J = -3, q = 22;$  and  $\beta = 0.9, J = -3, q = 44$ , respectively.

In Figure 7, the magnetic susceptibility from equation (4.4) is plotted as a function of the magnetic field h for the following parameter sets:  $\beta = 0.01, J = 1.3, q = 17; \beta = 0.1, J = 1.3, q = 17; \beta = 0.9, J = -3, q = 22;$  and  $\beta = 0.9, J = -3, q = 44$ . All the graphs display peaks. As the temperature increases ( $\beta = 0.01; T = 100$ ), the magnetic susceptibility also increases. Notably, in the antiferromagnetic case with J = -3, the peak shifts to the right. Another key observation is the significant influence of the q-spin value on the graph behavior. Comparing the graphs for q = 22 and q = 44, while keeping  $\beta = 0.9$  and J = -3 constant, reveals substantial differences between the two cases. In [14], the authors demonstrated that the peaks of susceptibility indicate a higher maximum of  $\chi(T, H)$  at lower temperatures. Contrary to their findings, our model reveals that the peak value of magnetic susceptibility increases with rising temperature.

In summary, for the 1D q-SMPM, the surface curves corresponding to thermodynamic functions along the T-axis (or  $\beta$ ) and h-axis provide a powerful way to study the intricate behaviors of systems in statistical mechanics and thermodynamics, including phase transitions, critical points, and the response to external fields.

4.4. Specific heat capacity. Another important critical exponent of the 1D q-SMPM is associated with the heat capacity, defined as:

$$C(h, \beta, J, q) = k_B \beta^2 \frac{\partial^2}{\partial \beta^2} \ln \lambda_{\max}.$$

Alternatively, it can be expressed as:

(4.5) 
$$C(T,h,J,q) = -T\frac{\partial^2 F}{\partial T^2} = \frac{4e^{2h+\frac{2J}{T}}J^2(q-1)}{\left(1+(q-1)e^{2\left(h+\frac{J}{T}\right)}\right)^2 T^2}.$$



FIGURE 8. (Color online) Three-dimensional plots of the heat capacity (4.3) are presented for the following parameter ranges: (a) J = -1.26, q = 26 with  $T \in [0.001, 1]$  and  $h \in [-8, 15]$ ; (b) J = -3.5, q = 7 with  $T \in [0.001, 1.9]$  and  $h \in [-3, 25]$ .

In Figure 8, we present three-dimensional plots of the heat capacity (4.3) for the following parameter ranges: J = -1.26, q = 26 with  $T \in [0.001, 1]$  (8(a)) and  $h \in [-8, 15]$ , and J = -3.5, q = 7 with  $T \in [0.001, 1.9]$  and  $h \in [-3, 25]$  (8(b)). These plots clearly demonstrate that at critical temperatures T, the function either attains a local maximum or displays asymptotic behavior.

In Figure 9, we display behavior of the heat capacity function plots (4.5) as a function of temperature T: (left) for h = 4, J = -0.66, q = 20; (right) for h = 3, J = -5.2, q = 20. The locations of the peaks in the function change depending on the values of h, J, and q. As shown in the figures, the parameters influence the heat capacity behavior. In the first two cases, where the system is considered antiferromagnetic (J < 0), the shapes can be adjusted based on the positive and negative external magnetic field h. Additionally, as q decreases, the value of the peak increases.

#### 5. Conclusions

As demonstrated in [26], constructing partial partition functions for the q-state Potts model on the Cayley tree results in non-linear recurrence equation systems, making it impractical to directly derive the transition matrix from these equations. This limitation necessitates the exploration of alternative methods beyond the cavity method. In [31], the free energy for an interactive Ising model with nearest



FIGURE 9. (Color online) Temperature dependence of the heat capacity function (4.5): (a) for h = 4, J = -0.66, q = 20; (b) for h = 3, J = -5.2, q = 20; and (c) for h = -8, J = 9, q = 10.

and extended next-nearest neighbor interactions on a Cayley tree was successfully calculated using the cavity method.

In this study, we introduced a 1D q-SMPM under the influence of an external field. The analysis was performed using two complementary approaches. First, we employed the cavity method to derive a system of iterative equations, whose solutions determine the Gibbs measure for the model. In the one-dimensional case, this system is linear, allowing us to directly obtain the corresponding transition matrix. Second, we constructed the transition matrix using the Markov chain rule and utilized its largest eigenvalue to calculate the partition function. This partition function was then used to derive the model's free energy function.

By taking partial derivatives of the free energy function with respect to relevant parameters, we obtained exact expressions for key thermodynamic quantities, including free energy, entropy, magnetization, susceptibility, and specific heat capacity. Additionally, numerical calculations were performed to visualize these thermodynamic functions and compare them with previously reported results.

It is well-known that one-dimensional models with short-range interactions do not exhibit phase transitions when the single-spin space is finite, as formalized in van Hove's theorem [12]. However, Khakimov proved in [36] that a phase transition can occur in the one-dimensional SOS model with countable state under the influence of an external field. In our work, the existence of a phase transition for the one-dimensional q-SMPM in the countable case will be examined in future research. Accordingly, our future studies will investigate the phase transition behavior of this model on finite-order Cayley trees using the cavity method and the Kolmogorov consistency theorem. Additionally, the physical relevance of this model remains an open question that warrants further investigation.

The application of the transition matrix method to analyze thermodynamic functions in lattice models is a fundamental concept commonly introduced in undergraduate statistical mechanics courses. We believe the findings of this study will be of significant interest to researchers in the field of statistical mechanics.

### Acknowledgment

Part of this research was conducted during my time at ICTP in 2023, and I am grateful for the support I received from ICTP.

## Funding

This research received no external funding.

## Data Availability Statement

The statistical data presented in the article do not require copyright. They are freely available and are listed at the reference address in the bibliography.

## **Conflicts of Interest**

The author declare no conflict of interest.

### References

- [1] Baxter, R.J. Exactly solved models in statistical mechanics. Elsevier (2016).
- [2] Preston, C.J. Gibbs states on Countable Sets, Cambridge University Press, London (1974).
- [3] Potts, R.B. Some generalized order-disorder transformations, Mathematical Proc. Cambridge Philos. Soc. 48(1), 106-109 (1952). https://doi.org/10.1017/S0305004100027419
- [4] Graner, F., & Glazier, J. A. Simulation of biological cell sorting using a two-dimensional extended Potts model, Physical Review Letters, 69(13), 2013. https://doi.org/10.1103/PhysRevLett.69.2013
- [5] Suzuki, M. Relationship between d-Dimensional Quantal Spin Systems and (d+1)-Dimensional Ising Systems: Equivalence, Critical Exponents and Systematic Approximants of the Partition Function and Spin Correlations, Progress of Theoretical Physics, 56 (5), 1454-1469 (1976). https://doi.org/10.1143/PTP.56.1454
- Bhattacharjee, S.M. Complex dynamics approach to dynamical quantum phase transitions: The Potts model, Physical Review B, 109(3) (2024) 035130. https://doi.org/10.1103/PhysRevB.109.035130
- [7] Ising, E. Beitrag zur Theorie des Ferromagnetismus, Z. Phys. **31** (1925) 253–258. https://doi.org/10.1007/2FBF02980577
- [8] Wu, F.Y. The Potts model, Rev. Modern Phys. 54 (1982) 235-268. https://doi.org/10.1103/RevModPhys.54.235
- [9] Cannas, S.A. and de Magalhães, A. C. N. The one-dimensional Potts model with long-range interactions: a renormalization group approach, Journal of Physics A: Mathematical and General, **30**, 3345-3354 (1997). https://doi.org/10.1088/0305-4470/30/10/014
- [10] Kramers, H.A. Wannier, G.H. Statistics of the two-dimensional ferromagnet. Part I. Phys. Rev. 60, 252 (1941). https://doi.org/10.1103/PhysRev.60.252
- [11] Kramers, H.A. Wannier, G.H. Statistics of the two-dimensional ferromagnet. Part II. Phys. Rev. 60, 263 (1941). https://doi.org/10.1103/PhysRev.60.263
- [12] Cuesta, J.A., Sánchez, A. General non-existence theorem for phase transitions in one-dimensional systems with short range interactions, and physical examples of such transitions, Journal of Statistical Physics, 115, 869-893 (2004). https://doi.org/10.1023/B:JOSS.0000022373.63640.4e
- [13] Folk, R. Holovatch, Y. Ising's roots and the transfer-matrix eigenvalues, Entropy 2024, 26, 459. https://doi.org/10.3390/e26060459
- [14] Amin, M. E., Mubark, M., & Amin, Y. Investigation of the finite size properties of the Ising model under various boundary conditions, Zeitschrift für Naturforschung A, 75 (2), 175-182 (2020). https://doi.org/10.1515/zna-2019-0227.

- [15] Amin, M. E., Mubark, M., & Amin, Y. On the critical behavior of the spin-s ising model, Revista Mexicana de Fisica, 69 021701 1-6 (2023). https://doi.org/10.31349/RevMexFis.69.021701
- [16] Semkin, S.V., Smagin, V.P. Free energy, entropy, and magnetization of a one-dimensional Ising model of a diluted magnet, Theor Math Phys., 217, 1788-1794 (2023). https://doi.org/10.1134/S0040577923110132
- [17] Wang, W., Diaz-Mendez, R., Capdevila, R. Solving the one-dimensional Ising chain via mathematical induction: An intuitive approach to the transfer matrix, Eur. J. Phys., 40, 065102 (2019). https://doi.org/10.1088/1361-6404/ab330c
- [18] Creswick, R. J., & Kim, S.-Y. Microcanonical transfer matrix study of the Q-state Potts model. Computer Physics Communications, 121-122, 26-29 (1999). https://doi.org/10.1016/S0010-4655(99)00271-4
- [19] Kryzhanovsky, V. Modified q-state Potts model with binarized synaptic coefficients. In: Kurková, V., Neruda, R., Koutník, J. (eds) Artificial Neural Networks - ICANN 2008. ICANN 2008. Lecture Notes in Computer Science, 5164, p 72-80. Springer, Berlin, Heidelberg.
- [20] Akın, H. Calculation of thermodynamic quantities of 1D Ising model with mixed spin-(s,(2t-1)/2) by means of transfer matrix, Axioms, 12(9), 880 (2023). https://doi.org/10.3390/ axioms12090880
- [21] Akın, H. The qualitative properties of 1D mixed-type Potts-SOS model with 1-spin and its dynamical behavior, Phys. Scr. 99 (5) (2024) 055231. https://doi.org/10.1088/1402-4896/ad3a29
- [22] Rahmatullaev, M.M., Rasulova, M.A. Extremality of translation-invariant Gibbs measures for the Potts-SOS model on the Cayley tree, J. Stat. Mech. 2021 (2021) 073201. https://doi.org/10.1088/1742-5468/ac08ff
- [23] Akın, H. Exploring the phase transition challenge by analyzing stability in a 5-D dynamical system linked to (2,1/2)-MSIM, Chinese Journal of Physics, 91 494-504 (2024). https://doi.org/10.1016/j.cjph.2024.08.008
- [24] Akın, H. The classification of disordered phases of mixed spin (2,1/2) Ising model and the chaoticity of the corresponding dynamical system, Chaos, Solitons & Fractals, 167 113086 (2023). https://doi.org/10.1016/j.chaos.2022.113086
- [25] Akın, H. Investigation of thermodynamic properties of mixed-spin (2,1/2) Ising and Blum-Capel models on a Cayley tree, Chaos, Solitons & Fractals, 184 114980, (2024). https://doi.org/10.1016/j.chaos.2024.114980.
- [26] Akın, H., Ulusoy, U. A new approach to studying the thermodynamic properties of the q-state Potts model on a Cayley tree, Chaos, Solitons & Fractals 174 (2023) 113811. https://doi.org/10.1016/j.chaos.2023.113811
- [27] Akın, H., Ulusoy, U. Limiting Gibbs measures of the q-state Potts model with competing interactions, Physica B: Condensed Matter 640 (2022) 413944. https://doi.org/10.1016/j.physb.2022.413944
- [28] Yang, Z. A modified recursion method for one-dimensional Potts model, Communications in Theoretical Physics, 4(2), 157 (1985). https://doi.org/10.1088/0253-6102/4/2/157
- [29] Chang, SC., Shrock, R. Measures of spin ordering in the Potts model with a generalized external magnetic field, Physica A, 613 (2023) 128532. https://doi.org/10.1016/j.physa.2023.128532
- Bish, D. L. & Wang, H-W. Phase transitions in natural zeolites and the importance of P H2O, Philosophical Magazine, 90 (17-18), 2425-2441 (2010). https://doi.org/10.1080/14786430903581288
- [31] Akın, H. Calculation of the free energy of the Ising model on a Cayley tree via the self-similarity method, Axioms 2022, 11, 703. https://doi.org/10.3390/ axioms11120703
- [32] Akın, H. Quantitative behavior of (1,1/2)-MSIM on a Cayley tree, Chinese Journal of Physics, 83, 501-514 (2023). https://doi.org/10.1016/j.cjph.2023.04.014
- [33] Georgii, H.O. Gibbs measures and phase transitions. Walter de Gruyter GmbH & Co. KG, Berlin (2011).
- [34] Wolfram Research, Inc., Mathematica, Version 8.0, Champaign, IL (2010).
- [35] Melnikov, N.B., Paradezhenko, G.V. Reser, B.I. Magnetic susceptibility in metals above the Curie temperature, Journal of Magnetism and Magnetic Materials, 525 (2021) 167559. https://doi.org/10.1016/j.jmmm.2020.167559.
- [36] Khakimov, O. Phase transitions for countable state 1D SOS model with external field J. Stat. Mech. (2023) 053201 https://doi.org/10.1088/1742-5468/accf05