Exact solutions for the moments of the binary collision integral and its relation to the relaxation-time approximation in leading-order anisotropic fluid dynamics

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We compute the moments of the nonlinear binary collision integral in the ultrarelativistic hard-sphere approximation for an arbitrary anisotropic distribution function in the local rest frame. This anisotropic distribution function has an angular asymmetry controlled by the parameter of anisotropy ξ, such that in the limit of a vanishing anisotropy $\lim_{\xi\to 0}\hat{f}_{0\mathbf{k}} = f_{0\mathbf{k}}$, approaches the spherically symmetric local equilibrium distribution function. The corresponding moments of the binary collision integral are obtained in terms of quadratic products of different moments of the anisotropic distribution function and couple to a well defined set of lower-order moments. To illustrate these results we compare the moments of the binary collision integral to the moments of the widely used relaxation-time approximation of Anderson and Witting in case of a spheroidal distribution function. We found that in an expanding system the nonlinear Boltzmann collision term leads to twice slower equilibration than the relaxation-time approximation. Furthermore we also show that including two dynamical moments helps to resolve the ambiguity which additional moment of the Boltzmann equation to choose to close the conservation laws.

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I. INTRODUCTION

The earliest application of fluid dynamics in relativistic particle collisions was formulated by Landau [1] in the early 1950's with the following assumptions [2]. The initial state of the system forms at the instant of the collision when a large number of particles with mean free paths smaller than the dimensions of the system is born in local equilibrium; i.e., local isotropic momentum-space distribution $f_{0\mathbf{k}}$, the Jüttner distribution [3, 4]. This system then expands as a relativistic ideal fluid until the particle interactions become gradually weaker and the mean free path becomes comparable to the dimensions of the system. The final or freeze-out stage happens when the fluid dynamical description is no longer appropriate and the particle nature of the system becomes of primary importance.

Over the past decades this so-called hydrodynamical model has been constantly improved and now relativistic fluid dynamics became an indispensable tool in the description of the space-time evolution of the matter created in relativistic heavy-ion collisions, astrophysical phenomena as well as of the early universe [5–9]. Among the most important recent advancements in relativistic fluid dynamics has been the treatment of irreversible non-equilibrium phenomena that includes the transport properties of matter, such as shear and bulk viscosities, charge diffusivity, thermal and/or electric conductivity, and magnetization. These novel fluid dynamical theories extended beyond the constraints of ideal fluid dynamics and hence the requirement that the fluid is in local thermodynamic equilibrium, but still assume that the deviations from local equilibrium are relatively small, i.e., $|\delta f_{\mathbf{k}}/f_{0\mathbf{k}}| \ll 1$. This means that the non-equilibrium corrections to ideal fluid dynamics may be approximated order by order through the corresponding irreducible moments of $\delta f_{\mathbf{k}}$ representing the dissipative quantities. While the first-order or relativistic Navier-Stokes theory may be acausal and unstable [9–15], the well known and widely used equations of second-order dissipative fluid dynamics of Müller [10], and of Israel and Stewart [11], relax on finite timescales to their corresponding Navier-Stokes values. Henceforth under certain conditions second-order theories avoid the problems of acausality and instability through hyperbolic equations of motion for the dissipative fields.

Even so, in the case of a rapid longitudinal expansion as in the early stages of heavy-ion collisions the local momentum anisotropies can be very large, hence the series expansion of the distribution function near equilibrium is no longer an appropriate approximation. To overcome these limitations of traditional fluid dynamical theories in the late 1980's Barz, Kämpfer *et al.* [16, 17] proposed an energy-momentum tensor which incorporated the momentum anisotropy in terms of a space-like four-vector l^μ, introducing an anisotropic decomposition of the isotropic pressure. As matter of fact, anisotropic matter distributions have been studied in general relativity already in the early 1970's in the seminal work by Bowers and Liang [18]. On the other hand, it was also known that the strongly directed nature of various dynamical processes in heavy-ion collisions such as the hardonization and the freeze-out in hydrodynamical models, also favors an anisotropic distribution function over a locally isotropic distribution function [19, 20].

In the early 2010's the framework of (leading-order) anisotropic fluid-dynamics for ultrarelativistic heavy-ion collisions was rediscovered by two different groups: Florkowski and Ryblewski [21–25] and Martinez and Strickland [26–28].

This fluid dynamical framework extends beyond ideal fluid dynamics, i.e., the simplification that $f_{\mathbf{k}} \equiv f_{0\mathbf{k}}$ is a local isotropic distribution function, and instead it is based on a local anisotropic distribution function $f_{\mathbf{k}} \equiv \hat{f}_{0\mathbf{k}}$ which incorporates an anisotropy parameter ξ and a new spacelike four-vector l^μ in the direction of the momentum-space anisotropy. Therefore the corresponding moment equations derived from the relativistic Boltzmann equation for such an anisotropic distribution function form the framework of leading-order anisotropic fluid dynamics. This means that leading-order anisotropic fluid dynamics describes a specific class of dissipative fluids with one additional (when compared to ideal fluids) independent variable specified by $\hat{f}_{0\mathbf{k}}$. This leading-order framework can also be improved further by generalizing the expansion of the distribution function as $f_{\mathbf{k}} \equiv \hat{f}_{0\mathbf{k}} + \delta\hat{f}_{\mathbf{k}}$, and hence similarly to the method formulated in second-order dissipative fluid dynamics it leads to a more complete theory of anisotropic dissipative fluid dynamics [29–31]. Furthermore, taking into account the coupling to the external electromagnetic field naturally extends the equations of leading-order anisotropic fluid dynamics to electrically conducting fluids and leads to a novel theory of dissipative and resistive anisotropic magnetohydrodynamics [32].

In the derivation of fluid dynamical theories from the relativistic Boltzmann equation, the moments of the collision term reveal the time scales over which the matter approaches local equilibrium. However, unlike in traditional dissipative fluid dynamical theories, so far leading-order anisotropic fluid dynamical theories employed the moments of the simplified relaxation-time approximation (RTA) instead of the moments the binary the collision integral. In this paper we go beyond this approximation and present a simple and at the same time general projection method to calculate the moments of the nonlinear binary collision term. As expected the corresponding moments of the binary collision integral are found to be quadratic products of anisotropic thermodynamic integrals which can be computed exactly and in some cases analytically in terms of various moments of the underlying anisotropic distribution function. Therefore these new results also extend and generalize the well known Bobylev-Krook-Wu (BKW) [33–35] model based on the homogeneous and isotropic solutions of the non-relativistic Boltzmann equation to anisotropic distribution functions. The relativistic BKW model, the Bazow-Denicol-Heinz-Martinez-Noronha (BDHMN) [36, 37] also obtained exact solutions to the relativistic Boltzmann equation through an infinite set of nonlinear ordinary differential equations for the moments of an isotropic single-particle distribution function and the corresponding moments of the nonlinear binary collision integral computed for an isotropic distribution function.

The equations of leading-order fluid dynamics derived from the relativistic Boltzmann equation correspond to an infinite hierarchy of coupled moment equations, where the moments of the binary collision term now couple to a well defined set of lower-order moments in the hierarchy. These nonlinear couplings between moments of different order represent the nonlinear dependence on the distribution function of the binary collision integral. In order to be practical one must truncate and close the hierarchy of moment equations of leading-order anisotropic fluid dynamics. But even after a well defined truncation there remains an ambiguity, which higher-order moment to use for closure. In this paper we also go beyond a single dynamical moment to close the conservation equations [38, 39], and show that including two dynamical moments helps to resolve this ambiguity and improves the solutions to the fluid dynamical equations and the approximation to the anisotropic distribution function.

As an important application in heavy-ion collision we have used the well known spheroidal distribution function of Romatschke and Strickland (RS) [40] to explicitly study the moments of the binary collision integral and the fluid dynamical evolution of the system. We found that different anisotropic moments approach equilibrium on different time-scales, alike in second-order fluid dynamical theories. Furthermore, the widely used relativistic relaxation-time approximation of Anderson and Witting (AW) [41] for the collision integral is also included for comparison. We found that the collision term in the RTA drives the system about two times faster towards equilibrium than the binary collision integral irrespective of the choice of the dynamical moment(s). Finally, it is shown that the re-scaling of the relaxation time of the RTA, based on the asymptotic values of the ratio of the binary collision integral and the RTA collision integral, captures reasonably well the characteristics of the non-linear binary collision integral.

This paper is organized as follows. A brief introduction to leading-order anisotropic fluid dynamics is given in Sec. II. In Sec. III we define the moments of the binary collision integral in the general case of an arbitrary anisotropic distribution function. Then in Sec.III A and Sec.III B we compute the moments of the corresponding loss and gain terms for an arbitrary anisotropic distribution function. These results are then applied in Sec. IV A to the spheroidal distribution function introduced by Romatschke and Strickland. The evolution of matter in $(0 + 1)$-dimensional boost invariant expansion is studied using the moments of the binary collision integral as well as using its well known relativistic approximation, the relaxation-time approximation of Anderson and Witting, in Sec. IV B and in Sec. IV C. The differences between the collision terms are analyzed and the re-scaling of the relaxation times to reduce these differences is presented in Sec. IV D. We conclude this work in Sec. V. Technical details and additional computations are relegated to the appendixes.

A. Notation and definitions

In this paper we work in natural units by setting $\hbar = c = k_B = 1$, and in flat space-time with metric tensor $g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. The contravariant fluid four-velocity $u^\mu(t,\mathbf{x}) = \gamma(1,\mathbf{v})$ with $\gamma = (1-\mathbf{v}^2)^{-1/2}$, has timelike normalization $u^\mu u_\mu \equiv c^2 = 1$. The unit four-vector in the direction of the momentum anisotropy $l^\mu(t,\mathbf{x})$ is chosen orthogonal to the fluid four-velocity; i.e., $u^\mu l_\mu = 0$, and has spacelike normalization $l^\mu l_\mu \equiv -l^2 = -1$. For example in the context of heavy-ion collision choosing l^μ in the direction of the beam axis sets $l^\mu = \gamma_z(v_z,0,0,1)$ with $\gamma_z = (1-v_z^2)^{-1/2}$. In the local rest (LR) frame of the fluid $u^\mu_{LR} = (1,0,0,0)$ and $l^\mu_{LR} = (0,0,0,1)$.

The symmetric rank-two projection operator orthogonal to u^μ; i.e. $\Delta^{\mu\nu}u_\nu = 0$, is defined as in Refs. [6, 42, 43]

$$\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu\,, \tag{1}$$

while the symmetric rank-two projection operator that is orthogonal to both u^μ and l^μ; i.e., $\Xi^{\mu\nu}u_\nu = \Xi^{\mu\nu}l_\nu = 0$, is defined as in Refs. [6, 44–46]

$$\Xi^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu + l^\mu l^\nu = \Delta^{\mu\nu} + l^\mu l^\nu\,. \tag{2}$$

The particle four-momentum $k^\mu = \left(k^0,\mathbf{k}\right)$ is normalized to the rest mass squared of the particle, $k^\mu k_\mu = m_0^2$. The energy $E_{\mathbf{k}u}$, and the momentum in the direction of the anisotropy $E_{\mathbf{k}l}$, are

$$E_{\mathbf{k}u} \equiv k^\mu u_\mu\,, \quad E_{\mathbf{k}l} \equiv -k^\mu l_\mu\,. \tag{3}$$

The four-momentum orthogonal to the four-velocity u^μ is denoted by $k^{\langle\mu\rangle} \equiv \Delta^{\mu\nu}k_\nu$, while the four-momentum orthogonal to both u^μ and l^μ is denoted by $k^{\{\mu\}} \equiv \Xi^{\mu\nu}k_\nu$, such that $k^{\langle\mu\rangle} = E_{\mathbf{k}l}l^\mu + k^{\{\mu\}}$ and thus

$$k^\mu \equiv E_{\mathbf{k}u}u^\mu + k^{\langle\mu\rangle} = E_{\mathbf{k}u}u^\mu + E_{\mathbf{k}l}l^\mu + k^{\{\mu\}}\,. \tag{4}$$

The total momentum in binary collision is denoted by

$$P^\mu_T \equiv k^\mu + k'^\mu = p^\mu + p'^\mu\,, \tag{5}$$

and it is normalized to the square of the center of mass energy, $s \equiv P^\mu_T P_{T,\mu}$. The symmetric rank-two projection operator orthogonal to the total momentum; i.e., $\Delta^{\mu\nu}_T P_{T,\nu} = 0$, is defined similarly to Eq. (1) as

$$\Delta^{\mu\nu}_T \equiv g^{\mu\nu} - \frac{P^\mu_T P^\nu_T}{s}\,, \tag{6}$$

and therefore the particle four-momentum can be decomposed as $k^\mu = P^\mu_T(P^\nu_T p_\nu)/s + \Delta^{\mu\nu}_T k_\nu$.

The local distribution function of an anisotropic state as a function of $\hat\alpha$, $\hat\beta_u$, and $\hat\beta_l$, as well as of $E_{\mathbf{k}u}$, and $E_{\mathbf{k}l}$, is denoted by $\hat f_{0\mathbf{k}}\left(\hat\alpha, \hat\beta_u E_{\mathbf{k}u}, \hat\beta_l E_{\mathbf{k}l}\right)$. Such anisotropic distribution functions distinguish particle momenta parallel and perpendicular to the direction of the anisotropy, and hence are commonly interpreted by two different "temperatures". Here $\hat\beta_l^{-1}$ is the inverse temperature in the direction of the anisotropy, and $\hat\beta_u^{-1}$ is in the direction perpendicular to the anisotropy, while $\hat\alpha$ is related to the chemical potential. In the limit of vanishing anisotropy parameter $\hat\beta_l \to 0$, the anisotropic distribution converges to the distribution function in local equilibrium,

$$\lim_{\hat\beta_l\to 0} \hat f_{0\mathbf{k}}\left(\hat\alpha, \hat\beta_u E_{\mathbf{k}u}, \hat\beta_l E_{\mathbf{k}l}\right) = f_{0\mathbf{k}}\left(\hat\alpha, \hat\beta_u E_{\mathbf{k}u}\right)\,, \tag{7}$$

where the Jüttner distribution [3, 4] defines local thermodynamic equilibrium,

$$f_{0\mathbf{k}}(\alpha, \beta E_{\mathbf{k}u}) \equiv \left[\exp(\beta E_{\mathbf{k}u} - \alpha) + a\right]^{-1}\,. \tag{8}$$

Here $\alpha = \mu\beta$, such that μ is the chemical potential and $\beta = 1/T$ is the inverse temperature, while $a = \pm 1$ for fermions/bosons and $a = 0$ for classical Boltzmann particles. The equilibrium distribution function also defines [47] the inverse temperature four-vector $\beta^\mu \equiv \beta u^\mu = u^\mu/T$ normalized to $\beta^\mu\beta_\mu \equiv \beta^2 = 1/T^2$ and so $\beta^\mu k_\mu \equiv \beta E_{\mathbf{k}u}$.

Note that in principle there are infinitely many different distribution functions which are anisotropic in momentum-space according to Eq. (7). A straightforward approach is to stretch or squeeze an isotropic distribution function in some direction or along a particular axis. Thereby along the direction of the momentum anisotropy we may introduce a "longitudinal temperature", T_l, and consequently the inverse temperature four-vector now is $\beta^\mu \equiv \hat\beta_u u^\mu + \hat\beta_l l^\mu$ and it is normalized to $\beta^\mu\beta_\mu = 1/T_u^2 - 1/T_l^2 > 0$. Here $\hat\beta_u \equiv \beta$ is the reciprocal of the Jüttner temperature, and

$\hat{\beta}_l \equiv 1/T_l = \beta\xi$ is the inverse temperature along the anisotropy. The temperature ratio, $\xi = \hat{\beta}_l/\hat{\beta}_u = T_u/T_l$, defines the so-called anisotropy parameter that quantifies the strength of the anisotropy. This straightforward anisotropic decomposition of β^μ defines an anisotropic Jüttner distribution function, $\hat{f}_{0\mathbf{k}} = \left[\exp\left(\hat{\beta}_u E_{\mathbf{k}u} - \hat{\beta}_l E_{\mathbf{k}l} - \hat{\alpha}\right) + a\right]^{-1}$, which in the limit of $\hat{\beta}_l \to 0$ leads to the local equilibrium distribution function (8).

Another simple anisotropic distribution function can be constructed by modifying the argument of the equilibrium distribution function $\beta^\mu k_\mu \to \sqrt{k^\mu \Omega_{\mu\nu} k^\mu}$, where $\Omega^{\mu\nu} = u^\mu u^\nu + \xi l^\mu l^\nu$, hence $\hat{\beta}_u = \beta$ and $\hat{\beta}_l = \beta\sqrt{\xi}$. This then yields the well known spheroidal or RS distribution function [40], which will be used and discussed in this paper. Other anisotropic distributions functions are well known in plasma physics, where the momentum anisotropy is induced by the presence of an external magnetic field, e.g., the bi-Maxwellian distribution function, etc. Note that the main results in this paper, i.e., Sec. III, are valid for arbitrary anisotropic distribution function satisfying Eq. (7), and we only explicitly use the RS distribution function starting from Sec. IV.

The moments of tensor rank n of the anisotropic distribution function $\hat{f}_{0\mathbf{k}}$ are defined as

$$\begin{aligned}\hat{\mathcal{I}}^{\mu_1\cdots\mu_n}_{ij} &\equiv \int dK E^i_{\mathbf{k}u} E^j_{\mathbf{k}l} k^{\mu_1}\cdots k^{\mu_n} \hat{f}_{0\mathbf{k}} \\ &= \sum_{q=0}^{\lfloor n/2 \rfloor} \sum_{r=0}^{n-2q} (-1)^q b_{nrq} \hat{I}_{i+j+n,j+r,q} \Xi^{(\mu_1\mu_2}\cdots\Xi^{\mu_{2q-1}\mu_{2q}} l^{\mu_{2q+1}}\cdots l^{\mu_{2q+r}} u^{\mu_{2q+r+1}}\cdots u^{\mu_n)}\,, \end{aligned} \tag{9}$$

where $i \geq -1$ and $j \geq 0$ define the power of $E_{\mathbf{k}u}$ and $E_{\mathbf{k}l}$, while $dK = g d^3\mathbf{k}/\left[(2\pi)^3 k^0\right]$ is the invariant measure and g is the degeneracy of the state. Here $\lfloor n/2 \rfloor$ denotes the greatest integer less than equal to $n/2$; i.e., the floor function, and the round brackets around indices denote the symmetrization of indices, i.e., $T^{(\mu_1\cdots\mu_n)} = \frac{1}{n!}\sum T^{\mu_1\mu_2\cdots\mu_n}$.

The anisotropic thermodynamic integrals $\hat{I}_{nrq}$ are defined as in Refs. [31, 38]

$$\hat{I}_{nrq}\left(\hat{\alpha}, \hat{\beta}_u, \hat{\beta}_l\right) \equiv \frac{(-1)^q}{(2q)!!} \int dK E^{n-r-2q}_{\mathbf{k}u} E^r_{\mathbf{k}l} \left(\Xi^{\mu\nu} k_\mu k_\nu\right)^q \hat{f}_{0\mathbf{k}}\,, \tag{10}$$

where $(2q)!! = 2^q q!$ is the double factorial of even numbers, while the b_{nrq} coefficient counts the distinct terms of the symmetrized product $\Xi^{(\mu_1\mu_2}\cdots l^{\mu_{2q+r}}\cdots u^{\mu_n)}$,

$$b_{nrq} \equiv \frac{n!}{2^q q! r! (n-r-2q)!} = \frac{n!\,(2q-1)!!}{(2q)! r! (n-r-2q)!}\,, \tag{11}$$

where double factorial of odd numbers is $(2q-1)!! = (2q)!/\left(2^q q!\right)$.

The equilibrium moments of tensor rank n and of power r in energy $E_{\mathbf{k}u}$ are defined as

$$\begin{aligned}\mathcal{I}^{\mu_1\cdots\mu_n}_{i} &\equiv \int dK E^i_{\mathbf{k}u} k^{\mu_1}\cdots k^{\mu_n} f_{0\mathbf{k}} \\ &= \sum_{q=0}^{\lfloor n/2 \rfloor} (-1)^q b_{nq} I_{i+n,q} \Delta^{(\mu_1\mu_2}\cdots\Delta^{\mu_{2q-1}\mu_{2q}} u^{\mu_{2q+1}}\cdots u^{\mu_n)}\,, \end{aligned} \tag{12}$$

where $i \geq -1$ and the equilibrium thermodynamic integrals are defined as

$$I_{nq}(\alpha, \beta) \equiv \frac{(-1)^q}{(2q+1)!!} \int dK E^{n-2q}_{\mathbf{k}u} \left(\Delta^{\mu\nu} k_\mu k_\nu\right)^q f_{0\mathbf{k}}\,, \tag{13}$$

while the number of distinct terms in the symmetrized tensor product (12) is given by Eq. (11) for $r = 0$,

$$b_{nq} \equiv b_{n0q} = \frac{n!}{2^q q! (n-2q)!}\,. \tag{14}$$

The anisotropic thermodynamic integrals in equilibrium $I_{nrq}(\alpha, \beta)$ are defined using Eq. (10) with $\hat{f}_{0\mathbf{k}} \to f_{0\mathbf{k}}$ and hence from Eq. (7) it also follows that $I_{nrq} = \lim_{\hat{\beta}_l \to 0} \hat{I}_{nrq}$. Furthermore, replacing Eq. (2) into Eq. (13) we find the following relation between the equilibrium thermodynamic integrals, $I_{nq} = \frac{1}{(2q+1)!!}\sum_{r=0}^{q} \frac{q!}{r!} 2^{q-r} I_{n,2r,q-r}$.

Note that the local-equilibrium distribution function is isotropic in momentum space, thus the orthogonal projections of the equilibrium moments vanish for any tensor rank larger than one; i.e., for $n \geq 1$ we have $\mathcal{I}^{\langle\mu_1\rangle\cdots\langle\mu_n\rangle}_r = 0$. Similarly, the orthogonal projections with respect to both u^μ and l^μ of the anisotropic moments also vanish $\hat{\mathcal{I}}^{\{\mu_1\}\cdots\{\mu_n\}}_{ij} = 0$.

II. THE CONSERVATION EQUATIONS OF LEADING-ORDER ANISOTROPIC FLUID DYNAMICS

The relativistic Boltzmann equation for the space-time evolution of the single-particle distribution function of classical indistinguishable particles corresponding to $f_{\mathbf{k}} \equiv \hat{f}_{0\mathbf{k}}\left(\hat{\alpha}, \hat{\beta}_u E_{\mathbf{k}u}, \hat{\beta}_l E_{\mathbf{k}l}\right)$ reads,

$$k^\mu \partial_\mu \hat{f}_{0\mathbf{k}} = C\left[\hat{f}_0\right] , \tag{15}$$

where $C\left[\hat{f}_0\right]$ is the Boltzmann collision term specified in Sec. III.

Relativistic fluid dynamics is an effective theory for the long-wavelength and low-frequency dynamics of macroscopic systems. The fluid-dynamical equations of motion are derived from the Boltzmann equation by integrating over the microscopic equation (15) with respect to the particle momentum. This leads to an infinite hierarchy of moment equations being equivalent to the Boltzmann equation. The lower-order moments of the distribution function, i.e., lower power of $E_{\mathbf{k}u}$ and $E_{\mathbf{k}l}$, describe lower frequency or longer wavelength dynamics while higher-order moments capture the higher frequency or shorter wavelength dynamics. For an anisotropic distribution function we obtain,

$$\partial_\nu \hat{\mathcal{I}}_{00}^{\mu_1\cdots\mu_n\nu} = \hat{C}_{00}^{\mu_1\cdots\mu_n} , \tag{16}$$

where the rank n moment of the collision integral is defined as

$$\hat{C}_{ij}^{\mu_1\cdots\mu_n} \equiv \int dK E^i_{\mathbf{k}u} E^j_{\mathbf{k}l} k^{\mu_1} \cdots k^{\mu_n} C\left[\hat{f}_0\right] . \tag{17}$$

The general equations of motion which follow from Eq. (16) in terms of scalar moments $\hat{\mathcal{I}}_{ij}$ were derived in Refs. [32, 38] and for reasons of completeness are presented in Appendix A.

The particle four-current conservation, and the energy-momentum conservation follow from Eq. (16),

$$\partial_\mu \hat{N}^\mu \equiv \partial_\mu \hat{\mathcal{I}}^\mu_{00} = \hat{C}_{00} = 0 , \tag{18}$$

$$\partial_\mu \hat{T}^{\mu\nu} \equiv \partial_\mu \hat{\mathcal{I}}^{\mu\nu}_{00} = \hat{C}^\nu_{00} = 0 , \tag{19}$$

where the conservation of particle number or charge, the conservation of energy, the conservation of momentum in the direction of the momentum anisotropy as well as transverse to it leads to,

$$\hat{C}_{00} = 0 , \quad \hat{C}_{10} = 0 , \quad \hat{C}_{01} = 0 , \quad \hat{C}_{00}^{\{\mu\}} = 0 . \tag{20}$$

The tensor decompositions (9) of the primary or the lowest order moments of the anisotropic distribution function in leading-order anisotropic fluid dynamics give

$$\hat{N}^\mu \equiv \hat{\mathcal{I}}^\mu_{00} = \hat{n} u^\mu + \hat{n}_l l^\mu , \tag{21}$$

$$\hat{T}^{\mu\nu} \equiv \hat{\mathcal{I}}^{\mu\nu}_{00} = \hat{e} u^\mu u^\nu + 2\hat{M} u^{(\mu} l^{\nu)} + \hat{P}_l l^\mu l^\nu - \hat{P}_\perp \Xi^{\mu\nu} , \tag{22}$$

where the particle density and energy density are $\hat{n}$ and $\hat{e}$, respectively. The particle diffusion current along the l^μ direction is denoted by $\hat{n}_l$, while the energy-momentum diffusion current is denoted by $\hat{M}$. The pressure in the direction of the anisotropy is $\hat{P}_l$, while the pressure in the direction transverse to it is denoted by $\hat{P}_\perp$. These quantities are expressed either in terms of different tensor projections or equivalently through Eq. (9) as

$$\hat{n} \equiv \hat{N}^\mu u_\mu = \hat{\mathcal{I}}_{10} = \hat{I}_{100} , \tag{23}$$

$$\hat{n}_l \equiv -\hat{N}^\mu l_\mu = \hat{\mathcal{I}}_{01} = \hat{I}_{110} , \tag{24}$$

$$\hat{e} \equiv \hat{T}^{\mu\nu} u_\mu u_\nu = \hat{\mathcal{I}}_{20} = \hat{I}_{200} , \tag{25}$$

$$\hat{M} \equiv -\hat{T}^{\mu\nu} u_\mu l_\nu = \hat{\mathcal{I}}_{11} = \hat{I}_{210} , \tag{26}$$

$$\hat{P}_l \equiv \hat{T}^{\mu\nu} l_\mu l_\nu = \hat{\mathcal{I}}_{02} = \hat{I}_{220} , \tag{27}$$

$$\hat{P}_\perp \equiv -\frac{1}{2} \hat{T}^{\mu\nu} \Xi_{\mu\nu} = -\frac{1}{2}\left(m_0^2 \hat{\mathcal{I}}_{00} - \hat{\mathcal{I}}_{20} + \hat{\mathcal{I}}_{02}\right) = \hat{I}_{201} , \tag{28}$$

while the isotropic pressure is $\hat{P} \equiv -\frac{1}{3}\hat{T}^{\mu\nu}\Delta_{\mu\nu} = \frac{1}{3}\left(\hat{P}_l + 2\hat{P}_\perp\right)$. The fluid dynamical four-velocity and the local rest frame is usually chosen according to the definition of Landau and Lifshitz [48]; i.e., $u^\mu = \hat{T}^{\mu\nu} u_\nu / (u_\alpha \hat{T}^{\alpha\beta} u_\beta)$ which leads to a vanishing energy diffusion current; i.e., $\hat{M} = 0$.

The particle four-current and the energy-momentum tensor in local thermodynamic equilibrium are

$$N_0^\mu \equiv \mathcal{I}_0^\mu = n_0 u^\mu \,, \tag{29}$$
$$T_0^{\mu\nu} \equiv \mathcal{I}_0^{\mu\nu} = e_0 u^\mu u^\nu - P_0 \Delta^{\mu\nu} \,, \tag{30}$$

where the particle density, the energy density, and the pressure in local thermal equilibrium are obtained either by different projections of the conserved quantities or through Eq. (12) leading to,

$$n_0 \equiv N_0^\mu u_\mu = \mathcal{I}_1 = I_{10} \,, \tag{31}$$
$$e_0 \equiv T_0^{\mu\nu} u_\mu u_\nu = \mathcal{I}_2 = I_{20} \,, \tag{32}$$
$$P_0 \equiv -\frac{1}{3} T_0^{\mu\nu} \Delta_{\mu\nu} = -\frac{1}{3}\left(m_0^2 \mathcal{I}_0 - \mathcal{I}_2\right) = I_{21} \,. \tag{33}$$

The pressure $P_0 \equiv P_0\,(e_0, n_0)$ is specified by an equation of state (EoS) and hence the temperature $T(e_0, n_0)$ and chemical potential $\mu(e_0, n_0)$ are also determined. In an arbitrary anisotropic state, $\hat{\alpha}$, and $\hat{\beta}_u$, are obtained from the Landau matching conditions, $\left(\hat{N}^\mu - N_0^\mu\right) u_\mu \equiv \hat{n} - n_0 = 0$, and $\left(\hat{T}^{\mu\nu} - T_0^{\mu\nu}\right) u_\mu u_\nu \equiv \hat{e} - e_0 = 0$ leading to $\mu(\hat{\alpha}, \hat{\beta}_u, \hat{\beta}_l)$ and $T(\hat{\alpha}, \hat{\beta}_u, \hat{\beta}_l)$, while the new parameter $\hat{\beta}_l$ must be determined from an additional equation of motion. Note that the conservation equations, $\partial_\mu \hat{N}^\mu = 0$, and $\partial_\mu \hat{T}^{\mu\nu} = 0$, provide only five constraints for the six independent variables, and hence we need an additional equation of motion to determine the remaining variable. Naturally, this can be provided by choosing other equation(s) from the infinite hierarchy of moment equations of the Boltzmann equation (16). This is studied and discussed in detail in Sec. IV B and Sec. IV C.

III. THE MOMENTS OF THE BINARY COLLISION INTEGRAL

The collision term of the relativistic Boltzmann equation (15) in case of binary elastic collisions is defined as [6, 42, 43]

$$C\left[\hat{f}_0\right] \equiv \frac{1}{2} \int dK' dP dP' W_{\mathbf{kk}' \to \mathbf{pp}'} \left(\hat{f}_{0\mathbf{p}} \hat{f}_{0\mathbf{p}'} - \hat{f}_{0\mathbf{k}} \hat{f}_{0\mathbf{k}'}\right) \,, \tag{34}$$

where the 1/2 factor removes the double counting due to the indistinguishability of identical particles. The invariant transition rate satisfies detailed balance, $W_{\mathbf{kk}' \to \mathbf{pp}'} = W_{\mathbf{pp}' \to \mathbf{kk}'}$, and it is also symmetric with respect to the exchange of particle four-momentum in binary collisions, $W_{\mathbf{kk}' \to \mathbf{pp}'} = W_{\mathbf{k}'\mathbf{k} \to \mathbf{pp}'} = W_{\mathbf{kk}' \to \mathbf{p}'\mathbf{p}}$. The invariant transition rate for elastic binary collisions is defined as

$$g^2 W_{\mathbf{kk}' \to \mathbf{pp}'} \equiv s(2\pi)^6 \frac{d\sigma(s,\Omega)}{d\Omega} \delta(k^\mu + k'^\mu - p^\mu - p'^\mu) \,, \tag{35}$$

where the conservation of the energy and momentum in binary collision is enforced by the Dirac delta function. The transition rate depends on the total center-of-momentum (CM) energy squared $s \equiv (k^\mu + k'^\mu)^2 = (p^\mu + p'^\mu)^2$, while the transport cross section integrated over the solid angle Ω is defined as

$$\sigma_T(s) \equiv \frac{2\pi}{2} \int_0^\pi d\theta_s \sin\theta_s \frac{d\sigma(s,\Omega)}{d\Omega} \,, \tag{36}$$

where $\theta_s \equiv \arccos \frac{(k^\mu - k'^\mu)(p^\mu - p'^\mu)}{(k^\mu - k'^\mu)^2}$ is the scattering angle. From now on the so-called hard-sphere approximation will be used assuming that the transport cross section is isotropic and independent of the total CM energy,

$$\sigma_T \equiv 2\pi \frac{d\sigma(s,\Omega)}{d\Omega} = \frac{1}{\hat{n}\lambda_{\mathrm{mfp}}} \,, \tag{37}$$

where $\hat{n}$ is the particle density and λ_{mfp} is the mean free path between collisions.

The moments (17) of tensor rank n of the binary collision integral 34) naturally separates into gain and loss parts,

$$\hat{C}_{ij}^{\mu_1 \cdots \mu_n} \equiv \int dK E_{\mathbf{k}u}^i E_{\mathbf{k}l}^j k^{\mu_1} \cdots k^{\mu_n} C\left[\hat{f}_0\right] = \hat{G}_{ij}^{\mu_1 \cdots \mu_n} - \hat{L}_{ij}^{\mu_1 \cdots \mu_n} \,, \tag{38}$$

where using Eq. (34) and the symmetries of the transition rate the corresponding gain and loss moments of tensor rank n are defined as

$$\hat{G}_{ij}^{\mu_1\cdots\mu_n} \equiv \frac{1}{2}\int_{KK'}\int_{PP'} W_{\mathbf{kk}'\to\mathbf{pp}'}\hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}E^i_{\mathbf{p}u}E^j_{\mathbf{p}l}p^{\mu_1}\cdots p^{\mu_n}\,, \tag{39}$$

$$\hat{L}_{ij}^{\mu_1\cdots\mu_n} \equiv \frac{1}{2}\int_{KK'}\int_{PP'} W_{\mathbf{kk}'\to\mathbf{pp}'}\hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}E^i_{\mathbf{k}u}E^j_{\mathbf{k}l}k^{\mu_1}\cdots k^{\mu_n}\,, \tag{40}$$

where we introduced the shorthand notation, $\int dKdK' = \int_{KK'}$ and $\int dPdP' = \int_{PP'}$.

In order to evaluate these 12-dimensional integrals, first we need to compute the following simpler 6-dimensional integrals over P and P',

$$\begin{aligned}\mathcal{P}_{ij}^{\mu_1\cdots\mu_n} &\equiv \frac{1}{2}\int_{PP'} W_{\mathbf{kk'}\to\mathbf{pp'}}E^i_{\mathbf{p}u}E^j_{\mathbf{p}l}p^{\mu_1}\cdots p^{\mu_n} \\ &= (-1)^j\, u_{\mu_1}\cdots u_{\mu_i} l_{\mu_{i+1}}\cdots l_{\mu_{i+j}}\Theta^{\mu_1\cdots\mu_{n+i+j}}\,,\end{aligned} \tag{41}$$

which in turn require the calculation of the following unprojected auxiliary integrals, see also Refs. [49, 50],

$$\begin{aligned}\Theta^{\mu_1\cdots\mu_n} &\equiv \frac{1}{2}\int_{PP'} W_{\mathbf{kk'}\to\mathbf{pp'}}p^{\mu_1}\cdots p^{\mu_n} \\ &= \sum_{q=0}^{\lfloor n/2\rfloor}(-1)^q\, b_{nq}\mathcal{B}_{nq}\Delta_T^{(\mu_1\mu_2}\cdots\Delta_T^{\mu_{2q-1}\mu_{2q}}P_T^{\mu_{2q+1}}\cdots P_T^{\mu_n)}\,.\end{aligned} \tag{42}$$

Here the b_{nq} coefficient was defined in Eq. (14), while the $\mathcal{B}_{nq}$ coefficient is defined similarly to Eq. (13), see the Appendix B for more details, and it reads,

$$\begin{aligned}\mathcal{B}_{nq} &\equiv \frac{(-1)^q}{2\,(2q+1)!!}\int_{PP'} W_{\mathbf{kk'}\to\mathbf{pp'}}\left(\frac{P_T^{\mu}p_\mu}{s}\right)^{n-2q}\left(\Delta_T^{\mu\nu}p_\mu p_\nu\right)^q \\ &= \frac{\sigma_T}{2^{n+1}\,(2q+1)!!}\sqrt{s}\left(\sqrt{s-4m_0^2}\right)^{(2q+1)}\,,\end{aligned} \tag{43}$$

which in the ultrarelativistic limit when $m_0\to 0$ and hence when $s\equiv 2k^\mu k'_\mu$, leads to

$$\lim_{m_0\to 0}\mathcal{B}_{nq} = \frac{\sigma_T}{2^{n+1}\,(2q+1)!!}\left(\sqrt{s}\right)^{2+2q}\,. \tag{44}$$

Note that the $\mathcal{P}_{ij}^{\mu_1\cdots\mu_n}$ integrals are built contracting the $\Theta^{\mu_1\cdots\mu_n}$ tensors by the four-vectors u^μ and/or l^μ. Therefore integrals containing negative powers of energy, e.g., $E^{-1}_{\mathbf{p}u}$, are not possible to obtain through these tensor projections.

A. The loss terms

Using the results of the previous section, the loss term from Eq. (40) is easily computed in the massless limit, noting that for $i=j=0$ the P and P' integral from Eq. (41) leads to $\mathcal{P}_{00}\equiv\Theta = \sigma_T s/2 \equiv \sigma_T k^\lambda k'_\lambda$, and hence the remaining K and K' integrals evaluate to

$$\begin{aligned}\hat{L}_{ij}^{\mu_1\cdots\mu_n} &\equiv \int_{KK'}\hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}E^i_{\mathbf{k}u}E^j_{\mathbf{k}l}k^{\mu_1}\cdots k^{\mu_n}\mathcal{P}_{00} \\ &= \sigma_T\int dK\,\hat{f}_{0\mathbf{k}}E^i_{\mathbf{k}u}E^j_{\mathbf{k}l}k^{\mu_1}\cdots k^{\mu_n}k^\lambda\int dK'\,\hat{f}_{0\mathbf{k}'}k'_\lambda \\ &= \sigma_T\hat{\mathcal{I}}_{ij}^{\mu_1\cdots\mu_n\lambda}\hat{\mathcal{I}}_{00,\lambda}\,,\end{aligned} \tag{45}$$

where in the last step we used the definition of the moments of the anisotropic distribution function from Eq. (9). Now using the decomposition of the particle four-current from Eq. (21) we obtain, see the Appendix C for the details of the derivation,

$$\begin{aligned}\hat{L}_{ij}^{\mu_1\cdots\mu_n} = \sigma_T&\sum_{q=0}^{\lfloor n/2\rfloor}\sum_{r=0}^{n-2q}(-1)^q\, b_{nrq}\left(\hat{I}_{i+j+n+1,j+r,q}\hat{I}_{100}-\hat{I}_{i+j+n+1,j+r+1,q}\hat{I}_{110}\right) \\ &\times\Xi^{(\mu_1\mu_2}\cdots\Xi^{\mu_{2q-1}\mu_{2q}}l^{\mu_{2q+1}}\cdots l^{\mu_{2q+r}}u^{\mu_{2q+r+1}}\cdots u^{\mu_n)}\,.\end{aligned} \tag{46}$$

Note that in leading-order anisotropic fluid dynamics we only need to compute the scalar and vector moments of the collision integral. However, we can easily show that the orthogonal projection of the rank-one moment loss term vanishes equivalently $\Xi^{\mu}_{\alpha}\hat{L}^{\alpha}_{ij} \equiv \hat{L}^{\{\mu\}}_{ij} = 0$. Recalling that the vector moment of the loss term, $\hat{L}^{\alpha}_{ij} = (\cdots)u^{\alpha} + (\cdots)l^{\alpha}$, is proportional to both u^{α} and l^{α}, consequently when contracted by the elementary projection operator Ξ^{μ}_{α} which is orthogonal to both u^{α} and l^{α}, leads to zero. Furthermore, the rank-one gain term $\hat{G}^{\alpha}_{ij}$ shares the same tensor structure as the loss term, therefore in summary, the orthogonal projection of the rank-one moment of the collision integral vanishes in leading-order anisotropic fluid dynamics; i.e., $\Xi^{\mu}_{\alpha}\hat{C}^{\alpha}_{ij} \equiv \hat{C}^{\{\mu\}}_{ij} = 0$.

Thus for the purposes of this paper dealing with leading-order anisotropic fluid dynamics we only need to compute the scalar moments of the collision integral, i.e., when $n = r = q = 0$, and hence all scalar loss terms read,

$$\begin{aligned}\hat{L}_{ij} &\equiv \frac{1}{2}\int_{KK'}\int_{PP'} W_{\mathbf{kk'}\to\mathbf{pp'}}\hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k'}}E^{i}_{\mathbf{k}u}E^{j}_{\mathbf{k}l} \\ &= \sigma_T\left(\hat{I}_{i+j+1,j,0}\hat{n} - \hat{I}_{i+j+1,j+1,0}\hat{n}_l\right) . \end{aligned} \tag{47}$$

We are only interested in the first few entries of the collision matrix which we will compute here explicitly. For the values $i = 0, 1, 2, 3, 4$ and $j = 0$ the loss term leads to

$$\hat{L}_{00} = \sigma_T\left(\hat{n}^2 - \hat{n}_l^2\right) , \tag{48}$$
$$\hat{L}_{10} = \sigma_T\left(\hat{e}\hat{n} - \hat{M}\hat{n}_l\right) , \tag{49}$$
$$\hat{L}_{20} = \sigma_T\left(\hat{I}_{300}\hat{n} - \hat{I}_{310}\hat{n}_l\right) , \tag{50}$$
$$\hat{L}_{30} = \sigma_T\left(\hat{I}_{400}\hat{n} - \hat{I}_{410}\hat{n}_l\right) , \tag{51}$$
$$\hat{L}_{40} = \sigma_T\left(\hat{I}_{500}\hat{n} - \hat{I}_{510}\hat{n}_l\right) . \tag{52}$$

Similarly for $i = 0, 1, 2, 3$ and $j = 1$ we have

$$\hat{L}_{01} = \sigma_T\left(\hat{M}\hat{n} - \hat{P}_l\hat{n}_l\right) , \tag{53}$$
$$\hat{L}_{11} = \sigma_T\left(\hat{I}_{310}\hat{n} - \hat{I}_{320}\hat{n}_l\right) , \tag{54}$$
$$\hat{L}_{21} = \sigma_T\left(\hat{I}_{410}\hat{n} - \hat{I}_{420}\hat{n}_l\right) , \tag{55}$$
$$\hat{L}_{31} = \sigma_T\left(\hat{I}_{510}\hat{n} - \hat{I}_{520}\hat{n}_l\right) , \tag{56}$$

and finally for $i = 0, 1, 2$ and $j = 2$ we have

$$\hat{L}_{02} = \sigma_T\left(\hat{I}_{320}\hat{n} - \hat{I}_{330}\hat{n}_l\right) , \tag{57}$$
$$\hat{L}_{12} = \sigma_T\left(\hat{I}_{420}\hat{n} - \hat{I}_{430}\hat{n}_l\right) , \tag{58}$$
$$\hat{L}_{22} = \sigma_T\left(\hat{I}_{520}\hat{n} - \hat{I}_{530}\hat{n}_l\right) . \tag{59}$$

Note that the loss terms, $\hat{L}_{00}$ and $\hat{L}_{10}$ correspond to the moments of the particle number and energy conservation equations, while $\hat{L}_{01}$ to the moment of the momentum conservation in the direction of the anisotropy.

B. The gain terms

The gain terms are substantially more difficult to calculate than the loss terms due to the complicated structure of the P and P' integrals; i.e., now we need to compute $\mathcal{P}_{ij}$ for $i > 0$ and $j \geq 0$, while we only need $\mathcal{P}_{00}$ for the loss terms. For the purposes of this paper dealing with leading-order anisotropic distribution functions we only need to evaluate the scalar gain term $\mathcal{P}_{ij}$ hence now Eq. (39) reads,

$$\hat{G}_{ij} \equiv \int_{KK'}\hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k'}}\mathcal{P}_{ij} = (-1)^j\int_{KK'}\hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k'}}\Theta^{\mu_1\cdots\mu_{i+j}}u_{\mu_1}\cdots u_{\mu_i}l_{\mu_{i+1}}\cdots l_{\mu_{i+j}} , \tag{60}$$

where the unprojected auxiliary integrals were defined in Eq. (42). Note that although it is possible to compute the gain terms to arbitrary tensor rank, for practical purposes here we only compute a small subset of scalar gain terms explicitly. These follow from various projections with up to tensor rank four of the auxiliary integrals $\Theta^{\mu_1\mu_2\mu_3\mu_4}$ by the four-vectors u^μ and/or l^μ.

We start by computing the simplest scalar gain term which only contains projections in the direction of the fluid four-velocity and no projections in the direction of the anisotropy. Therefore substituting $i = n$ and $j = 0$ into Eq. (60), see Appendix D 1 for the derivation, we obtain

$$\begin{aligned}\hat{G}_{n0} &\equiv \int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\mathcal{P}_{n0} = \int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\Theta^{\mu_1\cdots\mu_n}u_{\mu_1}\cdots u_{\mu_n} \\ &= \int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'} \sum_{q=0}^{\lfloor n/2\rfloor} (-1)^q\, b_{nq}\mathcal{B}_{nq}\left(P_T^\mu u_\mu\right)^{n-2q}\left(\Delta_T^{\mu\nu}u_\mu u_\nu\right)^q \ . \end{aligned} \tag{61}$$

Now using this result in the massless limit when $s \equiv 2k^\mu k'_\mu$ together with the definitions,

$$P_T^\mu u_\mu = E_{\mathbf{k}u} + E_{\mathbf{k}'u} \ , \tag{62}$$

$$\Delta_T^{\mu\nu}u_\mu u_\nu = 1 - s^{-1}\left(E_{\mathbf{k}u} + E_{\mathbf{k}'u}\right)^2 \ , \tag{63}$$

we get,

$$\hat{G}_{n0} = \frac{\sigma_T}{2^n}\int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}k^\mu k'_\mu \sum_{q=0}^{\lfloor n/2\rfloor} \frac{(-1)^q\, b_{nq}}{(2q+1)!!}\left(E_{\mathbf{k}u} + E_{\mathbf{k}'u}\right)^{n-2q}\left(2k^\nu k'_\nu - \left(E_{\mathbf{k}u} + E_{\mathbf{k}'u}\right)^2\right)^q \ . \tag{64}$$

Thus for $n = 0$ and $n = 1$, also see Appendix D 1 for more details, we obtain

$$\hat{G}_{00} = \sigma_T\left(\hat{n}^2 - \hat{n}_l^2\right) \ , \tag{65}$$

$$\hat{G}_{10} = \sigma_T\left(\hat{e}\hat{n} - \hat{M}\hat{n}_l\right) \ . \tag{66}$$

These gain terms precisely correspond to loss terms in Eqs. (48) and (49) as they should according to the conservation of particle number and energy in binary collisions. The gain terms for $n = 2$ and $n = 3$ are

$$\hat{G}_{20} = \frac{\sigma_T}{6}\left(4\hat{I}_{300}\hat{n} - 4\hat{I}_{310}\hat{n}_l + 3\hat{e}^2\right) - \frac{\sigma_T}{6}\left(2\hat{M}^2 + \hat{P}_l^2 + 2\hat{P}_\perp^2\right) \ , \tag{67}$$

$$\hat{G}_{30} = \frac{\sigma_T}{2}\left(\hat{I}_{400}\hat{n} - \hat{I}_{410}\hat{n}_l + 2\hat{I}_{300}\hat{e}\right) - \frac{\sigma_T}{2}\left(\hat{I}_{310}\hat{M} + \hat{I}_{320}\hat{P}_l + 2\hat{I}_{301}\hat{P}_\perp\right) \ , \tag{68}$$

and finally for $n = 4$ we have,

$$\begin{aligned}\hat{G}_{40} = &\frac{\sigma_T}{5}\left(2\hat{I}_{500}\hat{n} - 2\hat{I}_{510}\hat{n}_l + 5\hat{I}_{400}\hat{e} - 2\hat{I}_{410}\hat{M} - 3\hat{I}_{420}\hat{P}_l - 6\hat{I}_{401}\hat{P}_\perp\right) \\ &+ \frac{\sigma_T}{20}\left(13\hat{I}_{300}^2 - 3\hat{I}_{310}^2 - 9\hat{I}_{320}^2 - \hat{I}_{330}^2 - 18\hat{I}_{301}^2 - 6\hat{I}_{311}^2\right) \ . \end{aligned} \tag{69}$$

The next set of gain terms from Eq. (60) is defined for $i = n$ and $j = 1$, and thus contains an additional projection with respect to l^μ. This additional projection along the direction of the anisotropy will give birth to new scalar products when compared to the previous case in Eq. (61), see Appendix D 2 for the derivation,

$$\begin{aligned}\hat{G}_{n1} &\equiv \int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\mathcal{P}_{n1} = -\int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\Theta^{\mu_1\cdots\mu_{n+1}}u_{\mu_1}\cdots u_{\mu_n}l_{\mu_{n+1}} \\ &= -\int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\left(l_{\mu_{n+1}}P_T^{\mu_{n+1}}\right) \sum_{q=0}^{\lfloor n/2\rfloor} (-1)^q\, b_{nq}\mathcal{B}_{n+1,q}\left(P_T^\mu u_\mu\right)^{n-2q}\left(\Delta_T^{\mu\nu}u_\mu u_\nu\right)^q \\ &- \int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\left(l_{\mu_{n+1}}\Delta_T^{\mu_{n+1}\mu_n}u_{\mu_n}\right) \sum_{q=1}^{\lfloor (n+1)/2\rfloor} (-1)^q\left(b_{n+1,q} - b_{nq}\right)\mathcal{B}_{n+1,q}\left(P_T^\mu u_\mu\right)^{n+1-2q}\left(\Delta_T^{\mu\nu}u_\mu u_\nu\right)^{q-1} \ , \end{aligned} \tag{70}$$

where the new invariant scalars are,

$$P_T^\mu l_\mu = -\left(E_{\mathbf{k}l} + E_{\mathbf{k}'l}\right) \ , \tag{71}$$

$$\Delta_T^{\mu\nu}u_\mu l_\nu = s^{-1}\left(E_{\mathbf{k}l} + E_{\mathbf{k}'l}\right)\left(E_{\mathbf{k}u} + E_{\mathbf{k}'u}\right) \ . \tag{72}$$

Now using this latter result for $n=0$ we obtain,

$$\hat{G}_{01} = \sigma_T \left(\hat{M}\hat{n} - \hat{P}_l \hat{n}_l \right) , \tag{73}$$

which is precisely the same result as previously obtained for the loss term in Eq. (53), respecting the conservation of momentum in the direction of the anisotropy. Similarly, using Eq. (70) for $n=1$ and $n=2$ we get,

$$\hat{G}_{11} = \frac{2\sigma_T}{3} \left(\hat{I}_{310}\hat{n} - \hat{I}_{320}\hat{n}_l + \hat{M}\hat{e} - \hat{M}\hat{P}_l \right) , \tag{74}$$

$$\hat{G}_{21} = \frac{\sigma_T}{6} \left(3\hat{I}_{410}\hat{n} - 3\hat{I}_{420}\hat{n}_l + 5\hat{I}_{310}\hat{e} - 4\hat{I}_{320}\hat{M} \right) + \frac{\sigma_T}{6} \left(3\hat{I}_{300}\hat{M} - 3\hat{I}_{310}\hat{P}_l - \hat{I}_{330}\hat{P}_l - 2\hat{I}_{311}\hat{P}_\perp \right) , \tag{75}$$

while the result for $n=3$ is

$$\hat{G}_{31} = \frac{2\sigma_T}{5} \left(\hat{I}_{510}\hat{n} - \hat{I}_{520}\hat{n}_l + \hat{I}_{400}\hat{M} - \hat{I}_{410}\hat{P}_l \right) + \frac{3\sigma_T}{10} \left(3\hat{I}_{410}\hat{e} - 2\hat{I}_{420}\hat{M} - \hat{I}_{430}\hat{P}_l - 2\hat{I}_{411}\hat{P}_\perp \right)$$
$$+ \frac{3\sigma_T}{10} \left(3\hat{I}_{300}\hat{I}_{310} - 2\hat{I}_{310}\hat{I}_{320} - \hat{I}_{320}\hat{I}_{330} - 2\hat{I}_{301}\hat{I}_{311} \right) . \tag{76}$$

Finally, the last gain term which we explicitly calculate follows from Eq. (60) for $i=n$ and $j=2$, therefore generalizing our previous results by including two projections in the direction of the anisotropy, see Appendix D 3 for details of the derivation of this expression,

$$\hat{G}_{n2} \equiv \int_{KK'} \hat{f}_{0\mathbf{k}} \hat{f}_{0\mathbf{k}'} \mathcal{P}_{n2} = \int_{KK'} \hat{f}_{0\mathbf{k}} \hat{f}_{0\mathbf{k}'} \Theta^{\mu_1 \cdots \mu_{n+2}} u_{\mu_1} \cdots u_{\mu_n} l_{\mu_{n+1}} l_{\mu_{n+2}}$$
$$= \int_{KK'} \hat{f}_{0\mathbf{k}} \hat{f}_{0\mathbf{k}'} \left(l_{\mu_{n+2}} P_T^{\mu_{n+2}} \right) \left(l_{\mu_{n+1}} P_T^{\mu_{n+1}} \right) \sum_{q=0}^{\lfloor n/2 \rfloor} (-1)^q \, b_{nq} \mathcal{B}_{n+2,q} \left(P_T^\mu u_\mu \right)^{n-2q} \left(\Delta_T^{\mu\nu} u_\mu u_\nu \right)^q$$
$$+ \int_{KK'} \hat{f}_{0\mathbf{k}} \hat{f}_{0\mathbf{k}'} \left(l_{\mu_{n+2}} P_T^{\mu_{n+2}} \right) \left(l_{\mu_{n+1}} \Delta_T^{\mu_{n+1}\mu_n} u_{\mu_n} \right)$$
$$\times \sum_{q=1}^{\lfloor (n+2)/2 \rfloor} (-1)^q \, 2 \left(b_{n+1,q} - b_{nq} \right) \mathcal{B}_{n+2,q} \left(P_T^\mu u_\mu \right)^{n+1-2q} \left(\Delta_T^{\mu\nu} u_\mu u_\nu \right)^{q-1}$$
$$+ \int_{KK'} \hat{f}_{0\mathbf{k}} \hat{f}_{0\mathbf{k}'} \left(l_{\mu_{n+2}} \Delta_T^{\mu_{n+2}\mu_{n+1}} l_{\mu_{n+1}} \right) \sum_{q=1}^{\lfloor (n+2)/2 \rfloor} (-1)^q \, b_{n,q-1} \mathcal{B}_{n+2,q} \left(P_T^\mu u_\mu \right)^{n+2-2q} \left(\Delta_T^{\mu\nu} u_\mu u_\nu \right)^{q-1}$$
$$+ \int_{KK'} \hat{f}_{0\mathbf{k}} \hat{f}_{0\mathbf{k}'} \left(l_{\mu_{n+2}} \Delta_T^{\mu_{n+2}\mu_n} u_{\mu_n} \right) \left(l_{\mu_{n+1}} \Delta_T^{\mu_{n+1}\mu_{n-1}} u_{\mu_{n-1}} \right)$$
$$\times \sum_{q=2}^{\lfloor (n+2)/2 \rfloor} (-1)^q \, 2 \left(q-1 \right) b_{n,q-1} \mathcal{B}_{n+2,q} \left(P_T^\mu u_\mu \right)^{n+2-2q} \left(\Delta_T^{\mu\nu} u_\mu u_\nu \right)^{q-2} , \tag{77}$$

where the new invariant scalar is,

$$\Delta_T^{\mu\nu} l_\mu l_\nu = -1 - s^{-1} \left(E_{\mathbf{k}l} + E_{\mathbf{k}'l} \right)^2 . \tag{78}$$

Using this result for $n=0$ and $n=1$, we obtain,

$$\hat{G}_{02} = \frac{\sigma_T}{6} \left(4\hat{I}_{320}\hat{n} - 4\hat{I}_{330}\hat{n}_l + \hat{e}^2 \right) + \frac{\sigma_T}{6} \left(2\hat{M}^2 - 3\hat{P}_l^2 + 2\hat{P}_\perp^2 \right) , \tag{79}$$

$$\hat{G}_{12} = \frac{\sigma_T}{6} \left(3\hat{I}_{420}\hat{n} - 3\hat{I}_{430}\hat{n}_l + 4\hat{I}_{310}\hat{M} - 5\hat{I}_{320}\hat{P}_l \right) + \frac{\sigma_T}{6} \left(3\hat{I}_{320}\hat{e} - 3\hat{I}_{330}\hat{M} + \hat{I}_{300}\hat{e} + 2\hat{I}_{301}\hat{P}_\perp \right) , \tag{80}$$

and finally the result for $n=2$ reads,

$$\hat{G}_{22} = \frac{\sigma_T}{10} \left(4\hat{I}_{520}\hat{n} - 4\hat{I}_{530}\hat{n}_l + 7\hat{I}_{420}\hat{e} - 7\hat{I}_{420}\hat{P}_l + 4\hat{I}_{320}\hat{I}_{300} - 4\hat{I}_{330}\hat{I}_{310} \right)$$
$$+ \frac{\sigma_T}{10} \left(6\hat{I}_{410}\hat{M} - 6\hat{I}_{430}\hat{M} + \hat{I}_{400}\hat{e} - \hat{I}_{440}\hat{P}_l + 2\hat{I}_{401}\hat{P}_\perp - 2\hat{I}_{421}\hat{P}_\perp \right)$$
$$+ \frac{\sigma_T}{60} \left(5\hat{I}_{300}^2 - 5\hat{I}_{330}^2 + 33\hat{I}_{310}^2 - 33\hat{I}_{320}^2 + 6\hat{I}_{301}^2 - 6\hat{I}_{311}^2 \right) . \tag{81}$$

Note that the corresponding moments of the collision integral $\hat{C}_{ij} = \hat{G}_{ij} - \hat{L}_{ij}$ are computed and listed in Appendix E.

These results show that all loss and all gain terms, and hence all the moments of the binary collision integral in leading-order anisotropic fluid dynamics are expressed as quadratic products of anisotropic thermodynamic integrals $\hat{I}_{nrq}$ defined in Eq. (10). These nonlinear couplings between moments of different order represent the nonlinear dependence on the distribution function of the binary collision integral. It is important to observe that the number of nonlinear terms increases with the power of energy $E_{\mathbf{k}u}$ and the power of momentum in the direction of the anisotropy $E_{\mathbf{k}l}$, corresponding to specific projections of higher rank tensor moments. This also means that at any given order the collision term couples only to a well defined set of lower-order moments in the hierarchy.

The method presented here is not only applicable but also essential to the calculation of arbitrary rank moment of the binary collision integral $\hat{C}_{ij}^{\mu_1\cdots\mu_n}$ in more complete theories of anisotropic fluid dynamics [31, 32]. These also include $\delta\hat{f}_{\mathbf{k}}$ corrections to improve the leading-order anisotropic framework further as $f_{\mathbf{k}} \equiv \hat{f}_{0\mathbf{k}} + \delta\hat{f}_{\mathbf{k}}$ and hence require the computation of $C\left[\hat{f}_0 + \delta\hat{f}\right]$ and corresponding irreducible moments of the collision integral. Nonetheless, for the sake of simplicity in this paper we only focus on leading-order anisotropic fluid dynamics.

IV. APPLICATIONS

A. The moments of the collision integral for the Romatschke-Strickland distribution function

In this section we study the moments of collision integral using the spheroidal distribution function of Ref. [40]. The RS distribution function represents the rescaling of the momentum in the direction of the anisotropy,

$$\hat{f}_{RS}(\alpha_{RS}, \beta_{RS}, \xi) \equiv \exp\left(-\alpha_{RS} + \beta_{RS}\sqrt{E^2_{\mathbf{k}u} + \xi E^2_{\mathbf{k}l}}\right) , \tag{82}$$

where ξ denotes the anisotropy parameter. In the LR frame $u^\mu_{LR} = (1, 0, 0, 0)$, thus $E_{\mathbf{k}u} = k^0$ and $E_{\mathbf{k}l} = k_z$, hence with respect to the $z-$axis in momentum space $\hat{f}_{RS}$ is a prolate spheroid for $\xi < 0$, while it is an oblate spheroid for $\xi > 0$. The spherically symmetric equilibrium distribution is obtained in case $\xi = 0$.

The anisotropic thermodynamic integrals corresponding to the RS distribution function $\hat{I}^{RS}_{nrq}(\alpha_{RS}, \beta_{RS}, \xi)$ are given replacing $\hat{f}_{0\mathbf{k}} \to \hat{f}_{RS}$ in Eq. (10),

$$\hat{I}^{RS}_{nrq} \equiv \frac{(-1)^q}{(2q)!!}\int dK E^{n-r-2q}_{\mathbf{k}u} E^r_{\mathbf{k}l} \left(\Xi^{\mu\nu}k_\mu k_\nu\right)^q \hat{f}_{RS} , \tag{83}$$

which in the ultrarelativistic limit $m_0 \to 0$ of a Boltzmann gas leads to the following factorization [26],

$$\hat{I}^{RS}_{nrq}(\alpha_{RS}, \beta_{RS}, \xi) = I_{nq}(\alpha_{RS}, \beta_{RS}) R_{nrq}(\xi) . \tag{84}$$

All relevant thermodynamic integrals and ratios R_{nrq} are explicitly listed in Appendix F. Furthermore, for the RS distribution function all ratios and thus all anisotropic thermodynamic integrals vanish equivalently for all odd r,

$$\hat{I}^{RS}_{nrq} \equiv R_{nrq} = 0 , \quad \forall r \in \text{odd} , \tag{85}$$

hence due to this property the corresponding moments of the binary collision integral $\hat{C}^{RS}_{ir} = \hat{L}^{RS}_{ir} = \hat{G}^{RS}_{ir} = 0$ also vanish equivalently for all odd r. Note that now the fluid dynamical flow velocity is also independent on the choice of the LR frame, since not only the energy-momentum diffusion current, $\hat{M} \equiv \hat{I}^{RS}_{210} = 0$, but also the particle diffusion current in the direction of the anisotropy vanishes $\hat{n}_l \equiv \hat{I}^{RS}_{110} = 0$ equivalently. The former corresponds to Landau's definition of the LR frame [48], while the latter corresponds to Eckart's definition of the LR frame [51].

The first and second moments of the RS distribution function are

$$\hat{N}^\mu_{RS} = \hat{n}u^\mu , \quad \hat{T}^{\mu\nu}_{RS} = \hat{e}u^\mu u^\nu + \hat{P}_l l^\mu l^\nu - \hat{P}_\perp \Xi^{\mu\nu} , \tag{86}$$

and with the help of Eq. (84) the primary fluid dynamical quantities read

$$\hat{n} \equiv \hat{I}^{RS}_{100} = n_0(\alpha_{RS}, \beta_{RS}) R_{100}(\xi) , \tag{87}$$

$$\hat{e} \equiv \hat{I}^{RS}_{200} = e_0(\alpha_{RS}, \beta_{RS}) R_{200}(\xi) , \tag{88}$$

$$\hat{P}_l \equiv \hat{I}^{RS}_{220} = e_0(\alpha_{RS}, \beta_{RS}) R_{220}(\xi) , \tag{89}$$

$$\hat{P}_\perp \equiv \hat{I}^{RS}_{201} = P_0(\alpha_{RS}, \beta_{RS}) R_{201}(\xi) . \tag{90}$$

For and ideal gas of massless particles, $P_0(n_0, e_0) \equiv e_0/3 = n_0 T$, where $\mu(n_0, e_0)$ is the chemical potential and $T(n_0, e_0)$ is the temperature. Therefore the bulk viscous pressure, defined according to $\hat{\Pi} \equiv \hat{P} - P_0$, where $\hat{P} = (\hat{P}_l + 2\hat{P}_\perp)/3$ is the isotropic pressure, also vanishes in the massless limit.

The parameters of the anisotropic distribution function, α_{RS}, β_{RS}, and ξ, can be expressed in terms of a fictitious equilibrium state specified by $\alpha = \mu/T$, and $\beta = 1/T$, through the so-called Landau matching conditions [41],

$$\hat{n}\left(\alpha_{RS}, \beta_{RS}, \xi\right) = n_0\left(\alpha, \beta\right) , \quad \hat{e}\left(\alpha_{RS}, \beta_{RS}, \xi\right) = e_0\left(\alpha, \beta\right) , \tag{91}$$

and hence from Eqs. (87, 88), and the corresponding fugacities, $\lambda \equiv \exp\left(\alpha\right)$, and $\lambda_{RS} = \exp\left(\mu_{RS}\beta_{RS}\right)$, we obtain

$$\lambda = \lambda_{RS} \frac{\left[R_{100}\left(\xi\right)\right]^4}{\left[R_{200}\left(\xi\right)\right]^3} , \quad \beta = \beta_{RS} \frac{R_{100}\left(\xi\right)}{R_{200}\left(\xi\right)} . \tag{92}$$

Using these results together with Eq. (84) leads to the following general relation between the anisotropic thermodynamic integrals of the RS distribution function and the equilibrium thermodynamic integrals [38],

$$\hat{I}^{RS}_{nrq}\left(\alpha_{RS}, \beta_{RS}, \xi\right) = I_{nq}\left(\alpha, \beta\right) R_{nrq}\left(\xi\right) \frac{\left[R_{200}\left(\xi\right)\right]^{1-n}}{\left[R_{100}\left(\xi\right)\right]^{2-n}} . \tag{93}$$

Now taking into account the properties of the RS distribution function from Eq. (85) together with the matching condition $\hat{e}\left(\alpha_{RS}, \beta_{RS}, \xi\right) = e_0\left(\alpha, \beta\right)$, the moments of the binary collision integral from Appendix E lead to:

$$\hat{C}^{RS}_{20} = -\frac{1}{\tau_R}\left(\frac{\hat{n}}{3n_0}\right)\hat{I}^{RS}_{300} + \frac{1}{\tau_R}\left[\frac{1}{6n_0}\left(3e_0^2 - \hat{P}_l^2 - 2\hat{P}_\perp^2\right)\right] , \tag{94}$$

$$\hat{C}^{RS}_{30} = -\frac{1}{\tau_R}\left(\frac{\hat{n}}{2n_0}\right)\hat{I}^{RS}_{400} + \frac{1}{\tau_R}\left[\frac{1}{2n_0}\left(2e_0\hat{I}^{RS}_{300} - \hat{P}_l\hat{I}^{RS}_{320} - 2\hat{P}_\perp\hat{I}^{RS}_{301}\right)\right] , \tag{95}$$

and

$$\begin{aligned}\hat{C}^{RS}_{40} &= -\frac{1}{\tau_R}\left(\frac{3\hat{n}}{5n_0}\right)\hat{I}^{RS}_{500} + \frac{1}{\tau_R}\left[\frac{1}{5n_0}\left(5e_0\hat{I}^{RS}_{400} - 3\hat{P}_l\hat{I}^{RS}_{420} - 6\hat{P}_\perp\hat{I}^{RS}_{401}\right)\right] \\ &+ \frac{1}{\tau_R}\left[\frac{1}{20n_0}\left(13\left(\hat{I}^{RS}_{300}\right)^2 - 9\left(\hat{I}^{RS}_{320}\right)^2 - 18\left(\hat{I}^{RS}_{301}\right)^2\right)\right] .\end{aligned} \tag{96}$$

Similarly the other relevant moments of the binary collision integral are

$$\hat{C}^{RS}_{02} = -\frac{1}{\tau_R}\left(\frac{\hat{n}}{3n_0}\right)\hat{I}^{RS}_{320} + \frac{1}{\tau_R}\left[\frac{1}{6n_0}\left(e_0^2 - 3\hat{P}_l^2 + 2\hat{P}_\perp^2\right)\right] , \tag{97}$$

$$\hat{C}^{RS}_{12} = -\frac{1}{\tau_R}\left(\frac{\hat{n}}{2n_0}\right)\hat{I}^{RS}_{420} + \frac{1}{\tau_R}\left[\frac{1}{6n_0}\left(e_0\hat{I}^{RS}_{300} + \left(3e_0 - 5\hat{P}_l\right)\hat{I}^{RS}_{320} + 2\hat{P}_\perp\hat{I}^{RS}_{301}\right)\right] , \tag{98}$$

and finally

$$\begin{aligned}\hat{C}^{RS}_{22} &= -\frac{1}{\tau_R}\left(\frac{3\hat{n}}{5n_0}\right)\hat{I}^{RS}_{520} + \frac{1}{\tau_R}\left[\frac{1}{10n_0}\left(e_0\hat{I}^{RS}_{400} + \left(7e_0 - 7\hat{P}_l\right)\hat{I}^{RS}_{420} - \hat{P}_l\hat{I}^{RS}_{440} + 2\hat{P}_\perp\hat{I}^{RS}_{401} - 2\hat{P}_\perp\hat{I}^{RS}_{421}\right)\right] \\ &+ \frac{1}{\tau_R}\left[\frac{1}{60n_0}\left(5\left(\hat{I}^{RS}_{300}\right)^2 + 24\hat{I}^{RS}_{320}\hat{I}^{RS}_{300} - 33\left(\hat{I}^{RS}_{320}\right)^2 + 6\left(\hat{I}^{RS}_{301}\right)^2\right)\right] .\end{aligned} \tag{99}$$

Note that here we have replaced the transport cross-section by $\sigma_T = 1/(n_0\tau_R)$, which introduces the mean free path or mean free time between collisions based on the equilibrium particle density. However according to Eq. (37) there is freedom to chose $\sigma_T = 1/(\hat{n}\hat{\tau}_R)$ based on the non-equilibrium particle density with a different mean free time $\hat{\tau}_R$ between collisions. Nevertheless, in case the number of particles or charge(s) remain conserved, such as is in the case of binary elastic collisions, then it is required that $\hat{n}\left(\alpha_{RS}, \beta_{RS}, \xi\right) = n_0\left(\alpha, \beta\right)$ and hence $\hat{\tau}_R = \tau_R$.

For the sake of comparison, here we recall the relativistic relaxation-time approximation model of Anderson and Witting (AW) [41], which is the relativistic generalization of the RTA model of Bhatnagar-Gross-Krook (BGK) [52]. The RTA assumes that the nonlinear binary collision integral from Eq. (15) is approximated by

$$\hat{C}\left[\hat{f}_0\right] \approx \hat{C}_{AW}\left[\hat{f}_0\right] = -\frac{E_{\mathbf{k}u}}{\tau_R}\left(\hat{f}_{0\mathbf{k}} - f_{0\mathbf{k}}\right) , \tag{100}$$

where the relaxation-time τ_R is an energy and momentum independent parameter, and represents the time scale on which the anisotropic distribution function approaches the local equilibrium distribution function. This is a free parameter that is chosen either a constant or according to some transport coefficient that relates to some intrinsic property of matter such as viscosity or diffusivity. For reasons of simplicity and aiming for direct comparison to the binary collision integral, in the following the relaxation time is defined through the transport cross section and the particle density, $\tau_R = 1/(\sigma_T n_0)$, and hence represents the mean free time between collisions.

Replacing the RS distribution function into Eq. (100), the corresponding scalar moments of the collision integral (38) in the RTA are given by the following simple formula,

$$\hat{C}^{RS}_{ij,AW} = -\frac{1}{\tau_R}\hat{I}^{RS}_{i+j+1,j,0} + \frac{1}{\tau_R} I_{i+j+1,j,0} \,. \tag{101}$$

Now comparing the moments of the nonlinear binary collision integral from Eqs. (94-96) and Eqs. (97-99) to the moments of the RTA approximation (101), reveals that the latter omits an ever increasing number of nonlinear couplings represented through the quadratic products of various anisotropic thermodynamic integrals. Even though the very first terms, i.e., $-\hat{I}^{RS}_{i+j+1,j,0}/\tau_R$, on the right hand sides appear in both cases, the numerical prefactors in case of the binary collision integral are consistently smaller than 1. However, the major differences between the moments of the binary collision integral and the moments in the RTA are explicitly given by all the remaining terms in square brackets in Eqs. (94-96) and Eqs. (97-99). In the relaxation-time approximation such terms are simply represented by an equilibrium thermodynamic integral, see the second term in Eq. (101). Hence due to these differences it should be expected that the approach to local equilibrium; i.e., $\xi \to 0$, will also happen on different times-scales for the different approaches. Since most terms in square brackets are non-equilibrium quantities being some function of the anisotropy parameter we expect that the system approaches equilibrium slower than in case of the RTA which directly and explicitly relaxes to local equilibrium on the shortest time-scale. This observation is in agreement with earlier results based on the exact solutions of the relativistic Boltzmann equation for homogeneous and isotropic solutions [36, 37]. Further detailed comparisons and discussions are presented in the next Sections.

B. 0+1 dimensional boost-invariant expansion and a judicious choice of moment

We now study the direct influence of the collision term on the solution of the fluid-dynamical equations of motion in the 0+1 dimensional boost-invariant expansion, known as Bjorken flow [53]. The space-time coordinates (t, z) are transformed to proper time $\tau = \sqrt{t^2 - z^2}$, and space-time rapidity $\eta_s = \frac{1}{2}\ln\frac{t+z}{t-z}$, where the inverse transformations are $t = \tau\cosh\eta_s$, and $z = \tau\sinh\eta_s$. The longitudinal fluid velocity is defined as $v_z \equiv z/t = \tanh\eta_s$, and hence now $u^\mu \equiv \left(\frac{t}{\tau}, 0, 0, \frac{z}{\tau}\right) = (\cosh\eta_s, 0, 0, \sinh\eta_s)$ and $l^\mu \equiv \left(\frac{z}{\tau}, 0, 0, \frac{t}{\tau}\right) = (\sinh\eta_s, 0, 0, \cosh\eta_s)$. Furthermore, $D \equiv u^\mu\partial_\mu = \frac{\partial}{\partial\tau}$, $D_l \equiv -l^\mu\partial_\mu = -\frac{\partial}{\tau\partial\eta_s}$, and $Du^\mu = Dl^\mu = 0$, $D_l u^\mu = -\frac{1}{\tau}l^\mu$, $D_l l^\mu = -\frac{1}{\tau}u^\mu$, while all thermodynamic quantities and variables are independent of η_s and only depend on the proper-time.

Applying these simplifications we obtain a hierarchy of coupled equations of motion represented by the following simple differential equation; see Eq. (53) in Ref. [38],

$$\frac{\partial\hat{I}^{RS}_{i+j,j,0}}{\partial\tau} + \frac{1}{\tau}\left[(j+1)\,\hat{I}^{RS}_{i+j,j,0} + (i-1)\,\hat{I}^{RS}_{i+j,j+2,0}\right] = \hat{C}^{RS}_{i-1,j} \approx \hat{C}^{RS}_{i-1,j,AW} \,, \tag{102}$$

where $i, j \geq 0$, and on the rhs we either use the corresponding moments of the binary collision integral $\hat{C}^{RS}_{i-1,j}$ or the relaxation-time approximation to the collision integral denoted by, $\hat{C}^{RS}_{i-1,j,AW}$.

The conservation of particle number and the conservation of energy are obtained from Eq. (102) for $i = 1, j = 0$ and $i = 2, j = 0$ respectively,

$$\frac{\partial n_0(\alpha,\beta)}{\partial\tau} + \frac{1}{\tau}n_0(\alpha,\beta) = 0 \,, \tag{103}$$

$$\frac{\partial e_0(\alpha,\beta)}{\partial\tau} + \frac{1}{\tau}\left[e_0(\alpha,\beta) + \hat{P}_l(\alpha_{RS},\beta_{RS},\xi)\right] = 0 \,, \tag{104}$$

where according to Eqs. (20) on the lhs we have $\hat{C}^{RS}_{00} \equiv \hat{C}^{RS}_{00,AW} = 0$ and $\hat{C}^{RS}_{10} \equiv \hat{C}^{RS}_{10,AW} = 0$.

The hierarchy of coupled moment equations (102) provides infinitely many possibilities to close the conservation equations and to determine the time evolution of the anisotropy parameter ξ. Here we will follow Ref. [38] and restrict ourselves to a few examples by choosing particular values for the indices i and j corresponding to the powers of $E_{\mathbf{k}u}$ and $E_{\mathbf{k}l}$ of the anisotropic integral $\hat{I}^{RS}_{i+j,j,0}$. Therefore the corresponding anisotropic moment selected for closure is now treated dynamically and represented by the corresponding differential equation (102). This way we also obtain the proper time evolution of the anisotropic distribution function (82) through its three parameters: $\alpha_{ij}^{RS}(\tau)$, $\beta_{ij}^{RS}(\tau)$, and $\xi_{ij}(\tau)$. Using these solutions now all other (non-dynamical) moments in the hierarchy are subsequently obtained algebraically, i.e., using Eq. (83) and leading to $\hat{I}_{nrq}^{RS}\left(\alpha_{ij}^{RS}, \beta_{ij}^{RS}, \xi_{ij}\right)$.

Here we list the indices corresponding to the moments of the binary collision integral calculated in Eqs. (94-96).

(i) $i = 3$, $j = 0$: This choice is analogous to the one of Israel and Stewart [11], and follows from the projection of the rank 3 tensor moment equation $\hat{\mathcal{I}}_{00}^{\mu_1\mu_2\mu_3}$ to close the conservation equations,

$$\frac{\partial \hat{I}_{300}^{RS}}{\partial \tau} + \frac{1}{\tau}\left(\hat{I}_{300}^{RS} + 2\hat{I}_{320}^{RS}\right) = \hat{C}_{20}^{RS} \approx \hat{C}_{20,AW}^{RS} \,, \tag{105}$$

(ii) $i = 4$, $j = 0$: This choice is analogous the previous but follows from a rank 4 tensor moment,

$$\frac{\partial \hat{I}_{400}^{RS}}{\partial \tau} + \frac{1}{\tau}\left(\hat{I}_{400}^{RS} + 3\hat{I}_{420}^{RS}\right) = \hat{C}_{30}^{RS} \approx \hat{C}_{30,AW}^{RS} \,, \tag{106}$$

(iii) $i = 5$, $j = 0$: This choice is analogous to both before and follows from a rank 5 tensor moment,

$$\frac{\partial \hat{I}_{500}^{RS}}{\partial \tau} + \frac{1}{\tau}\left(\hat{I}_{500}^{RS} + 4\hat{I}_{520}^{RS}\right) = \hat{C}_{40}^{RS} \approx \hat{C}_{40,AW}^{RS} \,. \tag{107}$$

Now we fix $j = 2$ and then choose i to correspond to Eqs. (97-99):

(iv) $i = 1$, $j = 2$: This also results from a rank 3 tensor moment but through a different projection than in case (i),

$$\frac{\partial \hat{I}_{320}^{RS}}{\partial \tau} + \frac{1}{\tau}\left(3\hat{I}_{320}^{RS}\right) = \hat{C}_{02}^{RS} \approx \hat{C}_{02,AW}^{RS} \,, \tag{108}$$

(v) $i = 2$, $j = 2$: This is similar to (ii) but a different projection than in case (ii)

$$\frac{\partial \hat{I}_{420}^{RS}}{\partial \tau} + \frac{1}{\tau}\left(3\hat{I}_{420}^{RS} + \hat{I}_{440}^{RS}\right) = \hat{C}_{12}^{RS} \approx \hat{C}_{12,AW}^{RS} \,, \tag{109}$$

(vi) $i = 3$, $j = 2$: This choice is similar to (iii) but projected differently than in case (iii)

$$\frac{\partial \hat{I}_{520}^{RS}}{\partial \tau} + \frac{1}{\tau}\left(3\hat{I}_{520}^{RS} + 2\hat{I}_{540}^{RS}\right) = \hat{C}_{22}^{RS} \approx \hat{C}_{22,AW}^{RS} \,. \tag{110}$$

Note that the choices (i) and (iv) were already studied in Ref. [38], while the choice $i = 0$, $j = 2$, i.e., the equation for the longitudinal pressure $\hat{P}_l\left(\alpha_{RS}, \beta_{RS}, \xi\right)$, was also found to represent the best match [38, 39, 54] to the exact numerical solution of the Boltzmann equation in the RTA,

$$\frac{\partial \hat{P}_l}{\partial \tau} + \frac{1}{\tau}\left(3\hat{P}_l - \hat{I}_{240}^{RS}\right) = \hat{C}_{-12}^{RS} \approx -\frac{1}{\tau_R}\left(\hat{P}_l - P_0\right) \,. \tag{111}$$

This choice is also special because both $\hat{P}_l$ and $\hat{I}_{240}^{RS}$ expressed through Eq. (93) remain formally unchanged and hence independent of whether we conserve particle number or not [38, 55]. Here we only make this choice in the RTA since the corresponding moments of the binary collision integral with negative powers of energy cannot be computed using the projection method. Nevertheless as shown in Ref. [38] as well as in what follows one can reasonably well approximate the solutions of $\hat{I}_{220}$ by the solutions of higher moments, i.e., $\hat{I}_{i+2,2,0}$ for $i > 0$.

In what follows we will focus on the solutions to the conservation equations, (103) and (104), closed by one of the moment equations (105-110) listed here. We will study the evolution of an ultrarelativistic anisotropic fluid either using the moments of the binary collision integral or the moments in the relaxation-time approximation of the collision integral. The Landau matching conditions (91) are used to infer the fugacity and temperature, while for the sake of comparison we also show the solutions of the ideal fluid dynamical equations, i.e., Eq. (103) and Eq. (104), where in the latter replacing $\hat{P}_l \to P_0(\alpha, \beta)$ leads to the ideal equation of motion, $\partial e_0\left(\alpha, \beta\right)/\partial\tau + \left[e_0\left(\alpha, \beta\right) + P_0\left(\alpha, \beta\right)\right]/\tau = 0$.

In all cases we have initialized the system at $\tau_0 = 1$ fm/c and with a temperature of $T(\tau_0) \equiv T_0 = 0.5$ GeV, and chemical potential of $\mu_0 = 0$ GeV, hence the initial fugacity is $\lambda(\tau_0) \equiv \lambda_0 = 1$. In all cases the relaxation time is fixed as constant, $\tau_R = 0.5$ fm/c. Furthermore, all plots in Fig. 1 correspond to an initially isotropic distribution, i.e., the initial value of anisotropy is $\xi(\tau_0) \equiv \xi_0 = 0$, while all plots in Fig. 2 correspond to an initially oblate spheroidal distribution, i.e., the initial anisotropy is $\xi(\tau_0) \equiv \xi_0 = 50$.

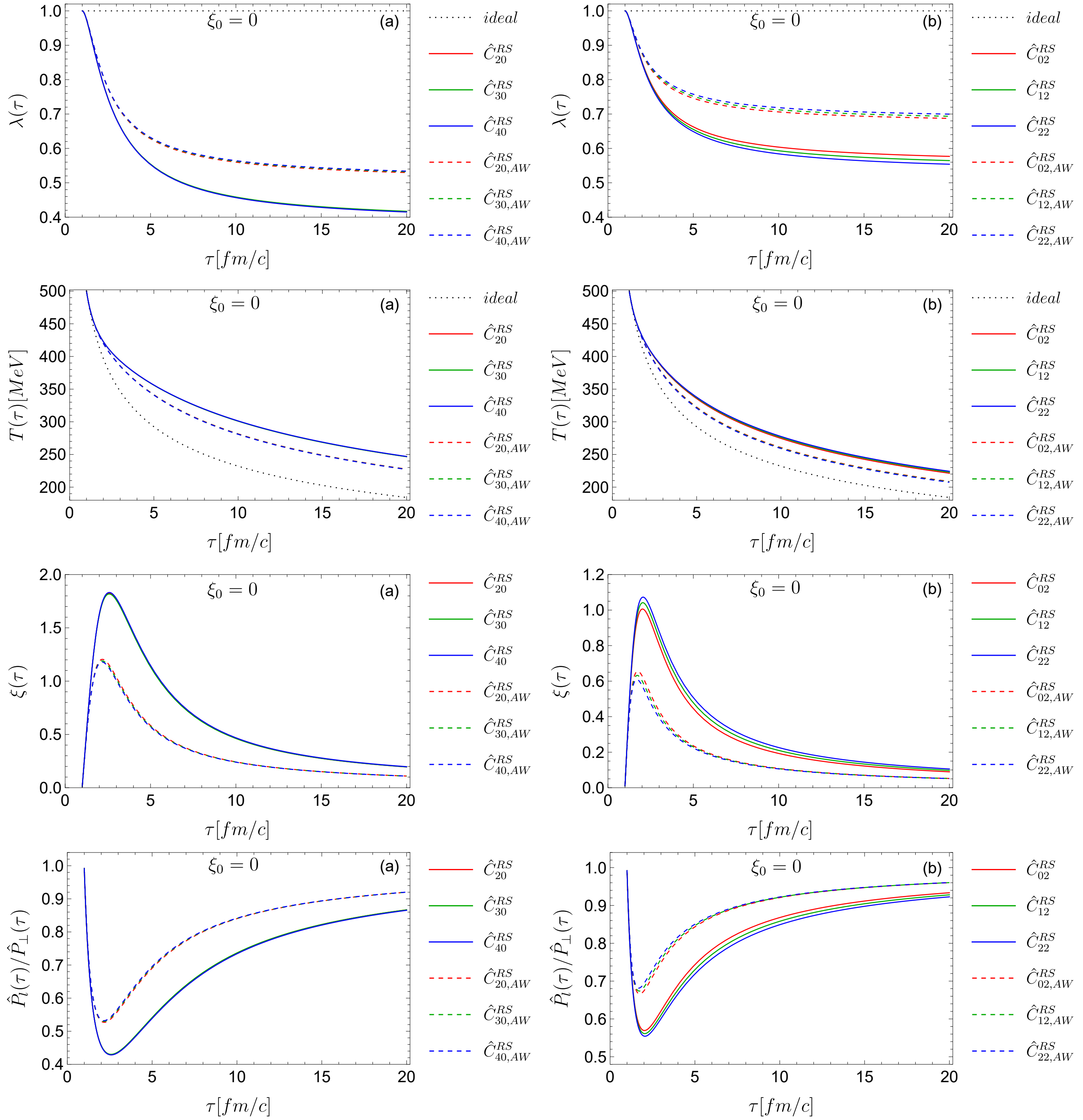

FIG. 1. From top to bottom as well as both left (a) and right (b) panels the initial anisotropy is $\xi_0 = 0$. The evolution of the fugacity $\lambda(\tau)$, the temperature $T(\tau)$, the anisotropy parameter $\xi(\tau)$, and the ratio of the longitudinal pressure and the transverse pressure $\hat{P}_l(\tau)/\hat{P}_\perp(\tau)$, as function of proper time. The black dotted lines represent the time evolution of the fugacity and temperature of an ideal fluid. The solid lines with red, with green, and with blue, represent the moments of the binary collision integral, while the dashed lines (with AW labels) of the same color represent the RTA of the collision integral. All left (a) panels show the solutions of the conservation equations closed by the moment equations, (105) with red, (106) with green, and (107) with blue. Similarly all right (b) panels show the solutions of the conservation equations closed by the moment equations, (108) with red, (109) with green, and (110) with blue. Note that all solutions presented in the left (a) panels are very similar and they differ by less than the thickness of the line hence they overlap and cover each other.

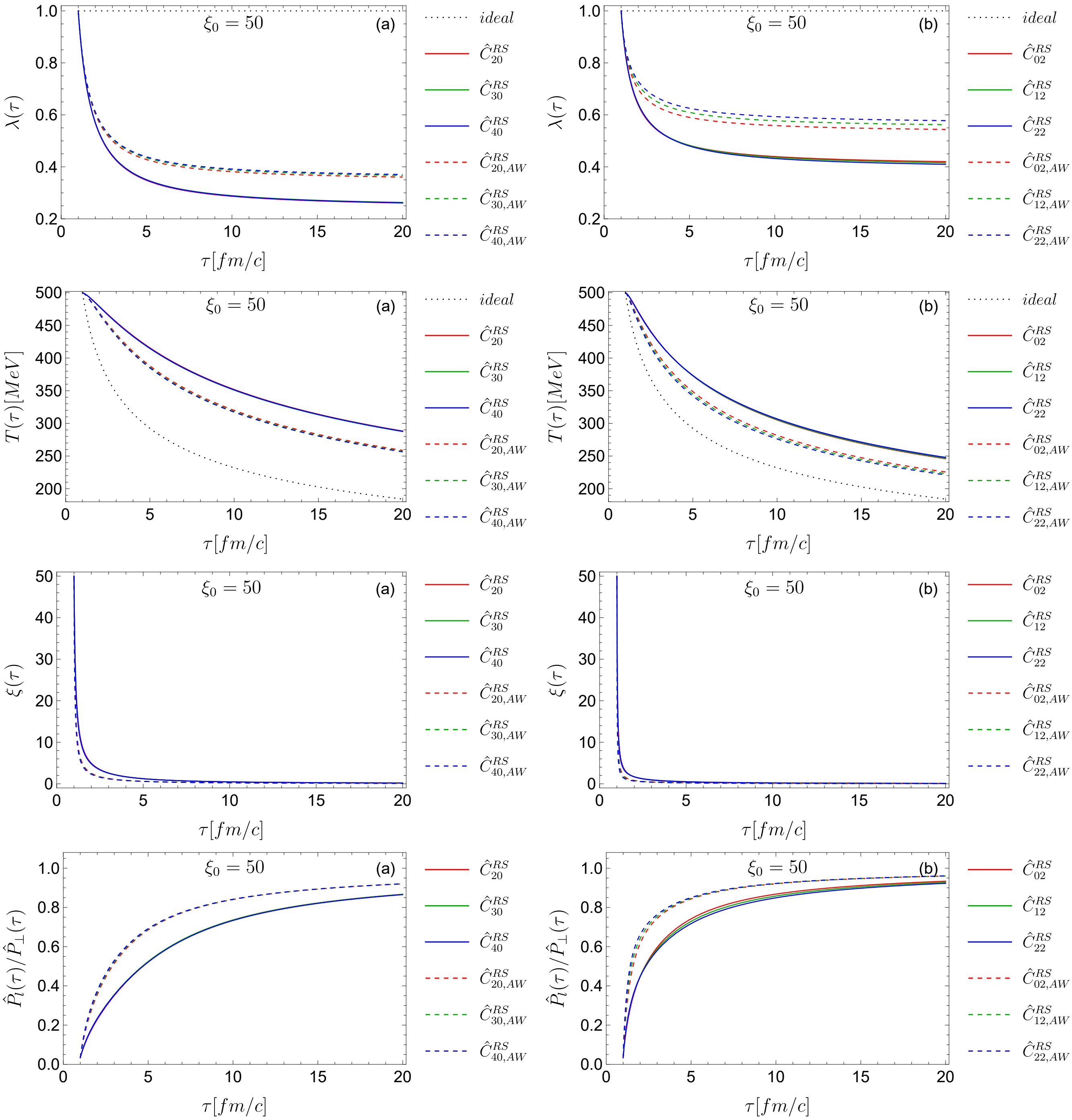

FIG. 2. The same as Fig. 1 but for an initial anisotropy of $\xi_0 = 50$.

Fig. 1 and Fig. 2 from top to bottom, show the evolution of the fugacity $\lambda(\tau)$, the temperature $T(\tau)$, the anisotropy parameter $\xi(\tau)$, and the ratio of the longitudinal pressure and the transverse pressure $\hat{P}_l(\tau)/\hat{P}_\perp(\tau)$ as a function of proper time. All figures in the left columns (a), i.e., Fig. 1(a) and Fig. 2(a), correspond to the solution of the conservation laws closed by one of the moment equations: Eq. (105), Eq. (106), or Eq. (107), with red, green, and blue lines respectively. Similarly all figures in the right columns (b), i.e., Fig. 1(b) and Fig. 2(b), correspond to the solutions obtained by one of the moment equations: Eq. (108), Eq. (109), or Eq. (110), with red, green, and blue lines respectively. In all cases the solid lines always represent the solutions of anisotropic fluid dynamics including the moments of the binary collision integral, while the dashed lines represent the solutions of the fluid dynamical equations including the moments of the RTA.

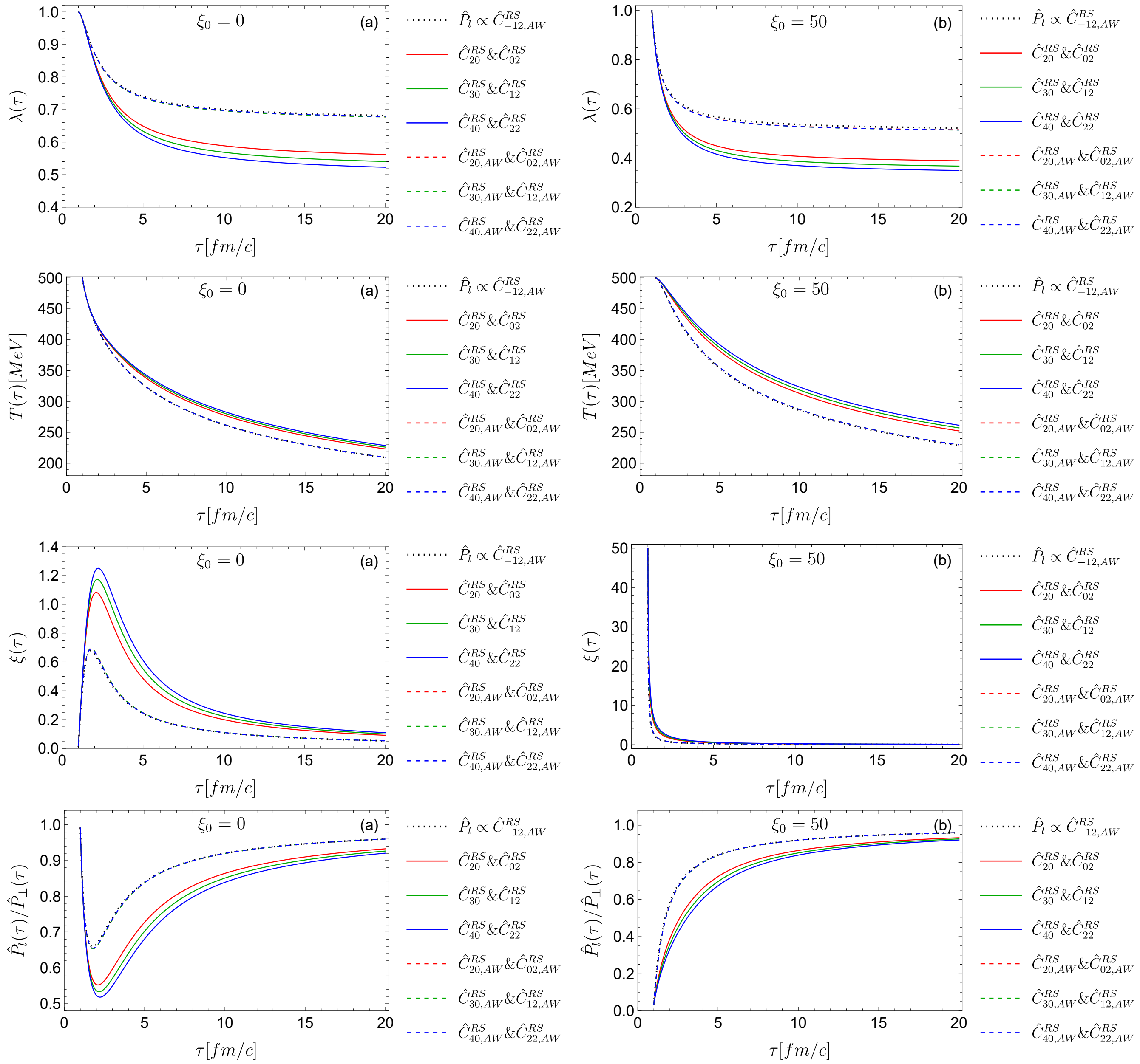

FIG. 3. Similar to Fig. 1 and Fig. 2. The evolution of the fugacity $\lambda(\tau)$, the temperature $T(\tau)$, the anisotropy parameter $\xi(\tau)$ and the ratio of the longitudinal and the transverse pressure components $\hat{P}_l(\tau)/\hat{P}_\perp(\tau)$. Here the black dotted lines represent the solution of the conservation equations closed by the equation of $\hat{P}_l$ in the RTA from Eq. (111). The solid red, green, and blue lines are the solutions corresponding to the binary collision terms, while the dashed lines (with AW labels) of the same color coding represents the RTA. The left (a) panels with $\xi_0 = 0$ show the solutions of the conservation equations closed by the moment equations, (113) with red, (114) with green, and (115) with blue lines. Similarly the right (b) panels show the solutions of the same dynamical equations but with an initial anisotropy of $\xi_0 = 50$. Note that all solutions in the RTA presented in both (a) and (b) panels are very similar and less than the thickness of the line hence they overlap and cover each other.

The system expands and cools down, therefore the fugacity and the temperature of the system decrease as time passes; see the first and the second rows of Fig. 1 and Fig. 2. This change is faster for the fugacity and slower for the temperature for a nonzero initial anisotropy. The reason is that any decrease in fugacity has to be compensated by an increase in temperature, hence a smaller $\lambda(\tau)$ requires a larger $T(\tau)$. An ideal fluid, without any dissipation, has the largest pressure, $P_0 = e_0/3 = n_0 T$, hence it expands and cools fastest. Consequently during the expansion the temperatures in anisotropic fluid dynamics are consistently higher than in ideal fluid dynamics, shown with a black dotted line in the figures.

The evolution of the anisotropy parameter ξ and the ratio of pressures $\hat{P}_l/\hat{P}_\perp$ are shown in the third and fourth rows of Fig. 1 and Fig. 2. In case the system was initially in equilibrium, i.e., isotropic distribution with $\xi_0 = 0$, the left and right columns of Fig. 1, the longitudinal expansion of the system drives the system out of equilibrium. This leads to an increase in the anisotropy parameter ξ which in turn leads to a decrease of the ratio of pressures $\hat{P}_l/\hat{P}_\perp$. This only lasts for about $2-3$ fm/c after which the longitudinal pressure builds up and the system starts to be driven towards a state of local equilibrium, i.e., $\xi \to 0$ and $\hat{P}_l \to \hat{P}_\perp \simeq P_0$. For an initially anisotropic configuration when $\xi_0 = 50$, the left and right columns of Fig. 2, the longitudinal pressure increases faster than for an initially isotropic configuration. Comparing both the left and the right columns of Fig. 1 to Fig. 2 reveals this distinction. Nevertheless, the late-time behavior is quite similar in both cases as the system eventually approaches an isotropic state, $\xi \to 0$.

These conclusions are generally valid for both (binary and RTA) collision terms as well as for the six different choices of closure for the conservation equations. As already discussed in detail in Ref. [38], the inclusion of some of these as well as other moments for closure (different from those considered here), also demonstrates a universal grouping of the corresponding results based on the power of the longitudinal momentum $E^j_{\mathbf{k}l}$. In the current cases of interest when $j = 0$, hence corresponding to the anisotropic moments $\hat{I}^{RS}_{i00}$ with $i = 3, 4, 5$, all the curves overlap entirely with differences that are less than the thickness of the line in case of the binary collision integral. Similarly, in the RTA the dashed lines are also overlapping for the most part; see the left columns of Fig. 1 and Fig. 2. The case when all the anisotropic moments include the second power, $j = 2$, of the longitudinal momentum, i.e., $\hat{I}^{RS}_{i+2,2,0}$ with $i = 1, 2, 3$, small differences, about the thickness of the line, between the solutions become visible, see the right columns of Fig. 1 and Fig. 2. This can be observed about all relevant quantities irrespective of the initial anisotropy of the system, hence the choice which closes the conservation equations strongly depends on the power of the longitudinal momentum and less on the power of energy. Here we showed that this universal grouping is present and persists regardless of the choice for the collision term.

The other important observation regarding these results are about the striking differences in the evolution of the system due to the differences between the nonlinear binary collision integral and the RTA; compare the corresponding solid lines to the dashed lines in Fig. 1 and Fig. 2. The relaxation-time approximation of the collision term drives the system towards equilibrium faster than the binary collision integral. This was expected based on the explicit formulas and discussions in Sec. IV A. We will return to this issue and explicitly study the behavior of the moments of the collision integral as a function of the anisotropy parameter in more detail in Sec. IV D.

C. The importance of two dynamical moments

The results of the previous section demonstrated the ambiguity in selecting a higher-order moment for closure, as different choices result in distinct groups of solutions based on the power of the longitudinal momentum. In Refs. [38, 39] the so-called "judicious choice" of moment, $\hat{P}_l = \hat{I}_{220}$, i.e., the driving force in the energy conservation equation, offered the best overall agreement between anisotropic fluid dynamics and the solution of the Boltzmann equation in the RTA. Furthermore, it was also shown that using other higher-order moments with the second power of the longitudinal momentum i.e., $\hat{I}_{i+2,2,0}$ with $i > 0$, also leads to a very good agreement to the solution of the Boltzmann equation, similarly to the equation for $\hat{P}_l$. However, when choosing other dynamical moments than $\hat{P}_l$ to close the conservation equations, the algebraically derived values for $\hat{P}_l$ deviate substantially from the exact solutions as shown and discussed in Ref. [38].

These indicate that in a coupled hierarchy of moment equations including more than one dynamical moment to close the conservation equations would be beneficial. Fundamentally, more is more, in our case this should translate to more complete and more precise. This is because the proper solution to the moment equations (102) requires that all (infinitely many) higher-order moments should be included to obtain the solution to the underlying microscopic equation, the Boltzmann equation. This is built on the premise that the lower-order moments of the distribution function describe the lower frequency dynamics while moments of higher-order capture the higher frequency dynamics. Although an exact representation of the distribution function is practically unattainable through the moment method, a well defined finite truncation of the coupled hierarchy is expected to yield a reasonably good approximation for the distribution function. Therefore including an increasing number of dynamical moments to close the conservation equations should also improve the solutions to the fluid dynamical equations (for the lower-order moments which are included) and the approximation to the anisotropic distribution function. A well known example is transient fluid dynamics which is based on a series expansion around equilibrium, hence including additional higher-order moments dynamically, also improve the fluid dynamical solutions when compared to the Boltzmann equation [56].

Inspecting the main equation of motion (102) for any given set of indices $i, j > 0$ we observe that the left hand side contains only two variables, $\hat{I}^{RS}_{i+j,j,0}$ and $\hat{I}^{RS}_{i+j,j+2,0}$. Thus rewriting this equation for indices corresponding to, $i \to i-2$ and $j \to j+2$, leads to a differential equation for $\hat{I}^{RS}_{i+j,j+2,0}$. From these two coupled differential equations we obtain the following equation of motion that governs the evolution of two dynamical moments

$$\frac{\partial \hat{I}^{RS}_{i+j,j,0}}{\partial \tau} - \frac{(i-1)}{(j+3)}\frac{\partial \hat{I}^{RS}_{i+j,j+2,0}}{\partial \tau} + \frac{1}{\tau}\left[(j+1)\,\hat{I}^{RS}_{i+j,j,0} - \frac{(i-1)(i-3)}{(j+3)}\hat{I}^{RS}_{i+j,j+4,0}\right]$$
$$= \hat{C}^{RS}_{i-1,j} - \frac{(i-1)}{(j+3)}\hat{C}^{RS}_{i-3,j+2} \approx \hat{C}^{RS}_{i-1,j,AW} - \frac{(i-1)}{(j+3)}\hat{C}^{RS}_{i-3,j+2,AW}\,. \tag{112}$$

This equation when contrasted to Eq. (102) contains two selected dynamical moments for the pair of indices i and j. This means that the proper time evolution of the anisotropic distribution function will be represented by three new parameters, $\alpha^{RS}_{ij}(\tau) \to \alpha^{RS}_{i,j,j+2}(\tau)$, $\beta^{RS}_{ij}(\tau) \to \beta^{RS}_{i,j,j+2}(\tau)$, and $\xi_{ij}(\tau) \to \xi_{i,j,j+2}(\tau)$.

Now, using the previously given equations of motion from Sec. IV B, here we list the newly obtained moment equations. Therefore recalling Eq. (105) and Eq. (108), or equivalently writing Eq. (112) for $i = 3$ and $j = 0$, leads to the following equation of motion to close the conservation equations,

$$\frac{\partial \hat{I}^{RS}_{300}}{\partial \tau} - \frac{2}{3}\frac{\partial \hat{I}^{RS}_{320}}{\partial \tau} + \frac{1}{\tau}\hat{I}^{RS}_{300} = \hat{C}^{RS}_{20} - \frac{2}{3}\hat{C}^{RS}_{02} \approx \hat{C}^{RS}_{20,AW} - \frac{2}{3}\hat{C}^{RS}_{02,AW}\,. \tag{113}$$

Then similarly, using Eq. (106) and Eq. (109), or similarly writing Eq. (112) for $i = 4$ and $j = 0$, we obtain

$$\frac{\partial \hat{I}^{RS}_{400}}{\partial \tau} - \frac{\partial \hat{I}^{RS}_{420}}{\partial \tau} + \frac{1}{\tau}\left(\hat{I}^{RS}_{400} - \hat{I}^{RS}_{440}\right) = \hat{C}^{RS}_{30} - \hat{C}^{RS}_{12} \approx \hat{C}^{RS}_{30,AW} - \hat{C}^{RS}_{12,AW}\,, \tag{114}$$

and finally, from Eq. (107) and Eq. (110), or from Eq. (112) for $i = 5$ and $j = 0$, we get

$$\frac{\partial \hat{I}^{RS}_{500}}{\partial \tau} - \frac{4}{3}\frac{\partial \hat{I}^{RS}_{520}}{\partial \tau} + \frac{1}{\tau}\left(\hat{I}^{RS}_{500} - \frac{8}{3}\hat{I}^{RS}_{540}\right) = \hat{C}^{RS}_{30} - \frac{4}{3}\hat{C}^{RS}_{22} \approx \hat{C}^{RS}_{30,AW} - \frac{4}{3}\hat{C}^{RS}_{22,AW}\,. \tag{115}$$

The solutions to the conservation equations (103,104) closed by these new moment equations are shown in Fig. 3. Similarly to the previous results in Fig. 1 and Fig. 2 this new figure also shows the evolution of the fugacity $\lambda(\tau)$, the temperature $T(\tau)$, the anisotropy parameter $\xi(\tau)$ and the ratio of the longitudinal and the transverse pressure components $\hat{P}_l(\tau)/\hat{P}_\perp(\tau)$ as a function of proper time.

All figures in the left (a) and right (b) columns of Fig. 3 present the solutions of the conservation laws closed by the moment equations corresponding to Eq. (113), Eq. (114), and Eq. (115); red, green and blue lines respectively. These are denoted in the figure captions by $\hat{C}^{RS}_{20}\&\hat{C}^{RS}_{02}$, $\hat{C}^{RS}_{30}\&\hat{C}^{RS}_{12}$ and $\hat{C}^{RS}_{40}\&\hat{C}^{RS}_{22}$. The solid lines always represent the solutions of anisotropic fluid dynamics including the moments of the binary collision integral, while the dashed lines represent the solutions of the fluid dynamical equations in the RTA and the captions have the additional AW label. The difference between the (a) and (b) columns are due to the different initial values of the anisotropy parameter that is $\xi_0 = 0$ and $\xi_0 = 50$ respectively. Furthermore, in Fig. 3 the black dotted lines represent the solution of the conservation equations closed by the moment equation for $\hat{P}_l$ in the RTA, i.e., Eq. (111).

As expected, now the solutions to the moment equations closed by two dynamical moments (113-115) show a very good agreement with the "judicious choice" which represents the best match to the exact numerical solution of the Boltzmann equation in the RTA [38, 39, 54]. These new solutions in the RTA presented in Fig. 3 differ from each other and from the best match by less than the thickness of the line hence they overlap and cover each other.

Similarly, also the solutions with the non-linear collision integral are in a better agreement with the solutions shown in the (b) panels of Fig. 1 and Fig. 2; i.e., the solutions for $\hat{I}_{i+2,2,0}$. To express it differently, the solutions corresponding to a single dynamical moment $\hat{I}_{i,0,0}$ shown in the (a) panels of Fig. 1 and Fig. 2, improve considerably by taking into account an additional dynamical moment of the second power of the longitudinal momentum, i.e., $\hat{I}_{i+2,2,0}$. Therefore, we also expect that using the general equation of motion (112) recursively to include additional dynamical moments would also improve the solutions for the higher-order moments when compared to the Boltzmann equation.

Even so important differences between the solutions corresponding to the binary collision integral and the collision term in the RTA remain; this is apparent when comparing the full lines to the dashed lines in all figures presented so far. In the next section we aim to minimize these differences through the re-scaling of the relaxation-time parameter that optimally matches the moments of the non-linear binary collision integral to the moments in the RTA.

D. Matching to the RTA and the re-scaling of the relaxation-time

To better understand the differences between the binary collision integral and the RTA we will study the moments of the collision integral $\hat{C}_{ij}$, multiplied by the relaxation time parameter τ_R, and divided by the corresponding equilibrium

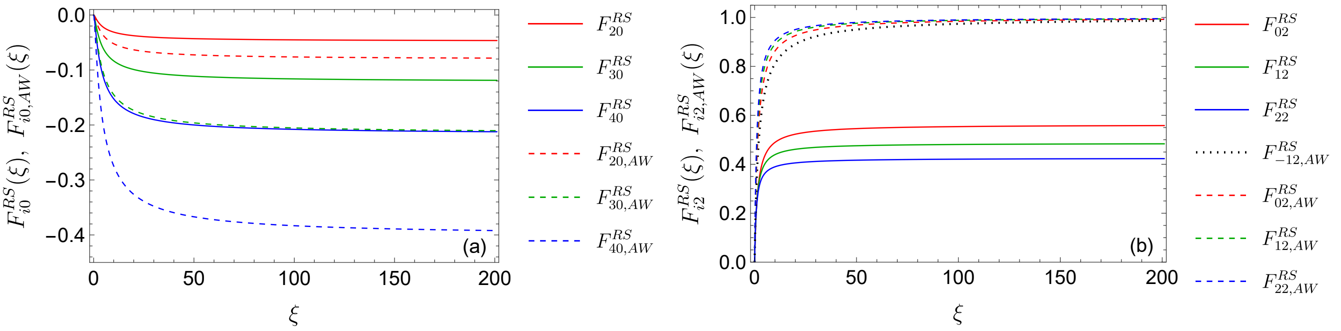

FIG. 4. The normalized and dimensionless collision terms F_{ij}^{RS} with solid lines, $F_{ij,AW}^{RS}$ with dashed lines, and $F_{-12,AW}^{RS}$ with dotted black line. Panel (a) on the left: the red solid line is F_{20}^{RS}, the green solid line is F_{30}^{RS}, while the blue solid line is F_{20}^{RS}. The dashed lines with the same color coding represent $F_{i0,AW}^{RS}$, i.e., the ratios in the RTA. Panel (b) on the right: F_{02}^{RS}, F_{12}^{RS}, and F_{22}^{RS}, with red, green, and blue lines. The dashed lines with red, green and blue represent, $F_{02,AW}^{RS}$, $F_{12,AW}^{RS}$, and $F_{22,AW}^{RS}$.

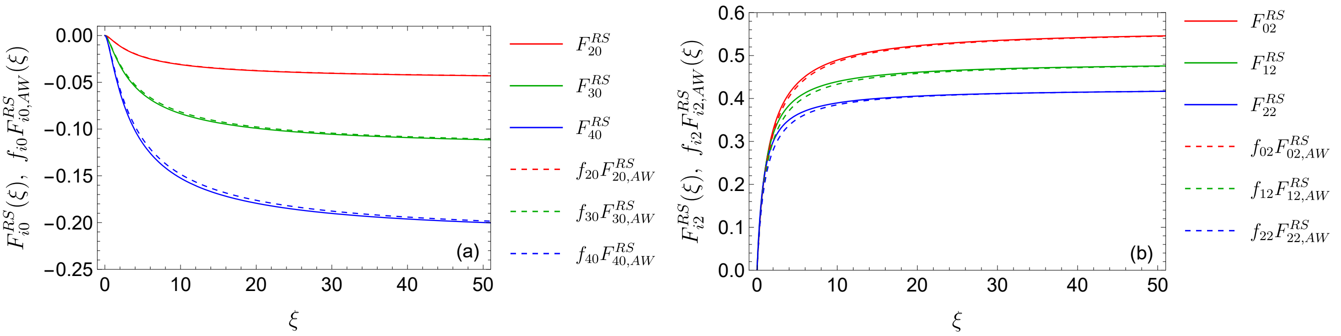

FIG. 5. The same as Fig. 4, but with scaled values, see Eq. (124), of the collision terms in the RTA; i.e., $f_{ij}F_{ij,AW}^{RS}$ with dashed lines, to match the moments of the binary collision integral; i.e., F_{ij}^{RS} with red, green, and blue lines.

moment $I_{i+j+1,j,0}$. Thereby we define the following dimensionless ratios,

$$F_{ij}^{RS}(\xi) \equiv \tau_R \frac{\hat{C}_{ij}^{RS}(\xi)}{I_{i+j+1,j,0}}, \tag{116}$$

$$F_{ij,AW}^{RS}(\xi) \equiv \tau_R \frac{\hat{C}_{ij,AW}^{RS}(\xi)}{I_{i+j+1,j,0}} = 1 - \frac{\hat{I}_{i+j+1,j,0}^{RS}}{I_{i+j+1,j,0}}, \tag{117}$$

where F_{ij}^{RS} is defined using the moments of the binary collision integral, while $F_{ij,AW}^{RS}$ makes use of Eq. (101). Note that these ratios are function of the anisotropy parameter alone and hence we can study the behavior of the moments of the collision integral as function of ξ.

Now recalling the moments of the nonlinear binary collision integral from Eqs. (94-96) and Eqs. (97-99) as well as the moments of the RTA approximation (101), we present the corresponding dimensionless ratios in Fig. 4. In Fig. 4(a) on the left: the red solid line is F_{20}^{RS}, the green solid line is F_{30}^{RS}, while the blue solid line is F_{40}^{RS}. The dashed lines with the same color coding show the corresponding ratio in the RTA, i.e., $F_{20,AW}^{RS}$, $F_{30,AW}^{RS}$, and $F_{40,AW}^{RS}$. Similarly, in Fig. 4(b) on the right: the red solid line is F_{02}^{RS}, the green solid line is F_{12}^{RS}, while the blue solid line is F_{22}^{RS}. The dashed lines with the same color coding represent the ratios in the RTA, i.e., $F_{i2,AW}^{RS}$.

First and foremost we observe that for all values of the anisotropy parameter the absolute value of the moments of the normalized binary collision integral, F_{ij}^{RS}, are always smaller than the absolute value corresponding to the same moments of the collision integral in the RTA, $F_{ij,AW}^{RS}$. This also means that the lhs of the moment equations is larger in the RTA which in turn drives the system faster to equilibrium. As expected for $\xi = 0$, the normalized moments of the collision integrals are also zero; $\lim_{\xi\to 0} F_{ij}^{RS}(\xi) = \lim_{\xi\to 0} F_{ij,AW}^{RS}(\xi) = 0$. Furthermore, by increasing $\xi > 0$ these ratios also increase and approach their corresponding maximums (asymptote) as a function of the anisotropy parameter.

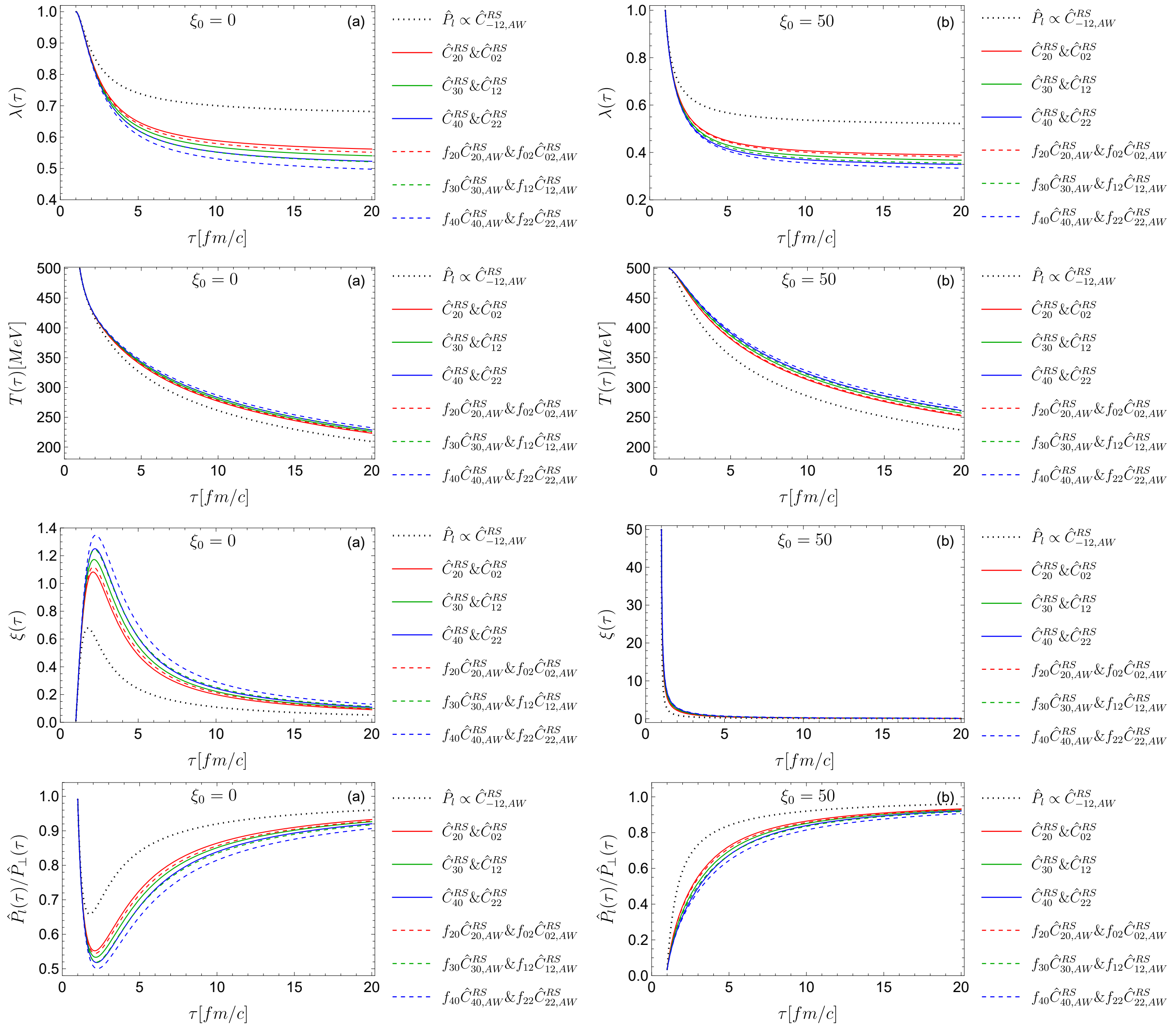

FIG. 6. Similar to Fig. 3, showing the evolution of the fugacity $\lambda(\tau)$, the temperature $T(\tau)$, the anisotropy parameter $\xi(\tau)$ and the ratio of the longitudinal and the transverse pressure components $\hat{P}_l(\tau)/\hat{P}_\perp(\tau)$. The black dotted lines represent the solution of the conservation equations closed by the equation of $\hat{P}_l$ from Eq. (111). The solid lines with, red, green, and blue represent solution of two dynamical moments and corresponding binary collision terms, while the dashed lines of the same color represent the solutions with two moments and re-scaled RTA. All (a) panels are for $\xi_0 = 0$ while all (b) panels show the solutions of the same equations but with an initial anisotropy of $\xi_0 = 50$.

In the asymptotic limit when $\xi \to \infty$, the ratios $F^{RS}_{i0,AW}(\xi)$ and $F^{RS}_{i0}(\xi)$ lead to the following finite values

$$\lim_{\xi\to\infty} F^{RS}_{20,AW}(\xi) = 1 - \frac{32}{3\pi^2}\,, \quad \lim_{\xi\to\infty} F^{RS}_{20}(\xi) = -\frac{512 - 45\pi^2}{144\pi^2}\,, \tag{118}$$

$$\lim_{\xi\to\infty} F^{RS}_{30,AW}(\xi) = 1 - \frac{12}{\pi^2}\,, \quad \lim_{\xi\to\infty} F^{RS}_{30}(\xi) = -\frac{6}{5\pi^2}\,, \tag{119}$$

$$\lim_{\xi\to\infty} F^{RS}_{40,AW}(\xi) = 1 - \frac{2048}{15\pi^4}\,, \quad \lim_{\xi\to\infty} F^{RS}_{40}(\xi) = -\frac{2816 - 189\pi^2}{45\pi^4}\,, \tag{120}$$

and similarly the $F^{RS}_{i2,AW}(\xi)$ and $F^{RS}_{i2}(\xi)$ lead to

$$\lim_{\xi\to\infty} F^{RS}_{02,AW}(\xi) = 1\,, \quad \lim_{\xi\to\infty} F^{RS}_{02}(\xi) = \frac{9}{16}\,, \tag{121}$$

$$\lim_{\xi\to\infty} F^{RS}_{12,AW}(\xi) = 1\,, \quad \lim_{\xi\to\infty} F^{RS}_{12}(\xi) = \frac{24}{5\pi^2}\,, \tag{122}$$

$$\lim_{\xi\to\infty} F^{RS}_{22,AW}(\xi) = 1\,, \quad \lim_{\xi\to\infty} F^{RS}_{22}(\xi) = \frac{6656+1215\pi^2}{450\pi^4}\,. \tag{123}$$

The ratio of the (normalized) collision terms in the asymptotic limit,

$$f_{ij} \equiv \lim_{\xi\to\infty} \frac{F^{RS}_{ij}(\xi)}{F^{RS}_{ij,AW}(\xi)} = \lim_{\xi\to\infty} \frac{\hat{C}^{RS}_{ij}}{\hat{C}^{RS}_{ij,AW}}\,, \tag{124}$$

defines the corresponding scaling factor of the relaxation time parameter τ_R of the RTA that optimally matches the asymptotic value of the binary collision integral as a function of the anisotropy parameter. Therefore, using the asymptotic scale factors we expect to improve the relaxation time approximation, similarly as in Eq. (101), but now with properly scaled relaxation-time parameter, $\tau_R \to \tau_{ij}$, as

$$\begin{aligned} \hat{C}^{RS}_{ij}(\xi) &\approx f_{ij}\hat{C}^{RS}_{ij,AW}(\xi) \equiv \frac{\tau_R}{\tau_{ij}}\hat{C}^{RS}_{ij,AW}(\xi) \\ &= -\frac{1}{\tau_{ij}}\hat{I}^{RS}_{i+j+1,j,0} + \frac{1}{\tau_{ij}} I_{i+j+1,j,0}\,, \end{aligned} \tag{125}$$

where the corresponding asymptotic relaxation times are

$$\tau_{ij} \equiv \tau_R f^{-1}_{ij} = \tau_R \lim_{\xi\to\infty} \frac{\hat{C}^{RS}_{ij,AW}(\xi)}{\hat{C}^{RS}_{ij}(\xi)}\,. \tag{126}$$

Henceforth the inverse of the nonlinear binary collision term is proportional to microscopic time scales and can be interpreted as an effective relaxation time proportional to $\tau_R = 1/(\sigma_T n_0)$. However, the relaxation times corresponding to the binary collision integral have different relaxation rates or collision frequencies f_{ij} for different moments. For example using the previously obtained asymptotic values, here we list the numerical results,

$$f^{-1}_{20} = \frac{1536-144\pi^2}{512-45\pi^2} \approx 1.691\,, \quad f^{-1}_{02} = \frac{16}{9} \approx 1.777\,, \tag{127}$$

$$f^{-1}_{30} = 10 - \frac{5\pi^2}{6} \approx 1.775\,, \quad f^{-1}_{12} = \frac{5\pi^2}{24} \approx 2.056\,, \tag{128}$$

$$f^{-1}_{40} = \frac{6144-45\pi^4}{2816-189\pi^2} \approx 1.852\,, \quad f^{-1}_{22} = \frac{450\pi^4}{6656+1215\pi^2} \approx 2.350\,. \tag{129}$$

These results show that the "proper" relaxation times of the binary collision integral τ_{ij} increase with the order of the moments, i.e., the power i of energy $E_{\mathbf{k}u}$ and the power j of momentum in the direction of the anisotropy $E_{\mathbf{k}l}$. This is in fact also well known from the linearized binary collision integral in transient fluid dynamics, where various physical phenomena, such as viscosity or diffusion processes are also described by different moments which by their nature relax on different timescales [50]. The nonlinear collision dynamics and hence the differences in time scales cannot be captured by the simplistic RTA with a single relaxation-time parameter [57, 58]. However, inspecting the moments of the nonlinear binary collision term we find that the time it takes to reach local equilibrium becomes longer for higher-order moments. This translates to slower relaxation time scales and hence the higher-order moments contributing to the higher-frequency dynamics should not be neglected from the fluid dynamical description. This behavior arises from coupling effects characteristic of the nonlinear Boltzmann collision term which leads to a spectrum of microscopic relaxation time scales even in the isotropic limit [36, 37].

The outcome of these modifications to the RTA using Eq. (125) is shown in Fig. 5, where we have plotted the moments of the binary collision integral $F^{RS}_{i0}(\xi)$ on the lhs (a) and $F^{RS}_{i2}(\xi)$ on rhs (b) with red, green, and blue, lines. The corresponding scaled values $f_{i0}F^{RS}_{i0}(\xi)$ are included on the lhs of Fig. 5 panel (a), while $f_{i2}F^{RS}_{i2}(\xi)$ on the rhs of Fig. 5 panel (b), with dashed lines, red, green, and blue. The overall agreement is excellent and only small differences are visible at small ξ, while the differences gradually diminish with the increase of the anisotropy parameter as we approach the asymptotic limit.

The evolution of the expanding system using the scaled relaxation times τ_{ij} in the RTA according to Eq. (125) are shown in Fig. 6. However, here for the sake of brevity we only present the fluid dynamical solutions in the case when the conservation laws were closed by two dynamical moments (112) as in the previous section. Similarly to Fig. 3, we present the evolution of the fugacity, the temperature, the anisotropy parameter, and the ratio of the longitudinal and the transverse pressure components as function of the proper time. The overall conclusions regarding the results presented in Fig. 6 are that using the asymptotic scaling of for the RTA matches the solutions obtained with the non-linear binary collision integral reasonably well. The small differences between the binary collision integral and the scaled RTA presented in Fig. 5, drives the evolution of the system in slightly different ways.

V. CONCLUSIONS AND OUTLOOK

In this paper we have computed the scalar moments of the nonlinear binary collision integral in the ultrarelativistic hard-sphere approximation assuming a constant and energy independent transport cross section. The moments of the collision integral are expressed in quadratic products of anisotropic thermodynamic integrals and represent the nonlinear dependence of the binary collision integral on the distribution function. These exact results are valid for arbitrary anisotropic distribution functions of the form $\hat{f}_{0\mathbf{k}}\left(\hat{\alpha}, \hat{\beta}_u E_{\mathbf{k}u}, \hat{\beta}_l E_{\mathbf{k}l}\right)$, which are commonly interpreted having distinct temperatures in the direction of the anisotropy $\hat{\beta}_l^{-1}$ and perpendicular to it $\hat{\beta}_u^{-1}$.

As a particular example we have chosen the well known spheroidal distribution function, also known as the Romatschke-Strickland distribution function [40], to study the moments of the binary collision integral. In this case the moments of the binary collision integral simplify considerably since the corresponding anisotropic thermodynamical integrals vanish, i.e., $\hat{I}^{RS}_{nrq} = 0$, for all odd powers of the momentum in the direction of the anisotropy $E^r_{\mathbf{k}l}$. Furthermore, the well known and widely used relativistic relaxation-time approximation of Anderson and Witting [41] for the collision integral was also included for comparison.

To better understand the differences between the nonlinear binary collision integral and the RTA we have studied the time evolution of an ultrarelativistic massless but particle conserving system in the well known longitudinal boost-invariant Bjorken expansion scenario. Then we discussed various possible closures for the conservation laws based on different higher-order moments of the Boltzmann equation. Additionally, we demonstrated that the ambiguity in selecting a higher-order moment for closure can be effectively resolved by including not one but two corresponding dynamical moments from the hierarchy. Although now the choice of moment(s) selected for the closure is no longer of primary concern, differences between the moments of the nonlinear binary collision integral and the corresponding moments computed in the relaxation-time approximation remain significant. These differences highlight the importance of nonlinear couplings between the moments and the limitations of the RTA.

Our main conclusion regarding the spheroidal distribution function is that the collision term in the RTA drives the system towards equilibrium faster than the binary collision integral irrespective of the choice of the dynamical moment(s) used to close the conservation laws. To remedy such inaccuracies inherent in the simplified relaxation-time approximation we have re-scaled the relaxation time of the RTA based on the asymptotic values of the ratio of the binary collision integral and the RTA collision integral. This scaled RTA reproduces reasonably well the properties of the non-linear binary collision integral in the case of the RS distribution function.

In closing we also mention that these exact analytic results are also important in case of a more complete anisotropic distribution function, $\hat{f}_{\mathbf{k}} = \hat{f}_{0\mathbf{k}} + \delta\hat{f}_{\mathbf{k}}$, since the same steps and principles as presented here are required to compute the moments of the collision integral, $C\left[\hat{f}_0 + \delta\hat{f}\right]$. Moreover our exact results are also useful for investigating the fluid dynamical evolution of a wide range of anisotropic distribution functions frequently utilized in plasma physics. Additionally, our exact results can be used to test various relativistic transport frameworks that incorporate the Boltzmann collision term [59, 60], and hence facilitate direct comparisons between leading-order anisotropic fluid dynamical models and relativistic transport models.

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Appendix A: The general equations of motion of leading-order anisotropic fluid dynamics

For reasons of completeness we recall here the general equations of anisotropic fluid dynamics from Refs. [38] see Eqs. (24) and (25) of Ref. [38],

$$\begin{aligned}\mathcal{C}_{i-1,j} &= D\hat{\mathcal{I}}_{ij} - D_l\hat{\mathcal{I}}_{i-1,j+1} - \left[i\hat{\mathcal{I}}_{i-1,j+1} + j\hat{\mathcal{I}}_{i+1,j-1}\right] l_\lambda D u^\lambda \\ &+ \left[(i-1)\,\hat{\mathcal{I}}_{i-2,j+2} + (j+1)\,\hat{\mathcal{I}}_{i,j}\right] l_\lambda D_l u^\lambda \\ &- \frac{1}{2}\tilde{\theta}\left[m_0^2\,(i-1)\,\hat{\mathcal{I}}_{i-2,j} - (i+1)\,\hat{\mathcal{I}}_{i,j} + (i-1)\,\hat{\mathcal{I}}_{i-2,j+2}\right] \\ &+ \frac{1}{2}\tilde{\theta}_l\left[m_0^2 j\hat{\mathcal{I}}_{i-1,j-1} - j\hat{\mathcal{I}}_{i+1,j-1} + (j+2)\,\hat{\mathcal{I}}_{i-1,j+1}\right], \end{aligned} \tag{A1}$$

and

$$\begin{aligned}\mathcal{C}^{\{\mu\}}_{i-1,j} &= \frac{1}{2}\tilde{\nabla}^\mu\left(m_0^2\hat{\mathcal{I}}_{i-1,j} - \hat{\mathcal{I}}_{i+1,j} + \hat{\mathcal{I}}_{i-1,j+2}\right) \\ &- \frac{(i-1)}{2}\left(m_0^2\hat{\mathcal{I}}_{i-2,j+1} - \hat{\mathcal{I}}_{i,j+1} + \hat{\mathcal{I}}_{i-2,j+3}\right) l_\lambda\tilde{\nabla}^\mu u^\lambda \\ &- \frac{j}{2}\left(m_0^2\hat{\mathcal{I}}_{i,j-1} - \hat{\mathcal{I}}_{i+2,j-1} + \hat{\mathcal{I}}_{i,j+1}\right) l_\lambda\tilde{\nabla}^\mu u^\lambda \\ &- \frac{1}{2}\left[m_0^2 i\hat{\mathcal{I}}_{i-1,j} - (i+2)\,\hat{\mathcal{I}}_{i+1,j} + i\hat{\mathcal{I}}_{i-1,j+2}\right]\Xi^\mu_\lambda D u^\lambda \\ &+ \frac{1}{2}\left[m_0^2 j\hat{\mathcal{I}}_{i,j-1} - j\hat{\mathcal{I}}_{i+2,j-1} + (j+2)\,\hat{\mathcal{I}}_{i,j+1}\right]\Xi^\mu_\lambda D l^\lambda \\ &+ \frac{1}{2}\left[m_0^2\,(i-1)\,\hat{\mathcal{I}}_{i-2,j+1} - (i+1)\,\hat{\mathcal{I}}_{i,j+1} + (i-1)\,\hat{\mathcal{I}}_{i-2,j+3}\right]\Xi^\mu_\lambda D_l u^\lambda \\ &- \frac{1}{2}\left[m_0^2\,(j+1)\,\hat{\mathcal{I}}_{i-1,j} - (j+1)\,\hat{\mathcal{I}}_{i+1,j} + (j+3)\,\hat{\mathcal{I}}_{i-1,j+2}\right]\Xi^\mu_\lambda D_l l^\lambda\,. \end{aligned} \tag{A2}$$

Here, $\partial_\mu \equiv u_\mu D + l_\mu D_l + \tilde{\nabla}_\mu$, such that $D \equiv u^\mu\partial_\mu$ denotes the comoving derivative and $D_l \equiv -l^\mu\partial_\mu$ is the derivative in the direction of the anisotropy, while the spatial gradient orthogonal to both u^μ and l^μ is $\tilde{\nabla}_\mu \equiv \Xi_{\mu\nu}\partial^\nu$. The expansion scalars are $\tilde{\theta} = \tilde{\nabla}_\mu u^\mu$ and $\tilde{\theta}_l = \tilde{\nabla}_\mu l^\mu$.

Now, noting that due to the conservation of the particle-number in binary collisions $\hat{C}_{00} = 0$, while $\hat{C}_{10} = 0$ and $\hat{C}_{01} = 0$ vanish due to energy conservation and momentum conservation in the direction of the anisotropy. Thus the particle-number conservation equation follows from Eq. (A1) by choosing $i = 1$ and $j = 0$,

$$0 = \partial_\mu\hat{N}^\mu \equiv D\hat{n} - D_l\hat{n}_l + \hat{n}\tilde{\theta} + \hat{n}_l\tilde{\theta}_l + \hat{n}l_\mu D_l u^\mu - \hat{n}_l l_\mu D u^\mu\,, \tag{A3}$$

while the energy-conservation equation is obtained by choosing $i = 2$ and $j = 0$,

$$0 = u_\nu\partial_\mu\hat{T}^{\mu\nu} \equiv D\hat{e} - D_l\hat{M} + \left(\hat{e} + \hat{P}_\perp\right)\tilde{\theta} + \hat{M}\tilde{\theta}_l + \left(\hat{e} + \hat{P}_l\right) l_\mu D_l u^\mu - 2\hat{M} l_\mu D u^\mu\,, \tag{A4}$$

and the conservation equation for the momentum in the direction of the anisotropy follows for $i = 1$ and $j = 1$,

$$0 = l_\nu\partial_\mu\hat{T}^{\mu\nu} \equiv D\hat{M} - D_l\hat{P}_l + \hat{M}\tilde{\theta} - \left(\hat{P}_\perp - \hat{P}_l\right)\tilde{\theta}_l + 2\hat{M} l_\mu D_l u^\mu - \left(\hat{e} + \hat{P}_l\right) l_\mu D u^\mu\,. \tag{A5}$$

Finally, due to the conservation of transverse momenta in microscopic collisions $\hat{C}^{\{\mu\}}_{00} = 0$, hence the conservation equation for the momentum transverse to the direction of the anisotropy follows from Eq. (A2) for $i = 1$ and $j = 0$,

$$\begin{aligned}0 = \Xi^\alpha_\nu\partial_\mu\hat{T}^{\mu\nu} &\equiv \left(\hat{e} + \hat{P}_\perp\right)(Du^\alpha + l^\alpha l_\nu D u^\nu) - \tilde{\nabla}^\alpha\hat{P}_\perp + \left(\hat{P}_\perp - \hat{P}_l\right)(D_l l^\alpha + u^\alpha l_\nu D_l u^\nu) \\ &+ \hat{M}\,(Dl^\alpha + u^\alpha l_\nu D u^\nu) - \hat{M}\,(D_l u^\alpha + l^\alpha l_\nu D_l u^\nu)\,. \end{aligned} \tag{A6}$$

Appendix B: The center of momentum frame

It is useful to define the center-of-momentum (CM) frame where the total momentum in binary collisions in Eq. (5) is $P_T^\mu \equiv \left(P_T^0, \mathbf{P}_T\right) \stackrel{\text{CM}}{=} (\sqrt{s}, \mathbf{0})$, such that

$$P_T^0 \stackrel{\text{CM}}{=} k^0 + k'^0 = p^0 + p'^0 = \sqrt{s}\,, \tag{B1}$$

$$\mathbf{P}_T \stackrel{\text{CM}}{=} \mathbf{k} + \mathbf{k}' = \mathbf{p} + \mathbf{p}' = \mathbf{0}\,. \tag{B2}$$

Furthermore, since $P_T^\mu P_{T,\mu} = s$, in the CM frame we also have,

$$P_T^\mu k_\mu \equiv P_T^\mu k'_\mu = m_0^2 + k^\mu k'_\mu \stackrel{\text{CM}}{=} \frac{s}{2}\,, \tag{B3}$$

$$\Delta_T^{\mu\nu} k_\mu k_\nu \equiv m_0^2 - \left(P_T^\mu k_\mu\right)^2 / s \stackrel{\text{CM}}{=} m_0^2 - \frac{s}{4}\,, \tag{B4}$$

and hence in the ultrarelativistic limit, when $k^\mu k_\mu = m_0^2 \to 0$, and $k^0 = |\mathbf{k}| \equiv k$, we also obtain,

$$s \equiv P_T^\mu P_{T,\mu} \underset{m_0\to 0}{=} 2k^\mu k'_\mu\,. \tag{B5}$$

Recalling Eq. (43) together with the definition of the transition rate from Eq. (35) we obtain the $\mathcal{B}_{nq}$ coefficient

$$\begin{aligned}
\mathcal{B}_{nq} &\equiv \frac{(-1)^q}{(2q+1)!!}\frac{(2\pi)^6}{2}\int_{PP'} s\frac{d\sigma(s,\Omega)}{d\Omega}\delta(k^\mu + k'^\mu - p^\mu - p'^\mu)\left(\frac{P_T^\mu p_\mu}{s}\right)^{n-2q}\left(\Delta_T^{\mu\nu}p_\mu p_\nu\right)^q \\
&= \frac{1}{(2q+1)!!}\frac{\sigma_T(s)}{2^n}\int \frac{dp}{\left(p^0\right)^2}p^2\delta(\sqrt{s}-2p^0)s\left(s-4m_0^2\right)^q \\
&= \frac{\sigma_T(s)}{2^{n+1}(2q+1)!!}\sqrt{s}\left(s-4m_0^2\right)^{(2q+1)/2}\,,
\end{aligned} \tag{B6}$$

where we used that $p^0 \equiv p'^0 = \sqrt{s}/2$ and $\int_{PP'}\delta(k^\mu + k'^\mu - p^\mu - p'^\mu) = 1/(2\pi)^5$. This is precisely the result obtained earlier; see for example Eq. (G6) in Ref. [49] and Eq. (D3) in Ref. [50]. Note however that both references contain the same typographical error in the definition of $\mathcal{B}_{nq}$ as an integral over P and P'. This is easily remedied by replacing $\sqrt{s} \to s$ in the term $(P_T^\mu p_\mu/\sqrt{s}) \to (P_T^\mu p_\mu/s)$ in those references. Also note that this typographical error does not effect the results of Refs. [49, 50] since they used the correct value of $\mathcal{B}_{nq}$ obtained after integration.

Using this result we obtain the $\Theta^{\mu_1\cdots\mu_n} \equiv \mathcal{P}_{00}^{\mu_1\cdots\mu_n}$ tensors defined in Eq. (42), hence for $n=0$ and $n=1$ we obtain,

$$\Theta \equiv \mathcal{P}_{00} = \sigma_T\frac{s}{2}\,, \tag{B7}$$

$$\Theta^\mu \equiv \mathcal{P}_{00}^\mu = \sigma_T\frac{s}{2^2}P_T^\mu\,, \tag{B8}$$

where we also used that $b_{00} = b_{10} = 1$. Similarly for $n=2$ and $n=3$, Eq. (42) leads to,

$$\Theta^{\mu\nu} \equiv \mathcal{P}_{00}^{\mu\nu} = \sigma_T\frac{s}{2^3}\left(P_T^\mu P_T^\nu - \frac{1}{3}s\Delta_T^{\mu\nu}\right)\,, \tag{B9}$$

$$\Theta^{\mu\nu\alpha} \equiv \mathcal{P}_{00}^{\mu\nu\alpha} = \sigma_T\frac{s}{2^4}\left(P_T^\mu P_T^\nu P_T^\alpha - s\Delta_T^{(\mu\nu}P_T^{\alpha)}\right)\,, \tag{B10}$$

where $b_{20} = b_{21} = b_{30} = 1$, and $b_{31} = 3$ was used. Finally for $n=4$ with $b_{40} = 1$, $b_{41} = 6$, and $b_{42} = 3$ we get,

$$\Theta^{\mu\nu\alpha\beta} \equiv \mathcal{P}_{00}^{\mu\nu\alpha\beta} = \sigma_T\frac{s}{2^5}\left(P_T^\mu P_T^\nu P_T^\alpha P_T^\beta - 2s\Delta_T^{(\mu\nu}P_T^\alpha P_T^{\beta)} + \frac{1}{5}s^2\Delta_T^{(\mu\nu}\Delta_T^{\alpha\beta)}\right)\,. \tag{B11}$$

For later use, here we also list the moments of the anisotropic distribution from Eq. (9) in the following cases,

$$\hat{\mathcal{I}}_{ij}^{\mu} = \hat{I}_{i+j+1,j,0}u^\mu + \hat{I}_{i+j+1,j+1,0}l^\mu\,, \tag{B12}$$

$$\hat{\mathcal{I}}_{ij}^{\mu\nu} = \hat{I}_{i+j+2,j,0}u^\mu u^\nu + 2\hat{I}_{i+j+2,j+1,0}u^{(\mu}l^{\nu)} + \hat{I}_{i+j+2,j+2,0}l^\mu l^\nu - \hat{I}_{i+j+2,j,1}\Xi^{\mu\nu}\,, \tag{B13}$$

and

$$\begin{aligned}
\hat{\mathcal{I}}_{ij}^{\mu\nu\lambda} &= \hat{I}_{i+j+3,j,0}u^\mu u^\nu u^\lambda + 3\hat{I}_{i+j+3,j+1,0}u^{(\mu}u^\nu l^{\lambda)} + 3\hat{I}_{i+j+3,j+2,0}u^{(\mu}l^\nu l^{\lambda)} \\
&\quad + \hat{I}_{i+j+3,j+3,0}l^\mu l^\nu l^\lambda - 3\hat{I}_{i+j+3,j,1}u^{(\mu}\Xi^{\nu\lambda)} - 3\hat{I}_{i+j+3,j+1,1}l^{(\mu}\Xi^{\nu\lambda)}\,.
\end{aligned} \tag{B14}$$

Appendix C: The loss terms

Substituting $n \to n+1$ into Eq. (9) the moments of the anisotropic distribution function are generalized to $n+1$ tensor indices

$$\hat{\mathcal{I}}_{ij}^{\mu_1\cdots\mu_{n+1}} \equiv \sum_{q=0}^{\lfloor (n+1)/2 \rfloor} \sum_{r=0}^{n+1-2q} (-1)^q \, b_{n+1,r,q} \hat{I}_{i+j+n+1,j+r,q} \Xi^{(\mu_1\mu_2} \cdots \Xi^{\mu_{2q-1}\mu_{2q}} l^{\mu_{2q+1}} \cdots l^{\mu_{2q+r}} u^{\mu_{2q+r+1}} \cdots u^{\mu_{n+1})} \,, \quad \text{(C1)}$$

where the symmetrized tensor product is

$$\begin{aligned}
&\Xi^{(\mu_1\mu_2} \cdots \Xi^{\mu_{2q-1}\mu_{2q}} l^{\mu_{2q+1}} \cdots l^{\mu_{2q+r}} u^{\mu_{2q+r+1}} \cdots u^{\mu_{n+1})} \\
&= \frac{1}{b_{n+1,r,q}} \sum_{\mathcal{P}_\mu^{n+1}} \Xi^{\mu_1\mu_2} \cdots \Xi^{\mu_{2q-1}\mu_{2q}} l^{\mu_{2q+1}} \cdots l^{\mu_{2q+r}} u^{\mu_{2q+r+1}} \cdots u^{\mu_{n+1}} \,. \qquad \text{(C2)}
\end{aligned}$$

The total number of permutations of $n+1$ tensor indices is $(n+1)!$. The number of index permutations on symmetric rank 2 projections $\Xi^{\mu_i\mu_j}$ equals to $2^q q!$, while the four-vectors, l^{μ_i} and u^{μ_j}, also have $r!$ and $(n+1-r-2q)!$ number of permutations, respectively. These index permutations will not lead to distinct terms, and hence one needs to divide the total number of index permutations by these factors. Therefore the number of distinct terms is set by the $b_{n+1,r,q}$ coefficient, which follows from Eq. (11) substituting $n \to n+1$.

The symmetrized rank $n+1$ tensor product from Eq. (C2) can be separated into three distinct parts where each part will contain a well defined symmetrized rank n tensor product of Ξ's, u's, and l's,

$$\begin{aligned}
\hat{\mathcal{I}}_{ij}^{\mu_1\cdots\mu_{n+1}} &= \sum_{q=0}^{\lfloor (n+1)/2 \rfloor} \sum_{r=0}^{n+1-2q} (-1)^q \, b_{n+1,r,q} \frac{(n+1-r-2q)}{(n+1)} \hat{I}_{i+j+n+1,j+r,q} \\
&\times u^{\mu_{n+1}} \Xi^{(\mu_1\mu_2} \cdots \Xi^{\mu_{2q-1}\mu_{2q}} l^{\mu_{2q+1}} \cdots l^{\mu_{2q+r}} u^{\mu_{2q+r+1}} \cdots u^{\mu_n)} \\
&+ \sum_{q=0}^{\lfloor (n+1)/2 \rfloor} \sum_{r=0}^{n-2q} (-1)^q \, b_{n+1,r+1,q} \frac{(r+1)}{(n+1)} \hat{I}_{i+j+n+1,j+r+1,q} \\
&\times l^{\mu_{n+1}} \Xi^{(\mu_1\mu_2} \cdots \Xi^{\mu_{2q-1}\mu_{2q}} l^{\mu_{2q+1}} \cdots l^{\mu_{2q+r}} u^{\mu_{2q+r+1}} \cdots u^{\mu_n)} \\
&+ \sum_{q=0}^{\lfloor (n+1)/2 \rfloor} \sum_{r=0}^{n+1-2q} (-1)^q \, b_{n+1,r,q} \frac{(2q)}{(n+1)} \hat{I}_{i+j+n+1,j+r,q} \\
&\times \Xi^{\mu_{n+1}(\mu_1} \cdots \Xi^{\mu_{2q-1}\mu_{2q}} l^{\mu_{2q+1}} \cdots l^{\mu_{2q+r}} u^{\mu_{2q+r+1}} \cdots u^{\mu_n)} \,. \qquad \text{(C3)}
\end{aligned}$$

Here the first term removes $u^{\mu_{n+1}}$ from the tensor product (C2) and therefore the number of distinct terms remaining under symmetrization is $b_{n+1,r,q}\,(n+1-r-2q)\,/\,(n+1) = b_{nrq}$. Similarly, the second term separates $l^{\mu_{n+1}}$ from a total of $r+1$ such space-like four-vectors, therefore the number of remaining terms under symmetrization is $b_{n+1,r+1,q}\,(r+1)\,/\,(n+1) = b_{nrq}$. The last term isolates one tensor index from the rank 2 projection operator leading to terms of type $\Xi^{\mu_{n+1}(\mu_1} \cdots u^{\mu_n)}$, and thus the number of distinct terms remaining under symmetrization is $b_{n+1,r,q}\,(2q)\,/\,(n+1)$. Furthermore using Eq. (11) one can show that, $b_{nrq} \equiv b_{n+1,r,q}\frac{(n+1-r-2q)}{(n+1)} = b_{n+1,r+1,q}\frac{(r+1)}{(n+1)}$, and then recalling the definition of thermodynamic integrals from Eq. (9) we obtain

$$\begin{aligned}
&\hat{\mathcal{I}}_{ij}^{\mu_1\cdots\mu_n\lambda} = u^\lambda \hat{\mathcal{I}}_{i+1,j}^{\mu_1\cdots\mu_n} + l^\lambda \hat{\mathcal{I}}_{i,j+1}^{\mu_1\cdots\mu_n} \\
&+ \sum_{q=0}^{\lfloor (n+1)/2 \rfloor} \sum_{r=0}^{n+1-2q} (-1)^q \, b_{n+1,r,q} \frac{(2q)}{(n+1)} \hat{I}_{i+j+n+1,j+r,q} \Xi^{\lambda(\mu_1} \cdots \Xi^{\mu_{2q-1}\mu_{2q}} l^{\mu_{2q+1}} \cdots l^{\mu_{2q+r}} u^{\mu_{2q+r+1}} \cdots u^{\mu_n)} \,. \qquad \text{(C4)}
\end{aligned}$$

Now, using the decomposition of the particle four-current, $\hat{\mathcal{I}}_{00,\lambda} = \hat{I}_{100} u_\lambda + \hat{I}_{110} l_\lambda$, while also taking into account that all terms found in the second line of Eq. (C4) are proportional to $\Xi^{\lambda\mu}$. Since all such terms are by definition orthogonal to both u_λ and l_λ, therefore all terms proportional to $\Xi^{\lambda\mu}$ vanish from the tensor product $\hat{\mathcal{I}}_{ij}^{\mu_1\cdots\mu_n\lambda} \hat{\mathcal{I}}_{00,\lambda}$, and thus we obtain Eq. (46).

Appendix D: The gain terms

1. The $\hat{G}_{n0}$ gain terms

The simplest gain term $\hat{G}_{n0}$ is computed by inserting Eq. (42) into Eq. (60) and hence for $i = n$ and $j = 0$ we have

$$\hat{G}_{n0} = \int_{KK'} \hat{f}_{0\mathbf{k}} \hat{f}_{0\mathbf{k}'} \sum_{q=0}^{\lfloor n/2 \rfloor} (-1)^q \, b_{nq} \mathcal{B}_{nq} \Delta_T^{(\mu_1\mu_2} \cdots \Delta_T^{\mu_{2q-1}\mu_{2q}} P_T^{\mu_{2q+1}} \cdots P_T^{\mu_n)} u_{\mu_1} \cdots u_{\mu_n} \, . \tag{D1}$$

Now noting that the contraction of the symmetrized tensor product by $u_{\mu_1} \cdots u_{\mu_n}$ leads to,

$$\begin{aligned} \Delta_T^{(\mu_1\mu_2} \cdots \Delta_T^{\mu_{2q-1}\mu_{2q}} P_T^{\mu_{2q+1}} \cdots P_T^{\mu_n)} u_{\mu_1} \cdots u_{\mu_n} &\equiv u_{\mu_1} \cdots u_{\mu_n} \frac{1}{b_{nq}} \sum_{\mathcal{P}_\mu^n} \Delta_T^{\mu_1\mu_2} \cdots \Delta_T^{\mu_{2q-1}\mu_{2q}} P_T^{\mu_{2q+1}} \cdots P_T^{\mu_n} \\ &= \left(P_T^{\mu} u_\mu\right)^{n-2q} \left(\Delta_T^{\mu\nu} u_\mu u_\nu\right)^q \, , \end{aligned} \tag{D2}$$

and hence we obtain,

$$\Theta^{\mu_1 \cdots \mu_n} u_{\mu_1} \cdots u_{\mu_n} = \sum_{q=0}^{\lfloor n/2 \rfloor} (-1)^q \, b_{nq} \mathcal{B}_{nq} \left(P_T^{\mu} u_\mu\right)^{n-2q} \left(\Delta_T^{\mu\nu} u_\mu u_\nu\right)^q \, , \tag{D3}$$

which then proves the formula in Eq. (61).

Now, recalling Eq. (44) in the massless limit when $s \equiv 2k^\mu k'_\mu$, the gain term $\hat{G}_{n0}$ from Eq. (61) for $n = 0$ leads to

$$\begin{aligned} \hat{G}_{00} &\equiv \int_{KK'} \hat{f}_{0\mathbf{k}} \hat{f}_{0\mathbf{k}'} \mathcal{P}_{00} = \int_{KK'} \hat{f}_{0\mathbf{k}} \hat{f}_{0\mathbf{k}'} \Theta \\ &= \int_{KK'} \hat{f}_{0\mathbf{k}} \hat{f}_{0\mathbf{k}'} \left[b_{00} \mathcal{B}_{00}\right] = \sigma_T \int_{KK'} \hat{f}_{0\mathbf{k}} \hat{f}_{0\mathbf{k}'} k^\mu k'_\mu \\ &= \sigma_T \hat{\mathcal{I}}^{\mu}_{00} \hat{\mathcal{I}}_{00,\mu} = \sigma_T \left(\hat{I}_{100} \hat{I}_{100} - \hat{I}_{110} \hat{I}_{110} \right) \, , \end{aligned} \tag{D4}$$

where $b_{00} = 1$, and $\mathcal{P}_{00} \equiv \mathcal{B}_{00} = \sigma_T s/2$, and we used Eq. (B12) for the decomposition of the tensor moments.

Similarly, Eq. (61) for $n = 1$, leads to

$$\begin{aligned} \hat{G}_{10} &\equiv \int_{KK'} \hat{f}_{0\mathbf{k}} \hat{f}_{0\mathbf{k}'} \mathcal{P}_{10} = \int_{KK'} \hat{f}_{0\mathbf{k}} \hat{f}_{0\mathbf{k}'} \Theta^{\mu_1} u_{\mu_1} \\ &= \int_{KK'} \hat{f}_{0\mathbf{k}} \hat{f}_{0\mathbf{k}'} \left[b_{10} \mathcal{B}_{10} \left(P_T^{\mu} u_\mu\right)\right] = \frac{\sigma_T}{2} \int_{KK'} \hat{f}_{0\mathbf{k}} \hat{f}_{0\mathbf{k}'} \left(E_{\mathbf{k}u} + E_{\mathbf{k}'u}\right) k^\mu k'_\mu \\ &= \sigma_T \hat{\mathcal{I}}^{\mu}_{10} \hat{\mathcal{I}}_{00,\mu} = \sigma_T \left(\hat{I}_{200} \hat{I}_{100} - \hat{I}_{210} \hat{I}_{110} \right) \, , \end{aligned} \tag{D5}$$

where $b_{10} = 1$, and $\mathcal{B}_{10} = \sigma_T s/4$, and we used Eq. (62) as well as Eq. (B12).

Likewise for $n = 2$ we have

$$\begin{aligned} \hat{G}_{20} &\equiv \int_{KK'} \hat{f}_{0\mathbf{k}} \hat{f}_{0\mathbf{k}'} \mathcal{P}_{20} = \int_{KK'} \hat{f}_{0\mathbf{k}} \hat{f}_{0\mathbf{k}'} \Theta^{\mu_1\mu_2} u_{\mu_1} u_{\mu_2} \\ &= \int_{KK'} \hat{f}_{0\mathbf{k}} \hat{f}_{0\mathbf{k}'} \left[b_{20} \mathcal{B}_{20} \left(P_T^{\mu} u_\mu\right)^2 - b_{21} \mathcal{B}_{21} \left(\Delta_T^{\mu\nu} u_\mu u_\nu\right)\right] \\ &= \frac{\sigma_T}{3} \int_{KK'} \hat{f}_{0\mathbf{k}} \hat{f}_{0\mathbf{k}'} \left(E_{\mathbf{k}u} + E_{\mathbf{k}'u}\right)^2 k^\mu k'_\mu - \frac{\sigma_T}{6} \int_{KK'} \hat{f}_{0\mathbf{k}} \hat{f}_{0\mathbf{k}'} \left(k^\mu k'_\mu\right)^2 \\ &= \frac{2\sigma_T}{3} \left(\hat{\mathcal{I}}^{\mu}_{20} \hat{\mathcal{I}}_{00,\mu} + \hat{\mathcal{I}}^{\mu}_{10} \hat{\mathcal{I}}_{10,\mu} \right) - \frac{\sigma_T}{6} \hat{\mathcal{I}}^{\mu\nu}_{00} \hat{\mathcal{I}}_{00,\mu\nu} \\ &= \frac{\sigma_T}{6} \left(4\hat{I}_{300} \hat{I}_{100} - 4\hat{I}_{310} \hat{I}_{110} + 3\hat{I}_{200} \hat{I}_{200} - 2\hat{I}_{210} \hat{I}_{210} - \hat{I}_{220} \hat{I}_{220} - 2\hat{I}_{201} \hat{I}_{201} \right) \, , \end{aligned} \tag{D6}$$

where $b_{20} = b_{21} = 1$, $\mathcal{B}_{20} = \sigma_T s/8$, and $\mathcal{B}_{21} = \sigma_T s^2/24$, while we also used Eqs. (62, 63) and Eqs. (B12, B13) for the tensor products.

Similarly for $n=3$ we obtain

$$
\begin{aligned}
\hat{G}_{30} &\equiv \int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\mathcal{P}_{30} = \int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\Theta^{\mu_1\mu_2\mu_3}u_{\mu_1}u_{\mu_2}u_{\mu_3} \\
&= \int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\left[b_{30}\mathcal{B}_{30}\left(P^{\mu}_{T}u_{\mu}\right)^3 - b_{31}\mathcal{B}_{31}\left(\Delta^{\mu\nu}_{T}u_{\mu}u_{\nu}\right)\left(P^{\mu}_{T}u_{\mu}\right)\right] \\
&= \frac{\sigma_T}{4}\int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\left(E_{\mathbf{k}u}+E_{\mathbf{k}'u}\right)^3 k^{\mu}k'_{\mu} - \frac{\sigma_T}{4}\int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\left(E_{\mathbf{k}u}+E_{\mathbf{k}'u}\right)\left(k^{\mu}k'_{\mu}\right)^2 \\
&= \frac{\sigma_T}{2}\left(\hat{\mathcal{I}}^{\mu}_{30}\hat{\mathcal{I}}_{00,\mu} + 3\hat{\mathcal{I}}^{\mu}_{20}\hat{\mathcal{I}}_{10,\mu}\right) - \frac{\sigma_T}{2}\hat{\mathcal{I}}^{\mu\nu}_{10}\hat{\mathcal{I}}_{10,\mu\nu} \\
&= \frac{\sigma_T}{2}\left(\hat{I}_{400}\hat{I}_{100} - \hat{I}_{410}\hat{I}_{110} + 2\hat{I}_{300}\hat{I}_{200} - \hat{I}_{310}\hat{I}_{210} - \hat{I}_{320}\hat{I}_{220} - 2\hat{I}_{301}\hat{I}_{201}\right),
\end{aligned} \tag{D7}
$$

where $b_{30}=1$, $b_{31}=3$, $\mathcal{B}_{30}=\sigma_T s/16$, and $\mathcal{B}_{31}=\sigma_T s^2/48$.

The last gain term follows from Eq. (61) for $n=4$,

$$
\begin{aligned}
\hat{G}_{40} &\equiv \int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\mathcal{P}_{40} = \int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\Theta^{\mu_1\mu_2\mu_3\mu_4}u_{\mu_1}u_{\mu_2}u_{\mu_3}u_{\mu_4} \\
&= \int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\left[b_{40}\mathcal{B}_{40}\left(P^{\mu}_{T}u_{\mu}\right)^4 - b_{41}\mathcal{B}_{41}\left(\Delta^{\mu\nu}_{T}u_{\mu}u_{\nu}\right)\left(P^{\mu}_{T}u_{\mu}\right)^2 + b_{42}\mathcal{B}_{42}\left(\Delta^{\mu\nu}_{T}u_{\mu}u_{\nu}\right)^2\right] \\
&= \frac{\sigma_T}{5}\int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\left(E_{\mathbf{k}u}+E_{\mathbf{k}'u}\right)^4 k^{\mu}k'_{\mu} - \frac{3\sigma_T}{10}\int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\left(E_{\mathbf{k}u}+E_{\mathbf{k}'u}\right)^2\left(k^{\mu}k'_{\mu}\right)^2 + \frac{\sigma_T}{20}\int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\left(k^{\mu}k'_{\mu}\right)^3 \\
&= \frac{2\sigma_T}{5}\left(\hat{\mathcal{I}}^{\mu}_{40}\hat{\mathcal{I}}_{00,\mu} + 4\hat{\mathcal{I}}^{\mu}_{30}\hat{\mathcal{I}}_{10,\mu} + 3\hat{\mathcal{I}}^{\mu}_{20}\hat{\mathcal{I}}_{20,\mu}\right) - \frac{3\sigma_T}{5}\left(\hat{\mathcal{I}}^{\mu\nu}_{20}\hat{\mathcal{I}}_{00,\mu\nu} + \hat{\mathcal{I}}^{\mu\nu}_{10}\hat{\mathcal{I}}_{10,\mu\nu}\right) + \frac{\sigma_T}{20}\hat{\mathcal{I}}^{\mu\nu\alpha}_{00}\hat{\mathcal{I}}_{00,\mu\nu\alpha} \\
&= \frac{\sigma_T}{5}\left(2\hat{I}_{500}\hat{I}_{100} - 2\hat{I}_{510}\hat{I}_{110} + 5\hat{I}_{400}\hat{I}_{200} - 2\hat{I}_{410}\hat{I}_{210} - 3\hat{I}_{420}\hat{I}_{220} - 6\hat{I}_{401}\hat{I}_{201}\right) \\
&+ \frac{\sigma_T}{20}\left(13\hat{I}_{300}\hat{I}_{300} - 3\hat{I}_{310}\hat{I}_{310} - 9\hat{I}_{320}\hat{I}_{320} - \hat{I}_{330}\hat{I}_{330} - 18\hat{I}_{301}\hat{I}_{301} - 6\hat{I}_{311}\hat{I}_{311}\right),
\end{aligned} \tag{D8}
$$

where $b_{40}=1$, $b_{41}=6$, $b_{42}=3$, while $\mathcal{B}_{40}=\sigma_T s/32$, $\mathcal{B}_{41}=\sigma_T s^2/96$, and $\mathcal{B}_{42}=\sigma_T s^3/480$.

2. The $\hat{G}_{n1}$ gain terms

The $\hat{G}_{n1}$ gain term differs from the $\hat{G}_{n0}$ gain term since it contains one projection in the direction of the anisotropy,

$$
\hat{G}_{n1} = -\int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\sum_{q=0}^{\lfloor (n+1)/2\rfloor}(-1)^q\, b_{n+1,q}\mathcal{B}_{n+1,q}\Delta^{(\mu_1\mu_2}_{T}\cdots\Delta^{\mu_{2q-1}\mu_{2q}}_{T}P^{\mu_{2q+1}}_{T}\cdots P^{\mu_{n+1})}_{T}u_{\mu_1}\cdots u_{\mu_n}l_{\mu_{n+1}}, \tag{D9}
$$

where the coefficient of distinct terms in the symmetrized tensor product is $b_{n+1,q}\equiv\frac{(n+1)!}{2^q q!(n+1-2q)!}=\frac{(n+1)}{(n+1-2q)}b_{nq}$. The symmetrized tensor product $\Delta^{(\mu_1\mu_2}_{T}\cdots P^{\mu_{n+1})}_{T}$ contains $(n+1)!$ index permutations, however the symmetric rank 2 projection operators have $q!2^q$ possible permutations while the remaining four-vectors also have $(n+1-2q)!$ permutations which do not lead to distinct terms.

Here we are interested in the scalar product of the symmetrized $n+1$ rank tensor $\Delta^{(\mu_1\mu_2}_{T}\cdots P^{\mu_{n+1})}_{T}$ by the product of four-vectors containing n four-velocities $u_{\mu_1}\cdots u_{\mu_n}$ and only 1 four-vector in the direction of the anisotropy $l_{\mu_{n+1}}$. Such scalar products can lead up to four distinct scalar terms. Here, similarly as before in Eq. (D2), the contraction by the four-velocities results in, $(P^{\mu}_{T}u_{\mu})$ and $(\Delta^{\mu\nu}_{T}u_{\mu}u_{\nu})$, while the contractions by the anisotropy four-vector lead to two new scalar products, $(P^{\mu}_{T}l_{\mu})$ and $(\Delta^{\mu\nu}_{T}u_{\mu}l_{\nu})$.

To compute the complete scalar product we first separate the symmetrized tensor product $\Delta^{(\mu_1\mu_2}_{T}\cdots P^{\mu_{n+1})}_{T}$ into two distinct parts as follows,

$$
\begin{aligned}
&\Theta^{\mu_1\cdots\mu_{n+1}}u_{\mu_1}\cdots u_{\mu_n}l_{\mu_{n+1}}\\
&\equiv \sum_{q=0}^{\lfloor (n+1)/2\rfloor}(-1)^q\, b_{n+1,q}\mathcal{B}_{n+1,q}\Delta_T^{(\mu_1\mu_2}\cdots\Delta_T^{\mu_{2q-1}\mu_{2q}}\,P_T^{\mu_{2q+1}}\cdots P_T^{\mu_{n+1})}u_{\mu_1}\cdots u_{\mu_n}l_{\mu_{n+1}}\\
&= \sum_{q=0}^{\lfloor (n+1)/2\rfloor}(-1)^q\, b_{n+1,q}\mathcal{B}_{n+1,q}\frac{(n+1-2q)}{(n+1)}l_{\mu_{n+1}}P_T^{\mu_{n+1}}\Delta_T^{(\mu_1\mu_2}\cdots\Delta_T^{\mu_{2q-1}\mu_{2q}}\,P_T^{\mu_{2q+1}}\cdots P_T^{\mu_n)}u_{\mu_1}\cdots u_{\mu_n}\\
&+ \sum_{q=0}^{\lfloor (n+1)/2\rfloor}(-1)^q\, b_{n+1,q}\mathcal{B}_{n+1,q}\frac{(2q)}{(n+1)}l_{\mu_{n+1}}\Delta_T^{\mu_{n+1}(\mu_1}\cdots\Delta_T^{\mu_{2q-1}\mu_{2q}}\,P_T^{\mu_{2q+1}}\cdots P_T^{\mu_n)}u_{\mu_1}\cdots u_{\mu_n}\\
&= \left(l_{\mu_{n+1}}P_T^{\mu_{n+1}}\right)\sum_{q=0}^{\lfloor n/2\rfloor}(-1)^q\, b_{nq}\mathcal{B}_{n+1,q}\left(P_T^{\mu}u_\mu\right)^{n-2q}\left(\Delta_T^{\mu\nu}u_\mu u_\nu\right)^q\\
&+ \left(l_{\mu_{n+1}}\Delta_T^{\mu_{n+1}\mu_n}u_{\mu_n}\right)\sum_{q=1}^{\lfloor (n+1)/2\rfloor}(-1)^q\left(b_{n+1,q}-b_{nq}\right)\mathcal{B}_{n+1,q}\left(P_T^{\mu}u_\mu\right)^{n+1-2q}\left(\Delta_T^{\mu\nu}u_\mu u_\nu\right)^{q-1}\ ,
\end{aligned}
\tag{D10}
$$

where in the last line we used that $b_{n+1,q}-b_{nq}\equiv\frac{2q}{(n+1-2q)}b_{nq}=\frac{2q}{(n+1)}b_{n+1,q}$.

The first term, i.e., $P_T^{\mu_{n+1}}\Delta_T^{(\mu_1\mu_2}\cdots P_T^{\mu_n)}$ now contains only n symmetrized indices, hence by removing one four-vector $P_T^{\mu_{n+1}}$ from the total of $n+1-2q$ different four-vectors changes the number of distinct terms to $b_{n+1,q}\left(n+1-2q\right)/\left(n+1\right)=b_{nq}$, as it should be. The remaining terms can be collected into a symmetrized tensor product of the following type, $\Delta_T^{\mu_{n+1}(\mu_1}\cdots P_T^{\mu_n)}$. This product is symmetric with respect to n indices such that one index from a rank 2 projection operator is not included. Since there are only $2q$ different indices on elementary projection operators, then removing one tensor index changes the number of distinct terms under symmetrization and it is counted by the following formula, $b_{n+1,q}\left(2q\right)/\left(n+1\right)$.

Here we will calculate the following $\hat{G}_{n1}$ gain terms of interest. The gain term from Eq. (70) for $n=0$ leads to

$$
\begin{aligned}
\hat{G}_{01} &\equiv \int_{KK'}\hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\mathcal{P}_{01} = -\int_{KK'}\hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\Theta^{\mu_1}l_{\mu_1}\\
&= -\int_{KK'}\hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\left[b_{00}\mathcal{B}_{10}\left(l_\mu P_T^\mu\right)\right] = \frac{\sigma_T}{2}\int_{KK'}\hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\left(E_{\mathbf{k}l}+E_{\mathbf{k}'l}\right)k^\mu k'_\mu\\
&= \sigma_T\hat{\mathcal{I}}_{01}^{\mu}\hat{\mathcal{I}}_{00,\mu} = \sigma_T\left(\hat{I}_{210}\hat{I}_{100}-\hat{I}_{220}\hat{I}_{110}\right)\ ,
\end{aligned}
\tag{D11}
$$

where $b_{00}=1$ and $\mathcal{B}_{10}=\sigma_T s/4$, and we used Eq. (71) as well as Eq. (B12).

Similarly for $n=1$ we obtain the following gain term,

$$
\begin{aligned}
\hat{G}_{11} &\equiv \int_{KK'}\hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\mathcal{P}_{11} = -\int_{KK'}\hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\Theta^{\mu_1\mu_2}u_{\mu_1}l_{\mu_2}\\
&= -\int_{KK'}\hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\left[\left(l_{\mu_2}P_T^{\mu_2}\right)b_{10}\mathcal{B}_{20}\left(P_T^{\mu_1}u_{\mu_1}\right)-\left(l_{\mu_2}\Delta_T^{\mu_2\mu_1}u_{\mu_1}\right)\left(b_{21}-b_{11}\right)\mathcal{B}_{21}\right]\\
&= \frac{\sigma_T}{3}\int_{KK'}\hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\left(E_{\mathbf{k}u}+E_{\mathbf{k}'u}\right)\left(E_{\mathbf{k}l}+E_{\mathbf{k}'l}\right)k^\mu k'_\mu\\
&= \frac{2\sigma_T}{3}\left(\hat{\mathcal{I}}_{11}^{\mu}\hat{\mathcal{I}}_{00,\mu}+\hat{\mathcal{I}}_{10}^{\mu}\hat{\mathcal{I}}_{01,\mu}\right) = \frac{2\sigma_T}{3}\left(\hat{I}_{310}\hat{I}_{100}-\hat{I}_{320}\hat{I}_{110}+\hat{I}_{200}\hat{I}_{210}-\hat{I}_{210}\hat{I}_{220}\right)\ ,
\end{aligned}
\tag{D12}
$$

where $b_{10}=b_{21}=1$ and $b_{11}=0$, while $\mathcal{B}_{20}=\sigma_T s/8$, and $\mathcal{B}_{21}=\sigma_T s^2/24$.

Furthermore, from Eq. (70) for $n=2$ we get,

$$
\begin{aligned}
\hat{G}_{21} &\equiv \int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\mathcal{P}_{21} = -\int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\Theta^{\mu_1\mu_2\mu_3}u_{\mu_1}u_{\mu_2}l_{\mu_3} \\
&= -\int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\left(l_{\mu_3}P_T^{\mu_3}\right)\left[b_{20}\mathcal{B}_{30}\left(P_T^{\mu}u_{\mu}\right)^2 - b_{21}\mathcal{B}_{31}\left(\Delta_T^{\mu\nu}u_{\mu}u_{\nu}\right)\right] \\
&+ \int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\left(l_{\mu_3}\Delta_T^{\mu_3\mu_2}u_{\mu_2}\right)\left(b_{31}-b_{21}\right)\mathcal{B}_{31}\left(P_T^{\mu}u_{\mu}\right) \\
&= \frac{\sigma_T}{4}\int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\left(E_{\mathbf{k}u}+E_{\mathbf{k}'u}\right)^2\left(E_{\mathbf{k}l}+E_{\mathbf{k}'l}\right)k^{\mu}k'_{\mu} - \frac{\sigma_T}{12}\int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\left(E_{\mathbf{k}l}+E_{\mathbf{k}'l}\right)k^{\mu}k^{\nu}k'_{\mu}k'_{\nu} \\
&= \frac{\sigma_T}{2}\left(\hat{\mathcal{I}}^{\mu}_{21}\hat{\mathcal{I}}_{00,\mu} + 2\hat{\mathcal{I}}^{\mu}_{11}\hat{\mathcal{I}}_{10,\mu} + \hat{\mathcal{I}}^{\mu}_{01}\hat{\mathcal{I}}_{20,\mu}\right) - \frac{\sigma_T}{6}\hat{\mathcal{I}}^{\mu\nu}_{01}\hat{\mathcal{I}}_{00,\mu\nu} \\
&= \frac{\sigma_T}{6}\left(3\hat{I}_{410}\hat{I}_{100} - 3\hat{I}_{420}\hat{I}_{110} + 5\hat{I}_{310}\hat{I}_{200} - 4\hat{I}_{320}\hat{I}_{210}\right) + \frac{\sigma_T}{6}\left(3\hat{I}_{300}\hat{I}_{210} - 3\hat{I}_{310}\hat{I}_{220} - \hat{I}_{330}\hat{I}_{220} - 2\hat{I}_{311}\hat{I}_{201}\right),
\end{aligned}
\tag{D13}
$$

where $b_{31}=3$ and $b_{21}=1$, and $\mathcal{B}_{30}=\sigma_T s/16$ and $\mathcal{B}_{31}=\sigma_T s^2/48$.

Finally the gain term for $n=3$ leads to

$$
\begin{aligned}
\hat{G}_{31} &\equiv \int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\mathcal{P}_{31} = -\int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\Theta^{\mu_1\mu_2\mu_3\mu_4}u_{\mu_1}u_{\mu_2}u_{\mu_3}l_{\mu_4} \\
&= -\int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\left(l_{\mu_4}P_T^{\mu_4}\right)\left[b_{30}\mathcal{B}_{40}\left(P_T^{\mu}u_{\mu}\right)^3 - b_{31}\mathcal{B}_{41}\left(P_T^{\mu}u_{\mu}\right)\left(\Delta_T^{\mu\nu}u_{\mu}u_{\nu}\right)\right] \\
&+ \int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\left(l_{\mu_4}\Delta_T^{\mu_4\mu_3}u_{\mu_3}\right)\left[\left(b_{41}-b_{31}\right)\mathcal{B}_{41}\left(P_T^{\mu}u_{\mu}\right)^2 - \left(b_{42}-b_{32}\right)\mathcal{B}_{42}\left(\Delta_T^{\mu\nu}u_{\mu}u_{\nu}\right)\right] \\
&= \frac{\sigma_T}{5}\int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\left(E_{\mathbf{k}u}+E_{\mathbf{k}'u}\right)^3\left(E_{\mathbf{k}l}+E_{\mathbf{k}'l}\right)k^{\mu}k'_{\mu} - \frac{3\sigma_T}{20}\int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\left(E_{\mathbf{k}u}+E_{\mathbf{k}'u}\right)\left(E_{\mathbf{k}l}+E_{\mathbf{k}'l}\right)k^{\mu}k^{\nu}k'_{\mu}k'_{\nu} \\
&= \frac{2\sigma_T}{5}\left(\hat{\mathcal{I}}^{\mu}_{31}\hat{\mathcal{I}}_{00,\mu} + 3\hat{\mathcal{I}}^{\mu}_{21}\hat{\mathcal{I}}_{10,\mu} + 3\hat{\mathcal{I}}^{\mu}_{11}\hat{\mathcal{I}}_{20,\mu} + \hat{\mathcal{I}}^{\mu}_{30}\hat{\mathcal{I}}_{01,\mu}\right) - \frac{3\sigma_T}{10}\left(\hat{\mathcal{I}}^{\mu\nu}_{11}\hat{\mathcal{I}}_{00,\mu\nu} + \hat{\mathcal{I}}^{\mu\nu}_{10}\hat{\mathcal{I}}_{01,\mu\nu}\right) \\
&= \frac{2\sigma_T}{5}\left(\hat{I}_{510}\hat{I}_{100} - \hat{I}_{520}\hat{I}_{110} + \hat{I}_{400}\hat{I}_{210} - \hat{I}_{410}\hat{I}_{220}\right) + \frac{3\sigma_T}{10}\left(3\hat{I}_{410}\hat{I}_{200} - 2\hat{I}_{420}\hat{I}_{210} - \hat{I}_{430}\hat{I}_{220} - 2\hat{I}_{411}\hat{I}_{201}\right) \\
&+ \frac{3\sigma_T}{10}\left(3\hat{I}_{300}\hat{I}_{310} - 2\hat{I}_{310}\hat{I}_{320} - \hat{I}_{320}\hat{I}_{330} - 2\hat{I}_{301}\hat{I}_{311}\right),
\end{aligned}
\tag{D14}
$$

where we used that $b_{30}=1$, $b_{31}=b_{42}=3$, $b_{32}=0$, $b_{41}=6$, $\mathcal{B}_{40}=\sigma_T s/32$, $\mathcal{B}_{41}=\sigma_T s^2/96$ and $\mathcal{B}_{42}=\sigma_T s^3/480$.

3. The $\hat{G}_{n2}$ gain terms

The last gain term of interest is a generalization of our previous results, now including two projections in the direction of the anisotropy,

$$
\hat{G}_{n2} = \int dK dK' \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'} \sum_{q=0}^{\lfloor (n+2)/2 \rfloor} (-1)^q\, b_{n+2,q}\mathcal{B}_{n+2,q}\Delta_T^{(\mu_1\mu_2}\cdots\Delta_T^{\mu_{2q-1}\mu_{2q}}\, P_T^{\mu_{2q+1}}\cdots P_T^{\mu_{n+2})}u_{\mu_1}\cdots u_{\mu_n}l_{\mu_{n+1}}l_{\mu_{n+2}}\,,
\tag{D15}
$$

where now the coefficient of the symmetrized tensor product is $b_{n+2,q} \equiv \frac{(n+2)}{(n+2-2q)}b_{n+1,q} = \frac{(n+2)}{(n+2-2q)}\frac{(n+1)}{(n+1-2q)}b_{nq}$. The scalar product of the symmetrized tensor $\Delta_T^{(\mu_1\mu_2}\cdots P_T^{\mu_{n+2})}$ by the product of four-vectors containing n four-velocities $u_{\mu_1}\cdots u_{\mu_n}$, and 2 four-vectors in the direction of the anisotropy $l_{\mu_{n+1}}l_{\mu_{n+2}}$ leads to only five different type of scalars. The first four scalar products are the same as obtained before in Eq. (D9), namely $(P_T^{\mu}u_{\mu})$, $(\Delta_T^{\mu\nu}u_{\mu}u_{\nu})$, $(P_T^{\mu}l_{\mu})$, and $(\Delta_T^{\mu\nu}u_{\mu}l_{\nu})$, while the fifth term is a new scalar product $(\Delta_T^{\mu\nu}l_{\mu}l_{\nu})$ which follows from the projection by two four-vectors in the direction of anisotropy.

To calculate the complete scalar product we separate the symmetrized tensor product $\Delta_T^{(\mu_1\mu_2}\cdots P_T^{\mu_{n+2})}$ iteratively.

First similarly to Eq. (D10) we construct symmetrized rank $n+1$ tensor products,

$$
\begin{aligned}
&\Theta^{\mu_1\cdots\mu_{n+2}}u_{\mu_1}\cdots u_{\mu_n}l_{\mu_{n+1}}l_{\mu_{n+2}} \\
&\equiv \sum_{q=0}^{\lfloor (n+2)/2\rfloor} (-1)^q\, b_{n+2,q}\mathcal{B}_{n+2,q}\Delta_T^{(\mu_1\mu_2}\cdots\Delta_T^{\mu_{2q-1}\mu_{2q}}\,P_T^{\mu_{2q+1}}\cdots P_T^{\mu_{n+2})}u_{\mu_1}\cdots u_{\mu_n}l_{\mu_{n+1}}l_{\mu_{n+2}} \\
&= \sum_{q=0}^{\lfloor (n+2)/2\rfloor} (-1)^q\, b_{n+2,q}\mathcal{B}_{n+2,q}\frac{(n+2-2q)}{(n+2)}l_{\mu_{n+2}}P_T^{\mu_{n+2}}\Delta_T^{(\mu_1\mu_2}\cdots\Delta_T^{\mu_{2q-1}\mu_{2q}}\,P_T^{\mu_{2q+1}}\cdots P_T^{\mu_{n+1})}u_{\mu_1}\cdots u_{\mu_n}l_{\mu_{n+1}} \\
&+ \sum_{q=0}^{\lfloor (n+2)/2\rfloor} (-1)^q\, b_{n+2,q}\mathcal{B}_{n+2,q}\frac{(2q)}{(n+2)}l_{\mu_{n+2}}\Delta_T^{\mu_{n+2}(\mu_1}\,\Delta_T^{\mu_2\mu_3}\cdots\Delta_T^{\mu_{2q-1}\mu_{2q}}\,P_T^{\mu_{2q+1}}\cdots P_T^{\mu_{n+1})}u_{\mu_1}\cdots u_{\mu_n}l_{\mu_{n+1}}\,, \qquad \text{(D16)}
\end{aligned}
$$

where the coefficients of the symmetrized tensors are obtained by substituting $n\to n+1$ in Eq. (D10). Then, in the subsequent step we once again separate both terms into symmetrized rank n tensor products, and hence we obtain four different terms,

$$
\begin{aligned}
&\Theta^{\mu_1\cdots\mu_{n+2}}u_{\mu_1}\cdots u_{\mu_n}l_{\mu_{n+1}}l_{\mu_{n+2}} \\
&= \sum_{q=0}^{\lfloor (n+2)/2\rfloor} (-1)^q\,\mathcal{B}_{n+2,q}b_{n+2,q}\frac{(n+2-2q)}{(n+2)}\times\frac{(n+1-2q)}{(n+1)} \\
&\times l_{\mu_{n+2}}P_T^{\mu_{n+2}}l_{\mu_{n+1}}P_T^{\mu_{n+1}}\Delta_T^{(\mu_1\mu_2}\cdots\Delta_T^{\mu_{2q-1}\mu_{2q}}\,P_T^{\mu_{2q+1}}\cdots P_T^{\mu_n)}u_{\mu_1}\cdots u_{\mu_n} \\
&+ \sum_{q=0}^{\lfloor (n+2)/2\rfloor} (-1)^q\,\mathcal{B}_{n+2,q}b_{n+2,q}\frac{(n+2-2q)}{(n+2)}\times\frac{(2q)}{(n+1)}\times 2 \\
&\times l_{\mu_{n+2}}P_T^{\mu_{n+2}}l_{\mu_{n+1}}\Delta_T^{\mu_{n+1}(\mu_1}\,\Delta_T^{\mu_2\mu_3}\cdots\Delta_T^{\mu_{2q-1}\mu_{2q}}\,P_T^{\mu_{2q+1}}\cdots P_T^{\mu_n)}u_{\mu_1}\cdots u_{\mu_n} \\
&+ \sum_{q=0}^{\lfloor (n+2)/2\rfloor} (-1)^q\,\mathcal{B}_{n+2,q}b_{n+2,q}\frac{(2q)}{(n+2)}\times\frac{1}{(n+1)} \\
&\times l_{\mu_{n+2}}\Delta_T^{\mu_{n+2}\mu_{n+1}}l_{\mu_{n+1}}\Delta_T^{(\mu_1\mu_2}\cdots\Delta_T^{\mu_{2q-1}\mu_{2q}}\,P_T^{\mu_{2q+1}}\cdots P_T^{\mu_n)}u_{\mu_1}\cdots u_{\mu_n} \\
&+ \sum_{q=0}^{\lfloor (n+2)/2\rfloor} (-1)^q\,\mathcal{B}_{n+2,q}b_{n+2,q}\frac{(2q)}{(n+2)}\times\frac{2\,(q-1)}{(n+1)} \\
&\times l_{\mu_{n+2}}\Delta_T^{\mu_{n+2}(\mu_1}\,\Delta_T^{\mu_2\mu_3}\cdots\Delta_T^{\mu_{2q-1}\mu_{2q}}\,P_T^{\mu_{2q+1}}\cdots P_T^{\mu_{n-1}}\Delta_T^{\mu_n)\mu_{n+1}}l_{\mu_{n+1}}u_{\mu_1}\cdots u_{\mu_n}\,. \qquad \text{(D17)}
\end{aligned}
$$

Now using the identities, $b_{n+1,q}-b_{nq}\equiv b_{n+2,q}\frac{(n+2-2q)}{(n+2)}\frac{(2q)}{(n+1)}$, and $b_{n,q-1}\equiv b_{n+2,q}\frac{(2q)}{(n+2)}\frac{1}{(n+1)}$, we obtain

$$
\begin{aligned}
&\Theta^{\mu_1\cdots\mu_{n+2}}u_{\mu_1}\cdots u_{\mu_n}l_{\mu_{n+1}}l_{\mu_{n+2}} \\
&= \left(l_{\mu_{n+2}}P_T^{\mu_{n+2}}\right)\left(l_{\mu_{n+1}}P_T^{\mu_{n+1}}\right)\sum_{q=0}^{\lfloor n/2\rfloor}(-1)^q\, b_{nq}\mathcal{B}_{n+2,q}\left(P_T^{\mu}u_\mu\right)^{n-2q}\left(\Delta_T^{\mu\nu}u_\mu u_\nu\right)^q \\
&+ \left(l_{\mu_{n+2}}P_T^{\mu_{n+2}}\right)\left(l_{\mu_{n+1}}\Delta_T^{\mu_{n+1}\mu_n}u_{\mu_n}\right)\sum_{q=1}^{\lfloor (n+2)/2\rfloor}(-1)^q\,2\left(b_{n+1,q}-b_{nq}\right)\mathcal{B}_{n+2,q}\left(P_T^{\mu}u_\mu\right)^{n+1-2q}\left(\Delta_T^{\mu\nu}u_\mu u_\nu\right)^{q-1} \\
&+ \left(l_{\mu_{n+2}}\Delta_T^{\mu_{n+2}\mu_{n+1}}l_{\mu_{n+1}}\right)\sum_{q=1}^{\lfloor (n+2)/2\rfloor}(-1)^q\, b_{n,q-1}\mathcal{B}_{n+2,q}\left(P_T^{\mu}u_\mu\right)^{n+2-2q}\left(\Delta_T^{\mu\nu}u_\mu u_\nu\right)^{q-1} \\
&+ \left(l_{\mu_{n+2}}\Delta_T^{\mu_{n+2}\mu_n}u_{\mu_n}\right)\left(l_{\mu_{n+1}}\Delta_T^{\mu_{n+1}\mu_{n-1}}u_{\mu_{n-1}}\right)\sum_{q=2}^{\lfloor (n+2)/2\rfloor}(-1)^q\,2\,(q-1)\,b_{n,q-1}\mathcal{B}_{n+2,q}\left(P_T^{\mu}u_\mu\right)^{n+2-2q}\left(\Delta_T^{\mu\nu}u_\mu u_\nu\right)^{q-2}\,.
\end{aligned}
$$

(D18)

The gain term form Eq. (77) for $n=0$ leads to,

$$
\begin{aligned}
\hat{G}_{02} &\equiv \int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\mathcal{P}_{02} = \int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\Theta^{\mu_1\mu_2} l_{\mu_1} l_{\mu_2} \\
&= \int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'} \left[b_{00}\mathcal{B}_{20} \left(l_\mu P^\mu_T\right)^2 - b_{00}\mathcal{B}_{21}\left(\Delta^{\mu\nu}_T l_\mu l_\nu\right)\right] \\
&= \frac{\sigma_T}{3}\int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'} \left(E_{\mathbf{k}l} + E_{\mathbf{k}'l}\right)^2 k^\mu k'_\mu + \frac{\sigma_T}{6}\int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'} k^\mu k^\nu k'_\mu k'_\nu \\
&= \frac{2\sigma_T}{3}\left(\hat{\mathcal{I}}^\mu_{02}\hat{\mathcal{I}}_{00,\mu} + \hat{\mathcal{I}}^\mu_{01}\hat{\mathcal{I}}_{01,\mu}\right) + \frac{\sigma_T}{6}\hat{\mathcal{I}}^{\mu\nu}_{00}\hat{\mathcal{I}}_{00,\mu\nu} \\
&= \frac{\sigma_T}{6}\left(4\hat{I}_{320}\hat{I}_{100} - 4\hat{I}_{330}\hat{I}_{110} + \hat{I}_{200}\hat{I}_{200} + 2\hat{I}_{210}\hat{I}_{210} - 3\hat{I}_{220}\hat{I}_{220} + 2\hat{I}_{201}\hat{I}_{201}\right) . \qquad \text{(D19)}
\end{aligned}
$$

where $b_{00}=1$, $\mathcal{B}_{20}=\sigma_T s/8$, and $\mathcal{B}_{21}=\sigma_T s^2/24$. Similarly, the next gain term for $n=1$ leads to,

$$
\begin{aligned}
\hat{G}_{12} &\equiv \int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\mathcal{P}_{12} = \int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\Theta^{\mu_1\mu_2\mu_3} u_{\mu_1} l_{\mu_2} l_{\mu_3} \\
&= \int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'} \left(l_{\mu_3}P^{\mu_3}_T\right)\left(l_{\mu_2}P^{\mu_2}_T\right) b_{10}\mathcal{B}_{30}\left(P^\mu_T u_\mu\right) - \int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'} \left(l_{\mu_3}P^{\mu_3}_T\right)\left(l_{\mu_2}\Delta^{\mu_2\mu_1}_T u_{\mu_1}\right) 2\left(b_{21}-b_{11}\right)\mathcal{B}_{31} \\
&- \int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'} \left(l_{\mu_3}\Delta^{\mu_3\mu_2}_T l_{\mu_2}\right) b_{10}\mathcal{B}_{31}\left(P^\mu_T u_\mu\right) \\
&= \frac{\sigma_T}{4}\int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'} \left(E_{\mathbf{k}l} + E_{\mathbf{k}'l}\right)^2 \left(E_{\mathbf{k}u} + E_{\mathbf{k}'u}\right) k^\mu k'_\mu + \frac{\sigma_T}{12}\int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'} \left(E_{\mathbf{k}u} + E_{\mathbf{k}'u}\right) k^\mu k^\nu k'_\mu k'_\nu \\
&= \frac{\sigma_T}{2}\left(\hat{\mathcal{I}}^\mu_{12}\hat{\mathcal{I}}_{00,\mu} + 2\hat{\mathcal{I}}^\mu_{11}\hat{\mathcal{I}}_{01,\mu} + \hat{\mathcal{I}}^\mu_{02}\hat{\mathcal{I}}_{10,\mu}\right) + \frac{\sigma_T}{6}\hat{\mathcal{I}}^{\mu\nu}_{10}\hat{\mathcal{I}}_{00,\mu\nu} \\
&= \frac{\sigma_T}{6}\left(3\hat{I}_{420}\hat{I}_{100} - 3\hat{I}_{430}\hat{I}_{110} + 4\hat{I}_{310}\hat{I}_{210} - 5\hat{I}_{320}\hat{I}_{220}\right) + \frac{\sigma_T}{6}\left(3\hat{I}_{320}\hat{I}_{200} - 3\hat{I}_{330}\hat{I}_{210} + \hat{I}_{300}\hat{I}_{200} + 2\hat{I}_{301}\hat{I}_{201}\right) , \qquad \text{(D20)}
\end{aligned}
$$

where $b_{10}=b_{21}=1$, $b_{11}=0$, $\mathcal{B}_{30}=\sigma_T s/16$ and $\mathcal{B}_{31}=\sigma_T s^2/48$. Finally the last gain term for $n=2$ gives,

$$
\begin{aligned}
\hat{G}_{22} &\equiv \int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\mathcal{P}_{22} = \int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'}\Theta^{\mu_1\mu_2\mu_3\mu_4} u_{\mu_1} u_{\mu_2} l_{\mu_3} l_{\mu_4} \\
&= \int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'} \left(l_{\mu_4}P^{\mu_4}_T\right)\left(l_{\mu_3}P^{\mu_3}_T\right)\left[b_{20}\mathcal{B}_{40}\left(P^\mu_T u_\mu\right)^2 - b_{21}\mathcal{B}_{41}\left(\Delta^{\mu\nu}_T u_\mu u_\nu\right)\right] \\
&- \int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'} \left(l_{\mu_4}P^{\mu_4}_T\right)\left(l_{\mu_3}\Delta^{\mu_3\mu_2}_T u_{\mu_2}\right)\left[2\left(b_{31}-b_{21}\right)\mathcal{B}_{41}\left(P^\mu_T u_\mu\right)\right] \\
&- \int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'} \left(l_{\mu_4}\Delta^{\mu_4\mu_3}_T l_{\mu_3}\right)\left[b_{20}\mathcal{B}_{41}\left(P^\mu_T u_\mu\right)^2 - b_{21}\mathcal{B}_{42}\left(\Delta^{\mu\nu}_T u_\mu u_\nu\right)\right] \\
&+ \int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'} \left(l_{\mu_4}\Delta^{\mu_4\mu_2}_T u_{\mu_2}\right)\left(l_{\mu_3}\Delta^{\mu_3\mu_1}_T u_{\mu_1}\right) 2b_{21}\mathcal{B}_{42} , \qquad \text{(D21)}
\end{aligned}
$$

where replacing $b_{20}=1$, $b_{21}=1$, $b_{31}=3$, $\mathcal{B}_{40}=\sigma_T s/32$, $\mathcal{B}_{41}=\sigma_T s^2/96$ and $\mathcal{B}_{42}=\sigma_T s^3/480$, leads to

$$
\begin{aligned}
\hat{G}_{22} &= \frac{\sigma_T}{5}\int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'} \left(E_{\mathbf{k}u} + E_{\mathbf{k}'u}\right)^2 \left(E_{\mathbf{k}l} + E_{\mathbf{k}'l}\right)^2 k^\mu k'_\mu - \frac{\sigma_T}{60}\int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'} k^\mu k^\nu k^\alpha k'_\mu k'_\nu k'_\alpha \\
&- \frac{\sigma_T}{20}\int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'} \left(E_{\mathbf{k}l} + E_{\mathbf{k}'l}\right)^2 k^\mu k^\nu k'_\mu k'_\nu + \frac{\sigma_T}{20}\int_{KK'} \hat{f}_{0\mathbf{k}}\hat{f}_{0\mathbf{k}'} \left(E_{\mathbf{k}u} + E_{\mathbf{k}'u}\right)^2 k^\mu k^\nu k'_\mu k'_\nu \\
&= \frac{2\sigma_T}{5}\left(\hat{\mathcal{I}}^\mu_{22}\hat{\mathcal{I}}_{00,\mu} + \hat{\mathcal{I}}^\mu_{02}\hat{\mathcal{I}}_{20,\mu} + 2\hat{\mathcal{I}}^\mu_{12}\hat{\mathcal{I}}_{10,\mu} + 2\hat{\mathcal{I}}^\mu_{21}\hat{\mathcal{I}}_{01,\mu} + 2\hat{\mathcal{I}}^\mu_{11}\hat{\mathcal{I}}_{11,\mu}\right) \\
&- \frac{\sigma_T}{10}\left(\hat{\mathcal{I}}^{\mu\nu}_{02}\hat{\mathcal{I}}_{00,\mu\nu} + \hat{\mathcal{I}}^{\mu\nu}_{01}\hat{\mathcal{I}}_{01,\mu\nu}\right) + \frac{\sigma_T}{10}\left(\hat{\mathcal{I}}^{\mu\nu}_{20}\hat{\mathcal{I}}_{00,\mu\nu} + \hat{\mathcal{I}}^{\mu\nu}_{10}\hat{\mathcal{I}}_{10,\mu\nu}\right) - \frac{\sigma_T}{60}\hat{\mathcal{I}}^{\mu\nu\alpha}_{00}\hat{\mathcal{I}}_{00,\mu\nu\alpha} \\
&= \frac{\sigma_T}{10}\left(4\hat{I}_{520}\hat{I}_{100} - 4\hat{I}_{530}\hat{I}_{110} + 7\hat{I}_{420}\hat{I}_{200} - 7\hat{I}_{420}\hat{I}_{220} + 4\hat{I}_{320}\hat{I}_{300} - 4\hat{I}_{330}\hat{I}_{310}\right) \\
&+ \frac{\sigma_T}{10}\left(6\hat{I}_{410}\hat{I}_{210} - 6\hat{I}_{430}\hat{I}_{210} + \hat{I}_{400}\hat{I}_{200} - \hat{I}_{440}\hat{I}_{220} + 2\hat{I}_{401}\hat{I}_{201} - 2\hat{I}_{421}\hat{I}_{201}\right) \\
&+ \frac{\sigma_T}{60}\left(5\hat{I}_{300}\hat{I}_{300} - 5\hat{I}_{330}\hat{I}_{330} + 33\hat{I}_{310}\hat{I}_{310} - 33\hat{I}_{320}\hat{I}_{320} + 6\hat{I}_{301}\hat{I}_{301} - 6\hat{I}_{311}\hat{I}_{311}\right) . \qquad \text{(D22)}
\end{aligned}
$$

Appendix E: The moments of the binary collision integrals

Here we write down the full collision integrals of interest. Using Eqs. (50-52) and Eqs. (67-69) we obtain

$$\hat{C}_{20} = -\frac{\sigma_T}{6}\left(2\hat{I}_{300}\hat{n} - 2\hat{I}_{310}\hat{n}_l - 3\hat{e}^2\right) - \frac{\sigma_T}{6}\left(2\hat{M}^2 + \hat{P}_l^2 + 2\hat{P}_\perp^2\right) , \tag{E1}$$

$$\hat{C}_{30} = -\frac{\sigma_T}{2}\left(\hat{I}_{400}\hat{n} - \hat{I}_{410}\hat{n}_l - 2\hat{I}_{300}\hat{e}\right) - \frac{\sigma_T}{2}\left(\hat{I}_{310}\hat{M} + \hat{I}_{320}\hat{P}_l + 2\hat{I}_{301}\hat{P}_\perp\right) , \tag{E2}$$

and

$$\begin{aligned}\hat{C}_{40} = &-\frac{\sigma_T}{5}\left(3\hat{I}_{500}\hat{n} - 3\hat{I}_{510}\hat{n}_l - 5\hat{I}_{400}\hat{e} + 2\hat{I}_{410}\hat{M} + 3\hat{I}_{420}\hat{P}_l + 6\hat{I}_{401}\hat{P}_\perp\right) \\ &+ \frac{\sigma_T}{20}\left(13\hat{I}_{300}^2 - 3\hat{I}_{310}^2 - 9\hat{I}_{320}^2 - \hat{I}_{330}^2 - 18\hat{I}_{301}^2 - 6\hat{I}_{311}^2\right) .\end{aligned} \tag{E3}$$

Similarly, using Eqs. (54-56) and Eqs. (74-76) we get

$$\hat{C}_{11} = -\frac{\sigma_T}{3}\left(\hat{I}_{310}\hat{n} - \hat{I}_{320}\hat{n}_l - 2\hat{M}\hat{e} + 2\hat{M}\hat{P}_l\right) , \tag{E4}$$

$$\hat{C}_{21} = -\frac{\sigma_T}{6}\left(3\hat{I}_{410}\hat{n} - 3\hat{I}_{420}\hat{n}_l - 5\hat{I}_{310}\hat{e} + 4\hat{I}_{320}\hat{M}\right) + \frac{\sigma_T}{6}\left(3\hat{I}_{300}\hat{M} - 3\hat{I}_{310}\hat{P}_l - \hat{I}_{330}\hat{P}_l - 2\hat{I}_{311}\hat{P}_\perp\right) , \tag{E5}$$

and

$$\begin{aligned}\hat{C}_{31} = &-\frac{\sigma_T}{5}\left(3\hat{I}_{510}\hat{n} - 3\hat{I}_{520}\hat{n}_l - 2\hat{I}_{400}\hat{M} + 2\hat{I}_{410}\hat{P}_l\right) + \frac{3\sigma_T}{10}\left(3\hat{I}_{410}\hat{e} - 2\hat{I}_{420}\hat{M} - \hat{I}_{430}\hat{P}_l - 2\hat{I}_{411}\hat{P}_\perp\right) \\ &+ \frac{3\sigma_T}{10}\left(3\hat{I}_{300}\hat{I}_{310} - 2\hat{I}_{310}\hat{I}_{320} - \hat{I}_{320}\hat{I}_{330} - 2\hat{I}_{301}\hat{I}_{311}\right) .\end{aligned} \tag{E6}$$

Finally, using Eqs. (57-59) and Eqs. (79-81) we obtain

$$\hat{C}_{02} = -\frac{\sigma_T}{6}\left(2\hat{I}_{320}\hat{n} - 2\hat{I}_{330}\hat{n}_l - \hat{e}^2\right) + \frac{\sigma_T}{6}\left(2\hat{M}^2 - 3\hat{P}_l^2 + 2\hat{P}_\perp^2\right) , \tag{E7}$$

$$\hat{C}_{12} = -\frac{\sigma_T}{6}\left(3\hat{I}_{420}\hat{n} - 3\hat{I}_{430}\hat{n}_l - 4\hat{I}_{310}\hat{M} + 5\hat{I}_{320}\hat{P}_l\right) + \frac{\sigma_T}{6}\left(3\hat{I}_{320}\hat{e} - 3\hat{I}_{330}\hat{M} + \hat{I}_{300}\hat{e} + 2\hat{I}_{301}\hat{P}_\perp\right) , \tag{E8}$$

and

$$\begin{aligned}\hat{C}_{22} = &-\frac{\sigma_T}{10}\left(6\hat{I}_{520}\hat{n} - 6\hat{I}_{530}\hat{n}_l - 7\hat{I}_{420}\hat{e} + 7\hat{I}_{420}\hat{P}_l - 4\hat{I}_{320}\hat{I}_{300} + 4\hat{I}_{330}\hat{I}_{310}\right) \\ &+ \frac{\sigma_T}{10}\left(6\hat{I}_{410}\hat{M} - 6\hat{I}_{430}\hat{M} + \hat{I}_{400}\hat{e} - \hat{I}_{440}\hat{P}_l + 2\hat{I}_{401}\hat{P}_\perp - 2\hat{I}_{421}\hat{P}_\perp\right) \\ &+ \frac{\sigma_T}{60}\left(5\hat{I}_{300}^2 - 5\hat{I}_{330}^2 + 33\hat{I}_{310}^2 - 33\hat{I}_{320}^2 + 6\hat{I}_{301}^2 - 6\hat{I}_{311}^2\right) .\end{aligned} \tag{E9}$$

Appendix F: Thermodynamic integrals in the massless Boltzmann limit

In the LR frame $u^\mu_{LR} = (1, 0, 0, 0)$ and $l^\mu_{LR} = (0, 0, 0, 1)$, hence $E_{\mathbf{k}u,LR} = k^0$ and $E_{\mathbf{k}l,LR} = k_z$. Assuming Boltzmann statistics, the corresponding Maxwell-Jüttner distribution function from Eq. (8) reads

$$f_{0\mathbf{k}} = \lambda \exp\left(-\beta\sqrt{m_0^2 + k^2}\right) , \tag{F1}$$

where $\lambda = \exp\alpha$ is the fugacity. Similarly, the RS distribution function from Eq. (82) now reads

$$\hat{f}_{RS} = \lambda_{RS} \exp\left(-\beta_{RS}\sqrt{m_0^2 + k^2 + \xi k_z^2}\right) , \tag{F2}$$

where $\lambda_{RS} = \exp\alpha_{RS}$ is corresponding the fugacity.

In the massless limit of Eq. (13), i.e., $\lim_{m_0\to 0}\sqrt{m_0^2+k^2}=|k|$, we obtain the following result for the thermodynamic integrals

$$\lim_{m_0\to 0} I_{nq} \equiv \lambda \frac{(-1)^{2q}\, 4\pi A_0}{(2q+1)!!}\int_0^\infty dk k^{n+1}\exp\left(-\beta k\right) = \lambda \frac{4\pi A_0\,(n+1)!}{\beta^{n+2}\,(2q+1)!!}\,. \tag{F3}$$

Here we list those thermodynamic integrals that are need explicitly,

$$I_{10}\left(\alpha,\beta\right) \equiv I_{100} = \lambda\frac{8\pi A_0}{\beta^3} = n_0\,, \tag{F4}$$

$$I_{20}\left(\alpha,\beta\right) \equiv I_{200} = \lambda\frac{24\pi A_0}{\beta^4} = e_0 = 3\frac{n_0}{\beta}\,, \tag{F5}$$

$$I_{21}\left(\alpha,\beta\right) \equiv I_{201} = I_{220} = P_0 = \frac{1}{3}e_0 = \frac{n_0}{\beta}\,, \tag{F6}$$

and

$$I_{30}\left(\alpha,\beta\right) \equiv I_{300} = \lambda\frac{96\pi A_0}{\beta^5} = 12\frac{n_0}{\beta^2}\,, \tag{F7}$$

$$I_{40}\left(\alpha,\beta\right) \equiv I_{400} = \lambda\frac{480\pi A_0}{\beta^6} = 60\frac{n_0}{\beta^3}\,, \tag{F8}$$

$$I_{50}\left(\alpha,\beta\right) \equiv I_{500} = \lambda\frac{2880\pi A_0}{\beta^7} = 360\frac{n_0}{\beta^4}\,, \tag{F9}$$

$$I_{31}\left(\alpha,\beta\right) \equiv I_{301} = I_{320} = \frac{1}{3}I_{30}\left(\alpha,\beta\right)\,, \tag{F10}$$

$$I_{41}\left(\alpha,\beta\right) \equiv I_{401} = I_{420} = \frac{1}{3}I_{40}\left(\alpha,\beta\right)\,, \tag{F11}$$

$$I_{51}\left(\alpha,\beta\right) \equiv I_{501} = I_{520} = \frac{1}{3}I_{50}\left(\alpha,\beta\right)\,. \tag{F12}$$

Using the RS distribution function in the massless limit leads to the following anisotropic thermodynamical integrals

$$\lim_{m_0\to 0}\hat{I}^{RS}_{nrq} \equiv \lambda_{RS}\frac{(-1)^{2q}\,2\pi A_0}{(2q)!!}\int_0^\infty dk k^{n+1}\int_{-1}^{1} dx x^r\left(1-x^2\right)^q \exp\left(-\beta_{RS}k\sqrt{1+\xi x^2}\right) = \lambda_{RS}\frac{2\pi A_0\,(n+1)!}{\beta_{RS}^{n+2}\,(2q)!!}\int_{-1}^{1} dx\frac{x^r\left(1-x^2\right)^q}{\left(1+\xi x^2\right)^{\frac{n+2}{2}}}\,, \tag{F13}$$

while the ratio between the anisotropic and equilibrium thermodynamical integrals defined in Eq. (84) reads; see Eq. (A11) in Ref. [38],

$$R_{nrq} = \frac{(2q+1)!!}{2\,(2q)!!}\int_0^\pi d\theta\frac{\cos^r\theta\sin^{2q+1}\theta}{\left(1+\xi\cos^2\theta\right)^{\frac{n+2}{2}}}\,. \tag{F14}$$

The values of $R_{nrq}(\xi)$ that correspond to Eqs. (87)-(90) and to Eq. (111) are

$$R_{100}\left(\xi\right) = \frac{1}{\sqrt{1+\xi}}\,, \tag{F15}$$

$$R_{200}\left(\xi\right) = \frac{1}{2}\left(\frac{1}{1+\xi}+\frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right)\,, \tag{F16}$$

$$R_{201}\left(\xi\right) = \frac{3}{2\xi}\left(\frac{1}{1+\xi}-\left(1-\xi\right)R_{200}\left(\xi\right)\right)\,, \tag{F17}$$

$$R_{220}\left(\xi\right) = -\frac{1}{\xi}\left(\frac{1}{1+\xi}-R_{200}\left(\xi\right)\right)\,, \tag{F18}$$

$$R_{240}\left(\xi\right) = \frac{1}{\xi^2}\left(\frac{3+\xi}{1+\xi}-3R_{200}\left(\xi\right)\right)\,. \tag{F19}$$

Furthermore, for the moment equations listed in Eqs. (105-107), and Eqs. (108-110), we also need the specify the following $R_{nrq}(\xi)$ ratios

$$R_{300}\left(\xi\right) = \frac{3+2\xi}{3\left(1+\xi\right)^{3/2}}\,, \tag{F20}$$

$$R_{301}\left(\xi\right) = R_{100}\left(\xi\right)\,, \tag{F21}$$

$$R_{320}\left(\xi\right) = \frac{1}{3\left(1+\xi\right)^{3/2}}\,, \tag{F22}$$

and

$$R_{400}\left(\xi\right) = \frac{1}{8}\left(\frac{5+3\xi}{(1+\xi)^2} + 3\frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right)\,, \tag{F23}$$

$$R_{401}\left(\xi\right) = \frac{3}{16\xi}\left(\frac{1+3\xi}{(1+\xi)} - \left(1-3\xi\right)\frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right)\,, \tag{F24}$$

$$R_{420}\left(\xi\right) = -\frac{1}{8\xi}\left(\frac{1-\xi}{(1+\xi)^2} - \frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right)\,, \tag{F25}$$

$$R_{421}\left(\xi\right) = \frac{3}{16\xi^2}\left(\frac{3+\xi}{(1+\xi)} - \left(3-\xi\right)\frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right)\,, \tag{F26}$$

$$R_{440}\left(\xi\right) = -\frac{1}{8\xi^2}\left(\frac{3+5\xi}{(1+\xi)^2} - 3\frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right)\,, \tag{F27}$$

and, finally

$$R_{500}\left(\xi\right) = \frac{15+4\xi\left(5+2\xi\right)}{15\left(1+\xi\right)^{5/2}}\,, \tag{F28}$$

$$R_{520}\left(\xi\right) = \frac{5+2\xi}{15\left(1+\xi\right)^{5/2}}\,, \tag{F29}$$

$$R_{540}\left(\xi\right) = \frac{1}{5\left(1+\xi\right)^{5/2}}\,. \tag{F30}$$

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