Long-time dynamics for the Kelvin-Helmholtz equations close to circular vortex sheets

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Abstract

We consider the Kelvin-Helmholtz system describing the evolution of a vortex-sheet near the circular stationary solution. Answering previous numerical conjectures in the 90s physics literature, we prove an almost global existence result for small-amplitude solutions. We first establish the existence of a linear stability threshold for the Weber number, which represents the ratio between the square of the background velocity jump and the surface tension. Then, we prove that for almost all values of the Weber number below this threshold any small solution lives for almost all times, remaining close to the equilibrium. Our analysis reveals a remarkable stabilization phenomenon: the presence of both non-zero background velocity jump and capillarity effects enables to prevent nonlinear instability phenomena, despite the inherently unstable nature of the classical Kelvin-Helmholtz problem. This long-time existence would not be achievable in a setting where capillarity alone provides linear stabilization, without the richer modulation induced by the velocity jump. Our proof exploits the Hamiltonian nature of the equations. Specifically, we employ Hamiltonian Birkhoff normal form techniques for quasi-linear systems together with a general approach for paralinearization of non-linear singular integral operators. This approach allows us to control resonances and quasi-resonances at arbitrary order, ensuring the desired long-time stability result.

Keywords: Kelvin-Helmholtz, vortex sheets, paradifferential calculus, Birkhoff normal form.

Contents

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1 Presentation of the problem and main result

The Kelvin-Helmholtz (KH) equations is a classic of fluid dynamics, modeling the intricate behavior of vortex sheets at the interface between fluids with different velocities. Since their introduction by Lord Kelvin and Hermann von Helmholtz in the nineteenth century [44,45,65,66], these equations have provided crucial insights into fundamental hydrodynamic phenomena, from the formation of ocean waves to atmospheric turbulence. The classic KH problem addresses the instability of a plane vortex sheet, where the jump in tangential velocity across an interface drives the system linearly, generating the well-known instability that defines Kelvin-Helmholtz phenomena. Of particular interest is the interplay between background velocity jump ($\mathtt{b}$) and capillarity effects (γ), which together determine critical stability thresholds and equilibrium states. This work explores the mathematical structures emerging from these interactions, with special focus on the Weber number $\beta \triangleq \mathtt{b}^2/\gamma$, which naturally emerges in the equilibrium spectrum and governs the system's stabilization properties.

We consider a planar Euler system for two irrotational fluids with same density (constant equal to 1) separated by an interface $\Gamma(t)$ homeomorphic to a circle and parametrized by $z(t,\cdot):\mathbb{T}\to\mathbb{R}^2$. This interface divides the plane into two open components $\Omega^\pm(t)$ with $\Omega^-(t)$ bounded and $\Omega^+(t)$ unbounded. Given two functions $f^\pm:\Omega^\pm(t)\to\mathbb{R}$ we define

$$[\![f^\pm]\!] \triangleq f^- - f^+.$$

The evolutionary system is thus composed of the following equations

$$\begin{cases} u_t^\pm + u^\pm\cdot\nabla u^\pm + \nabla p^\pm = 0, & \text{in } \Omega^\pm(t),\\ \left(z_t - u^\pm|_{\Gamma(t)}\right)\cdot z_x = 0, & \text{at } \Gamma(t),\\ [\![p^\pm]\!]\big|_{\Gamma(t)} = \gamma \Bbbk(z), & \text{at } \Gamma(t),\\ u^+(t,\boldsymbol{x}) \to 0, & \text{as } |\boldsymbol{x}|\to+\infty,\\ \nabla\cdot u^\pm = 0, & \text{in } \Omega^\pm(t),\\ \nabla^\perp\cdot u^\pm = 0, & \text{in } \Omega^\pm(t). \end{cases} \tag{1.1}$$

In the above set of equations, the quantities $u^\pm, p^\pm$ are respectively the velocity field and pressure inside the domain $\Omega^\pm$. The parameter $\gamma \geqslant 0$ is the surface tension coefficient and $\Bbbk(z)$ is the curvature defined by

$$\Bbbk(z) \triangleq -\frac{z_x^\perp\cdot z_{xx}}{|z_x|^3}.$$

The last equation in (1.1) implies that the vorticity distribution $\boldsymbol{\omega}$ is localized on the curve $\Gamma(t)$ at time t, namely

$$\boldsymbol{\omega}(t,\boldsymbol{x}) = \omega(t,x)\delta\big(\boldsymbol{x}-z(t,x)\big), \qquad \omega \triangleq [\![u^\pm]\!]\cdot z_x, \qquad \boldsymbol{x}\in\mathbb{R}^2, \quad x\in\mathbb{T}. \tag{1.2}$$

In the case in which

$$z(t,x) = r(t,x)\begin{bmatrix}\cos(x+\Omega t)\\ \sin(x+\Omega t)\end{bmatrix}, \qquad (t,x,\Omega)\in I\times\mathbb{T}\times\mathbb{R}, \qquad r(t,x)\triangleq\sqrt{1+2\eta(t,x)}, \tag{1.3}$$

where I is a given interval of time, the system (1.1) was recast, in [59], as the *Contour Dynamic Equation (CDE)*

$$\begin{cases} \eta_t = \Omega\eta_x - \dfrac{1}{2}\mathcal{H}(\eta)\,\omega, \\ \omega_t = \Omega\omega_x - \left(\dfrac{\omega}{2}\mathcal{D}_0(\eta)\,\omega\right)_x - \gamma\left(\mathcal{K}(\eta)\right)_x, \end{cases} \tag{1.4}$$

with

$$\mathcal{H}(\eta)\,\omega \triangleq \int_{\mathbb{T}} \frac{\eta_x(x)\left[1-\sqrt{\frac{1+2\eta(y)}{1+2\eta(x)}}\cos(x-y)\right] + \sqrt{1+2\eta(x)}\sqrt{1+2\eta(y)}\sin(x-y)}{1+\eta(x)+\eta(y)-\sqrt{1+2\eta(x)}\sqrt{1+2\eta(y)}\cos(x-y)}\,\omega(y)\,\mathrm{d}y,$$

$$\mathcal{D}_0(\eta)\,\omega \triangleq \int_{\mathbb{T}} \frac{1-\sqrt{\frac{1+2\eta(y)}{1+2\eta(x)}}\cos(x-y)}{1+\eta(x)+\eta(y)-\sqrt{1+2\eta(x)}\sqrt{1+2\eta(y)}\cos(x-y)}\,\omega(y)\,\mathrm{d}y, \tag{1.5}$$

$$\mathcal{K}(\eta) \triangleq \frac{\eta_{xx}-(1+2\eta)-3\left(\frac{\eta_x}{\sqrt{1+2\eta}}\right)^2}{\left(1+2\eta+\left(\frac{\eta_x}{\sqrt{1+2\eta}}\right)^2\right)^{\frac{3}{2}}}.$$

Throughout the document, we use the notation

$$\int_{\mathbb{T}} f(x)\mathrm{d}x \triangleq \frac{1}{2\pi}\text{p.v.}\int_{-\pi}^{\pi} f(x)\mathrm{d}x.$$

We refer the reader to [59] for a derivation of (1.4) from (1.1). We also warn the reader with the change of notation for the surface tension and mean vorticity with respect to [59]. Also, here we write the system in a rotating frame with angular velocity Ω but one can easily follow the changes. An explicit computation shows that

$$\text{for any } \mathtt{b}\in\mathbb{R}, \quad (\eta,\omega) = (0,\mathtt{b}) \quad \text{is a solution of (1.4).} \tag{1.6}$$

Let us define the background velocity jump

$$\mathtt{b} \triangleq \int_{\mathbb{T}} \omega(x)\,\mathrm{d}x,$$

which is time independent according to (1.4). We can define the invertible change of variables

$$\psi_x \triangleq \omega - \mathtt{b}, \qquad\qquad \psi = \partial_x^{-1}(\omega-\mathtt{b}). \tag{1.7}$$

The change of variables (1.7) allows us to determine the evolution equation for ψ modulo a real, time-dependent constant, which is

$$\psi_t = -\frac{\psi_x+\mathtt{b}}{2}\mathcal{D}_0(\eta)\left[\psi_x+\mathtt{b}\right] - \gamma\mathcal{K}(\eta) + c(t). \tag{1.8}$$

Since the system (1.4) depends only on (η,ψ_x), the projection onto the zero-th mode in (1.8), which includes the constant $c(t)$, does not influence its dynamics. Therefore, we disregard $c(t)$ and we impose that ψ belongs to the homogeneous Sobolev space $\dot{H}^s(\mathbb{T};\mathbb{R}) \triangleq H^s(\mathbb{T};\mathbb{R})/\mathbb{R}$, where $H^s(\mathbb{T};\mathbb{R})$ denotes the classical Sobolev space of periodic real-valued functions, see Section 3. We can now rewrite the system (1.4) in terms of the variables (η,ψ) using (1.8) and get

$$\begin{cases} \eta_t = \Omega\eta_x - \dfrac{1}{2}\mathcal{H}(\eta)\left[\psi_x+\mathtt{b}\right], \\ \psi_t = \Omega\psi_x - \left(\dfrac{\psi_x+\mathtt{b}}{2}\mathcal{D}_0(\eta)\left[\psi_x+\mathtt{b}\right]\right) - \gamma\mathcal{K}(\eta). \end{cases} \tag{1.9}$$

Notice that

$$
\begin{aligned}
\mathcal{H}(\eta)[\omega](x) &= 2\,\mathrm{BR}(z)[\omega]\cdot z_x^{\perp}(x)\\
&= 2\int_{\mathbb{T}} \frac{(z(x)-z(y))^{\perp}\cdot z_x^{\perp}(x)}{|z(x)-z(y)|^2}\omega(y)\mathrm{d}y\\
&= 2\int_{\mathbb{T}} \frac{(z(x)-z(y))\cdot z_x(x)}{|z(x)-z(y)|^2}\omega(y)\mathrm{d}y\\
&= \int_{\mathbb{T}} \partial_x\Big[\log\big(|z(x)-z(y)|^2\big)\Big]\omega(y)\mathrm{d}y.
\end{aligned}
\tag{1.10}
$$

Therefore, the first equation of (1.9) preserves the average and the natural phase space for η is

$$
H_0^s(\mathbb{T};\mathbb{R}) \triangleq \left\{\eta\in H^s(\mathbb{T};\mathbb{R}) \quad \text{s.t.} \quad \int_{\mathbb{T}}\eta(x)\,\mathrm{d}x = 0\right\}.
$$

In the absence of capillarity ($\gamma = 0$), it is now understood that the problem is ill-posed in Sobolev regularity [28, 53, 68], but it admits weak solutions [35], while one has to require at least analytic regularity on the initial data in order to have a satisfactory local well-posedness theory [47, 63, 64]. It is also well-established [54, Chap. 9.3] that the presence of a background velocity jump ($\mathtt{b}$) across the interface drives linear instability in the system, generating the classical Kelvin-Helmholtz phenomena. Conversely, surface tension (γ) exerts a stabilizing effect at the linear level, enabling local-in-time solutions to the full nonlinear equations [3–5, 32, 51]. When stabilization is induced solely by capillarity, the existence time is limited to $T\sim\epsilon^{-1}$, where ϵ represents the magnitude of the initial datum. Besides, in the physics literature [48] the authors perform several numerical simulations for (1.4) varying the Weber number We which corresponds, up to a period factor of 2π, with the parameter

$$
\beta \triangleq \frac{\mathtt{b}^2}{\gamma}
$$

that captures the interplay between the previous two mentioned opposite phenomena. In particular, despite the slightly different geometry considered, it is numerically conjectured in [48] that the behavior of the system below a critical Weber number, which is very close to (1.11), "*...is quite predictable by linear theory, even over long times...*", [48, p. 1939]. The aim of our work is to give a rigorous mathematical proof of this numerical conjecture. To do so, we exploit the Hamiltonian nature of the quasi-linear system (1.9). Conversely to the classical formulations using the Dirichlet-Neumann operator, here the system is quite explicit and related to singular integral operators, see (1.5). By introducing a novel framework that combines the Hamiltonian Birkhoff normal form procedure for quasi-linear Hamiltonian systems [25] with a generalization of the paralinearization method for singular integral operators developed in [17], we prove that the lifespan extends to $T\sim\epsilon^{-(N+1)}$ for any $N\in\mathbb{N}$. This result enables us to establish stability way beyond the classical local lifespan results of [3–5, 32, 51], leading to what we refer to as *almost global well-posedness*. The precise statement is given in the following theorem.

Theorem 1.1 (**Almost global existence of nearly circular vortex-sheets**). *Let*

$$
0<\beta_1<\beta_2<4\left(2+\sqrt{3}\right).
\tag{1.11}
$$

There exists a zero measure set $\mathcal{B}\subset[\beta_1,\beta_2]$ such that for any values $\gamma\in(0,\infty)$ of the surface tension and $\mathtt{b}\in\mathbb{R}$ of the background velocity jump with

$$
\frac{\mathtt{b}^2}{\gamma}\in[\beta_1,\beta_2]\setminus\mathcal{B},
$$

for any $N\in\mathbb{N}$, there exists $s_0>0$ such that for any $s\geqslant s_0$ there exist $\varepsilon_0, c, C>0$, such that for any $0<\varepsilon<\varepsilon_0$ and any initial datum

$$
(\eta_0,\psi_0)\in H_0^{s+\frac14}(\mathbb{T};\mathbb{R})\times\dot{H}^{s-\frac14}(\mathbb{T};\mathbb{R}), \qquad \text{with} \qquad \|\eta_0\|_{H_0^{s+\frac14}(\mathbb{T};\mathbb{R})}+\|\psi_0\|_{\dot{H}^{s-\frac14}(\mathbb{T};\mathbb{R})}\leqslant\varepsilon,
$$

the system (1.9) *admits a unique classical solution*

$$
(\eta,\psi)\in C^0\left([-T_\varepsilon,T_\varepsilon];\,H_0^{s+\frac14}(\mathbb{T};\mathbb{R})\times\dot{H}^{s-\frac14}(\mathbb{T};\mathbb{R})\right), \qquad T_\varepsilon\geqslant c\varepsilon^{-(N+1)},
$$

with initial datum (η_0, ψ_0) *and size*

$$\sup_{t\in[-T_\varepsilon,T_\varepsilon]}\left(\|\eta(t,\cdot)\|_{H_0^{s+\frac{1}{4}}(\mathbb{T};\mathbb{R})}+\|\psi(t,\cdot)\|_{\dot{H}^{s-\frac{1}{4}}(\mathbb{T};\mathbb{R})}\right)\leqslant C\varepsilon.$$

Remark 1.2. Let us make the following remarks about the previous theorem.

1. *The almost-global well-posedness result of Theorem 1.1* cannot *be achieved in settings where capillarity serves only a stabilizing parameter*, i.e. when $\beta = 0$. The Weber number β, which emerges naturally in the equilibrium spectrum $\{\omega_{\gamma,\mathtt{b}}(j)\}_{j\in\mathbb{Z}^*}$ (see (1.12)), captures the interplay between capillarity stabilization and Kelvin-Helmholtz instability. Remarkably, the incorporation of the (physically relevant) interplay between background velocity jump and capillarity is the key factor that generates stability, in particular we can pass from a Sobolev ill-posed problem ($\gamma = 0$) to a almost-global well-posed one when γ is arbitrarily small and $\mathtt{b}^2$ is comparable. The Weber number β modulates the linear frequencies $\omega_{\gamma,\mathtt{b}}(j)$ in a non-trivial fashion and enables us to exclude resonances

$$\omega_{\gamma,\mathtt{b}}(j_1)\pm\ldots\pm\omega_{\gamma,\mathtt{b}}(j_N)\neq 0$$

   taking β outside a suitable zero-measure resonant set $\mathcal{B}$. The non-resonance condition is a fundamental requirement for implementing the Hamiltonian Birkhoff normal form procedure in PDEs [7–11, 14–16, 25, 27, 37, 38, 49].

2. Referring again to the work [48] the authors notice, at page 1939, that

   > We had hoped to see some repartition of energy from the $k = 1$ mode to smaller scales over large times. However, for $We = 10.0$ only a very slow increase is observed, if any, of the width of the active spatial spectrum.

   This unexpected localization phenomenon is consistent with our analysis. Indeed we prove that, after a suitable change of variables, each Fourier mode u_j can exchange energy only with u_{-j} for very long times, as we shall explain later. This is a consequence of the non-resonance conditions and the Hamiltonian structure of the equation which gives the conservation of super-actions (see (1.19)).

3. We identify a threshold for the linear stability in (1.11).

   This constraint is imposed to ensure that the spectrum of the linearized equation at the trivial state $(\eta,\psi) = (0,0)$ is purely imaginary (see Section 2.2), a necessary condition for implementing the Hamiltonian Birkhoff normal form argument. While this condition can be relaxed by restricting the phase space to $\mathbf{m}$-fold solutions for sufficiently large $\mathbf{m}$, such a restriction significantly narrows the class of admissible solutions. For more details, we refer the reader to Section 2.2 and [59]. Although one could adapt the following analysis by verifying the $\mathbf{m}$-fold preserving properties of the transformations along the scheme, following a similar approach to [42].

4. The parameter Ω in (1.4) represents the speed of rotation of the reference frame. Since the Kelvin-Helmholtz problem is invariant under rotations, Ω can be chosen arbitrarily without altering the shape of the solutions. We choose Ω as in (2.21). This choice simplifies some computations and does not affect generality.

5. Global-in-time solutions with specific structures can be constructed, as demonstrated in [59], where we identified families of globally defined, uniformly rotating solutions. We also refer to [29, 30, 40, 56, 57, 59, 60, 67] for the construction of families of steady solutions in slightly different settings. However, it remains unclear whether the almost global existence solutions stated in Theorem 1.1 are actually global in time. The quasi-linear structure of the Kelvin-Helmholtz system, combined with the absence of dispersion due to the periodic boundary conditions, makes the global existence of the Cauchy problem currently out of reach.

Ideas of the proof While the Dirichlet-Neumann operator approach pioneered by Zakharov, Craig, and Sulem in [34, 69] has become standard for Water-Waves problems-and also employed to the two phase setting [51]- we employ an alternative formulation. Following previous works such as [31, 33], we utilize the Birkhoff-Roth integral operator formulation, which exploits the Dirac-δ structure of the vorticity (1.2) in conjunction with the Biot-Savart law to express the KH equations as a CDE. A crucial aspect of our approach is establishing that the KH equations thus derived possess a *Hamiltonian* structure (Section 2.1), which is the following

$$\begin{bmatrix}\eta_t\\ \psi_t\end{bmatrix} = \boldsymbol{J}\nabla H(\eta,\psi), \qquad \boldsymbol{J} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix},$$

where the Hamiltonian is related to the pseudo kinetic energy $\mathcal{E}_{\mathtt{b}}$, the length $\mathcal{L}$ of the free boundary and the angular momentum $\mathcal{M}$ through

$$H(\eta,\psi) \triangleq \mathcal{E}_{\mathtt{b}}(\eta,\psi) + \gamma\mathcal{L}(\eta) + \Omega\mathcal{M}(\eta,\psi).$$

In different geometrical contexts the Hamiltonian structure of the KH system was already presented by Benjamin-Bridges [12, 13]. Once this Hamiltonian formulation is established, we are methodologically committed to working with the specific equations that arise from it, as any deviation would compromise the Hamiltonian property, which is essential for our analysis. This Hamiltonian structure is a fundamental requirement for obtaining the almost global well-posedness result Theorem 1.1 using the Hamiltonian Birkhoff normal form of [25]. Since we are looking for a stability result near the trivial state $(\eta,\psi) = (0,0)$, a quantity of interest is the linearization at this stationary solution. The linearized KH system there writes

$$\begin{bmatrix}\eta_t\\ \psi_t\end{bmatrix} = \boldsymbol{L}_{\gamma,\mathtt{b}}(D)\begin{bmatrix}\eta\\ \psi\end{bmatrix}, \qquad \boldsymbol{L}_{\gamma,\mathtt{b}}(\xi) \triangleq \begin{bmatrix}0 & -\frac{|\xi|}{2}\\ \gamma|\xi|^2 - \frac{\mathtt{b}^2}{2}|\xi| - \left(\gamma - \mathtt{b}^2\right) & 0\end{bmatrix}.$$

The associated spectrum is given by $\lambda^{\pm}_{\gamma,\mathtt{b}}(\xi) = \pm\mathrm{i}\omega_{\gamma,\mathtt{b}}(\xi)$, with

$$\omega_{\gamma,\mathtt{b}}(\xi) = \sqrt{\frac{\gamma|\xi|}{2}}\sqrt{|\xi|^2 - \frac{\beta}{2}|\xi| + \beta - 1}, \qquad \beta \triangleq \frac{\mathtt{b}^2}{\gamma}. \tag{1.12}$$

Here we see appearing the parameter β that modulates the equilibrium frequencies. This parameter is homogeneous to a wave number (inverse of a length). The modulation is fundamental for avoiding resonances later in the Hamiltonian Birkhoff normal form. Let us mention that the modulation of the linear frequencies by an external or geometrical parameter has been used to avoid resonances and construct quasi-periodic solutions for fluid models, see [6, 20–22, 26, 41–43, 46, 61]. Observe that $\omega_{\gamma,\mathtt{b}}(\xi)$ is real for any $|\xi| \geqslant 1$ provided that

$$0 < \beta < 4\left(2 + \sqrt{3}\right).$$

There emerges our linear stability threshold.This means that one gets linear stability for typically small oscillations at small scales where the stabilizing effects of the surface tension are dominant. Also notice that the asymptotic of the linear frequencies is superlinear, namely as $|\xi| \to \infty$

$$\omega_{\gamma,\mathtt{b}}(\xi) \sim \sqrt{\frac{\gamma}{2}}|\xi|^{\frac{3}{2}}.$$

Our purpose is to prove a nonlinear stability result near the circular interface corresponding to $(\eta,\psi) = (0,0)$. To do so, we are able to obtain a suitable energy estimate of the form, for any $N \in \mathbb{N}$,

$$\|(\eta,\psi)(t)\|_s^2 \leqslant C(s)\left(\|(\eta,\psi)(0)\|_s^2 + \int_0^t \|(\eta,\psi)(\tau)\|_{s_0}^{N+1}\|(\eta,\psi)(\tau)\|_s^2\,\mathrm{d}\tau\right), \qquad \forall 0 < t < T, \tag{1.13}$$

where we used the notation

$$\|(\eta,\psi)\|_s \triangleq \|\eta\|_{H_0^{s+\frac{1}{4}}(\mathbb{T};\mathbb{R})} + \|\psi\|_{\dot{H}^{s-\frac{1}{4}}(\mathbb{T};\mathbb{R})}.$$

Then, a bootstrap argument allows us to get an existence time of the form $T \geqslant c\varepsilon^{-N-1}$ where ε is the size of the initial datum. Notice that the above estimate is highly non-trivial for two reasons:

1. The right-hand side contains the same number of derivatives as the left-hand side, which is particularly delicate given the quasi-linear nature of the equations.
2. The integral term exhibits high homogeneity, a non-trivial property considering the quadratic nonlinearity of the equations.

To address the first obstacle, we perform a paralinearization of the Kelvin-Helmholtz system (1.9) together with a paradifferential reduction procedure to remove the space dependance in the positive order part, which allows to remain at the same level of regularity. The paralinearization result is given in Theorem 4.2 and writes as follows

$$\begin{bmatrix}\eta_t\\ \psi_t\end{bmatrix} = \mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{Q}_{\gamma,\mathtt{b}}\left(\eta,\psi;x,\xi\right) + \boldsymbol{B}_{\mathtt{b}}\left(\eta,\psi;x\right)|\xi| - \mathrm{i}V_{\mathtt{b}}\left(\eta,\psi;x\right)\mathrm{Id}_{\mathbb{R}^2}\,\xi + \boldsymbol{A}_{[0]}\left(\eta,\psi;x,\xi\right)\right)\begin{bmatrix}\eta\\ \psi\end{bmatrix} + \boldsymbol{R}\left(\eta,\psi\right)\begin{bmatrix}\eta\\ \psi\end{bmatrix}, \quad (1.14)$$

where $\mathrm{Op}^{\mathrm{BW}}(\cdot)$ denotes the Bony-Weyl quantization in (3.9). Each term of positive order has an explicit formula with

$$\begin{aligned}
\boldsymbol{Q}_{\gamma,\mathtt{b}}\left(\eta,\psi;x,\xi\right) &\triangleq \begin{bmatrix} 0 & -\frac{|\xi|}{2}\\ \gamma\left(1+\mathtt{f}\left(\eta;x\right)\right)\left(|\xi|^2-1\right) - \left(\frac{\mathtt{b}^2}{2} + w_{\mathtt{b}}\left(\eta,\psi;x\right)\right)|\xi| + \frac{\mathtt{b}^2}{(1+2\eta)} & 0\end{bmatrix},\\
\mathtt{f}\left(\eta;x\right) &\triangleq \left(\frac{1+2\eta}{\left(1+2\eta\right)^2+\eta_x^2}\right)^{\frac{3}{2}} - 1 \qquad w_{\mathtt{b}}\left(\eta,\psi;x\right) \triangleq \frac{1}{2}\left(\left(\left(\psi_x+\mathtt{b}\right)\frac{1+2\eta}{\left(1+2\eta\right)^2+\eta_x^2}\right)^2 - \mathtt{b}^2\right),\\
\boldsymbol{B}_{\mathtt{b}}\left(\eta,\psi;x\right) &\triangleq \frac{1}{2}\begin{bmatrix} B_{\mathtt{b}}\left(\eta,\psi;x\right) & 0\\ B_{\mathtt{b}}^2\left(\eta,\psi;x\right) & -B_{\mathtt{b}}\left(\eta,\psi;x\right)\end{bmatrix}, \qquad B_{\mathtt{b}}\left(\eta,\psi;x\right) \triangleq \left(\psi_x+\mathtt{b}\right)\frac{2\eta_x}{\left(1+2\eta\right)^2+\eta_x^2},\\
V_{\mathtt{b}}\left(\eta,\psi;x\right) &\triangleq \frac{1}{2}\mathcal{D}_0\left(\eta\right)\left[\mathtt{b}+\psi_x\right] - \frac{\mathtt{b}}{2}.
\end{aligned} \quad (1.15)$$

The matrix operator $\boldsymbol{A}_{[0]}$ is of order zero and $\boldsymbol{R}$ is a regularizing matrix operator up to a sufficiently large order. We believe that this paralinearization result is itself of interest for maybe future purposes. In our equations, the Dirichlet-Neumann operator does not explicitly appear, contrary to what occurs in the one-phase flat Water-Waves problem, see [34] and [52, Chap. 1]. Instead, the equation derived in (2.5) features nonlinearities of convolution type with nonlinear singular convolution kernels. This fundamental difference in structure necessitates the development of a specialized paralinearization technique adapted to these convolution operators—a technique we develop in this work (Section 4) building on the previous work of the last author and collaborators in [17] for the less challenging case of α-SQG patches. Let us now expose our method to paralinearize the KH system. The key insight of our approach is that terms that resist standard paralinearization techniques (i.e. paraproducts, Bony paralinearization formula and composition of paradifferential operators) share a common structure of convolution type in the form

$$H\left(\eta\right)g\left(x\right) \triangleq \mathrm{p.v.}\int_{-\pi}^{\pi} K\left(\eta;x,z\right)\frac{g\left(x-z\right)}{z}\mathrm{d}z,$$

where the function $K\left(\eta;x,z\right)$ exhibits regularity at $z=0$ comparable to η. By Taylor expanding the function $z\mapsto K\left(\eta;x,z\right)$ at $z=0$ and applying paraproduct expansions, we derive

$$H\left(\eta\right)g\left(x\right) = \sum_{\mathtt{j}=0}^{\mathtt{J}}\mathrm{Op}^{\mathrm{BW}}\left(K_{\mathtt{j}}\left(\eta;x\right)\right)\mathrm{p.v.}\int_{-\pi}^{\pi} z^{\mathtt{j}-1}g\left(x-z\right)\mathrm{d}z \quad (1.16a)$$

$$+\int_{-\pi}^{\pi}\mathrm{Op}^{\mathrm{BW}}\left(R\left(\eta;x,z\right)\right)g\left(x-z\right)\mathrm{d}z + \text{ l.o.t.}, \quad (1.16b)$$

with $J\in\mathbb{N}$, for any $j\in\{0,\ldots,J\}$, K_j being a z-independant function of x and R being a remainder satisfying $\left|R\left(\eta:x,z\right)\right| = \mathcal{O}\left(z^{\mathtt{J}+1}\right)$. This transformation reduces our analysis to two specific categories of terms, namely

- *Terms in the right-hand side of* (1.16a)*:* Classical theory (see [62, p. 355]) establishes that for any $\mathtt{j}\in\{0,\ldots,\mathtt{J}\}$,

$$\mathrm{p.v.}\int_{-\pi}^{\pi} z^{\mathtt{j}-1}g\left(x-z\right)\mathrm{d}z = m_{\mathtt{j}}\left(D\right)g,$$

where m_{j} represents a Fourier multiplier of order j. Thus, through composition theorems for paradifferential operators, we obtain

$$\mathrm{Op}^{\mathrm{BW}}\left(K_{\mathrm{j}}(\eta;x)\right)\,\mathrm{p.v.}\int_{-\pi}^{\pi} z^{\mathrm{j}-1} g(x-z)\,\mathrm{d}z = \mathrm{Op}^{\mathrm{BW}}\left(K_{\mathrm{j}}(\eta;x)\, m_{\mathrm{j}}(\xi)\right) g + \text{ bounded terms.}$$

- *Terms in* (1.16b)*:* We leverage the decay properties of R as $z \to 0$ to establish that these terms constitute paradifferential operators of order $-(\mathrm{J}+1)$ modulo smoothing operators, as detailed in Proposition 3.28.

The methodology outlined above is elaborated in detail in Section 4 and formalized in Theorem 4.2, representing one of the manuscript's principal contributions. Our work demonstrates that effective paralinearization is possible even when using the Birkhoff-Roth formulation rather than the Dirichlet-Neumann operator approach. The recovered paradifferential structure in (1.14) exhibits similarities with pure-capillarity one-phase Water-Waves equations, which allows us to derive several established results concerning vortex sheets, including the necessity of capillarity for system stabilization (cf. [3]). Furthermore, we identify purely nonlinear, unstable terms characteristic of the KH equations (cf. the term w_{b} in (1.15), which is non-nil even when $\mathrm{b}=0$), highlighting the enhanced instability of KH compared to Water-Waves systems. For further analysis of these distinctive unstable terms, we direct the interested reader to Remark 4.3.
Once the paralinearization obtained, our goal is to remove the x-dependence of the positive order terms in order to run an energy estimate in the same Sobolev space, namely without loss of derivatives. This is done through a classical paradifferential reduction procedure requiring to reformulate the problem with complex variables. Defining the complex coordinates

$$U = \begin{bmatrix} u \\ \bar{u} \end{bmatrix}, \qquad u \triangleq \mathrm{m}_{\gamma,\mathrm{b}}(D)^{-1}\eta + \mathrm{i}\,\mathrm{m}_{\gamma,\mathrm{b}}(D)\psi, \qquad \mathrm{m}_{\gamma,\mathrm{b}}(\xi) \triangleq \sqrt{\frac{|\xi|}{2\omega_{\gamma,\mathrm{b}}(\xi)}},$$

the paralinearized KH system is equivalent to the complex Hamiltonian system

$$U_t = \boldsymbol{J}_{\mathbb{C}}\,\mathrm{Op}^{\mathrm{BW}}\Big(\boldsymbol{A}_{\frac{3}{2}}(U;x)\ \omega_{\gamma,\mathrm{b}}(\xi) + \boldsymbol{A}_1(U;x,\xi) + \boldsymbol{A}_{\frac{1}{2}}(U;x)\ |\xi|^{\frac{1}{2}} + \boldsymbol{A}_{[0]}(U;x,\xi)\Big) U + \boldsymbol{R}(U)\,U. \tag{1.17}$$

In the above system, each $\boldsymbol{A}_m$ corresponds a matrix of x-dependent symbols of order m, while $\boldsymbol{R}$ is a matrix of regularizing operators up to any fixed order. Then we follow the Hamiltonian method developed by the first author and collaborators [25] (see also [17, 18, 24, 39, 55, 58] for non Hamiltonian approach) which consists to perform a series of transformations:

i we first perform an *Alinhac Good Unknown* transformation—a nilpotent matrix-valued paradifferential change of variable introduced in [2, 50]. This transformation eliminates the unbounded terms $\boldsymbol{B}_{\mathrm{b}}$ in (1.15), which constitute the only unbounded contributions in the one-phase gravity water wave system, thus proving essential for developing local well-posedness theory in the pure gravity setting;

ii we then *diagonalize* and *reduce to constant coefficients* the resulting system at arbitrary order, modulo smoothing operators (whose regularizing effects depend on the initial data's regularity). This technique, initially developed in [18], has become standard for implementing normal-form techniques in quasi-linear systems [19, 25, 58]. The method involves conjugation with flows generated by paradifferential operators, where generators are selected based on desired cancellations. We reduce the equation to a diagonal, paradifferential constant-coefficient form by iterative application on the degrees of the paradifferential operators.

At the end of such procedure we are able to define a transformed, equivalent (in Sobolev) variable

$$W \triangleq \boldsymbol{B}(U)U, \qquad W = \begin{bmatrix} w \\ \bar{w} \end{bmatrix}$$

that satisfies a *constant-coefficient, scalar* equation given by

$$W_t = \mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}\Big(\mathrm{i}\Big(\big(1+v(U;t)\big)\omega_{\gamma,\mathrm{b}}(\xi) + \mathcal{V}_{\mathrm{b}}(U;t)\xi + \mathscr{b}_{\frac{1}{2}}(U;t)|\xi|^{\frac{1}{2}} + \mathscr{b}_0(U;t,\xi)\Big)\Big) W + \mathsf{R}(U;t)W, \tag{1.18}$$

up to a smoothing remainder R (see Proposition 6.1 and (3.11) for the definition of $\mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}(\cdot)$). The transformed equation (1.18) has the crucial property that its para-differential part is in constant-coefficient form. To overcome the second obstacle Item 2, we aim to implement a Hamiltonian Birkhoff normal form up to homogeneity degree N. However, unlike the original complex system (1.17), (1.18) no longer possesses the fundamental Hamiltonian structure. This structure is essential to ensure that certain *non-trivial* resonant terms do not contribute to energy estimates, as explained below. Recovering this structure is the purpose of the Darboux symplectic corrector as designed in [25]. To understand why this correction is needed, we first examine the role of non-resonance conditions. The non-resonance conditions in Section 2.3 ensure the exclusion of resonances, meaning that

$$\sigma_1\omega_{\gamma,\mathtt{b}}(j_1)+\cdots+\sigma_N\omega_{\gamma,\mathtt{b}}(j_N)\neq 0$$

unless the indices $(\sigma_1,\ldots,\sigma_N)\in\{\pm\}^N$ and $(j_1,\ldots,j_N)\in\mathbb{Z}^N$ are *super-action preserving* (see Definition 2.5). This can happen when $N=2p$ is even, with

$$\sigma_1=\cdots=\sigma_p=+,\qquad \sigma_{p+1}=\cdots=\sigma_{2p}=-,$$

and either

$$\text{(i) } j_\ell=j_{p+\ell}\qquad\text{or}\qquad \text{(ii) } j_\ell=-j_{p+\ell}.$$

Case (i) corresponds to *trivial resonances.* The associated monomials in the vector fields of (1.18) take the form

$$|u_{j_1}|^2\ldots|u_{j_{p-1}}|^2u_{j_p}e^{\mathrm{i}j_px}.$$

Proving that these terms do not contribute to Sobolev energy estimates is typically straightforward, as it suffices to show that their coefficients are purely imaginary. In contrast, case (ii) involves monomials of the form

$$u_{j_1}\overline{u_{-j_1}}\ldots u_{j_{p-1}}\overline{u_{-j_{p-1}}}u_{j_p}e^{-\mathrm{i}j_px}.$$

These terms couple different Fourier modes, making it more challenging to show that they do not affect energy estimates. However, if the vector field possesses the strong algebraic property of being Hamiltonian, these monomials—called *super-action preserving*—automatically admit infinitely many conservation laws, known as *super-actions.* Specifically, for any $n\in\mathbb{N}$, the quantities

$$J_n(u)\triangleq|u_n|^2+|u_{-n}|^2 \tag{1.19}$$

are conserved. As a consequence, a Hamiltonian, super-action preserving vector field remains transparent to any Sobolev energy estimate. However, the system (1.18) for W lacks this Hamiltonian structure, preventing direct application of these conservation laws. To overcome this issue, we introduce the Darboux symplectic correction, as detailed in Proposition 7.1, restoring the necessary structure and allowing us to exploit these properties effectively. Since the map $\boldsymbol{B}(U)$ satisfies the hypotheses of [25, Theorem 7.1], we apply it to obtain a new constant-coefficient equation:

$$\partial_tZ_0=\mathrm{i}\boldsymbol{\omega}_{\gamma,\mathtt{b}}(D)Z_0+\mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}\Big(\mathrm{i}(d_{\frac32})_{\leqslant N}(Z_0;\xi)+\mathrm{i}(d_{\frac32})_{>N}(U;t,\xi)\Big)Z_0+\boldsymbol{R}_{\leqslant N}(Z_0)Z_0+\boldsymbol{R}_{>N}(U;t)U. \tag{1.20}$$

This equation is Hamiltonian up to homogeneity N, meaning that

$$\mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}\Big(\mathrm{i}(d_{\frac32})_{\leqslant N}(Z_0;\xi)\Big)Z_0+\boldsymbol{R}_{\leqslant N}(Z_0)Z_0=\boldsymbol{J}_{\mathbb{C}}\nabla H_{\leqslant N}$$

for some real Hamiltonian function $H_{\leqslant N}$. At this point, we begin the algorithmic procedure of reducing the degrees of homogeneity (see Proposition 7.9). The final outcome is a super-action preserving Hamiltonian equation of the form

$$\partial_tZ=\mathrm{i}\boldsymbol{\omega}_{\gamma,\mathtt{b}}(D)Z+\boldsymbol{J}_{\mathbb{C}}\nabla H^{(\mathtt{SAP})}_{\frac32}(Z)+\boldsymbol{J}_{\mathbb{C}}\nabla H^{(\mathtt{SAP})}_{-\varrho}(Z)+\mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}\Big(\mathrm{i}(d_{\frac32})_{>N}(U;t,\xi)\Big)Z+\boldsymbol{R}_{>N}(U;t)U. \tag{1.21}$$

Since the super-action preserving Hamiltonian terms $\mathrm{i}\boldsymbol{\omega}_{\gamma,\mathtt{b}}(D)Z$, $\boldsymbol{J}_{\mathbb{C}}\nabla H^{(\mathtt{SAP})}_{\frac32}(Z)$, and $\boldsymbol{J}_{\mathbb{C}}\nabla H^{(\mathtt{SAP})}_{-\varrho}(Z)$ do not contribute to the energy estimate, we obtain, for small solutions $\|Z\|\sim\|U\|_s\lesssim\varepsilon$, the energy bound

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Z\|_s^2\lesssim\varepsilon^{N+3},$$

which allows us to prove Theorem 1.1.

Structure of the manuscript The Hamiltonian formulation can be found in Section 2.1 while the non-resonance conditions are proved in Section 2.3. The paralinearization of the system (1.9) is carried out in Section 4 and the final result is stated in Theorem 4.2. The next step is to reformulate the results using the complex notation, see Section 5. Then, we implement a reducibility procedure to get rid of the space dependence of the positive order part. This part is now rather classical and the corresponding final result is given in Proposition 6.1. With this in hand, one can perform the Hamiltonian Birkhoff normal form. It is done in Section 7.2 and requires non-resonance conditions for frequency vectors composed with the equilibrium spectrum. Such conditions are checked in Section 2.3 and are the reasons for the introduction of the zero measure set $\mathcal{B}$.

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2 Hamiltonian structure and non-resonance conditions

Here we highlight the Hamiltonian nature of the system (1.9). Then, we study the associated linearization at the trivial solution $(\eta,\psi)=(0,0)$ and discuss the non-resonance property of the corresponding eigenvalues. This latter fact is crucial for implementing the Birkhoff normal form in Section 7.

2.1 Derivation of the Hamiltonian formulation

Let us now exhibit the Hamiltonian nature of the Kelvin-Helmholtz system (1.9).

Proposition 2.1. *The system* (1.9) *is Hamiltonian. More precisely, let us consider*

$$H(\eta,\psi)\triangleq\mathcal{E}_{\mathtt{b}}(\eta,\psi)+\gamma\mathcal{L}(\eta)+\Omega\mathcal{M}(\eta,\psi),\tag{2.1}$$

where $\mathcal{E}_{\mathtt{b}}(\eta,\psi)$, $\mathcal{L}(\eta)$ *and* $\mathcal{M}(\eta,\psi)$ *are the pseudo kinetic energy, the length of the free boundary and the angular momentum, respectively defined by*

$$\mathcal{E}_{\mathtt{b}}(\eta,\psi)\triangleq-\frac{1}{4}\int_{\mathbb{T}}\int_{\mathbb{T}}\big(\psi_x(x)+\mathtt{b}\big)\big(\psi_x(y)+\mathtt{b}\big)\log\big(|z(x)-z(y)|^2\big)\mathrm{d}y\mathrm{d}x,\tag{2.2}$$

$$\mathcal{L}(\eta)\triangleq\int_{\mathbb{T}}|z_x(x)|\mathrm{d}x,\tag{2.3}$$

$$\mathcal{M}(\eta,\psi)\triangleq\int_{\mathbb{T}}\psi_x(x)\eta(x)\mathrm{d}x.\tag{2.4}$$

Then, the equations (1.9) *are equivalent to*

$$\begin{bmatrix}\eta_t\\ \psi_t\end{bmatrix}=\begin{bmatrix}-\nabla_\psi H(\eta,\psi)\\ \nabla_\eta H(\eta,\psi)\end{bmatrix}=\boldsymbol{J}\,\nabla H\big(\eta,\psi\big),\qquad \boldsymbol{J}\triangleq\begin{bmatrix}0&-1\\1&0\end{bmatrix}.\tag{2.5}$$

In addition, the Hamiltonian H is resversible and invariant under translations, namely defining the transformations

$$\mathrm{S}\begin{bmatrix}\eta\\ \psi\end{bmatrix}(x)=\begin{bmatrix}\eta\\ -\psi\end{bmatrix}(-x),\qquad \mathrm{t}_\varsigma\begin{bmatrix}\eta\\ \psi\end{bmatrix}(x)=\begin{bmatrix}\eta\\ \psi\end{bmatrix}(x+\varsigma),$$

we have

$$H\circ\mathrm{S}=H=H\circ\mathrm{t}_\varsigma,\qquad\forall\varsigma\in\mathbb{T}.\tag{2.6}$$

Proof. ► *The pseudo kinetic part:*
We compute the variation of $\mathcal{E}_{\mathtt{b}}(\eta,\psi)$ with respect to ψ. Using integration by parts and (1.10), we get

$$\mathrm{d}_\psi\mathcal{E}_{\mathtt{b}}(\eta,\psi)[\hat\psi]=\frac{1}{2}\int_{\mathbb{T}}\left(\int_{\mathbb{T}}\big(\psi_y(y)+\mathtt{b}\big)\partial_x\log\big(|z(x)-z(y)|^2\big)\mathrm{d}y\right)\hat\psi(x)\mathrm{d}x$$

$$= \frac{1}{2} \int_{\mathbb{T}} \mathcal{H}(\eta)[\partial_x \psi + \mathtt{b}](x) \hat{\psi}(x) \mathrm{d}x \tag{2.7}$$

From (2.7), we deduce

$$\nabla_\psi \mathcal{E}_{\mathtt{b}}(\eta, \psi) = \frac{1}{2} \mathcal{H}(\eta)[\partial_x \psi] + \frac{\mathtt{b}}{2} \mathcal{H}(\eta)[1]. \tag{2.8}$$

Now we turn to the differentiation of $\mathcal{E}_{\mathtt{b}}(\eta, \psi)$ with respect to η. Recall the notation in (1.3). Differentiating (2.2) with respect to η, we infer

$$\mathrm{d}_\eta \mathcal{E}_{\mathtt{b}}(\eta, \psi)[\hat{\eta}] = -\frac{1}{2} \int_{\mathbb{T}} \int_{\mathbb{T}} \big(\psi_x(x) + \mathtt{b}\big)\big(\psi_x(y) + \mathtt{b}\big) \frac{\hat{\eta}(x) + \hat{\eta}(y) - \big(\hat{\eta}(x) \frac{r(y)}{r(x)} + \hat{\eta}(y) \frac{r(x)}{r(y)}\big) \cos(x - y)}{|z(x) - z(y)|^2} \mathrm{d}x \mathrm{d}y.$$

By symmetry, we can reduce this expression to

$$\begin{aligned} \mathrm{d}_\eta \mathcal{E}_{\mathtt{b}}(\eta, \psi)[\hat{\eta}] &= -\int_{\mathbb{T}} \big(\psi_x(x) + \mathtt{b}\big) \left(\int_{\mathbb{T}} \big(\psi_x(y) + \mathtt{b}\big) \frac{1 - \frac{r(y)}{r(x)} \cos(x - y)}{|z(x) - z(y)|^2} \hat{\eta}(x) \mathrm{d}y \right) \mathrm{d}x \\ &= -\int_{\mathbb{T}} \frac{1}{2} \big(\psi_x(x) + \mathtt{b}\big) \mathcal{D}_0(\eta)[\partial_x \psi + \mathtt{b}](x) \hat{\eta}(x) \mathrm{d}x, \end{aligned}$$

which implies in turn

$$\nabla_\eta \mathcal{E}_{\mathtt{b}}(\eta, \psi) = -\frac{\psi_x + \mathtt{b}}{2} \mathcal{D}_0(\eta)[\partial_x \psi + \mathtt{b}]. \tag{2.9}$$

► *The length part:*
Recall that $z(x) = \big(1 + h(x)\big) e^{\mathrm{i}x}$ with $h = \sqrt{1 + 2\eta} - 1$. Therefore,

$$z_x(x) = \Big(h_x(x) + \mathrm{i}\big(1 + h(x)\big)\Big) e^{\mathrm{i}x}, \qquad |z_x(x)|^2 = h_x^2(x) + \big(1 + h(x)\big)^2.$$

Thus,

$$\mathcal{L}(\eta) = \int_{\mathbb{T}} \sqrt{h_x^2(x) + \big(1 + h(x)\big)^2} \mathrm{d}x \triangleq \widetilde{\mathcal{L}}(h).$$

Differentiating, we obtain

$$d_h \widetilde{\mathcal{L}}(h)[\hat{h}] = \int_{\mathbb{T}} \frac{h_x(x) \hat{h}_x(x) + \big(1 + h(x)\big) \hat{h}(x)}{\sqrt{h_x^2(x) + \big(1 + h(x)\big)^2}} \mathrm{d}x.$$

Integrating by parts, we get

$$\begin{aligned} d_h \widetilde{\mathcal{L}}(h)[\hat{h}] &= \int_{\mathbb{T}} \left[\frac{\big(1 + h(x)\big)}{\sqrt{h_x^2(x) + \big(1 + h(x)\big)^2}} - \partial_x \left(\frac{h_x(x)}{\sqrt{h_x^2(x) + \big(1 + h(x)\big)^2}} \right) \right] \hat{h}(x) \mathrm{d}x \\ &= -\int_{\mathbb{T}} \big(1 + h(x)\big) \frac{\big(1 + h(x)\big)\big(h_{xx}(x) - 1 - h(x)\big) - 2h_x^2(x)}{\Big(h_x^2(x) + \big(1 + h(x)\big)^2\Big)^{\frac{3}{2}}} \hat{h}(x) \mathrm{d}x. \end{aligned}$$

It is easy to see that

$$\frac{(1 + h)(h_{xx} - 1 - h) - 2h_x^2}{\big(h_x^2 + (1 + h)^2\big)^{\frac{3}{2}}} = \frac{\eta_{xx} - (1 + 2\eta) - 3\left(\frac{\eta_x}{\sqrt{1 + 2\eta}}\right)^2}{\left(1 + 2\eta + \left(\frac{\eta_x}{\sqrt{1 + 2\eta}}\right)^2\right)^{\frac{3}{2}}} = \mathcal{K}(\eta).$$

As a consequence,

$$\nabla_h \widetilde{\mathcal{L}}(h) = -(1 + h) \mathcal{K}(\eta).$$

Applying the chain rule, we deduce that

$$\nabla_\eta \mathcal{L}(\eta) = \nabla_h \widetilde{\mathcal{L}}(h) \cdot \nabla_\eta h = -(1 + h) \mathcal{K}(\eta) \cdot \frac{1}{1 + h} = -\mathcal{K}(\eta). \tag{2.10}$$

► *The momentum part:*
One readily has

$$d_\eta \mathcal{M}(\eta,\psi)[\hat\eta] = \int_{\mathbb{T}} \psi_x(x)\hat\eta(x)dx$$

and, via integration by parts

$$d_\psi \mathcal{M}(\eta,\psi)[\hat\psi] = \int_{\mathbb{T}} \hat\psi_x(x)\eta(x)dx = -\int_{\mathbb{T}} \eta_x(x)\hat\psi(x)\mathrm{d}x.$$

Hence,

$$\nabla_\eta \mathcal{M}(\eta,\psi) = \psi_x, \qquad \nabla_\psi \mathcal{M}(\eta,\psi) = -\eta_x. \tag{2.11}$$

Gathering (2.1), (2.8) and (2.11), we get

$$\begin{aligned}\nabla_\psi H(\eta,\psi) &= \nabla_\psi \mathcal{E}_{\mathtt{b}}(\eta,\psi) + \Omega\nabla_\psi \mathcal{M}(\eta,\psi)\\ &= -\Omega\eta_x + \frac{1}{2}\mathcal{H}(\eta)[\partial_x\psi + \mathtt{b}].\end{aligned} \tag{2.12}$$

Putting together (2.1), (2.9), (2.10) and (2.11) yields

$$\begin{aligned}\nabla_\eta H(\eta,\psi) &= \nabla_\eta \mathcal{E}_{\mathtt{b}}(\eta,\psi) + \gamma\nabla_\eta \mathcal{A}(\eta) + \Omega\nabla_\eta \mathcal{M}(\eta,\psi)\\ &= \Omega\psi_x - \frac{\psi_x + \mathtt{b}}{2}\mathcal{D}_0(\eta)[\partial_x\psi + \mathtt{b}] - \gamma\mathcal{K}(\eta).\end{aligned} \tag{2.13}$$

Comparing (1.9) with (2.12) and (2.13) concludes the desired result.
► *Invariances :* The properties (2.6) are easily obtained by changes of variables $x \mapsto -x$ and $x \mapsto x + \varsigma$. This ends the proof of Proposition 2.1. □

2.2 Analysis of the linearization of (1.9)

Definition 2.2. Let $m \in \mathbb{R}$, we define the space of Fourier multipliers of order m, $\tilde\Gamma_0^m$, as the space of smooth functions from $\mathbb{R}\setminus\{0\}$ to $\mathbb{C}$ of the form $\xi \mapsto a(\xi)$ such that

$$\left|\partial_\xi^\alpha a(\xi)\right| \leqslant C_\beta \langle\xi\rangle^{m-\alpha}, \qquad \forall\alpha\in\mathbb{N}, |\xi| \geqslant 1/2.$$

Following [59], the linearization of (1.9) around $(\eta,\psi) = (0,0)$ is given by

$$\begin{cases}\eta_t = \left(\Omega - \frac{\mathtt{b}}{2}\right)\eta_x - \frac{|D|}{2}\psi,\\ \psi_t = \left(\gamma|D|^2 - \frac{\mathtt{b}^2}{2}|D| - \left(\gamma - \mathtt{b}^2\right)\right)\eta + \left(\Omega - \frac{\mathtt{b}}{2}\right)\psi_x.\end{cases} \tag{2.14}$$

Namely, we can write (2.14) as

$$\begin{bmatrix}\eta_t\\ \psi_t\end{bmatrix} = \boldsymbol{L}_{\gamma,\mathtt{b}}(D)\begin{bmatrix}\eta\\ \psi\end{bmatrix}, \qquad \boldsymbol{L}_{\gamma,\mathtt{b}}(\xi) \triangleq \begin{bmatrix}\mathrm{i}\left(\Omega - \frac{\mathtt{b}}{2}\right)\xi & -\frac{|\xi|}{2}\\ \gamma|\xi|^2 - \frac{\mathtt{b}^2}{2}|\xi| - \left(\gamma - \mathtt{b}^2\right) & \mathrm{i}\left(\Omega - \frac{\mathtt{b}}{2}\right)\xi\end{bmatrix}. \tag{2.15}$$

The eigenvalues of $\boldsymbol{L}_{\gamma,\mathtt{b}}(\xi)$ are given by

$$\lambda^\pm_{\gamma,\mathtt{b}}(\xi) \triangleq \mathrm{i}\left(\Omega - \frac{\mathtt{b}}{2}\right)\xi \pm \sqrt{-\frac{|\xi|}{2}\left(\gamma|\xi|^2 - \frac{\mathtt{b}^2}{2}|\xi| - \left(\gamma - \mathtt{b}^2\right)\right)} \in \tilde\Gamma_0^{3/2}. \tag{2.16}$$

We want the eigenvalues in (2.16) to be purely imaginary, this happens if and only if

$$\gamma|\xi|^2 - \frac{\mathtt{b}^2}{2}|\xi| - \left(\gamma - \mathtt{b}^2\right) > 0, \tag{2.17}$$

The condition (2.17) is satisfied for any $|\xi| \geqslant 1$ if

$$\beta \triangleq \frac{\mathtt{b}^2}{\gamma} \in [0,\beta_+), \qquad \beta_+ \triangleq 4(2+\sqrt{3}) \approx 14,928...$$

Indeed, for $|\xi| \in [1,2]$, one has

$$\gamma |\xi|^2 - \frac{\mathtt{b}^2}{2} |\xi| - (\gamma - \mathtt{b}^2) \geqslant \min\left\{\frac{\mathtt{b}^2}{2}, 2\gamma\right\} > 0$$

for any $\beta > 0$. While, for $|\xi| > 2$, it reduces to

$$\beta < \min_{|\xi|>2} \left(\frac{|\xi|^2 - 1}{\frac{|\xi|}{2} - 1}\right) = 4(2+\sqrt{3}). \tag{2.18}$$

Remark 2.3. We consider the restriction in (2.17) for $|\xi| \geqslant 1$. If we further restrict to $\xi \in \mathbb{Z}^*$, we can improve β_+ from $4(2+\sqrt{3})$ to $\beta_+ = 15$. However, in Section 2.3, we extend the function $\lambda^{\pm}_{\gamma,\mathtt{b}}(\xi)$ to apply the Delort-Szeftel Theorem 2.7. We believe that the argument in Proposition 2.8 could be modified to maintain the slightly less restrictive condition $\beta_+ = 15$. Nevertheless, to preserve the simplicity of our approach, we do not pursue this further analysis.

Then we obtain that

$$\lambda^{\pm}_{\gamma,\mathtt{b}}(\xi) = \mathrm{i}\left(\Omega - \frac{\mathtt{b}}{2}\right)\xi \pm \mathrm{i}\,\omega_{\gamma,\mathtt{b}}(\xi), \qquad \omega_{\gamma,\mathtt{b}}(\xi) \triangleq \sqrt{\frac{|\xi|}{2}\left(\gamma\left(|\xi|^2 - 1\right) - \mathtt{b}^2\left(\frac{|\xi|}{2} - 1\right)\right)} \in \widetilde{\Gamma}^{3/2}_0. \tag{2.19}$$

Notice that we can expand $\omega_{\gamma,\mathtt{b}}(\xi)$ and obtain that

$$\omega_{\gamma,\mathtt{b}}(\xi) = \sqrt{\frac{\gamma}{2}}\,|\xi|^{\frac{3}{2}} - \frac{1}{\sqrt{2\gamma}}\left(\frac{\mathtt{b}}{2}\right)^2 |\xi|^{\frac{1}{2}} + \omega_{\gamma,\mathtt{b};-\frac{1}{2}}(\xi), \qquad \omega_{\gamma,\mathtt{b};-\frac{1}{2}}(\xi) \in \widetilde{\Gamma}^{-\frac{1}{2}}_0. \tag{2.20}$$

In the sequel, we make the following natural choice for Ω

$$\Omega \triangleq \frac{\mathtt{b}}{2}. \tag{2.21}$$

Remark 2.4. Already at linear level standard computations show that in order to close energy estimates for the system (2.14) we need a discrepancy in regularity between η and ψ, namely we can close the energy estimates on (2.14) in the case in which $\eta \in H^{s+\frac{1}{4}}_0(\mathbb{T};\mathbb{R})$ and $\psi \in \dot{H}^{s-\frac{1}{4}}(\mathbb{T};\mathbb{R})$, such regularity gap shall persist at nonlinear level as well. This is not unexpected, and the same behavior is present for the one-phase water waves problem, both at linear and nonlinear level, cf. [1].

2.3 Non-resonance conditions

In this subsection, we study the non-resonances between the frequencies. This is needed in the application of the normal form algorithm performed in Section 7.2. Specifically, we prove that there are no resonances between linear frequencies, except for the *super-action-preserving* ones, as defined below.

Definition 2.5 (SAP multi-index)**.** A multi-index $(\alpha,\beta) \in \mathbb{N}^{\mathbb{Z}^*} \times \mathbb{N}^{\mathbb{Z}^*}$ is *super-action preserving* if

$$\alpha_n + \alpha_{-n} = \beta_n + \beta_{-n}, \qquad \forall n \in \mathbb{N}. \tag{2.22}$$

A super-action preserving multi-index (α,β) satisfies $|\alpha| = |\beta|$ where $|\alpha| \triangleq \sum_{j\in\mathbb{Z}^*} \alpha_j$. If a multi-index $(\alpha,\beta) \in \mathbb{N}^{\mathbb{Z}^*} \times \mathbb{N}^{\mathbb{Z}^*}$ is not super-action preserving, then the set

$$\mathfrak{N}(\alpha,\beta) \triangleq \Big\{n \in \mathbb{N} \quad \text{s.t.} \quad \alpha_n + \alpha_{-n} - \beta_n - \beta_{-n} \neq 0\Big\} \tag{2.23}$$

is not empty and, since

$$\mathfrak{N}(\alpha,\beta) \subset \Big\{n \in \mathbb{N} \quad \text{s.t.} \quad \alpha_n + \alpha_{-n} + \beta_n + \beta_{-n} \neq 0\Big\},$$

its cardinality satisfies

$$|\mathfrak{N}(\alpha,\beta)| \leqslant |\alpha + \beta| = |\alpha| + |\beta|. \tag{2.24}$$

The main result of the present section is the following:

Proposition 2.6. *Let $M\in\mathbb{N}^*$ and $0<\beta_1<\beta_2<4\left(2+\sqrt{3}\right)$. Then, there exist $\tau,\delta>0$ and a zero measure set $\mathcal{B}\subset[\beta_1,\beta_2]$ such that for any $\beta\in[\beta_1,\beta_2]\setminus\mathcal{B}$ the following holds: there is $\nu>0$ such that for any multi-index $(\alpha,\alpha')\in(\mathbb{N}^*)^{\mathbb{Z}^*}\times(\mathbb{N}^*)^{\mathbb{Z}^*}$ of length $|\alpha+\alpha'|\leqslant M$, which is not super-action preserving (in the sense of Definition 2.5), one has*

$$\left|\vec{\omega}_{\gamma,\mathtt{b}}\cdot(\alpha-\alpha')\right|\geqslant\frac{\nu}{\left(\displaystyle\max_{j\in\operatorname{supp}(\alpha\cup\alpha')}|j|\right)^{\tau}},$$

where

$$\vec{\omega}_{\gamma,\mathtt{b}}\triangleq\left(\omega_{\gamma,\mathtt{b}}(j)\right)_{j\in\mathbb{Z}^*}.$$

The rest of Section 2.3 is dedicated to the proof of Proposition 2.6. This latter is a consequence of the Delort-Szeftel Theorem that we recall here for the convenience of the reader. For its proof, we refer to [36, Theorem 5.1].

Theorem 2.7. *Let $d\in\mathbb{N}^*$, $r_0>0$ and $\beta_1,\beta_2\in\mathbb{R}$. We denote $B_{r_0}\left(\mathbb{R}^d\right)\subset\mathbb{R}^d$ the ball centered at the origin and of radius r_0. Consider $f:B_{r_0}\left(\mathbb{R}^d\right)\times[\beta_1,\beta_2]\to\mathbb{R}$ a continuous sub-analytic function and $\rho:B_{r_0}\left(\mathbb{R}^d\right)\to\mathbb{R}$ a non-zero real-analytic function. We assume the following facts.*

1. *The function f is real-analytic on $\{x\in B_{r_0}\left(\mathbb{R}^d\right)\quad\text{s.t.}\quad\rho(x)\neq0\}\times[\beta_1,\beta_2]$.*
2. *For any $\bar{x}\in B(0,r_0)$ with $\rho(\bar{x})\neq0$, the equation $f(\bar{x},\beta)=0$ admit finitely many solutions in $[\beta_1,\beta_2]$.*

Then, there exist $N_0\in\mathbb{N}$ and $\alpha_0,\delta,C>0$ such that for any $\alpha\in(0,\alpha_0]$, any integer $N\geqslant N_0$ and any $x\in B(0,r_0)$ with $\rho(x)\neq0$, we have

$$\left|\left\{\beta\in[\beta_1,\beta_2]\quad\text{s.t.}\quad|f(x,y)|\leqslant\alpha\left|\rho(x)\right|^N\right\}\right|\leqslant C\alpha^{\delta}\left|\rho(x)\right|^{N\delta}.$$

First observe that we can write

$$\omega_{\gamma,\mathtt{b}}(\xi)=\sqrt{\frac{\gamma|\xi|}{2}}\sqrt{|\xi|^2-\frac{\beta}{2}|\xi|+\beta-1},\qquad\beta\triangleq\frac{\mathtt{b}^2}{\gamma}.\tag{2.25}$$

Notice that $\omega_{\gamma,\mathtt{b}}(2)=\sqrt{3\gamma}$ is independent of β and that

$$\omega_{\gamma,\mathtt{b}}(5)-\sqrt{5}\omega_{\gamma,\mathtt{b}}(3)\equiv0.$$

The above relation shows that there no resonance between the modes 3 and 5. However, the couple $(3,5)$ appears to be singular in the analysis of the non-degeneracy for the application of the Delort-Szeftel Theorem. That's why we need to treat it separately.

Application of Theorem 2.7

The application of Theorem 2.7 allows us to control quasi-resonances at arbitrary order via polynomial bounds, thus inducing a finite, but recoverable, loss of derivatives. The result we obtain is the following one:

Proposition 2.8. *Fix $0<\beta_1<\beta_2<4\left(2+\sqrt{3}\right)$, $\mathtt{A}\in\mathbb{N}$ and $\mathtt{M}\in\mathbb{N}^*$. Then, there exist $\nu_0,\tau,\delta>0$, depending on $\mathtt{A}$ and $\mathtt{M}$, such that for any $\tilde{\nu}\in(0,\nu_0)$, there exists a set $\mathcal{B}_{\tilde{\nu}}\subset[\beta_1,\beta_2]$ of measure $O(\tilde{\nu}^{\delta})$ such that for any $\beta\in[\beta_1,\beta_2]\setminus\mathcal{B}_{\tilde{\nu}}$, the following holds: denote $n_0=2$, $n_1=3$ and for any distinct integers $n_2,\ldots,n_{\mathtt{A}}\in\mathbb{N}\setminus\{2,3,5\}$ and any $\vec{c}=(c_0,c_1,\ldots,c_{\mathtt{A}})\in\mathbb{R}^{\mathtt{A}+1}\setminus\{0\}$ with $\max_{a=0,\ldots,\mathtt{A}}|c_a|\leqslant\mathtt{M}$, we have*

$$\left|\sum_{a=0}^{\mathtt{A}}c_a\omega_{\gamma,\mathtt{b}}(n_a)\right|\geqslant\tilde{\nu}\left(\sum_{a=0}^{\mathtt{A}}n_a\right)^{-\tau}.\tag{2.26}$$

Proof. We denote $\vec{n}=(n_0,n_1,\ldots,n_{\mathtt{A}})\in(\mathbb{N}^*)^{\mathtt{A}+1}$ with $n_2,\ldots,n_{\mathtt{A}}\notin\{2,3,5\}$ and distinct, we introduce the notations

$$x_0(\vec{n})\triangleq\left(\sum_{a=0}^{\mathtt{A}}n_a\right)^{-1},\qquad x_a(\vec{n})\triangleq x_0(\vec{n})\sqrt{n_a-1}.\tag{2.27}$$

We note that

$$0 \leqslant |x_a(\vec{n})| \leqslant 1, \quad \text{for any } a = 0, \ldots, \mathtt{A}. \tag{2.28}$$

The condition (2.26) is equivalent to

$$\left|f_{\vec{c}}(x,\beta)\right| \geqslant \frac{\widetilde{\nu}}{\sqrt{\gamma}} x_0^{\tau+3}, \quad x = (x_0(\vec{n}), x_1(\vec{n}), \ldots, x_{\mathtt{A}}(\vec{n})) \tag{2.29}$$

where

$$f_{\vec{c}}(x,\beta) \triangleq \sum_{a=1}^{\mathtt{A}} r_a \lambda(x_a, x_0, \beta) + c_0\sqrt{3}x_0^3, \qquad x = (x_0, x_1, \ldots, x_{\mathtt{A}}) \in B_1\left(\mathbb{R}^{\mathtt{A}+1}\right), \tag{2.30}$$

with

$$r_a \triangleq c_a\sqrt{x_a^2 + x_0^2}, \qquad \lambda(y, x_0, \beta) \triangleq \sqrt{y^4 + \left(2 - \frac{\beta}{2}\right)x_0^2 y^2 + \frac{\beta}{2}x_0^4}.$$

The function $f_{\vec{c}}\colon B_1\left(\mathbb{R}^{\mathtt{A}+1}\right) \times [\beta_1, \beta_2] \to \mathbb{R}$ is continuous and sub-analytic. Remark that for any $l \in \mathbb{N}^*$,

$$\partial_\beta^l \lambda(y, x_0, \beta) = \binom{\frac{1}{2}}{l}\mu^l(y, x_0, \beta)\lambda(y, x_0, \beta), \qquad \mu(y, x_0, \beta) \triangleq \frac{x_0^2\left(x_0^2 - y^2\right)}{x_0^2\left(x_0^2 - y^2\right)\beta + 2y^2\left(y^2 + 2x_0^2\right)}. \tag{2.31}$$

Note that, since $0 < \beta < \beta_2 < 4(2+\sqrt{3})$, the denominator in (2.31) satisfies

$$x_0^2\left(x_0^2 - y^2\right)\beta + 2y^2\left(y^2 + 2x_0^2\right) > 0 \quad \text{for any } y \in [0,1],\ x_0 \in (0,1]. \tag{2.32}$$

We introduce the polynomial function $\rho : [-1,1]^{\mathtt{A}+1} \to \mathbb{R}$

$$\rho(x_0, x_1, \ldots, x_{\mathtt{A}}) \triangleq x_0 \prod_{a=1}^{\mathtt{A}} \left(x_a^2 - x_0^2\right) \prod_{1 \leqslant a < b \leqslant \mathtt{A}} \left(\left(x_0^2 - x_a^2\right)x_b^2\left(x_b^2 + 2x_0^2\right) - \left(x_0^2 - x_b^2\right)x_a^2\left(x_a^2 + 2x_0^2\right)\right).$$

The condition (2.32) implies that the function $f_{\vec{c}}$ is real-analytic on $\{\rho \neq 0\} \times [\beta_1, \beta_2]$. We fix now $\bar{x} = (\bar{x}_0, \ldots \bar{x}_{\mathtt{A}})$ such that $\rho(\bar{x}) \neq 0$. Then $\bar{x}_0 \neq 0$ and, in view also of (2.32), one has

$$\mu\left(\bar{x}_a, \bar{x}_0, \beta\right) \neq \mu\left(\bar{x}_b, \bar{x}_0, \beta\right), \quad \text{for any } \beta \in \left(0, 4(2+\sqrt{3})\right). \tag{2.33}$$

The following Lemma ensures that the function $f_{\vec{c}}(x,\beta)$ fullfills the second assumption of Delort-Szeftel Theorem 2.7.

Lemma 2.9. *The solutions of the equation*

$$f_{\vec{c}}\left(\bar{x}, \beta\right) = 0,$$

if any, are finitely many.

Proof. Since the function $\beta \to f_{\vec{c}}\left(\bar{x}, \beta\right)$ is analytic for $\beta \in [\beta_1, \beta_2]$, we are only left to prove that it is not identically zero. To do so we fix $\bar{\beta} \in (\beta_1, \beta_2)$ and we note that

$$\forall l = 1, \ldots, \mathtt{A}, \quad \partial_\beta^l f\left(\bar{x}, \bar{\beta}\right) = 0 \qquad \Leftrightarrow \qquad A(\bar{x})\vec{r} = 0,$$

where $\vec{r} \triangleq (r_1, \ldots, r_{\mathtt{A}})$ and

$$A(\bar{x}) \triangleq \begin{pmatrix} \mu\left(\bar{x}_1, \bar{x}_0, \bar{\beta}\right)\lambda\left(\bar{x}_1, \bar{x}_0, \bar{\beta}\right) & \ldots & \mu\left(\bar{x}_{\mathtt{A}}, \bar{x}_0, \bar{\beta}\right)\lambda\left(\bar{x}_{\mathtt{A}}, \bar{x}_0, \bar{\beta}\right) \\ \mu^2\left(\bar{x}_1, \bar{x}_0, \bar{\beta}\right)\lambda\left(\bar{x}_1, \bar{x}_0, \bar{\beta}\right) & \ldots & \mu^2\left(\bar{x}_{\mathtt{A}}, \bar{x}_0, \bar{\beta}\right)\lambda\left(\bar{x}_{\mathtt{A}}, \bar{x}_0, \bar{\beta}\right) \\ \vdots & & \vdots \\ \mu^{\mathtt{A}}\left(\bar{x}_1, \bar{x}_0, \bar{\beta}\right)\lambda\left(\bar{x}_1, \bar{x}_0, \bar{\beta}\right) & \ldots & \mu^{\mathtt{A}}\left(\bar{x}_{\mathtt{A}}, \bar{x}_0, \bar{\beta}\right)\lambda\left(\bar{x}_{\mathtt{A}}, \bar{x}_0, \bar{\beta}\right) \end{pmatrix}.$$

Since $\vec{c} \neq 0$ and $\rho(\bar{x}) \neq 0$ we have also $\bar{x}_0 \neq 0$ and $\vec{r} \neq 0$. Then $\beta \mapsto f(\bar{x}, \beta) \equiv 0$ implies that $\det(A(\bar{x})) = 0$. Besides, by $\mathtt{A}$-linearity of the determinant and recognizing a Vandermonde determinant, we find

$$\det(A(\bar{x})) = \prod_{a=1}^{\mathtt{A}} \mu\left(\bar{x}_a, \bar{x}_0, \bar{\beta}\right)\lambda\left(\bar{x}_a, \bar{x}_0, \bar{\beta}\right) \prod_{1 \leqslant a < b \leqslant \mathtt{A}} \left(\mu\left(\bar{x}_a, \bar{x}_0, \bar{\beta}\right) - \mu\left(\bar{x}_b, \bar{x}_0, \bar{\beta}\right)\right).$$

By construction, since $\bar{x} \in \{\rho \neq 0\}$, one has that $\det(A(\bar{x})) \neq 0$ at the given $\overline{\beta}$. So $\beta \mapsto f(\bar{x}, \beta)$ cannot be identically zero proving Lemma 2.9. □

We thus conclude that there are $N_0 \in \mathbb{N}^*$, $\alpha_0, \delta, C > 0$, such that for any $\alpha \in (0, \alpha_0]$, any $N \in \mathbb{N}^*$, $N \geqslant N_0$, any $x \in X$ with $\rho(x) \neq 0$,

$$\left|\left\{\beta \in [\beta_1, \beta_2] \quad \text{s.t.} \quad \left|f_{\vec{c}}(x,\beta)\right| \leqslant \alpha \left|\rho(x)\right|^N\right\}\right| \leqslant C\alpha^\delta \left|\rho(x)\right|^{N\delta}. \tag{2.34}$$

To conclude the proof we need the following:

Lemma 2.10. *Let* $\tau_1 \triangleq 2\mathtt{A} + 1 + 8\binom{\mathtt{A}}{2}$. *Then*

$$\left(\sum_{a=1}^{\mathtt{A}} n_a\right)^{-\tau_1} \lesssim \left|\rho\left(x(\vec{n})\right)\right| \lesssim \left(\sum_{a=1}^{\mathtt{A}} n_a\right)^{-1}. \tag{2.35}$$

Proof. By definition (2.27), we have

$$\begin{aligned}&\left((x_0^2 - x_a^2)x_b^2(x_b^2 + 2x_0^2) - (x_0^2 - x_b^2)x_a^2(x_a^2 + 2x_0^2)\right)_{|x_0 = x_0(\vec{n}),\, x_a = x_a(\vec{n}),\, x_b = x_b(\vec{n})}\\ &= x_0^8(\vec{n})\left[(2 - n_a)\left(n_b^2 - 1\right) - (2 - n_b)\left(n_a^2 - 1\right)\right].\end{aligned}$$

The equation

$$(2 - n_a)\left(n_b^2 - 1\right) = (2 - n_b)\left(n_a^2 - 1\right)$$

is equivalent to

$$(2 - n_a)(n_b - n_a)\left(n_b - \frac{2n_a - 1}{n_a - 2}\right) = 0. \tag{2.36}$$

Since $n_a \neq 2$ and $n_b \neq n_a$, we must have $n_b = \frac{2n_a - 1}{n_a - 2}\cdot$ But

$$2n_a - 1 = 2(n_a - 2) + 3.$$

Hence

$$2n_a - 1 \equiv 3 \mod. [n_a - 2].$$

Therefore

$$n_a - 2 | 2n_a - 1 \quad \Leftrightarrow \quad n_a \in \{3, 5\}.$$

We conclude that the only non-trivial couples of integers solutions to (2.36) are

$$(3,5) \qquad \text{and} \qquad (5,3).$$

However, we have excluded the possibility $n_a = 5$ or $n_b = 5$. This implies that the equation (2.36) is not solved with our choice of $\vec{n}$. Thus,

$$\left|\left(\left(x_0^2 - x_a^2\right)x_b^2\left(x_b^2 + 2x_0^2\right) - \left(x_0^2 - x_b^2\right)x_a^2\left(x_a^2 + 2x_0^2\right)\right)_{|x_0 = x_0(\vec{n}),\, x_a = x_a(\vec{n}),\, x_b = x_b(\vec{n})}\right| \geqslant x_0^8(\vec{n}). \tag{2.37}$$

Moreover, since $n_a \neq 2$ for $a = 1, \ldots, \mathtt{A}$, one has

$$\left|x_a^2(\vec{n}) - x_0^2(\vec{n})\right| \geqslant x_0^2(\vec{n}). \tag{2.38}$$

Gathering Equations (2.37) and (2.38) we obtain the lower bound in (2.35). To prove the upper bound it is sufficient to use the bounds in (2.28). ◻

Consider the set

$$\begin{aligned}\mathcal{B}(\alpha, N) &\triangleq \bigcup_{\substack{\vec{n} = (n_2, \ldots, n_{\mathtt{A}}) \in \mathbb{N}^{\mathtt{A}},\, 1 \leqslant n_1 < \ldots < n_{\mathtt{A}} \\ \vec{c} \in (\mathbb{Z}^*)^{\mathtt{A}},\, |\vec{c}|_\infty \leqslant \mathtt{M}}} \mathcal{B}_{\vec{c},\vec{n}}(\alpha, N) \subset [\beta_1, \beta_2],\\ \mathcal{B}_{\vec{c},\vec{n}}(\alpha, N) &\triangleq \left\{\beta \in [\beta_1, \beta_2] \quad \text{s.t.} \quad \left|f_{\vec{c}}\left(x(\vec{n}), \beta\right)\right| \leqslant \alpha \left|\rho\left(x(\vec{n})\right)\right|^N\right\}.\end{aligned} \tag{2.39}$$

We fix now $\underline{N} \in \mathbb{N}$ such that $\bar{N}\delta > 1$ and $\mathcal{B}_\alpha \triangleq \mathcal{B}(\alpha, \underline{N})$. Then we get

$$|\mathcal{B}_\alpha| \leqslant C(\mathtt{A}, \mathtt{M})\alpha^\delta \sum_{n_2, \ldots, n_{\mathtt{A}} \in \mathbb{N}} \left(\sum_{a=0}^{\mathtt{A}} n_a\right)^{-N\delta} \leqslant C'(\mathtt{A}, \mathtt{M})\alpha^\delta \tag{2.40}$$

In conclusion, for any $\beta \in [\beta_1, \beta_2] \setminus \mathcal{B}_\alpha$, for any $\vec{n} = (n_0, n_1, \ldots, n_{\mathtt{A}})$ as in the hypothesis, any $\vec{c} \in (\mathbb{Z}^*)^{\mathtt{A}}$ with $|\vec{c}|_\infty \leqslant \mathtt{M}$, it results, by (2.39) and (2.35), that

$$\left| f_{\vec{c}}\left(x(\vec{n}), \beta\right)\right| > \alpha \left|\rho\left(x(\vec{n})\right)\right|^{\underline{N}} \geqslant c(\mathtt{A})\alpha \left(\sum_{a=1}^{\mathtt{A}} n_a\right)^{-\tau_1 \underline{N}}. \tag{2.41}$$

Recalling the definition of $f_{\vec{c}}$ in (2.30) and $x_0(\vec{n})$ in (2.27), the lower bound (2.41) implies (2.8) with $\tau \triangleq \tau_1 \underline{N} - 3$ (cfr. (2.29)) and re-denoting $\tilde{\nu} \triangleq \alpha c(\mathtt{A})$. □

Proof of Proposition 2.6

Fix a multi-index $(\alpha, \alpha') \in (\mathbb{N}^*)^{\mathbb{Z}^*} \times (\mathbb{N}^*)^{\mathbb{Z}^*}$ of length $|\alpha + \alpha'| \leqslant M$. We denote $\mathfrak{N}(\alpha, \alpha')$ as in (2.23). Since the couple (α, α') is not super-action preserving, then

$$\mathfrak{N}(\alpha, \alpha') \neq \varnothing. \tag{2.42}$$

Then, we can write

$$\begin{aligned} \vec{\omega}_{\gamma,\mathtt{b}} \cdot (\alpha - \alpha') &= \sum_{j \in \mathbb{Z}^*} \omega_{\gamma,\mathtt{b}}(|j|) \left(\alpha_j - \alpha'_j\right) = \sum_{n \in \mathbb{N}^*} \omega_{\gamma,\mathtt{b}}(n) \left(\alpha_n + \alpha_{-n} - \alpha'_n - \alpha'_{-n}\right) \\ &= \sum_{n \in \mathfrak{N}(\alpha,\alpha')} \omega_{\gamma,\mathtt{b}}(n) \left(\alpha_n + \alpha_{-n} - \alpha'_n - \alpha'_{-n}\right). \end{aligned}$$

We use the notation

$$\mathtt{A} \triangleq \left|\mathfrak{N}(\alpha, \alpha') \setminus \{2,3,5\}\right| + 1.$$

We also denote

$$\mathfrak{N}(\alpha, \alpha') \setminus \{2,3,5\} = \{n_2, \ldots n_{\mathtt{A}}\}.$$

Therefore,

$$\vec{\omega}_{\gamma,\mathtt{b}} \cdot (\alpha - \alpha') = \sum_{a=2}^{\mathtt{A}} c_a \omega_{\gamma,\mathtt{b}}(n_a) + c_0 \omega_{\gamma,\mathtt{b}}(2) + c_1 \omega_{\gamma,\mathtt{b}}(3),$$

with

$$c_0 \triangleq \alpha_2 + \alpha_{-2} - \alpha'_2 - \alpha'_{-2}, \qquad c_1 \triangleq \left(\alpha_3 + \alpha_{-3} - \alpha'_3 - \alpha'_{-3}\right) + \sqrt{5}\left(\alpha_5 + \alpha_{-5} - \alpha'_5 - \alpha'_{-5}\right)$$

and for any $a = 2, \ldots, \mathtt{A}$,

$$c_a \triangleq \alpha_{n_a} + \alpha_{-n_a} - \alpha'_{n_a} - \alpha'_{-n_a}.$$

Observe that for any $a = 0, \ldots, \mathtt{A}$, $|c_a| \leqslant 4M$, where M is defined in Proposition 2.6. Let us assume the absurd hypothesis $\vec{c} \triangleq (c_0, c_1, c_2, \ldots, c_{\mathtt{A}}) = 0$. By definition, under such absurd hypothesis, we obtain that $\mathfrak{N}(\alpha, \alpha') \subset \{3,5\}$. But $c_1 \in \mathbb{Z}[\sqrt{5}]$ so $c_1 = 0$ implies also that $3, 5 \not\in \mathfrak{N}(\alpha, \alpha')$. Hence, $\mathfrak{N}(\alpha, \alpha') = \varnothing$ which is a contradiction with (2.42), thus we can safely assume $\vec{c} \neq 0$. We consider the set

$$\mathcal{B} \triangleq \bigcap_{\tilde{\nu} \in (0, \nu_0)} \mathcal{B}_{\tilde{\nu}},$$

where ν_0 and $\mathcal{B}_{\tilde{\nu}}$ are introduced in Proposition 2.8. Hence, for any $\beta \in [\beta_1, \beta_2] \setminus \mathcal{B}$ there is $\tilde{\nu} \in (0, \nu_0)$ such that $\beta \in [\beta_1, \beta_2] \setminus \mathcal{B}_{\tilde{\nu}}$ and, applying Proposition 2.8 we get the desired result with $\nu \triangleq \frac{\tilde{\nu}}{M^\tau}$. □

3 Functional setting

Throughout the document, we shall use the notations

$$\mathbb{N} \triangleq \{0, 1, 2, \ldots\}, \qquad \mathbb{N}^* \triangleq \mathbb{N} \setminus \{0\}, \qquad \mathbb{Z} \triangleq \mathbb{N} \cup (-\mathbb{N}), \qquad \mathbb{Z}^* \triangleq \mathbb{Z} \setminus \{0\}.$$

Along the paper we deal with real parameters

$$s \geqslant s_0 \gg K \gg \varrho \gg N \geqslant 0, \tag{3.1}$$

where $N \in \mathbb{N}$. The values of s, s_0, K and ϱ may vary from line to line while still being true the relation (3.1). We expand a 2π-periodic function $u \in L^2(\mathbb{T};\mathbb{C})$ in Fourier series as

$$u(x) = \sum_{j\in\mathbb{Z}} \hat{u}\left(j\right) e^{\mathrm{i}jx}, \qquad \hat{u}\left(j\right) \triangleq \mathcal{F}_{x\to j}\left(j\right) \triangleq u_j \triangleq \int_{\mathbb{T}} u(x) e^{-\mathrm{i}jx}\,\mathrm{d}x.$$

The function u is real-valued if and only if $\overline{u_j} = u_{-j}$, for any $j \in \mathbb{Z}$. For any $s \in \mathbb{R}$, we define the Sobolev space $H^s \triangleq H^s(\mathbb{T};\mathbb{C})$ with norm

$$\|u\|_s \triangleq \|u\|_{H^s} = \left(\sum_{j\in\mathbb{Z}} \langle j\rangle^{2s} \left|\hat{u}\left(j\right)\right|^2\right)^{\frac{1}{2}}, \qquad \langle j\rangle \triangleq \max\{1, |j|\}.$$

We define $\Pi_0 u \triangleq u_0$ the average of u and

$$\Pi_0^{\perp} \triangleq \mathrm{Id} - \Pi_0.$$

We define H_0^s the subspace of zero average functions of H^s for which we also denote $\|u\|_s = \|u\|_{H^s} = \|u\|_{H_0^s}$, and with $\dot{H}^s \triangleq H^s/\mathbb{C}$ endowed with the norm $\|u\|_{\dot{H}^s} \triangleq \left\|\Pi_0^{\perp} u\right\|_s$. We define, on $\dot{L}^2(\mathbb{T};\mathbb{C}) \triangleq \dot{H}^0(\mathbb{T};\mathbb{C})$, the complex scalar product $\langle \cdot \mid \cdot\rangle_{\mathbb{C}}$ and the real symmetric bilinear form $\langle \cdot \mid \cdot\rangle_{\mathbb{R}}$ as follows: for any $u, v \in \dot{L}^2(\mathbb{T};\mathbb{C})$,

$$\langle u \mid v\rangle_{\mathbb{C}} \triangleq \int_{\mathbb{T}} \Pi_0^{\perp} u(x)\, \overline{\Pi_0^{\perp} v(x)}\,\mathrm{d}x, \qquad \langle u \mid v\rangle_{\mathbb{R}} \triangleq \int_{\mathbb{T}} \Pi_0^{\perp} u(x)\, \Pi_0^{\perp} v(x)\,\mathrm{d}x. \tag{3.2}$$

Moreover we define the real subspace of $\dot{H}^s(\mathbb{T};\mathbb{C}^2)$

$$\dot{H}^s_{\mathbb{R}}(\mathbb{T};\mathbb{C}^2) \triangleq \left\{ U = \begin{bmatrix} u^+ \\ u^- \end{bmatrix} \in \dot{H}^s(\mathbb{T};\mathbb{C}^2) \quad \text{s.t.} \quad u^- = \overline{u^+} \right\}.$$

We also denote

$$\dot{H}^\infty(\mathbb{T};\mathbb{C}^2) \triangleq \bigcap_{s\in\mathbb{R}} \dot{H}^s(\mathbb{T};\mathbb{C}^2), \qquad \dot{H}^\infty_{\mathbb{R}}(\mathbb{T};\mathbb{C}^2) \triangleq \bigcap_{s\in\mathbb{R}} \dot{H}^s_{\mathbb{R}}(\mathbb{T};\mathbb{C}^2).$$

Given an interval $I \subset \mathbb{R}$ symmetric with respect to $t = 0$ and $s \in \mathbb{R}$, we define the space

$$C^K_*\left(I; \dot{H}^s\left(\mathbb{T};\mathbb{C}^2\right)\right) \triangleq \bigcap_{k=0}^{K} C^k\left(I; \dot{H}^{s-\frac{3}{2}k}\left(\mathbb{T};\mathbb{C}^2\right)\right),$$

endowed with the norm

$$\sup_{t\in I} \|U(t,\cdot)\|_{K,s} \qquad \text{where} \qquad \|U(t,\cdot)\|_{K,s} \triangleq \sum_{k=0}^{K} \left\|\partial_t^k U(t,\cdot)\right\|_{\dot{H}^{s-\frac{3}{2}k}}. \tag{3.3}$$

We also consider its subspace

$$C^K_{*\mathbb{R}}\left(I, \dot{H}^s\left(\mathbb{T},\mathbb{C}^2\right)\right) \triangleq \left\{ U \in C^K_*\left(I, \dot{H}^s\left(\mathbb{T},\mathbb{C}^2\right)\right) \quad \text{s.t.} \quad U = \begin{pmatrix} u \\ \bar{u} \end{pmatrix} \right\}.$$

Given $r > 0$ we set $B^K_s(I;r)$ the ball of radius r in $C^K_*\left(I, \dot{H}^s\left(\mathbb{T},\mathbb{C}^2\right)\right)$ and by $B^K_{s,\mathbb{R}}(I;r)$ the ball of radius r in $C^K_{*\mathbb{R}}\left(I, \dot{H}^s\left(\mathbb{T},\mathbb{C}^2\right)\right)$.

A vector field $X(u)$ is *translation invariant* if

$$X \circ \mathrm{t}_\varsigma = \mathrm{t}_\varsigma \circ X, \qquad \forall \varsigma \in \mathbb{R},$$

where the translation operator t_ς is defined by

$$\mathrm{t}_\varsigma \colon u(x) \mapsto u(x+\varsigma).$$

Given a linear operator $R(u)[\cdot]$ acting on $L_0^2(\mathbb{T};\mathbb{C}) \triangleq H_0^0(\mathbb{T};\mathbb{C})$ we associate the linear operator defined by the relation $\overline{R(u)}v \triangleq \overline{R(u)\overline{v}}$ for any $v \in L_0^2(\mathbb{T};\mathbb{C})$. An operator $R(u)$ is *real* if $R(u) = \overline{R(u)}$ for any u real.

3.1 Paradifferential calculus

We introduce the para-differential calculus developed in [18, 25].

Classes of symbols. Roughly speaking the class $\widetilde{\Gamma}^m_p$ contains symbols of order m and homogeneity p in u, whereas the class $\Gamma^m_{K,K',p}$ contains non-homogeneous symbols of order m that vanish at degree at least p in u and that are $(K-K')$-times differentiable in t. We can think the parameter K' like the number of time derivatives of u that are contained in the symbols.

Definition 3.1 (Symbols)**.** Let $m \in \mathbb{R}$, $p, N \in \mathbb{N}$, $K, K' \in \mathbb{N}$ with $K' \leqslant K$, and $\epsilon_0 > 0$.

i) **p-Homogeneous symbols.** We denote by $\widetilde{\Gamma}^m_p$ the space of p-linear symmetric maps from $\left(\dot{H}^\infty(\mathbb{T};\mathbb{C}^2)\right)^p$ to $C^\infty(\mathbb{T}\times\mathbb{R};\mathbb{C})$, $(x,\xi) \mapsto a_p(U_1,\dots,U_p;x,\xi)$ whose associated polynomial has the form

$$a_p(U;x,\xi) \triangleq a_p(U,\dots,U;x,\xi) \triangleq \sum_{\substack{\vec{\jmath}\in\mathbb{Z}^p\\ \vec{\sigma}\in\{\pm\}^p}} a^{\vec{\sigma}}_{\vec{\jmath}}(\xi)\, u^{\vec{\sigma}}_{\vec{\jmath}}\, e^{\mathrm{i}(\vec{\sigma}\cdot\vec{\jmath})x}, \tag{3.4}$$

where $a^{\vec{\sigma}}_{\vec{\jmath}}(\xi)$ are complex valued Fourier multipliers, satisfying

$$a^{\vec{\sigma}}_{\vec{\jmath}}(\xi) \triangleq a^{\sigma_1,\dots,\sigma_p}_{j_1,\dots,j_p}(\xi) = a^{\sigma_{\pi(1)},\dots,\sigma_{\pi(p)}}_{j_{\pi(1)},\dots,j_{\pi(p)}}(\xi) \quad \text{for any } \pi \text{ permutation of } \{1,\dots,p\},$$

and for some $\mu \geqslant 0$,

$$|\partial^\beta_\xi a^{\vec{\sigma}}_{\vec{\jmath}}(\xi)| \leqslant C_\beta \langle\vec{\jmath}\rangle^\mu \langle\xi\rangle^{m-\beta}, \quad \forall \vec{\jmath}\in\mathbb{Z}^p,\ \vec{\sigma}\in\{\pm\}^p,\ \beta\in\mathbb{N}. \tag{3.5}$$

We have used the following notations for given $\vec{\jmath} = (j_1,\dots,j_p) \in \mathbb{Z}^p$ and $\vec{\sigma} = (\sigma_1,\dots,\sigma_p) \in \{\pm\}^p$,

$$\langle\vec{\jmath}\rangle \triangleq \max(1,|\vec{\jmath}|), \qquad |\vec{\jmath}| \triangleq \max(|j_1|,\dots,|j_p|), \qquad u^{\vec{\sigma}}_{\vec{\jmath}} \triangleq u^{\sigma_1}_{j_1}\dots u^{\sigma_p}_{j_p},$$

where for a fixed $u \in \mathbb{C}$, we denote

$$u^+ \triangleq u \qquad \text{and} \qquad u^- \triangleq \overline{u}.$$

We denote by $\widetilde{\Gamma}^m_0$ the space of constant coefficients symbols $\xi \mapsto a(\xi)$ which satisfy (3.5) with $\mu = 0$.

ii) **Non-homogeneous symbols.** We denote by $\Gamma^m_{K,K',p}[\epsilon_0]$ the space of functions $a(U;t,x,\xi)$, defined for $U \in B^{K'}_{s_0}(I;\epsilon_0)$ for some s_0 large enough, with complex values, such that for any $0 \leqslant k \leqslant K-K'$, any $s \geqslant s_0$, there is $0 < \epsilon_0(s) < \epsilon_0$ such that for any $\beta \in \mathbb{N}$ the following holds. There is $C \triangleq C_{s,\beta} > 0$ such that for any $U \in B^{K'}_{s_0}(I;\epsilon_0(s)) \cap C^{k+K'}_*\left(I;\dot{H}^s(\mathbb{T};\mathbb{C})\right)$ and $\alpha\in\mathbb{N}$, with $\alpha \leqslant s-s_0$ one has the estimate

$$\left|\partial^k_t\partial^\alpha_x\partial^\beta_\xi a(U;t,x,\xi)\right| \leqslant C\langle\xi\rangle^{m-\beta}\|U\|^{p-1}_{k+K',s_0}\|U\|_{k+K',s}. \tag{3.6}$$

If $p = 0$ the right hand side has to be replaced by $C\langle\xi\rangle^{m-\beta}$.

We say that a non-homogeneous symbol $a(U;x,\xi)$ is *real* if it is real valued for any $U \in B^{K'}_{s_0,\mathbb{R}}(I;\epsilon_0)$.

iii) **Symbols.** We denote by $\Sigma\Gamma^m_{K,K',p}[\epsilon_0,N]$ the space of symbols

$$a(U;t,x,\xi) = \sum_{q=p}^{N} a_q(U;x,\xi) + a_{>N}(U;t,x,\xi) \tag{3.7}$$

where a_q, $q = p,\dots,N$ are homogeneous symbols in $\widetilde{\Gamma}^m_q$ and $a_{>N}$ is a non-homogeneous symbol in $\Gamma^m_{K,K',N+1}$.

We say that a symbol $a(U;t,x,\xi)$ is *real* if it is real valued for any $U \in B^{K'}_{s_0,\mathbb{R}}(I;\epsilon_0)$.

We shall also denote with $\Sigma^N_p\Gamma^m_q$ the subspace of $\Sigma\Gamma^m_{K,K',p}[\epsilon_0,N]$ made of pluri-homogeneous symbols, namely symbols which expand as in (3.7) with $a_{>N} \equiv 0$.

Remark 3.2. Let us make the following remarks.

- Given a p-homogeneous symbol $a_p \in \widetilde{\Gamma}^m_p$, we shall often, with a slight abuse of notation, identify the associated p-homogeneous polynomial with the p-linear symbol itself

$$a_p(U_1,\dots,U_p;x,\xi) \quad \leftrightsquigarrow \quad a_p(U;x,\xi) \triangleq a_p(U,\dots,U;x,\xi).$$

  This identification is harmless due to the standard correspondence between p-homogeneous polynomials and symmetric p-linear maps.

- If $a(U;\cdot)$ is a homogeneous symbol in $\widetilde{\Gamma}^m_p$ then it belongs to the class of non-homogeneous symbols $\Gamma^m_{K,0,p}[\epsilon_0]$, for any $\epsilon_0 > 0$.

- The classical properties expected for a symbol hold: if a is a symbol in $\Sigma\Gamma^m_{K,K',p}[\epsilon_0,N]$ then $\partial_x a$ is in $\Sigma\Gamma^m_{K,K',p}[\epsilon_0,N]$ and $\partial_\xi a$ belongs to $\Sigma\Gamma^{m-1}_{K,K',p}[\epsilon_0,N]$. If in addition b is a symbol in $\Sigma\Gamma^{m'}_{K,K',p'}[\epsilon_0,N]$ then their product ab is a symbol in $\Sigma\Gamma^{m+m'}_{K,K',p+p'}[\epsilon_0,N]$.

We also define classes of functions in analogy with our classes of symbols.

Definition 3.3 (Functions)**.** Let $p,N\in\mathbb{N}$, $K,K'\in\mathbb{N}$ with $K'\leqslant K$, $\epsilon_0>0$. We denote by $\tilde{\mathcal{F}}_p$, resp. $\mathcal{F}_{K,K',p}[\epsilon_0]$, $\Sigma\mathcal{F}_{K,K',p}[\epsilon_0,N]$, the subspace of $\tilde{\Gamma}^0_p$, resp. $\Gamma^0_{K,K',p}[\epsilon_0]$, resp. $\Sigma\Gamma^0_{K,K',p}[\epsilon_0,N]$, made of those symbols which are independent of ξ. We write $\tilde{\mathcal{F}}^{\mathbb{R}}_p$, resp. $\mathcal{F}^{\mathbb{R}}_{K,K',p}[\epsilon_0]$, $\Sigma\mathcal{F}^{\mathbb{R}}_{K,K',p}[\epsilon_0,N]$, to denote functions in $\tilde{\mathcal{F}}_p$, resp. $\mathcal{F}_{K,K',p}[\epsilon_0]$, $\Sigma\mathcal{F}_{K,K',p}[\epsilon_0,N]$, which are real valued for any $u\in B^{K'}_{s_0}(I;\epsilon_0)$.

Paradifferential quantization. Given $p\in\mathbb{N}$, we consider functions $\chi_p\in C^\infty(\mathbb{R}^p\times\mathbb{R};\mathbb{R})$ and $\chi\in C^\infty(\mathbb{R}\times\mathbb{R};\mathbb{R})$, even with respect to each of their arguments, satisfying, for $0<\delta_0\leqslant\frac{1}{10}$,

$$\begin{aligned} &\operatorname{supp}\chi_p \subset \{(\xi',\xi)\in\mathbb{R}^p\times\mathbb{R} \quad \text{s.t.} \quad |\xi'|\leqslant\delta_0\langle\xi\rangle\}, \qquad \chi_p(\xi',\xi)\equiv 1 \text{ for } |\xi'|\leqslant\tfrac{1}{2}\delta_0\langle\xi\rangle, \\ &\operatorname{supp}\chi \subset \{(\xi',\xi)\in\mathbb{R}\times\mathbb{R} \quad \text{s.t.} \quad |\xi'|\leqslant\delta_0\langle\xi\rangle\}, \qquad \chi(\xi',\xi)\equiv 1 \text{ for } |\xi'|\leqslant\tfrac{1}{2}\delta_0\langle\xi\rangle. \end{aligned} \tag{3.8}$$

For $p=0$, we set $\chi_0\equiv 1$. We assume moreover that

$$\begin{aligned} |\partial_\xi^\ell\partial_{\xi'}^\beta\chi_p(\xi',\xi)| &\leqslant C_{\ell,\beta}\langle\xi\rangle^{-\ell-|\beta|}, \quad \forall\ell\in\mathbb{N},\ \beta\in\mathbb{N}^p, \\ |\partial_\xi^\ell\partial_{\xi'}^\beta\chi(\xi',\xi)| &\leqslant C_{\ell,\beta}\langle\xi\rangle^{-\ell-\beta}, \quad \forall\ell,\ \beta\in\mathbb{N}. \end{aligned}$$

If $a(x,\xi)$ is a smooth symbol we define its Weyl quantization as the operator acting on a 2π-periodic function u as

$$\mathrm{Op}^W(a)u(x) = \sum_{k\in\mathbb{Z}}\Big(\sum_{j\in\mathbb{Z}}\hat{a}\big(k-j,\tfrac{k+j}{2}\big)u_j\Big)e^{\mathrm{i}kx},$$

where $\hat{a}(k,\xi)$ is the $k^{th}-$Fourier coefficient of the $2\pi-$periodic function $x\mapsto a(x,\xi)$.

Definition 3.4. (Bony-Weyl quantization) If $a(U;x,\xi)$ is a symbol in $\widetilde{\Gamma}^m_p$, respectively in $\Gamma^m_{K,K',p}[\epsilon_0]$, we set

$$\begin{aligned} a_{\chi_p}(U;x,\xi) &\triangleq \sum_{\substack{\vec{\jmath}\in(\mathbb{Z}^*)^p\\ \vec{\sigma}\in\{\pm\}^p}} \chi_p(\vec{\jmath},\xi)a^{\vec{\sigma}}_{\vec{\jmath}}(\xi)u^{\vec{\sigma}}_{\vec{\jmath}}e^{\mathrm{i}(\vec{\sigma}\cdot\vec{\jmath})x}, \\ a_\chi(U;x,\xi) &\triangleq \sum_{j\in\mathbb{Z}^*}\chi(j,\xi)\hat{a}(U;j,\xi)e^{\mathrm{i}jx}, \end{aligned}$$

where in the last equality $\hat{a}(U;j,\xi)$ stands for j^{th} Fourier coefficient of $a(U;x,\xi)$ with respect to the x variable, and we define the *Bony-Weyl* quantization of $a(U;\cdot)$ as

$$\mathrm{Op}^{\mathrm{BW}}(a(U;\cdot))v(x) \triangleq \mathrm{Op}^W(a_{\chi_p}(U;\cdot))v(x) = \sum_{\substack{(\vec{\jmath},j,k)\in(\mathbb{Z}^*)^{p+2}\\ \vec{\sigma}\in\{\pm\}^p\\ \vec{\sigma}\cdot\vec{\jmath}+j=k}} \chi_p\left(\vec{\jmath},\frac{j+k}{2}\right)a^{\vec{\sigma}}_{\vec{\jmath}}\left(\frac{j+k}{2}\right)u^{\vec{\sigma}}_{\vec{\jmath}}v_je^{\mathrm{i}kx}, \tag{3.9}$$

$$\mathrm{Op}^{\mathrm{BW}}(a(U;\cdot))v(x) \triangleq \mathrm{Op}^W(a_\chi(U;\cdot))v(x) = \sum_{(j,k)\in(\mathbb{Z}^*)^2}\chi\left(k-j,\frac{j+k}{2}\right)\hat{a}\left(U;k-j,\frac{k+j}{2}\right)v_je^{\mathrm{i}kx}. \tag{3.10}$$

Note that if $\chi\left(k-j, \frac{k+j}{2}\right) \neq 0$ then $|k-j| \leqslant \delta_0 \langle \frac{j+k}{2} \rangle$ and therefore, for $\delta_0 \in (0,1)$,

$$\frac{1-\delta_0}{1+\delta_0}|k| \leqslant |j| \leqslant \frac{1+\delta_0}{1-\delta_0}|k|, \quad \forall j,k \in \mathbb{Z}.$$

This relation shows that the action of a paradifferential operator does not spread much the Fourier support of functions.
If a is a symbol in $\Sigma\Gamma^m_{K,K',p}[\epsilon_0, N]$, we define its *Bony-Weyl* quantization

$$\mathrm{Op}^{\mathrm{BW}}(a(U;\cdot)) = \sum_{q=p}^{N} \mathrm{Op}^{\mathrm{BW}}\left(a_q(U;\cdot)\right) + \mathrm{Op}^{\mathrm{BW}}(a_{>N}(U;\cdot)).$$

We define as well

$$\mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}(a(U;x,\xi)) \triangleq \mathrm{Op}^{\mathrm{BW}}\left(\begin{bmatrix} a(U;x,\xi) & 0 \\ 0 & \overline{a^\vee(U;x,\xi)} \end{bmatrix}\right), \qquad a^\vee(U;x,\xi) \triangleq a(U;x,-\xi). \tag{3.11}$$

Remark 3.5.
- The operator $\mathrm{Op}^{\mathrm{BW}}(a)$ maps functions with zero average in functions with zero average, and $\Pi_0^\perp \mathrm{Op}^{\mathrm{BW}}(a) = \mathrm{Op}^{\mathrm{BW}}(a)\Pi_0^\perp$.
- If a is a homogeneous symbol, the two definitions of quantization in (3.9)-(3.10) differ by a smoothing operator according to Definition 3.11 below, see [18, page 50].
- Definition 3.4 is independent of the cut-off functions χ_p, χ, up to smoothing operators (Definition 3.11).
- The action of $\mathrm{Op}^{\mathrm{BW}}(a)$ on the spaces $\dot{H}^s$ only depends on the values of the symbol $a(u;t,x,\xi)$ for $|\xi| \geqslant 1$. Therefore, we may identify two symbols $a(u;t,x,\xi)$ and $b(u;t,x,\xi)$ if they agree for $|\xi| \geqslant 1/2$. In particular, whenever we encounter a symbol that is not smooth at $\xi = 0$, such as, for example, $a = g(x)|\xi|^m$ for $m \in \mathbb{R}^*$, or $\operatorname{sgn}\xi$, we will consider its smoothed out version $\left(1-\chi(\xi)\right)a(x,\xi)$, where χ is defined in (3.8). Similarly for p-homogeneous symbols.

Remark 3.6. Given a paradifferential operator $A = \mathrm{Op}^{\mathrm{BW}}(a(x,\xi))$ it results

$$\overline{A} = \mathrm{Op}^{\mathrm{BW}}\left(\overline{a(x,-\xi)}\right), \qquad A^\intercal = \mathrm{Op}^{\mathrm{BW}}(a(x,-\xi)), \qquad A^* = \mathrm{Op}^{\mathrm{BW}}\left(\overline{a(x,\xi)}\right),$$

where $A^\intercal$ is the transposed operator with respect to the real scalar product $\langle \cdot \mid \cdot \rangle_{\mathbb{R}}$ in (3.2), and A^* denotes the adjoint operator with respect to the complex scalar product $\langle \cdot \mid \cdot \rangle_{\mathbb{C}}$ on $\dot{L}^2$ in (3.2). It results $A^* = \overline{A}^\intercal$.

- A paradifferential operator $A = \mathrm{Op}^{\mathrm{BW}}(a(x,\xi))$ is *real* (i.e. $A = \overline{A}$) if
$$\overline{a(x,\xi)} = a(x,-\xi). \tag{3.12}$$
- It is *symmetric* (i.e. $A = A^\intercal$) if
$$a(x,\xi) = a(x,-\xi).$$

We now provide the action of a paradifferential operator on Sobolev spaces, cf. [18, Prop. 3.8].

Lemma 3.7 (Action of a paradifferential operator)**.** *Let $m \in \mathbb{R}$.*

i) If $p \in \mathbb{N}$, there is $s_0 > 0$ such that for any symbol a in $\tilde{\Gamma}^m_p$, there is a constant $C > 0$, depending only on s and on (3.5) *with $\mathtt{b} = \beta = 0$, such that, for any $(U_1, \ldots, U_p)$, for $p \geqslant 1$,*

$$\left\| \mathrm{Op}^{\mathrm{BW}}\left(a(U_1,\ldots,U_p;\cdot)\right) u_{p+1} \right\|_{\dot{H}^{s-m}} \leqslant C \left\| U_1 \right\|_{\dot{H}^{s_0}} \cdots \left\| U_p \right\|_{\dot{H}^{s_0}} \left\| u_{p+1} \right\|_{\dot{H}^s}.$$

If $p = 0$ the above bound holds replacing the right hand side with $C \left\| u_{p+1} \right\|_{\dot{H}^s}$.

ii) Let $\epsilon_0 > 0$, $p \in \mathbb{N}$, $K' \leqslant K \in \mathbb{N}$, a in $\Gamma^m_{K,K',p}[\epsilon_0]$. There is $s_0 > 0$, and a constant C, depending only on s, ϵ_0, and on (3.6) *with $0 \leqslant \alpha \leqslant 2$, $\beta = 0$, such that, for any t in I, any $0 \leqslant k \leqslant K - K'$, any U in $B^K_{s_0}(I;\epsilon_0)$,*

$$\left\| \mathrm{Op}^{\mathrm{BW}}\left(\partial_t^k a(U;t,\cdot)\right) \right\|_{\mathcal{L}(\dot{H}^s, \dot{H}^{s-m})} \leqslant C \| U(t,\cdot) \|^p_{k+K',s_0}, \tag{3.13}$$

so that $\left\| \mathrm{Op}^{\mathrm{BW}}(a(U;t,\cdot))v(t) \right\|_{K-K',s-m} \leqslant C \| U(t,\cdot) \|^p_{K,s_0} \| v(t) \|_{K-K',s}$.

Classes of m-Operators and smoothing Operators. Given integers $(n_1,\dots,n_{p+1})\in(\mathbb{N}^*)^{p+1}$ we denote by $\max_2\{n_1,\dots,n_{p+1}\}$ the second largest among $n_1,\dots,n_{p+1}$. We now define m-operators which include the class of paradifferential operators of order m (see Remark 3.12) and allow to define the smoothing remainders when the order m is negative (see Definition 3.11). The class $\tilde{\mathcal{M}}_p^m$ denotes multilinear operators that lose m derivatives and are p-homogeneous in u, while the class $\mathcal{M}_{K,K',p}^m$ contains non-homogeneous operators which lose m derivatives, vanish at degree at least p in u, satisfy tame estimates and are $(K-K')$-times differentiable in t. The constant μ in (3.15) takes into account possible loss of derivatives in the "low" frequencies.

The following definition is taken from [25, Def. 2.5] (see also its Fourier characterization in [25, Lemma 2.9].

Definition 3.8 (Classes of m-operators)**.** Let $m\in\mathbb{R}$, $p,N\in\mathbb{N}$, $K,K'\in\mathbb{N}$ with $K'\leqslant K$, and $\epsilon_0>0$.

i) ***p*-homogeneous *m*-operators.** We denote by $\tilde{\mathcal{M}}_p^m$ the space of $(p+1)$-linear symmetric translation invariant operators from $\left(\dot H^\infty(\mathbb{T};\mathbb{C}^2)\right)^p\times\dot H^\infty(\mathbb{T};\mathbb{C})$ to $\dot H^\infty(\mathbb{T};\mathbb{C})$, whose associated polynomial has the form

$$M(U)v\triangleq M(U,\dots,U)\,v=\sum_{\substack{\vec\sigma\in\{\pm\}^p\\ k-j=\vec\sigma\cdot\vec\jmath}} M_{\vec\jmath,j,k}^{\vec\sigma}\,u_{\vec\jmath}^{\vec\sigma}\,v_j\,e^{\mathrm{i}kx} \tag{3.14}$$

with coefficients $M_{\vec\jmath,j,k}^{\vec\sigma}$ symmetric in $(j_1,\sigma_1),\dots,(j_p,\sigma_p)$, satisfying the following: there are $\mu\geqslant0$, $C>0$ such that, for any $\vec\jmath=(j_1,\dots,j_p)\in(\mathbb{Z}^*)^p$, $j,k\in\mathbb{Z}^*$, it results

$$\left|M_{\vec\jmath_p,j,k}^{\vec\sigma_p}\right|\leqslant C\max{}_2\{|j_1|,\dots,|j_p|,|j|\}^\mu\,\max\{|j_1|,\dots,|j_p|,|j|\}^m, \tag{3.15}$$

If $p=0$ the right hand side of (3.14) must be substituted with $\sum_{j\in\mathbf{Z}}M_j v_j e^{\mathrm{i}jx}$ with $|M_j|\leqslant C|j|^m$.

ii) **Non-homogeneous *m*-operators.** We denote by $\mathcal{M}_{K,K',p}^m[\epsilon_0]$ the space of operators $(U,t,v)\mapsto M(U;t)v$ defined on $B_{s_0}^{K'}(I;\epsilon_0)$ for some $s_0>0$, which are linear in the variable v and such that the following holds true. For any $s\geqslant s_0$ there are $C>0$ and $\epsilon_0(s)\in]0,\epsilon_0[$ such that for any $U\in B_{s_0}^{K'}(I;\epsilon_0)\cap C_*^K\left(I,\dot H^s(\mathbb{T};\mathbb{C})\right)$, any $v\in C_*^{K-K'}\left(I,\dot H^s(\mathbb{T};\mathbb{C})\right)$, any $0\leqslant k\leqslant K-K'$, $t\in I$, we have that

$$\left\|\partial_t^k\left(M(U;t)v\right)\right\|_{s-\frac32k-m}\leqslant C\sum_{k'+k''=k}\left(\|v\|_{k'',s}\|U\|_{k'+K',s_0}^p+\|v\|_{k'',s_0}\|U\|_{k'+K',s_0}^{p-1}\|U\|_{k'+K',s}\right). \tag{3.16}$$

In case $p=0$ we require the estimate $\|\partial_t^k(M(U;t)v)\|_{s-\frac32k-m}\leqslant C\|v\|_{k,s}$. We say that a non-homogeneous m-operator $M(U;t)$ is *real* if it is real valued for any $u\in B_{s_0}^{K'}(I;\epsilon_0)$.

iii) ***m*-Operators.** We denote by $\Sigma\mathcal{M}_{K,K',p}^m[\epsilon_0,N]$ the space of operators

$$M(U;t)v=\sum_{q=p}^N M_q(U)v+M_{>N}(U;t)v, \tag{3.17}$$

where M_q are homogeneous m-operators in $\tilde{\mathcal{M}}_q^m$, $q=p,\dots,N$ and $M_{>N}$ is a non–homogeneous m-operator in $\mathcal{M}_{K,K',N+1}^m[\epsilon_0]$. We say that a m-operator $M(u;t)$ is *real* if it is real valued for any $u\in B_{s_0}^{K'}(I;\epsilon_0)$. We shall also denote with $\Sigma_p^N\widetilde{\mathcal{M}}_q^m$ the subspace of $\Sigma\mathcal{M}_{K,K',p}^m[\epsilon_0,N]$ made of pluri-homogeneous m-operators, namely symbols which expand as in (3.7) with $M_{>N}\equiv0$.

Remark 3.9. By [25, Lemma 2.8], if $M(U_1,\dots,U_p)$ is a p–homogeneous m-operator in $\widetilde{\mathcal{M}}_p^m$ then $M(U)=M(U,\dots,U)$ is a non-homogeneous m-operator in $\mathcal{M}_{K,0,p}^m[\epsilon_0]$ for any $\epsilon_0>0$ and $K\in\mathbb{N}$. We shall say that $M(u)$ is in $\tilde{\mathcal{M}}_p^m$.

Notation 3.10.

- If $M(U_1,\dots,U_p)$ is a p-homogeneous m-operator, we shall often denote by $M(U)$ the associated p-homogeneous polynomial, as in (3.14), and write $M(U)\in\widetilde{\mathcal{M}}_p^m$. Conversely, a p-homogeneous polynomial can be represented by a $(p+1)$-linear form of the type $M(U_1,\dots,U_p)U_{p+1}$, which may not be symmetric in the first p variables. If this form satisfies the symmetric estimate (3.15), then it corresponds to an m-operator in $\widetilde{\mathcal{M}}_p^m$ obtained by symmetrizing the internal variables. In the sequel, we adopt this identification without further comment.

- given an operator $M(U;t)$ in $\Sigma\mathcal{M}^m_{K,K',p}[r,N]$ of the form (3.17) we denote by

$$\mathcal{P}_{\leqslant N}[M(U;t)] \triangleq \sum_{q=p}^{N} M_q(U)\,, \quad \text{resp.} \qquad \mathcal{P}_q[M(U;t)] \triangleq M_q(U)\,, \tag{3.18}$$

  the projections on the pluri-homogeneous, resp. homogeneous, operators in $\Sigma_p^N \widetilde{\mathcal{M}}^m_q$, resp. in $\widetilde{\mathcal{M}}^m_q$. Given an integer $p \leqslant p' \leqslant N$ we also denote

$$\mathcal{P}_{\geqslant p'}[M(U;t)] \triangleq \sum_{q=p'}^{N} M_q(U), \qquad \mathcal{P}_{\leqslant p'}[M(U;t)] \triangleq \sum_{q=p}^{p'} M_q(U)\,.$$

  The same notation will be also used to denote pluri-homogeneous/homogeneous components of symbols.

If $m \leqslant 0$ the operators in $\Sigma\mathcal{M}^m_{K,K',p}[\epsilon_0,N]$ are referred to as smoothing operators.

Definition 3.11 (Smoothing operators)**.** Let $\varrho \geqslant 0$. A $(-\varrho)$-operator $R(U)$ belonging to $\Sigma\mathcal{M}^{-\varrho}_{K,K',p}[\epsilon_0,N]$ is called a smoothing operator. We also denote

$$\widetilde{\mathcal{R}}^{-\varrho}_p \triangleq \widetilde{\mathcal{M}}^{-\varrho}_p\,, \quad \mathcal{R}^{-\varrho}_{K,K',p}[\epsilon_0] \triangleq \mathcal{M}^{-\varrho}_{K,K',p}[\epsilon_0]\,, \quad \Sigma\mathcal{R}^{-\varrho}_{K,K',p}[\epsilon_0,N] \triangleq \Sigma\mathcal{M}^{-\varrho}_{K,K',p}[\epsilon_0,N]\,, \quad \Sigma_p^N\widetilde{\mathcal{R}}^{-\varrho}_q \triangleq \Sigma_p^N\widetilde{\mathcal{M}}^{-\varrho}_q.$$

Remark 3.12. • Lemma 3.7 implies that, if $a(U;t,\cdot)$ is a symbol in $\Sigma\Gamma^m_{K,K',p}[\epsilon_0,N]$, $m \in \mathbb{R}$, then the associated paradifferential operator $\mathrm{Op}^{\mathrm{BW}}(a(U;t,\cdot))$ defines a m-operator in $\Sigma\mathcal{M}^m_{K,K',p}[\epsilon_0,N]$.

• The composition of smoothing operators $R_1 \in \Sigma\mathcal{R}^{-\varrho}_{K,K',p_1}[\epsilon_0,N]$ and $R_2 \in \Sigma\mathcal{R}^{-\varrho}_{K,K',p_2}[\epsilon_0,N]$ is a smoothing operator R_1R_2 in $\Sigma\mathcal{R}^{-\varrho}_{K,K',p_1+p_2}[\epsilon_0,N]$. This is a particular case of Proposition 3.17-(i) below.

Definition 3.13 (**Homogeneous vector fields**)**.** Let $m \in \mathbb{R}$ and $p,N \in \mathbb{N}$. We denote by $\widetilde{\mathfrak{X}}^m_{p+1}$ the space of $(p+1)$-homogeneous vector fields of the form $X(U) = M(U)U$ where $M(U)$ is a matrix of p-homogeneous m-operators in $\left(\widetilde{\mathcal{M}}^m_p\right)^{2\times 2}$. In particular, one has the Fourier expansion

$$X(U) = \begin{bmatrix} X(U)^+ \\ X(U)^- \end{bmatrix}, \qquad X(U)^\sigma \triangleq \sum_{(\vec{\jmath},k,\vec{\sigma},-\sigma)\in\mathfrak{T}_{p+2}} X^{\vec{\sigma},\sigma}_{\vec{\jmath},k}\, u^{\vec{\sigma}}_{\vec{\jmath}}\, e^{\mathrm{i}\sigma k},$$

where the Fourier restriction $\mathfrak{T}_{p+2}$ is the set of momentum preserving indices, defined, for a given $q \in \mathbb{N}^*$, as

$$\mathfrak{T}_q \triangleq \left\{(\vec{\jmath},\vec{\sigma}) \in \mathbb{Z}^q \times \{\pm\}^q \quad \text{s.t.} \quad \vec{\sigma}\cdot\vec{\jmath} = 0\right\}. \tag{3.19}$$

We denote $\Sigma_{p+1}^{N+1}\widetilde{\mathfrak{X}}^m_q$ the class of pluri-homogeneous vector fields. The vector fields in $\widetilde{\mathfrak{X}}^{-\varrho}_{p+1}$, $\varrho \geqslant 0$, are called smoothing.

Symbolic calculus. Let $\sigma(D_x,D_\xi,D_y,D_\eta) \triangleq D_\xi D_y - D_x D_\eta$ where $D_x \triangleq \frac{1}{\mathrm{i}}\partial_x$ and D_ξ, D_y, D_η are similarly defined. The following is Definition 3.11 in [18].

Definition 3.14 (Asymptotic expansion of composition symbol)**.** Let p, p' in $\mathbb{N}$, $K,K' \in \mathbb{N}$ with $K' \leqslant K$, $\varrho \geqslant 0$, $m,m' \in \mathbb{R}$, $\epsilon_0 > 0$. Consider symbols $a \in \Sigma\Gamma^m_{K,K',p}[\epsilon_0,N]$ and $b \in \Sigma\Gamma^{m'}_{K,K',p'}[\epsilon_0,N]$. For U in $B^K_\sigma(I;\epsilon_0)$ we define, for $\varrho < \sigma - s_0$, the symbol

$$(a\#_\varrho b)(U;t,x,\xi) \triangleq \sum_{k=0}^{\varrho} \frac{1}{k!}\left(\frac{\mathrm{i}}{2}\sigma\left(D_x,D_\xi,D_y,D_\eta\right)\right)^k \left[a(U;t,x,\xi)b(U;t,y,\eta)\right]_{|x=y,\xi=\eta}$$

modulo symbols in $\Sigma\Gamma^{m+m'-\varrho}_{K,K',p+p'}[\epsilon_0,N]$.

The symbol $a\#_\varrho b$ belongs to $\Sigma\Gamma^{m+m'}_{K,K',p+p'}[\epsilon_0,N]$. Moreover

$$a\#_\varrho b = ab + \frac{1}{2\mathrm{i}}\{a,b\}$$

up to a symbol in $\Sigma\Gamma^{m+m'-2}_{K,K',p+p'}[\epsilon_0,N]$, where

$$\{a,b\} \triangleq \partial_\xi a\, \partial_x b - \partial_x a\, \partial_\xi b$$

denotes the Poisson bracket. The following result is proved in Proposition 3.12 in [18].

Proposition 3.15 (Composition of Bony-Weyl operators). *Let $p,q,N,K,K' \in \mathbb{N}$ with $K' \leqslant K$, $\varrho \geqslant 0$, $m,m' \in \mathbb{R}$, $\epsilon_0 > 0$. Consider symbols $a \in \Sigma\Gamma^m_{K,K',p}[\epsilon_0,N]$ and $b \in \Sigma\Gamma^{m'}_{K,K',q}[\epsilon_0,N]$. Then*

$$\mathrm{Op}^{\mathrm{BW}}(a(U;t,x,\xi)) \circ \mathrm{Op}^{\mathrm{BW}}(b(U;t,x,\xi)) - \mathrm{Op}^{\mathrm{BW}}\big((a\#_\varrho b)(U;t,x,\xi)\big)$$

is a smoothing operator in $\Sigma\mathcal{R}^{-\varrho+m+m'}_{K,K',p+q}[\epsilon_0,N]$.

We have the following result, see e.g. Lemma 7.2 in [18].

Lemma 3.16 (Bony paraproduct decomposition). *Let u_1,u_2 be functions in $H^\sigma(\mathbb{T};\mathbb{C})$ with $\sigma > \frac{1}{2}$. Then*

$$u_1 u_2 = \mathrm{Op}^{\mathrm{BW}}(u_1)u_2 + \mathrm{Op}^{\mathrm{BW}}(u_2)u_1 + R_1(u_1)u_2 + R_2(u_2)u_1$$

where for $\mathrm{j}=1,2$, R_{j} is a homogeneous smoothing operator in $\widetilde{\mathcal{R}}^{-\varrho}_1$ for any $\varrho \geqslant 0$.

We now state other composition results for m-operators which follow as in [25, Proposition 2.15].

Proposition 3.17 (Compositions of m-operators). *Let $p,p',N,K,K' \in \mathbb{N}$ with $K' \leqslant K$ and $\epsilon_0 > 0$. Let $m,m' \in \mathbb{R}$. Then*

1. *If $M(U;t)$ is in $\Sigma\mathcal{M}^m_{K,K',p}[\epsilon_0,N]$ and $M'(U;t)$ is in $\Sigma\mathcal{M}^{m'}_{K,K',p'}[\epsilon_0,N]$ then the composition $M(u;t) \circ M'(U;t)$ is in $\Sigma\mathcal{M}^{m+\max(m',0)}_{K,K',p+p'}[\epsilon_0,N]$.*
2. *If $M(U)$ is a homogeneous m-operator in $\widetilde{\mathcal{M}}^m_p$ and $M^{(\ell)}(U;t)$, $\ell = 1,\dots,p+1$, are matrices of m_ℓ-operators in $\Sigma\mathcal{M}^{m_\ell}_{K,K',q_\ell}[\epsilon_0,N]$ with $m_\ell \in \mathbb{R}$, $q_\ell \in \mathbb{N}$, then*

$$M\big(M^{(1)}(U;t)u,\dots,M^{(p)}(U;t)U\big)\,M^{(p+1)}(U;t)$$

   belongs to $\Sigma\mathcal{M}^{m+\bar{m}}_{K,K',p+\bar{q}}[\epsilon_0,N]$ with $\bar{m} \triangleq \sum_{\ell=1}^{p+1}\max(m_\ell,0)$ and $\bar{q} \triangleq \sum_{\ell=1}^{p+1} q_\ell$.
3. *Let a be a symbol in $\Sigma\Gamma^m_{K,K',p}[\epsilon_0,N]$ with $m \geqslant 0$ and R a smoothing operator in $\Sigma\mathcal{R}^{-\varrho}_{K,K',p'}[\epsilon_0,N]$. Then*

$$\mathrm{Op}^{\mathrm{BW}}(a(U;t,\cdot)) \circ R(U;t),\;\; R(U;t) \circ \mathrm{Op}^{\mathrm{BW}}(a(U;t,\cdot)) \;\in \Sigma\mathcal{R}^{-\varrho+m}_{K,K',p+p'}[\epsilon_0,N].$$

4. *If a_p is in $\widetilde{\Gamma}^m_p$ and $M(U) \in \Sigma\mathcal{M}^{m'}_{K,K',p'}[r,N]$ then $a_p(M(U),U,\dots,U;x,\xi) \in \Sigma\Gamma^m_{K,K',p+p'}[r,N]$ and*

$$\mathrm{Op}^{\mathrm{BW}}(a(W,U,\dots,U;x,\xi))_{|W=M(U)} = \mathrm{Op}^{\mathrm{BW}}\big(a_p(M(U),U,\dots,U;x,\xi)\big) + R(U)$$

   where $R(U) \in \Sigma\mathcal{R}^{-\varrho}_{K,K',p+p'}[r,N]$, for any $\varrho > 0$. In particular if $a \in \Sigma\Gamma^m_{K,K',p}[r,N]$ then

$$\begin{aligned} &\textit{If}\quad \partial_t U = M_0(U)U, \quad M_0(U) \in \Sigma\mathcal{M}^{m'}_{K,K',0}[r,N],\\ &\textit{then}\quad \partial_t \mathrm{Op}^{\mathrm{BW}}(a(U;x,\xi)) = \mathrm{Op}^{\mathrm{BW}}\big(a_{[1]}(U;x,\xi)\big) + R(U), \end{aligned}$$

   where $a_{[1]} \in \Sigma\Gamma^m_{K,K'+1,p}[r,N]$ and $R(U) \in \Sigma\mathcal{R}^{-\varrho}_{K,K',p'}[r,N]$

Notation 3.18. In the sequel if $K'=0$ we denote a symbol $a(U;t,x,\xi)$ in $\Gamma^m_{K,0,p}[\epsilon_0]$ simply as $a(U;x,\xi)$, and a smoothing operator in $R(U;t)$ in $\Sigma\mathcal{R}^{-\varrho}_{K,0,p}[\epsilon_0,N]$ simply as $R(U)$, without writing the t-dependence.

We finally provide the Bony paralinearization formula of the composition operator whose proof is a combination of [18, Lemma 3.19].

Lemma 3.19 (Bony Paralinearization formula). *Let F be a smooth $\mathbb{C}$-valued function defined on a neighborhood of zero in $\mathbb{C}$, vanishing at zero at order $q \in \mathbb{N}$. Then there are $s_0, \epsilon_0 > 0$ such that if $u \in B_{H^{s_0}(\mathbb{T};\mathbb{R})}(\epsilon_0)$, then*

$$F(u) = \mathrm{Op}^{\mathrm{BW}}\big(F'(u)\big)u + R(u)\,u,$$

where $R(u)$ is a smoothing operator in $\Sigma\mathcal{R}^{-\varrho}_{K,0,q'}[\epsilon_0,N]$, $q' \triangleq \max(q-1,1)$, for any $\varrho \geqslant 0$.

3.2 Spectrally localized maps

In this section we introduce the class of spectrally localized map, needed to apply the Darboux symplectic correction (see Proposition 7.1). This class was introduced first in [25, Def. 2.15]

Definition 3.20 (Spectrally localized maps). Let $m \in \mathbb{R}, p, N \in \mathbb{N}, K, K' \in \mathbb{N}$ with $K' \leqslant K$ and $r > 0$.

i) **Spectrally localized p-homogeneous maps.** We denote by $\widetilde{\mathcal{S}}_p^m$ the subspace of m-operators $S(U)$ in $\widetilde{\mathcal{M}}_p^m$ whose coefficients $S_{\vec{\jmath},j,k}^{\vec{\sigma}}$ (see (3.14)) satisfying the following spectral condition: there are $\delta > 0, C > 1$ such that

$$S_{\vec{\jmath},j,k}^{\vec{\sigma}} \neq 0 \quad \Longrightarrow |\vec{\jmath}| \leqslant \delta |j|, \quad C^{-1}|k| \leqslant |j| \leqslant C|k|.$$

We denote $\widetilde{\mathcal{S}} \triangleq \bigcup_p \widetilde{\mathcal{S}}_p^m$ and by $\Sigma_p^N \widetilde{\mathcal{S}}_q^m$ the class of pluri-homogeneous spectrally localized maps of the form $\sum_{q=p}^N S_q$ with $S_q \in \widetilde{\mathcal{S}}_q^m$ and $\Sigma_p \widetilde{\mathcal{S}}_q^m \triangleq \bigcup_{N \in \mathbb{N}} \Sigma_p^N \widetilde{\mathcal{S}}_q^m$. For $p \geqslant N+1$ we mean that the sum is empty.

ii) **Non-homogeneous spectrally localized maps.** We denote $\mathcal{S}_{K,K',p}^m[\epsilon_0]$ the space of maps $(U,t,V) \mapsto S(U;t)V$ defined on $B_K^{K'}(I;r) \times I \times C^0(I, H^{s_0}(T,\mathbb{C}))$ for some $s_0 > 0$, which are linear in the variable V and such that the following holds true. For any $s \in \mathbb{R}$ there are $C > 0$ and $r(s) \in [0, \epsilon_0]$ such that for any $U \in B_K^{K'}(I; r(s)) \cap C^*(I, H^s(T;\mathbb{C}^2))$, any $V \in C^*_{K-K'}(I, H^s(T,\mathbb{C}))$, any $0 \leqslant k \leqslant K-K'$, $t \in I$, we have that

$$\|\partial_t^k (S(U;t)V)(t,\cdot)\|_{H^{s-\frac{3}{2}k-m}} \leqslant C \sum_{k'+k''=k} \|U\|_{k',s_0}^p \|V\|_{H^{k'',s}}, \qquad \text{if } p \geqslant 1,$$

$$\|\partial_t^k (S(U;t)V)\|_{H^{s-\frac{3}{2}k-m}} \leqslant C \|V\|_{k,s}, \qquad \text{if } p = 0.$$

We denote $S_{K,K',N}^m[\epsilon_0] \triangleq \bigcup_p \mathcal{S}_{K,K',p}^m[\epsilon_0]$.

iii) **Spectrally localized Maps.** We denote by $\Sigma\mathcal{S}_{K,K',p}^m[r,N]$, the space of maps $(U,t,V) \mapsto S(U;t)V$ of the form

$$S(U;t)V = \sum_{q=p}^{N} S_q(U)V + S_{>N}(U;t)V,$$

where S_q are spectrally localized homogeneous maps in $\widetilde{\mathcal{S}}_q^m$, $q = p,\dots,N$ and $S_{>N}$ is a non-homogeneous spectrally localized map in $\mathcal{S}_{K,K',N+1}^m[\epsilon_0]$. We denote by $\left(\Sigma_{K,K',p}^m[r,N]\right)^{2\times 2}$ the space of 2×2 matrices whose entries are spectrally localized maps in $\Sigma_{K,K',p}^m[r,N]$. We will use also the notation $\Sigma_{K,K',p}^m[r,N] \triangleq \bigcup_{l \geqslant 0} \Sigma_{K,K',p}^m[r,N+l]$.

3.3 z-dependent paradifferential calculus

The following “Kernel-functions", that depend on the "convolutive 2π-periodic variable" z, have to be considered as Taylor remainders of functions $K(u;x,z)$ at $z=0$ which are smooth in u and which have finite regularity in x and z. A Kernel function is a z-dependent family of functions (cfr. Definition 3.3) with coefficients of size proportional to $|z|_{\mathbb{T}}^n$. For $n > -1$ such singularity is integrable in z.

Definition 3.21 (Kernel functions). Let $n \in \mathbb{R}$, $p, N \in \mathbb{N}$, $K \in \mathbb{N}$, and $\epsilon_0 > 0$.

i) **p-homogeneous Kernel-functions.** If $p \in \mathbb{N}$ we denote $\widetilde{K\mathcal{F}}_p^n$ the space of z-dependent, p-homogeneous maps from $\dot{H}^\infty(\mathbb{T};\mathbb{C})$ to the space of x-translation invariant real functions $\kappa(u;x,z)$ of class $\mathcal{C}^\infty$ in $(x,z) \in \mathbb{T}^2$ with Fourier expansion

$$\kappa(u;x,z) = \sum_{j_1,\dots,j_p \in \mathbb{Z}^*} \kappa_{j_1,\dots,j_p}(z) u_{j_1} \cdots u_{j_p} e^{\mathrm{i}(j_1+\dots+j_p)x}, \quad z \in \mathbb{T}\setminus\{0\},$$

with coefficients $\kappa_{j_1,\dots,j_p}(z)$ of class $\mathcal{C}^\infty(\mathbb{T};\mathbb{C})$, symmetric in $(j_1,\dots,j_p)$, satisfying the reality condition $\overline{\kappa_{j_1,\dots,j_p}}(z) = \kappa_{-j_1,\dots,-j_p}(z)$ and the following: for any $l \in \mathbb{N}$, there exist $\mu > 0$ and a constant $C > 0$ such that

$$\left|\partial_z^l \kappa_{j_1,\dots,j_p}(z)\right| \leqslant C \left|\vec{\jmath}_p\right|^\mu |z|_{\mathbb{T}}^{n-l}, \quad \forall \vec{\jmath}_p = (j_1,\dots,j_p) \in (\mathbb{Z}^*)^p. \tag{3.20}$$

For $p=0$ we denote by $\widetilde{K\mathcal{F}}_0^n$ the space of maps $z \mapsto \kappa(z)$ which satisfy $\left|\partial_z^l \kappa(z)\right| \leqslant C\,|z|_{\mathbb{T}}^{n-l}$.

ii) **Non-homogeneous Kernel-functions.** We denote by $K\mathcal{F}^n_{K,0,p}[\epsilon_0]$ the space of z-dependent, real functions $\kappa(u;x,z)$, defined for $u \in B^0_{s_0}(I;\epsilon_0)$ for some s_0 large enough, such that for any $0 \leqslant k \leqslant K$ and $l \leqslant \max\{0, \lceil 1+n \rceil\}$, any $s \geqslant s_0$, there are $C > 0$, $0 < \epsilon_0(s) < \epsilon_0$ and for any $u \in B^K_{s_0}(I;\epsilon_0(s)) \cap C^k_*\left(I, \dot{H}^s(\mathbb{T};\mathbb{C})\right)$ and any $\mathtt{b} \in \mathbb{N}$, with $\alpha \leqslant s - s_0$, one has the estimate

$$\left|\partial_t^k \partial_x^\alpha \partial_z^l \kappa(u;x,z)\right| \leqslant C \|u\|^{p-1}_{k,s_0} \|u\|_{k,s} \, |z|^{n-l}_{\mathbb{T}}, \qquad z \in \mathbb{T} \setminus \{0\}. \tag{3.21}$$

If $p = 0$ the right hand side in (3.21) has to be replaced by $|z|^{n-l}_{\mathbb{T}}$.

iii) **Kernel-functions.** We denote by $\Sigma K\mathcal{F}^n_{K,0,p}[\epsilon_0, N]$ the space of real functions of the form

$$\kappa(u;x,z) = \sum_{q=p}^{N} \kappa_q(u;x,z) + \kappa_{>N}(u;x,z),$$

where $\kappa_q(u;x,z)$, $q = p,\dots,N$ are homogeneous Kernel functions in $\widetilde{K\mathcal{F}}^n_q$, and $\kappa_{>N}(u;x,z)$ is a non-homogeneous Kernel function in $K\mathcal{F}^n_{K,0,N+1}[\epsilon_0]$.

A Kernel function $\kappa(u;x,z)$ is *real* if it is real valued for any $u \in B^0_{s_0,\mathbb{R}}(I;\epsilon_0)$.

We list some properties of the Kernel functions. In view of the second point of Remark 3.2, a homogeneous Kernel function $\kappa(u;x,z)$ in $\widetilde{K\mathcal{F}}^n_p$ defines a non-homogeneous Kernel function in $K\mathcal{F}^n_{K,0,p}[\epsilon_0]$ for any $\epsilon_0 > 0$.

Remark 3.22. Let us make the following remarks.

- Let $\kappa(u;x,z)$ be a Kernel function in $\Sigma\mathcal{F}^n_{K,0,p}[\epsilon_0, N]$ with $n \geqslant 0$, which admits a continuous extension in $z = 0$. Then its trace $\kappa(u;x,0)$ at $z = 0$ is a function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,p}[\epsilon_0, N]$.
- If $\kappa(u;x,z)$ is a homogeneous Kernel function $\widetilde{K\mathcal{F}}^n_p$, the two definitions of quantization in (3.9) differ by a Kernel smoothing operator in $\widetilde{K\mathcal{R}}^{-\varrho,n}_p$, for any $\varrho > 0$, according to Definition 3.25 below.
- (Sum and product of Kernel functions) If $\kappa_1(u;x,z)$ is a Kernel function in $\Sigma K\mathcal{F}^{n_1}_{K,0,p_1}[\epsilon_0, N]$ and $\kappa_2(u;x,z)$ in $\Sigma K\mathcal{F}^{n_2}_{K,0,p_2}[\epsilon_0, N]$, then the sum $(\kappa_1 + \kappa_2)(u;x,z)$ is a Kernel function in $\Sigma K\mathcal{F}^{\min\{n_1,n_2\}}_{K,0,\min\{p_1,p_2\}}[\epsilon_0, N]$ and the product $(\kappa_1\kappa_2)(u;x,z)$ is a Kernel function in $\Sigma K\mathcal{F}^{n_1+n_2}_{K,0,p_1+p_2}[\epsilon_0, N]$.
- (Integral of Kernel functions) Let $\kappa(u;x,z)$ be a Kernel function in $\Sigma K\mathcal{F}^n_{K,0,p}[\epsilon_0, N]$ with $n > -1$. Then $⨍\kappa(u;x,z)\,\mathrm{d}z$ is a function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,p}[\epsilon_0, N]$. This follows directly integrating (3.20) and (3.21) in z.

The m-Kernel-operators defined below are a z-dependent family of m-operators (cfr. Definition 3.8) with coefficients of size proportional to $|z|^n_{\mathbb{T}}$. For $n > -1$ such singularity is integrable in z. A family of z-dependent paraproduct operators associated to Kernel functions defines a 0-Kernel operator, see Remark 3.24. The kind of operators will appear only in case $m < 0$, as smoothing operators in the composition of Bony-Weyl quantizations of Kernel-functions (see Definition 3.25).

Definition 3.23. Let $m, n \in \mathbb{R}$, $p, N \in \mathbb{N}$, $K \in \mathbb{N}$ with $\epsilon_0 > 0$.

i) ***p*-homogeneous *m*-Kernel-operator.** We denote by $\widetilde{K\mathcal{M}}^{m,n}_p$ the space of z-dependent, x-translation invariant homogeneous m-operators according to Definition 3.8, Item i, in which the constant C is substituted with $C|z|^n_{\mathbb{T}}$, equivalently

$$M(u;z)v(x) = \sum_{\substack{(\vec{\jmath}_p,j,k)\in\mathbb{Z}^{p+2} \\ j_1+\ldots+j_p+j=k}} M_{\vec{\jmath}_p,j,k}(z)\, u_{j_1}\ldots u_{j_p} v_j e^{\mathrm{i}kx}, \qquad z \in \mathbb{T} \setminus \{0\}, \tag{3.22}$$

with coefficients satisfying

$$|M_{\vec{\jmath}_p,j,k}(z)| \leqslant C \max_2\{|j_1|,\dots,|j_p|,|j|\}^\mu \max\{|j_1|,\dots,|j_p|,|j|\}^m |z|^n_{\mathbb{T}}. \tag{3.23}$$

If $p = 0$ the right hand side of (3.22) is replaced by $\sum_{j\in\mathbf{Z}} M_j(z)\, v_j e^{\mathrm{i}jx}$ with $|M_j(z)| \leqslant C|j|^m |z|^n_{\mathbb{T}}$.

ii) **Non-homogeneous m-Kernel-operator.** We denote by $K\mathcal{M}^{m,n}_{K,0,p}[\epsilon_0]$ the space of z-dependent, non-homogeneous operators $M(u;z)v$ defined for any $z\in\mathbb{T}\setminus\{0\}$, such that for any $0\leqslant k\leqslant K$

$$\left\|\partial_t^k\left(M(u;z)v\right)\right\|_{s-\alpha k-m}\leqslant C|z|^n_{\mathbb{T}}\sum_{k'+k''=k}\left(\|v\|_{k'',s}\|u\|^p_{k',s_0}+\|v\|_{k'',s_0}\|u\|^{p-1}_{k',s_0}\|u\|_{k',s}\right). \tag{3.24}$$

iii) **m-Kernel-Operator.** We denote by $\Sigma K\mathcal{M}^{m,n}_{K,0,p}[\epsilon_0,N]$ the space of operators of the form

$$M(u;z)v=\sum_{q=p}^{N}M_q(u)v+M_{>N}(u;z)v \tag{3.25}$$

where M_q are homogeneous m-Kernel operators in $\widetilde{K\mathcal{M}}^{m,n}_q$, $q=p,\dots,N$ and $M_{>N}$ is a non–homogeneous m-Kernel-operator in $\mathcal{M}^{m,n}_{K,0,N+1}[\epsilon_0]$. We denote by $\Sigma_p^N\widetilde{\mathcal{M}}^m_q$ the space pluri-homogeneous m-Kernel operators of the form (3.25) with $M_{>N}=0$.

Remark 3.24. Given a Kernel function $\kappa(u;x,z)$ in $\Sigma K\mathcal{F}^n_{K,0,p}[\epsilon_0,N]$ then $\mathrm{Op}^{\mathrm{BW}}(\kappa(u;x,z))$ is 0- Kernel operator in $\Sigma K\mathcal{M}^{0,n}_{K,0,p}[\epsilon_0,N]$.

Definition 3.25 (Kernel-smoothing operators)**.** Given $\varrho>0$ we define the homogeneous and non-homogeneous Kernel-smoothing operators as

$$\widetilde{K\mathcal{R}}^{-\varrho,n}_p\triangleq\widetilde{K\mathcal{M}}^{-\varrho,n}_p,\qquad K\mathcal{R}^{-\varrho,n}_{K,0,p}[\epsilon_0]\triangleq K\mathcal{M}^{-\varrho,n}_{K,0,p}[\epsilon_0],\qquad \Sigma K\mathcal{R}^{-\varrho,n}_{K,0,p}[\epsilon_0,N]\triangleq\Sigma K\mathcal{M}^{-\varrho,n}_{K,0,p}[\epsilon_0,N].$$

In view of [25, Lemma 2.8], if $M(u,\dots,u;z)$ is a homogeneous m-Kernel operator in $\widetilde{K\mathcal{M}}^{m,n}_p$ then $M(u,\dots,u;z)$ defines a non-homogeneous m-Kernel operator in $K\mathcal{M}^{m,n}_{K,0,p}[\epsilon_0]$ for any $\epsilon_0>0$ and $K\in\mathbb{N}$.

The classes of paraproducts associated to Kernel functions and m-Kernel-operators are closed w.r.t. compositions as we list below, cf. [17].

Proposition 3.26 (Composition of z-dependent operators)**.** *Let $m,n,m',n'\in\mathbb{R}$, and integers $K,p,p',N\in\mathbb{N}$ with $p,p'\leqslant N$.*

1. *Let $\kappa(u;x,z)\in\Sigma K\mathcal{F}^n_{K,0,p}[\epsilon_0,N]$ and $\kappa'(u;x,z)\in\Sigma K\mathcal{F}^{n'}_{K,0,p'}[\epsilon_0,N]$ be Kernel functions. Then*

$$\mathrm{Op}^{\mathrm{BW}}(\kappa(u;x,z))\circ\mathrm{Op}^{\mathrm{BW}}\left(\kappa'(u;x,z)\right)=\mathrm{Op}^{\mathrm{BW}}\left(\kappa\,\kappa'(u;x,z)\right)+R(u;z),$$

   where $R(u;z)$ is a Kernel-smoothing operator in $\Sigma K\mathcal{R}^{-\varrho,n+n'}_{K,0,p+p'}[\epsilon_0,N]$ for any $\varrho\geqslant 0$;

2. *Let $M(u;z)$ be a m-Kernel operator in $\Sigma K\mathcal{M}^{m,n}_{K,0,p}[\epsilon_0,N]$ and $M'(u;z)$ be an m'-operator belonging to $\Sigma K\mathcal{M}^{m',n'}_{K,0,p'}[\epsilon_0,N]$. Then $M(u;z)\circ M'(u;z)$ belongs to $\Sigma K\mathcal{M}^{m+\max(m',0),n+n'}_{K,0,p+p'}[\epsilon_0,N]$;*

3. *Let $\kappa(u;x,z)$ be a Kernel function in $\Sigma K\mathcal{F}^n_{K,0,p}[\epsilon_0,N]$ and $R(u;z)$ be a Kernel smoothing operator in $\Sigma K\mathcal{R}^{-\varrho,n'}_{K,0,p'}[\epsilon_0,N]$ then $\mathrm{Op}^{\mathrm{BW}}(\kappa(u;x,z))\circ R(u;z)$ and $R(u;z)\circ\mathrm{Op}^{\mathrm{BW}}(\kappa(u;x,z))$ are a Kernel smoothing operator in $\Sigma K\mathcal{R}^{-\varrho,n+n'}_{K,0,p+p'}[\epsilon_0,N]$;*

4. *Let $M(u;z)$ be an homogeneous m-Kernel operator in $\widetilde{K\mathcal{M}}^{m,n}_1$, and $M'(u;z)$ in $\Sigma K\mathcal{M}^{0,0}_{K,0,0}[\epsilon_0,N]$ then $M\left(M'(u;z)u;z\right)\in\Sigma K\mathcal{M}^{m,n}_{K,0,1}[\epsilon_0,N]$.*

Finally integrating (3.23) and (3.24) in z we deduce the following lemma.

Lemma 3.27 (Integrals of Kernel smoothing operators)**.** *Let $R(u;z)$ be a Kernel smoothing operator belonging to $\Sigma\mathcal{R}^{-\varrho,n}_{K,0,p}[\epsilon_0,N]$ with $n>-1$. Then*

$$\int_{\mathbb{T}}R(u;z)\,g(x-z)\,\mathrm{d}z=R_1(u)\,g,\qquad\int_{\mathbb{T}}R(u;z)\,\mathrm{d}z=R_2(u),$$

where $R_1(u)$, $R_2(u)$ are smoothing operators in $\Sigma\mathcal{R}^{-\varrho}_{K,0,p}[\epsilon_0,N]$.

The following proposition will be crucial in Section 4, and is proved in [17].

Proposition 3.28. *Let $n > -1$ and $\kappa(u; x, z)$ be a Kernel-function in $\Sigma K\mathcal{F}^{n}_{K,0,p}[\epsilon_0, N]$. Let us define the operator, for any $g \in \dot{H}^s(\mathbb{T};\mathbb{R})$, $s \in \mathbb{R}$,*

$$\left(\mathcal{T}_\kappa g\right)(x) \triangleq \int_{\mathbb{T}} \mathrm{Op}^{\mathrm{BW}}\left(\kappa(u; \bullet, z)\right) g(x-z)\, \mathrm{d}z.$$

Then there exists

- *a symbol $a(u; x, \xi)$ in $\Sigma\Gamma^{-(1+n)}_{K,0,p}[\epsilon_0, N]$ satisfying (3.12);*
- *a pluri-homogeneous smoothing operator $R(u)$ in $\Sigma^N_p \widetilde{\mathcal{R}}^{-\varrho}_q$ for any $\varrho > 0$;*

such that $\mathcal{T}_\kappa g = \mathrm{Op}^{\mathrm{BW}}\left(a(u; x, \xi)\right) g + R(u)\, g$.

4 Paralinearization of the Kelvin-Helmholtz system

Notation 4.1. In the present section we use the following notation

$$B^K_{s,\mathbb{R}}(I; r) \triangleq B_{C^K_*(I; \dot{H}^s(\mathbb{T};\mathbb{R}))}(0, r)\ .$$

We warn the reader that such notation is conflictive with the notation introduced at page 18, but we think that in the restricted context of the paralinearization procedure outlined here there is no risk of confusion and it helps to streamline the mathematical statements that we present.

In the present section we paralinearize the system Eq. (1.9) in the (η, ψ)-variables. The result we obtain is the following one.

Theorem 4.2. *Let $N \in \mathbb{N}$, $\gamma \geqslant 0$, $\mathtt{b} \in \mathbb{R}$ and $\varrho \geqslant 0$, for any $K \in \mathbb{N}$ there exists $s_0 > 0$ and $\epsilon_0 > 0$ such that if $\eta, \psi \in B^K_{s_0,\mathbb{R}}(I; \epsilon_0)$ is a solution of Eq. (1.9) then (η, ψ) solves the paradifferential equation*

$$\begin{bmatrix} \eta_t \\ \psi_t \end{bmatrix} = \mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{Q}_{\gamma,\mathtt{b}}(\eta,\psi;x,\xi) + \boldsymbol{B}_{\mathtt{b}}(\eta,\psi;x)|\xi| - \mathrm{i}V_{\mathtt{b}}(\eta,\psi;x)\,\mathrm{Id}_{\mathbb{R}^2}\,\xi + \boldsymbol{A}_{[0]}(\eta,\psi;x,\xi)\right)\begin{bmatrix} \eta \\ \psi \end{bmatrix} + \boldsymbol{R}(\eta,\psi)\begin{bmatrix} \eta \\ \psi \end{bmatrix}, \tag{4.1}$$

where

- *The matrix of symbols $\boldsymbol{Q}_{\gamma,\mathtt{b}}$ satisfy (3.12) and is given by*

$$\boldsymbol{Q}_{\gamma,\mathtt{b}}(\eta,\psi;x,\xi) \triangleq \begin{bmatrix} 0 & -\frac{|\xi|}{2} \\ \gamma\left(1+\mathtt{f}(\eta;x)\right)\left(|\xi|^2-1\right) - \left(\frac{\mathtt{b}^2}{2} + w_{\mathtt{b}}(\eta,\psi;x)\right)|\xi| + \frac{\mathtt{b}^2}{(1+2\eta)} & 0 \end{bmatrix} \in \left(\Sigma\Gamma^2_{K,0,0}[\epsilon_0,N]\right)^{2\times 2}, \tag{4.2}$$

  with

$$\begin{aligned} \mathtt{f}(\eta;x) &\triangleq \left(\frac{1+2\eta}{\left(1+2\eta\right)^2+\eta_x^2}\right)^{\frac{3}{2}} - 1 \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N], \\ w_{\mathtt{b}}(\eta,\psi;x) &\triangleq \frac{1}{2}\left(W^2_{\mathtt{b}}(\eta,\psi;x) - \mathtt{b}^2\right) \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N], \\ W_{\mathtt{b}}(\eta,\psi;x) &\triangleq \left(\psi_x+\mathtt{b}\right)\frac{1+2\eta}{\left(1+2\eta\right)^2+\eta_x^2} \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,0}[\epsilon_0,N]. \end{aligned} \tag{4.3}$$

  In particular, $W_{\mathtt{b}}(0,0;x) \equiv \mathtt{b}$;

- $$\boldsymbol{B}_{\mathtt{b}}(\eta,\psi;x) \triangleq \frac{1}{2}\begin{bmatrix} B_{\mathtt{b}}(\eta,\psi;x) & 0 \\ B^2_{\mathtt{b}}(\eta,\psi;x) & -B_{\mathtt{b}}(\eta,\psi;x) \end{bmatrix} \in \left(\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]\right)^{2\times 2}, \tag{4.4}$$

  where

$$B_{\mathtt{b}}(\eta,\psi;x) \triangleq \left(\psi_x+\mathtt{b}\right)\frac{\mathsf{J}^0(\eta;x)}{1+2\eta} \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N], \qquad \mathsf{J}^0(\eta;x) \triangleq \frac{2\eta_x\left(1+2\eta\right)}{\left(1+2\eta\right)^2+\eta_x^2} \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N].$$

  In particular $\boldsymbol{B}_{\mathtt{b}}$ satisfy (3.12);

- $V_{\mathtt{b}}(\eta,\psi;x) \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]$ *and is explicitly defined as*

$$V_{\mathtt{b}}(\eta,\psi;x) \triangleq \frac{1}{2}\mathcal{D}_0(\eta)\left[\mathtt{b}+\psi_x\right]-\frac{\mathtt{b}}{2}, \tag{4.5}$$

  where $\mathcal{D}_0(\eta)$ *is defined in* (1.5)*;*

- $\boldsymbol{A}_{[0]}(\eta,\psi;x,\xi) \in \left(\Sigma\Gamma^0_{K,0,1}[\epsilon_0,N]\right)^{2\times 2}$ *satisfies* (3.12) *and is explicitly defined as*

$$\boldsymbol{A}_{[0]}(\eta,\psi;x,\xi) \triangleq \begin{bmatrix} A^{\eta}_{[0]}(\eta,\psi;x,\xi) & A^{\eta}_{[-1]}(\eta,\psi;x,\xi) \\ A^{\psi}_{[0]}(\eta,\psi;x,\xi) & A^{\psi}_{[-1]}(\eta,\psi;x,\xi) \end{bmatrix},$$

  with $A^{\mathtt{u}}_{[m]}(\eta,\psi;x,\xi) \in \Sigma\Gamma^m_{K,0,1}[\epsilon_0,N]$ *for* $\mathtt{u}=\eta,\psi$ *and* $m\in\mathbb{R}$*;*

- $\boldsymbol{R}(\eta,\psi) \in \left(\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[\epsilon_0,N]\right)^{2\times 2}$ *and is real-valued.*

Remark 4.3. Notice that the quasilinear contribution $-\left(\frac{\mathtt{b}^2}{2}+w_{\mathtt{b}}(\eta,\psi;x)\right)|\xi| = \frac{W^2_{\mathtt{b}}(\eta,\psi;x)}{2}|\xi|$ is not nil when $\mathtt{b}=0$, as it is evident from Eqs. (4.68) and (4.69). This term, in particular, is an unstable contribution that is not present in the one-phase version of the present system, cf. [1, 18, 25].

Notation 4.4. Along this section, for $\eta \in B^K_{s_0,\mathbb{R}}(I;\epsilon_0)$ and $W \triangleq \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \left(B^K_{s_0,\mathbb{R}}(I;\epsilon_0)\right)^2$, we use the notation

1. For any $x,z\in\mathbb{T}$ (cf. (1.3))

$$r=r(x)=r(\eta;x)=\sqrt{1+2\eta(x)}\in\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N], \qquad \delta_z\eta \triangleq \eta(x)-\eta(x-z)\in\widetilde{K\mathcal{F}}^1_1; \tag{4.6}$$

2. $V(W;x)$ is a generic element in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]$ (cf. Definition 3.3) and $V^n(W;x,z)$ is a generic element in $\Sigma\mathcal{F}^n_{K,0,1}[\epsilon_0,N]$, $n>-1$ (cf. Definition 3.21);
3. $A_{[m]}(W;x,\xi)$ is a generic element in $\Sigma\Gamma^m_{K,0,1}[\epsilon_0,N]$ for $m\in\mathbb{R}$ that satisfies (3.12) and $\boldsymbol{A}_{[m]}(W;x,\xi)$ is a generic element in $\left(\Sigma\Gamma^m_{K,0,1}[\epsilon_0,N]\right)^{2\times 2}$ for $m\in\mathbb{R}$ whose entries satisfy (3.12);
4. $R(W)$ is a generic element in $\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[\epsilon_0,N]$ which is real-valued and $R(W;z)$ is a generic element in $\Sigma\mathcal{R}^{-\varrho,n}_{K,0,1}[\epsilon_0,N]$, $n>-1$ which is real-valued. Similarly $\boldsymbol{R}(W)$ is a generic element in $\left(\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[\epsilon_0,N]\right)^{2\times 2}$ which is real-valued and $\boldsymbol{R}(W;z)$ is a generic element in $\Sigma\mathcal{R}^{-\varrho,n}_{K,0,1}[\epsilon_0,N]$, $n>-1$ which is real-valued.

Remark 4.5. Accordingly to the notation introduced in Notation 4.4 we write

$$\boldsymbol{R}(W)\,W\cdot\begin{bmatrix}1\\1\end{bmatrix} = R_1(W)\,w_1+R_2(W)\,w_2,$$

where R_1 and R_2 are elements of the space $\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[\epsilon_0,N]$. The same holds when we write $\boldsymbol{R}(W;z)\,W\cdot\begin{bmatrix}1\\1\end{bmatrix}$.

Notice that from Equation (1.5) we derive the relation

$$\mathcal{H}(\eta)\,\omega = \eta_x\,\mathcal{D}_0(\eta)\,\omega+\mathcal{H}_0(\eta)\,\omega, \tag{4.7}$$

where

$$\mathcal{H}_0(\eta)\,\omega \triangleq \int_{\mathbb{T}} \frac{\sqrt{1+2\eta(x)}\sqrt{1+2\eta(y)}\sin(x-y)}{1+\eta(x)+\eta(y)-\sqrt{1+2\eta(x)}\sqrt{1+2\eta(y)}\cos(x-y)}\,\omega(y)\,\mathrm{d}y. \tag{4.8}$$

4.1 Paralinearization of $\mathcal{H}_0(\eta)$

The present section is dedicated to paralinearize the nonlocal operator $\mathcal{H}_0(\eta)$ given in (4.8).

Proposition 4.6. *Let $N \in \mathbb{N}$ and $\varrho \geqslant 0$, for any $K \in \mathbb{N}$ there exist $s_0 > 0$ and $\epsilon_0 > 0$ such that if $\eta, g \in B^K_{s_0,\mathbb{R}}(I;\epsilon_0)$, the nonlinear operator $\mathcal{H}_0(\eta)$ in (4.8) admits the following paralinearization*

1.

$$\begin{aligned}\mathcal{H}_0(\eta)\, g = {} & \mathrm{Op}^{\mathrm{BW}}\left(-\mathrm{i}\left(1+\mathsf{K}^0(\eta;x)\right)\operatorname{sgn}\xi + A_{[-2]}(\eta;x,\xi)\right) g \\ & + \mathrm{Op}^{\mathrm{BW}}\left(\frac{g}{1+2\eta}\mathsf{K}'^0(\eta;x)\,|\xi| + A_{[0]}(\eta,g;x,\xi)\right)\eta + \boldsymbol{R}(\eta,g)\begin{bmatrix}\eta\\ g\end{bmatrix}\cdot\begin{bmatrix}1\\ 1\end{bmatrix},\end{aligned} \tag{4.9}$$

where

$$\begin{aligned}\mathsf{K}^0(\eta;x) &\triangleq -\frac{2\eta_x^2}{(1+2\eta)^2+\eta_x^2} \in \mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N],\\ \mathsf{K}'^0(\eta;x) &\triangleq -\frac{4\eta_x(1+2\eta)^3}{\left((1+2\eta)^2+\eta_x^2\right)^2} \in \mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N],\end{aligned} \tag{4.10}$$

$A_{[m]}(\eta,g;x,\xi) \in \Sigma\Gamma^m_{K,0,1}[\epsilon_0,N]$ satisfies (3.12) *and $\boldsymbol{R}(\eta,g) \in \left(\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[\epsilon_0,N]\right)^{2\times 2}$ is real-valued.*

2.

$$\mathcal{H}_0(\eta)[1] = \mathrm{Op}^{\mathrm{BW}}\left(\frac{\mathsf{K}'^0(\eta;x)}{1+2\eta}\;|\xi| + A_{[0]}(\eta;x,\xi)\right)\eta + R(\eta)\,\eta, \tag{4.11}$$

where

- $\mathsf{K}'^0(\eta;x) \in \mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]$ *and is explicitly defined in* (4.10)*;*
- $A_{[m]}(\eta;x,\xi) \in \Sigma\Gamma^m_{K,0,1}[\epsilon_0,N]$ *satisfying* (3.12)*;*
- $R(\eta) \in \Sigma\mathcal{R}^{-\varrho}_{K,0,1}[\epsilon_0,N]$ *and is real-valued.*

Proof.

Part 1 (Proof of Item 1)**.** With the notation introduced in (4.6) we can rewrite (4.8) as

$$\mathcal{H}_0(\eta)\, g = \int_{\mathbb{T}} \mathsf{G}_z\left(\frac{\delta_z\eta}{r^2}\right) g(x-z)\,\mathrm{d}z = \mathcal{H}g + \int_{\mathbb{T}}\left(\mathsf{G}_z\left(\frac{\delta_z\eta}{r^2}\right) - \mathsf{G}_z(0)\right) g(x-z)\,\mathrm{d}z, \tag{4.12}$$

where

$$\mathsf{G}_z(\mathsf{X}) \triangleq \frac{\sqrt{1-2\mathsf{X}}\sin z}{1-\mathsf{X}-\sqrt{1-2\mathsf{X}}\cos z}.$$

Let us now define the desingularization of G_z

$$\mathsf{K}_z(\mathsf{X}) \triangleq \mathsf{G}_z(\mathsf{X}\,2\sin(z/2))\;2\tan(z/2) = \frac{\sqrt{1-4\mathsf{X}\,\sin(z/2)}\,(2\sin(z/2))^2}{1-2\mathsf{X}\,\sin(z/2)-\sqrt{1-4\mathsf{X}\,\sin(z/2)}\cos z}, \tag{4.13}$$

so that

$$\left(\mathsf{G}_z\left(\frac{\delta_z\eta}{r^2}\right)-\mathsf{G}_z(0)\right) g(x-z) = \left(\mathsf{K}_z\left(\frac{\Delta_z\eta}{r^2}\right)-2\right)\frac{g(x-z)}{2\tan(z/2)}. \tag{4.14}$$

Notice that from (4.13) we derive

$$\mathsf{K}'_z(\mathsf{X}) = -\frac{\mathsf{X}(2\sin(z/2))^4}{\left(1-2\mathsf{X}\sin(z/2)-\sqrt{1-4\mathsf{X}\sin(z/2)}\cos(z)\right)^2\sqrt{1-4\mathsf{X}\sin(z/2)}}. \tag{4.15}$$

We need the following technical result whose proof is postponed at page 34:

Lemma 4.7. *Let* $\mathsf{K}_z(\mathsf{X})$*, be as in Eq.* (4.13)*. Then*

$$\mathsf{K}_z\left(\frac{\Delta_z\eta}{1+2\eta}\right)-2\in\Sigma K\mathcal{F}^0_{K,0,1}[\epsilon_0,N], \qquad \mathsf{K}'_z\left(\frac{\Delta_z\eta}{1+2\eta}\right)\in\Sigma K\mathcal{F}^0_{K,0,1}[\epsilon_0,N], \tag{4.16}$$

are Kernel functions, which admit the expansions

$$\begin{aligned}\mathsf{K}_z\left(\frac{\Delta_z\eta}{1+2\eta}\right)-2&=\mathsf{K}^0(\eta;x)+\mathsf{K}^1(\eta;x)\ 2\tan(z/2)+V^2(\eta;x,z),\\ \mathsf{K}'_z\left(\frac{\Delta_z\eta}{1+2\eta}\right)&=\mathsf{K}'^0(\eta;x)+\mathsf{K}'^1(\eta;x)\ \sin z+V^2(\eta;x,z),\end{aligned} \tag{4.17}$$

where $\mathsf{K}^0(\eta;x)$ *and* $\mathsf{K}'^0(\eta;x)$ *are functions in* $\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]$ *having the expressions* (4.10)*.Then,* $\mathsf{K}^1(\eta;x)$ *and* $\mathsf{K}'^1(\eta;x)$ *are functions in* $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]$ *while* $V^2(\eta;x,z)$ *are Kernel-functions in* $\Sigma K\mathcal{F}^2_{K,0,1}[\epsilon_0,N]$ *as per Notation 4.4.*

Bony paraproduct decomposition (cf. Lemma 3.16) give us that

$$\begin{aligned}\left(\mathsf{K}_z\left(\frac{\Delta_z\eta}{r^2}\right)-2\right)g(x-z)&=\mathrm{Op}^{\mathrm{BW}}\left(\mathsf{K}_z\left(\frac{\Delta_z\eta}{r^2}\right)-2\right)g(x-z)+\mathrm{Op}^{\mathrm{BW}}\left(g(x-z)\right)\left[\mathsf{K}_z\left(\frac{\Delta_z\eta}{r^2}\right)-2\right]\\&\quad+R_1\left(\mathsf{K}_z\left(\frac{\Delta_z\eta}{r^2}\right)-2\right)g(x-z)+R_2\left(g(x-z)\right)\left[\mathsf{K}_z\left(\frac{\Delta_z\eta}{r^2}\right)-2\right].\end{aligned} \tag{4.18}$$

In view of (4.16) we can apply Proposition 3.26, Items 2 and 4 and obtain that

$$R_1\left(\mathsf{K}_z\left(\frac{\Delta_z\eta}{r^2}\right)-2\right)g(x-z)=R(\eta;z)\,g, \qquad R_2\left(g(x-z)\right)\left[\mathsf{K}_z\left(\frac{\Delta_z\eta}{r^2}\right)-2\right]=R(\eta;z)\,g, \tag{4.19}$$

for suitable $R(\bullet;z)\in\Sigma K\mathcal{R}^{-\varrho,0}_{K,0,1}[\epsilon_0,N]$. We use now Bony paralinearization formula of Lemma 3.19 and Bony paraproduct decomposition and obtain that

$$\begin{aligned}&\mathsf{K}_z\left(\frac{\Delta_z\eta}{r^2}\right)-2\\&=\mathrm{Op}^{\mathrm{BW}}\left(\mathsf{K}'_z\left(\frac{\Delta_z\eta}{r^2}\right)\right)\left[\mathrm{Op}^{\mathrm{BW}}(r^{-2})\Delta_z\eta+\mathrm{Op}^{\mathrm{BW}}(\Delta_z\eta)\left[r^{-2}-1\right]+R_1\left(r^{-2}-1\right)\Delta_z\eta+R_2\left(\Delta_z\eta\right)\left[r^{-2}-1\right]\right]\\&\quad+R\left(\frac{\Delta_z\eta}{r^2}\right)\frac{\Delta_z\eta}{r^2}.\end{aligned} \tag{4.20}$$

Then, we apply again Lemma 3.19, Proposition 3.26, Items 2 and 4 in order to get that

$$\mathrm{Op}^{\mathrm{BW}}\left(\mathsf{K}'_z\left(\frac{\Delta_z\eta}{r^2}\right)\right)\left(R_1\left(r^{-2}-1\right)\Delta_z\eta+R_2\left(\Delta_z\eta\right)\left[r^{-2}-1\right]\right)=R(\eta;z)\,\eta, \tag{4.21}$$

$$R\left(\frac{\Delta_z\eta}{r^2}\right)\frac{\Delta_z\eta}{r^2}=R(\eta;z)\,\eta. \tag{4.22}$$

Besides, combining Proposition 3.26, Item 1 and Lemma 3.19 we obtain that

$$\begin{aligned}&\mathrm{Op}^{\mathrm{BW}}\left(\mathsf{K}'_z\left(\frac{\Delta_z\eta}{r^2}\right)\right)\left(\mathrm{Op}^{\mathrm{BW}}(r^{-2})\Delta_z\eta+\mathrm{Op}^{\mathrm{BW}}(\Delta_z\eta)\left[r^{-2}-1\right]\right)\\&=\mathrm{Op}^{\mathrm{BW}}\left(r^{-2}\mathsf{K}'_z\left(\frac{\Delta_z\eta}{r^2}\right)\right)\Delta_z\eta+\mathrm{Op}^{\mathrm{BW}}\left(\tilde{V}^0(\eta;x,z)\right)\eta+R(\eta,g;z)\,\eta,\end{aligned} \tag{4.23}$$

where

$$\tilde{V}^0(\eta;x,z)\triangleq r^{-2}\left(K'_z(X)X\right)|_{X=\frac{\Delta_z\eta}{r^2}}.$$

We plug Eqs. (4.21) and (4.23) in Eq. (4.20) and the resulting equation and Eq. (4.19) in Eq. (4.18) and obtain, after using

$$\int_{\mathbb{T}}\frac{1}{\tan(z/2)}\mathrm{d}z=0$$

and applying Proposition 3.26, Item 1, that

$$\left(\mathsf{K}_z\left(\frac{\Delta_z\eta}{r^2}\right)-2\right)g(x-z)=\mathrm{Op}^{\mathrm{BW}}\left(\mathsf{K}_z\left(\frac{\Delta_z\eta}{r^2}\right)-2\right)g(x-z)+\mathrm{Op}^{\mathrm{BW}}\left(\frac{g(x-z)}{r^2}\mathsf{K}'_z\left(\frac{\Delta_z\eta}{r^2}\right)\right)\Delta_z\eta \\ +\mathrm{Op}^{\mathrm{BW}}\left(V^0(\eta,g;x,z)\right)\eta+\boldsymbol{R}(\eta,g;z)\begin{bmatrix}\eta\\ g\end{bmatrix}\cdot\begin{bmatrix}1\\ 1\end{bmatrix}, \tag{4.24}$$

where

$$V^0(\eta,g;x,z)\triangleq\tilde{V}^0(\eta;x,z)g(x-z).$$

We Taylor expand in z

$$V^0(\eta,g;x,z)=\overline{V}_0(\eta,g;x)+\overline{V}_1(\eta,g;x,z),\qquad \overline{V}_1\in\Sigma K\mathcal{F}^1_{K,0,1}[\epsilon_0,N].$$

We thus plug (4.24) in (4.14) and insert the resulting equation in (4.12) and, after application of Remark 3.22 and Lemma 3.27 we obtain that

$$\mathcal{H}_0(\eta)g=\mathcal{H}g+\int_{\mathbb{T}}\mathrm{Op}^{\mathrm{BW}}\left(\mathsf{K}_z\left(\frac{\Delta_z\eta}{r^2}\right)-2\right)\frac{g(x-z)}{2\tan(z/2)}\mathrm{d}z \\ +\int_{\mathbb{T}}\mathrm{Op}^{\mathrm{BW}}\left(\frac{g(x-z)}{r^2}\mathsf{K}'_z\left(\frac{\Delta_z\eta}{r^2}\right)\right)\frac{\delta_z\eta}{4\sin(z/2)\,\tan(z/2)}\mathrm{d}z \\ +\mathrm{Op}^{\mathrm{BW}}\left(V(\eta,g;x)\right)\eta+\boldsymbol{R}(\eta,g;z)\begin{bmatrix}\eta\\ g\end{bmatrix}\cdot\begin{bmatrix}1\\ 1\end{bmatrix}, \tag{4.25}$$

where

$$V(\eta,g;x)\triangleq\int_{\mathbb{T}}\frac{\overline{V}_1(\eta,g;x,z)}{2\tan(z/2)}\mathrm{d}z.$$

We use now the identity $\frac{1}{4\sin(z/2)\,\tan(z/2)}=\frac{1}{4\sin^2(z/2)}-\frac{1}{8\cos^2(z/4)}$ and Proposition 3.28 and Remark 3.22 in order to transform

$$\int_{\mathbb{T}}\mathrm{Op}^{\mathrm{BW}}\left(\frac{g(x-z)}{r^2}\mathsf{K}'_z\left(\frac{\Delta_z\eta}{r^2}\right)\right)\frac{\delta_z\eta}{4\sin(z/2)\,\tan(z/2)}\mathrm{d}z \\ =\int_{\mathbb{T}}\mathrm{Op}^{\mathrm{BW}}\left(\frac{g(x-z)}{r^2}\mathsf{K}'_z\left(\frac{\Delta_z\eta}{r^2}\right)\right)\frac{\delta_z\eta}{4\sin^2(z/2)}\mathrm{d}z+\mathrm{Op}^{\mathrm{BW}}\left(V(\eta,g;x)+A_{[-2]}(\eta,g;x,\xi)\right)\eta+R(\eta,g)\eta, \tag{4.26}$$

so that plugging (4.26) in (4.25) we obtain that

$$\mathcal{H}_0(\eta)g=\mathcal{H}g+\int_{\mathbb{T}}\mathrm{Op}^{\mathrm{BW}}\left(\mathsf{K}_z\left(\frac{\Delta_z\eta}{r^2}\right)-2\right)\frac{g(x-z)}{2\tan(z/2)}\mathrm{d}z \\ +\int_{\mathbb{T}}\mathrm{Op}^{\mathrm{BW}}\left(\frac{g(x-z)}{r^2}\mathsf{K}'_z\left(\frac{\Delta_z\eta}{r^2}\right)\right)\frac{\delta_z\eta}{4\sin^2(z/2)}\mathrm{d}z \\ +\mathrm{Op}^{\mathrm{BW}}\left(V(\eta,g;x)+A_{[-2]}(\eta,g;x,\xi)\right)\eta+\boldsymbol{R}(\eta,g;z)\begin{bmatrix}\eta\\ g\end{bmatrix}\cdot\begin{bmatrix}1\\ 1\end{bmatrix}. \tag{4.27}$$

Now, we can use the Taylor-like expansions (4.17), Proposition 3.28 and the fact that g has zero average and obtain that

$$\int_{\mathbb{T}}\mathrm{Op}^{\mathrm{BW}}\left(\mathsf{K}_z\left(\frac{\Delta_z\eta}{r^2}\right)-2\right)\frac{g(x-z)}{2\tan(z/2)}\mathrm{d}z=\mathrm{Op}^{\mathrm{BW}}\left(\mathsf{K}^0(\eta;x)\right)\mathcal{H}g+\mathrm{Op}^{\mathrm{BW}}\left(A_{[-2]}(\eta;x,\xi)\right)g+R(\eta,g)g. \tag{4.28}$$

Next, denoting

$$\tilde{\mathsf{K}}(\eta,g;x,z)\triangleq\frac{g(x-z)}{1+2\eta}\mathsf{K}'_z\left(\frac{\Delta_z\eta}{1+2\eta}\right)-\frac{g}{1+2\eta}\mathsf{K}'^0(\eta;x)\in\Sigma K\mathcal{F}^1_{K,0,1}[\epsilon_0,N],$$

we Taylor-expand $z\mapsto\tilde{\mathsf{K}}(\eta,g;x,z)$ obtaining

$$\tilde{\mathsf{K}}(\eta,g;x,z)=\tilde{\mathsf{K}}^1(\eta,g;x)\sin z+V^2(\eta,g;x,z),$$

so that applying Remark 3.22 and Proposition 3.28 we obtain that

$$\int_{\mathbb{T}} \mathrm{Op}^{\mathrm{BW}}\left(\frac{g(x-z)}{r^2}\mathsf{K}_z'\left(\frac{\Delta_z\eta}{r^2}\right)\right)\frac{\delta_z\eta}{4\sin^2(z/2)}\mathrm{d}z$$
$$= \mathrm{Op}^{\mathrm{BW}}\left(\frac{g}{1+2\eta}\mathsf{K}'^0(\eta;x)\right)|D|\eta + \mathrm{Op}^{\mathrm{BW}}\left(\tilde{\mathsf{K}}^1(\eta,g;x)\right)\mathcal{H}\eta + \mathrm{Op}^{\mathrm{BW}}\left(V(\eta,g;x) + A_{[-2]}(\eta,g;x,\xi)\right)\eta + R(\eta,g)\eta. \tag{4.29}$$

We plug Eqs. (4.28) and (4.29) in Eq. (4.27) and use Proposition 3.15 and obtain the desired quasi-linear expansion

$$\begin{aligned}\mathcal{H}_0(\eta)g = {}& \mathrm{Op}^{\mathrm{BW}}\left(1+\mathsf{K}^0(\eta;x)\right)\mathcal{H}g + \mathrm{Op}^{\mathrm{BW}}\left(\frac{g}{1+2\eta}\mathsf{K}'^0(\eta;x)\right)|D|\eta \\ & + \mathrm{Op}^{\mathrm{BW}}\left(A_{[0]}(\eta,g;x,\xi)\right)\eta + \mathrm{Op}^{\mathrm{BW}}\left(A_{[-2]}(\eta;x,\xi)\right)g + \boldsymbol{R}(\eta,g;z)\begin{bmatrix}\eta\\ g\end{bmatrix}\cdot\begin{bmatrix}1\\1\end{bmatrix},\end{aligned} \tag{4.30}$$

thus (4.9) can be derive applying Proposition 3.15 to (4.30) combined with $\mathcal{H} = \mathrm{Op}^{\mathrm{BW}}\left(-\mathrm{i}\,\mathrm{sgn}\,\xi\right)$.

Part 2 (Proof of Item 2)**.** From Eqs. (4.12) and (4.14) we have that

$$\mathcal{H}_0(\eta)[1] = \int_{\mathbb{T}}\left(\mathsf{K}_z\left(\frac{\Delta_z\eta}{r^2}\right)-2\right)\frac{\mathrm{d}z}{2\tan(z/2)}. \tag{4.31}$$

Thus, we apply Lemma 3.19 and obtain that

$$\mathsf{K}_z\left(\frac{\Delta_z\eta}{r^2}\right)-2 = \mathrm{Op}^{\mathrm{BW}}\left(\mathsf{K}_z'\left(\frac{\Delta_z\eta}{r^2}\right)\right)\left[\frac{\Delta_z\eta}{r^2}\right] + R\left(\frac{\Delta_z\eta}{r^2}\right)\left[\frac{\Delta_z\eta}{r^2}\right]. \tag{4.32}$$

Notice that from Proposition 3.26, Item 4 we have that $R\left(\frac{\Delta_z\eta}{r^2}\right) \in \Sigma K\mathcal{R}^{-\varrho,0}_{K,0,1}[\epsilon_0,N]$ and by Taylor expansion we have that

$$\tilde{R}(\eta;x,z) \triangleq R\left(\frac{\Delta_z\eta}{r^2}\right) - R\left(\frac{\eta_x}{r^2}\right) \in \Sigma K\mathcal{R}^{-\varrho,1}_{K,0,1}[\epsilon_0,N],$$

and, since $\frac{\Delta_z}{r^2} \in \Sigma K\mathcal{M}^{1,0}_{K,0,1}[\epsilon_0,N]$ applying Proposition 3.26, Item 2 we obtain that

$$\tilde{R}(\eta;x,z)\circ\frac{\Delta_z}{r^2} \in \Sigma K\mathcal{R}^{-\varrho+1,1}_{K,0,1}[\epsilon_0,N], \tag{4.33}$$

so that, thanks to (4.33), we have that

$$\int_{\mathbb{T}} R\left(\frac{\Delta_z\eta}{r^2}\right)\left[\frac{\Delta_z\eta}{r^2}\right]\frac{\mathrm{d}z}{2\tan(z/2)} = R\left(\frac{\eta_x}{r^2}\right)\left[\frac{1}{r^2}\int_{\mathbb{T}}\frac{\delta_z\eta}{4\sin(z/2)\tan(z/2)}\mathrm{d}z\right] + \int_{\mathbb{T}} R(\eta;x,z)\,\mathrm{d}z\,\eta.$$

Next we use the fact that

$$\int_{\mathbb{T}}\frac{\delta_z\eta}{4\sin(z/2)\tan(z/2)}\mathrm{d}z = m_1(D)\eta, \qquad \text{with} \qquad m_1(\xi)\in\widetilde{\Gamma}^1_0,$$

the fact that $r^{-2}m_1(D) \in \Sigma K\mathcal{M}^{1,0}_{K,0,1}[\epsilon_0,N]$, Proposition 3.26, Item 2 and Lemma 3.27 to obtain that

$$\int_{\mathbb{T}} R\left(\frac{\Delta_z\eta}{r^2}\right)\left[\frac{\Delta_z\eta}{r^2}\right]\frac{\mathrm{d}z}{2\tan(z/2)} = R(\eta)\eta. \tag{4.34}$$

Next, computations similar to the ones performed in Eqs. (4.20) and (4.23) allow us to deduce that

$$\begin{aligned}\int_{\mathbb{T}} \mathrm{Op}^{\mathrm{BW}}\left(\mathsf{K}_z'\left(\frac{\Delta_z\eta}{r^2}\right)\right)\left[\frac{\Delta_z\eta}{r^2}\right]\frac{\mathrm{d}z}{2\tan(z/2)} = {}& \int_{\mathbb{T}} \mathrm{Op}^{\mathrm{BW}}\left(r^{-2}\mathsf{K}_z'\left(\frac{\Delta_z\eta}{r^2}\right)\right)\left[\frac{\delta_z\eta}{4\sin(z/2)\tan(z/2)}\right]\mathrm{d}z \\ & + \mathrm{Op}^{\mathrm{BW}}\left(V(\eta;x)\right)\eta + R(\eta)\eta.\end{aligned} \tag{4.35}$$

Taylor-expanding as in (4.29) and using Proposition 3.15 we obtain that

$$\begin{aligned}&\int_{\mathbb{T}} \mathrm{Op}^{\mathrm{BW}}\left(r^{-2}\mathsf{K}_z'\left(\frac{\Delta_z\eta}{r^2}\right)\right)\left[\frac{\delta_z\eta}{4\sin(z/2)\tan(z/2)}\right]\mathrm{d}z \\ &= \mathrm{Op}^{\mathrm{BW}}\left(r^{-2}\mathsf{K}'^0(\eta;x)\right)|D|\eta + \mathrm{Op}^{\mathrm{BW}}\left(A_{[0]}(\eta;x,\xi)\right)\eta + R(\eta)\eta \\ &= \mathrm{Op}^{\mathrm{BW}}\left(r^{-2}\mathsf{K}'^0(\eta;x)|\xi| + A_{[0]}(\eta;x,\xi)\right)\eta + R(\eta)\eta.\end{aligned} \tag{4.36}$$

Combining Equations (4.32) and (4.34) to (4.36) we obtain (4.11).

□

Proof of Lemma 4.7. Let us prove at first (4.16). We prove the result for K_z only being the procedure for K'_z the same. Notice that from Eq. (4.13) we immediately have that the map $(z,\mathsf{X}) \mapsto \mathsf{K}_z(\mathsf{X})$ is analytic in $\mathbb{T}\times\left(-\frac{1}{4},\frac{1}{4}\right)$. Let us now define

$$\overline{\mathsf{K}}_z(\mathsf{x},\mathsf{y}) \triangleq \mathsf{K}_z\left(\frac{\mathsf{y}}{1+2\mathsf{x}}\right),$$

which is again analytic for $|\mathsf{x}|,|\mathsf{y}|<\varepsilon_1$ small. By analyticity we have that

$$\overline{\mathsf{K}}_z(\mathsf{x},\mathsf{y}) = \sum_{p_1,p_2=0}^{\infty} \underbrace{\frac{\partial_{\mathsf{x}}^{p_2}\partial_{\mathsf{y}}^{p_1}\overline{\mathsf{K}}_z(0,0)}{p_1!p_2!}}_{\triangleq \mathsf{k}_{p_1,p_2}(z)} \mathsf{x}^{p_2}\mathsf{y}^{p_1},$$

with

$$\left|\partial_{\mathsf{x}}^{p_2}\partial_{\mathsf{y}}^{p_1}\overline{\mathsf{K}}_z(0,0)\right| \leqslant C p_1!p_2!\,\varepsilon_1^{-(p_1+p_2)}.$$

From Eq. (4.13) it is immediate that

$$\mathsf{K}_z(0)=2,$$

so that

$$\begin{aligned}\mathsf{K}_z\left(\frac{\Delta_z\eta}{1+2\eta}\right) = \overline{\mathsf{K}}_z(\eta,\Delta_z\eta) &= 2+\sum_{p\geqslant 1}\underbrace{\sum_{\substack{p_1\geqslant 1\\ p_1+p_2=p}} \mathsf{k}_{p_1,p_2}(z)\;\eta^{p_2}\left(\Delta_z\eta\right)^{p_1}}_{=:\widetilde{\mathsf{K}}^p(f;x,z)}\\ &= 2+\sum_{p=1}^{N}\widetilde{\mathsf{K}}^p(\eta;x,z)+\underbrace{\sum_{p>N}\widetilde{\mathsf{K}}^p(\eta;x,z)}_{=:\mathsf{K}^{>N}(\eta;x,z)}.\end{aligned}$$

We claim that for any $p\in\mathbb{N}$ and $\ell=0,\ldots,7$,

$$\partial_z^{\ell}\widetilde{\mathsf{K}}^p(\eta;x,z) \in \widetilde{K\mathcal{F}}^0_p, \tag{4.37}$$

$$\partial_z^{\ell}\mathsf{K}^{>N}(\eta;x,z) \in K\mathcal{F}^0_{K,0,N+1}[\epsilon_0]. \tag{4.38}$$

The proof of Eqs. (4.37) and (4.38) is the same as the one performed in order to prove [17, Eqs. (4.34) and (4.35)] and is thus omitted. The proof for (4.16) for $\mathsf{K}'_z\left(\frac{\Delta_z\eta}{1+2\eta}\right)$ is the same as the one performed for $\mathsf{K}_z\left(\frac{\Delta_z\eta}{1+2\eta}\right)$ with the sole difference that $\mathsf{K}'_z(0)=0$.
The proof of (4.17) follows the lines of the proof of [17, Eq. (4.23)], and is generic enough to be applied in the present case. It remains to prove Eq. (4.10), we can Taylor-expand in $z=0$ and obtain that

$$\begin{aligned}\mathsf{K}_z(\mathsf{X}) &= \mathsf{K}_0(\mathsf{X})+\partial_z\mathsf{K}_z(\mathsf{X})|_{z=0}\,z+R_1(\mathsf{K};\mathsf{X})(z),\\ \mathsf{K}'_z(\mathsf{X}) &= \mathsf{K}'_0(\mathsf{X})+\partial_z\mathsf{K}'_z(\mathsf{X})\big|_{z=0}\,z+R_1\left(\mathsf{K}';\mathsf{X}\right)(z).\end{aligned}$$

Explicit computations show that

$$\begin{aligned}\mathsf{K}_0(\mathsf{X}) &\triangleq 2-\frac{2\mathsf{X}^2}{1+\mathsf{X}^2}, &\quad \partial_z\mathsf{K}_z(\mathsf{X})|_{z=0} &\triangleq -\frac{4\mathsf{X}^3}{\left(1+\mathsf{X}^2\right)^2},\\ \mathsf{K}'_0(\mathsf{X}) &\triangleq -\frac{4\mathsf{X}}{\left(1+\mathsf{X}^2\right)^2}, &\quad \partial_z\mathsf{K}'_z(\mathsf{X})\big|_{z=0} &\triangleq -\frac{4\mathsf{X}^2\left(3-\mathsf{X}^2\right)}{\left(1+\mathsf{X}^2\right)^3}.\end{aligned}$$

Hence setting $\mathsf{X} \triangleq \frac{\Delta_z \eta}{1+2\eta}$ and Taylor expanding we obtain that

$$\mathsf{K}_0\left(\frac{\Delta_z\eta}{1+2\eta}\right) = \underbrace{2-2\frac{\eta_x^2}{(1+2\eta)^2+\eta_x^2}}_{\triangleq \mathsf{K}^0(\eta;x)} + 4\frac{(1+2\eta)^2\eta_x\eta_{xx}}{\left((1+2\eta)^2+\eta_x^2\right)^2} z + V^2(\eta;x,z),$$

$$\mathsf{K}_0'\left(\frac{\Delta_z\eta}{1+2\eta}\right) = \underbrace{-\frac{4(1+2\eta)\eta_x}{\left((1+2\eta)^2+\eta_x^2\right)^2}}_{\triangleq \mathsf{K}'^0(\eta;x)} + 2\frac{\left((1+2\eta)^2-3\eta_x^2\right)(1+2\eta)^3\eta_{xx}}{\left((1+2\eta)^2+\eta_x^2\right)^3} z + V^2(\eta;x,z),$$

$$\partial_z \mathsf{K}_z\left(\frac{\Delta_x\eta}{1+2\eta}\right)\Big|_{z=0} \triangleq -\frac{4(1+2\eta)\eta_x^3}{\left((1+2\eta)^2+\eta_x^2\right)^2} + R_0\left(\partial_z\mathsf{K}_z\left(\frac{\Delta_x\eta}{1+2\eta}\right)\Big|_{z=0};x\right)(z),$$

$$\partial_z \mathsf{K}_z'\left(\frac{\Delta_x\eta}{1+2\eta}\right)\Big|_{z=0} \triangleq -\frac{4(1+2\eta)^2\eta_x^2\left(3(1+2\eta)^2-\eta_x^2\right)}{\left((1+2\eta)^2+\eta_x^2\right)^3} + R_0\left(\partial_z\mathsf{K}_z'\left(\frac{\Delta_x\eta}{1+2\eta}\right)\Big|_{z=0};x\right)(z),$$

so that defining

$$\mathsf{K}^1(\eta;x) \triangleq \frac{4(1+2\eta)^2\eta_x\eta_{xx}}{\left((1+2\eta)^2+\eta_x^2\right)^2} - \frac{4(1+2\eta)\eta_x^3}{\left((1+2\eta)^2+\eta_x^2\right)^2} = \frac{4(1+2\eta)\eta_x\left((1+2\eta)\eta_{xx}-\eta_x^2\right)}{\left((1+2\eta)^2+\eta_x^2\right)^2},$$

$$\mathsf{K}'^1(\eta;x) \triangleq \frac{2\left((1+2\eta)^2-3\eta_x^2\right)(1+2\eta)^3\eta_{xx} - 4(1+2\eta)^2\eta_x^2\left(3(1+2\eta)^2-\eta_x^2\right)}{\left((1+2\eta)^2+\eta_x^2\right)^3},$$

which are analytic applications w.r.t. η small in $W^{2,\infty}$, proving (4.17) and (4.10). ☐

4.2 Paralinearization of $\mathcal{D}_0(\eta)$

Here we perform the paralinearization of the nonlinear operator $\mathcal{D}_0(\eta)$ introduced in (1.5).

Proposition 4.8. *Let $N \in \mathbb{N}$ and $\varrho \geqslant 0$, for any $K \in \mathbb{N}$ there exists $s_0 > 0$ and $\epsilon_0 > 0$ such that if $\eta, g \in B^K_{s_0,\mathbb{R}}(I;\epsilon_0)$ and $\mathcal{D}_0(\eta)$ be as in* (1.5)*, we have that*

1.

$$\mathcal{D}_0(\eta)g = \mathrm{Op}^{\mathrm{BW}}\left(-\mathrm{i}\,\frac{\mathsf{J}^0(\eta;x)}{r^2}\,\mathrm{sgn}\,\xi + A_{[-2]}(\eta;x,\xi)\right)g + \mathrm{Op}^{\mathrm{BW}}\left(\frac{g}{r^4}\mathsf{J}'^0(\eta;x)|\xi| + A_{[0]}(\eta,g;x,\xi)\right)\eta + \boldsymbol{R}(\eta,g)\begin{bmatrix}\eta\\ g\end{bmatrix}\cdot\begin{bmatrix}1\\1\end{bmatrix}, \tag{4.39}$$

where

$$\mathsf{J}^0(\eta;x) \triangleq \frac{2\eta_x(1+2\eta)}{(1+2\eta)^2+\eta_x^2} \in \mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N], \tag{4.40}$$

$$\mathsf{J}'^0(\eta;x) \triangleq \frac{2(1+2\eta)^2\left((1+2\eta)^2-\eta_x^2\right)}{\left((1+2\eta)^2+\eta_x^2\right)^2} \in \mathcal{F}^{\mathbb{R}}_{K,0,0}[\epsilon_0,N], \tag{4.41}$$

$A_{[m]}(\eta,g;x,\xi) \in \Sigma\Gamma^m_{K,0,1}[\epsilon_0,N]$ and $\boldsymbol{R}(\eta,g) \in \left(\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[\epsilon_0,N]\right)^{2\times 2}$.

2.

$$\mathcal{D}_0(\eta)[1] = 1 + \mathrm{Op}^{\mathrm{BW}}\left(-\frac{2}{(1+2\eta)^2} + \frac{\mathsf{J}'^0(\eta;x)}{r^4}|\xi| + A_{[0]}(\eta;x,\xi)\right)\eta + R(\eta)\eta, \tag{4.42}$$

where

- $\mathsf{J}'^0(\eta;x)$ *is explicitly defined in* (4.41), *and is such that* $\mathsf{J}'^0(0;x)\equiv 2$;
- $A_{[0]}(\eta;x,\xi)\in\Sigma\Gamma^0_{K,0,1}[\epsilon_0,N]$;
- $R(\eta)\in\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[\epsilon_0,N]$.

Proof.

Part 1 (Proof of Proposition 4.8, Item 1)**.** Let us recall that $\mathcal{D}_0(\eta)$ is defined in (1.5). We proceed analogously as in the previous section. We have, using the auxiliary functions defined in (4.6) and the fact that g is of zero average, that

$$\mathcal{D}_0(\eta)\,g=\int_{\mathbb{T}}\frac{1}{r^2}\left(\mathsf{H}_z\left(\frac{\delta_z\eta}{r^2}\right)-\mathsf{H}_z(0)\right)\frac{g(x-z)}{2\sin(z/2)}\mathrm{d}z, \tag{4.43}$$

where

$$\mathsf{H}_z(\mathsf{X})\triangleq\frac{1-\sqrt{1-2\mathsf{X}}\cos z}{1-\mathsf{X}-\sqrt{1-2\mathsf{X}}\cos z}\;2\sin(z/2).$$

We define

$$\mathsf{J}_z(\mathsf{X})\triangleq\mathsf{H}_z(\mathsf{X}\,2\sin(z/2)). \tag{4.44}$$

We have the following technical result:

Lemma 4.9. *Let* $\mathsf{J}_z(\mathsf{X})$, *be as in Eq.* (4.44). *Then*

$$\mathsf{J}_z\left(\frac{\Delta_z\eta}{1+2\eta}\right)_{\geqslant 1}\triangleq\mathsf{J}_z\left(\frac{\Delta_z\eta}{1+2\eta}\right)-2\sin(z/2)\in\Sigma K\mathcal{F}^0_{K,0,1}[\epsilon_0,N],\qquad \mathsf{J}'_z\left(\frac{\Delta_z\eta}{1+2\eta}\right)\in\Sigma K\mathcal{F}^0_{K,0,0}[\epsilon_0,N], \tag{4.45}$$

are Kernel functions, which admit the expansion

$$\begin{aligned}\mathsf{J}_z\left(\frac{\Delta_z\eta}{1+2\eta}\right)-2\sin(z/2)&=\mathsf{J}^0(\eta;x)+\mathsf{J}^1(\eta;x)\;2\sin(z/2)+V^2(\eta;x,z),\\ \mathsf{J}'_z\left(\frac{\Delta_z\eta}{1+2\eta}\right)&=\mathsf{J}'^0(\eta;x)+\mathsf{J}'^1(\eta;x)\sin z+V^2(\eta;x,z),\end{aligned} \tag{4.46}$$

where $\mathsf{J}^0(\eta;x)$, $\mathsf{J}'^0(\eta;x)$ *are defined in* (4.40)-(4.41) *and* $\mathsf{J}^1(\eta;x)$, $\mathsf{J}'^1(\eta;x)\in\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]$.

The proof of Lemma 4.9 follows exactly the same lines of the proof of Lemma 4.7 (cf. page 34) and is thus omitted for the sake of brevity.

We apply Lemma 3.16 and obtain that

$$\begin{aligned}\frac{1}{r^2}\mathsf{J}_z\left(\frac{\Delta_z\eta}{r^2}\right)_{\geqslant 1}g(x-z)=&\operatorname{Op}^{\mathrm{BW}}\left(\frac{1}{r^2}\mathsf{J}_z\left(\frac{\Delta_z\eta}{r^2}\right)\right)_{\geqslant 1}g(x-z)+\operatorname{Op}^{\mathrm{BW}}(g(x-z))\left[\frac{1}{r^2}\mathsf{J}_z\left(\frac{\Delta_z\eta}{1+2\eta}\right)_{\geqslant 1}\right]\\ &+R_1\left(\frac{1}{r^2}\mathsf{J}_z\left(\frac{\Delta_z\eta}{1+2\eta}\right)_{\geqslant 1}\right)g(x-z)+R_2(g(x-z))\left[\frac{1}{r^2}\mathsf{J}_z\left(\frac{\Delta_z\eta}{1+2\eta}\right)_{\geqslant 1}\right].\end{aligned} \tag{4.47}$$

In view of (4.45) we can apply Proposition 3.26, Items 2 and 4 and obtain that

$$R_1\left(\frac{1}{r^2}\mathsf{J}_z\left(\frac{\Delta_z\eta}{1+2\eta}\right)_{\geqslant 1}\right)g(x-z)=R(\eta,g;z)\,g,\qquad R_2(g(x-z))\left[\frac{1}{r^2}\mathsf{J}_z\left(\frac{\Delta_z\eta}{1+2\eta}\right)_{\geqslant 1}\right]=R(\eta,g;z)\,\eta. \tag{4.48}$$

Then, we apply Lemmas 3.16 and 3.19 and Proposition 3.15 in a similar fashion to the procedure detailed in the previous section and obtain that

$$\operatorname{Op}^{\mathrm{BW}}(g(x-z))\left[\frac{1}{r^2}\mathsf{J}_z\left(\frac{\Delta_z\eta}{1+2\eta}\right)_{\geqslant 1}\right]=\operatorname{Op}^{\mathrm{BW}}\left(\frac{g(x-z)}{r^4}\mathsf{J}'_z\left(\frac{\Delta_z\eta}{r^2}\right)\right)\Delta_z\eta+\operatorname{Op}^{\mathrm{BW}}(V^0(\eta,g;x,z))\eta+R(\eta,g;z)\,\eta. \tag{4.49}$$

Next, we use the expansions of Lemma 4.9 and obtain that

$$\begin{aligned}\operatorname{Op}^{\mathrm{BW}}\left(\frac{1}{r^2}\mathrm{J}_z\left(\frac{\Delta_z\eta}{1+2\eta}\right)_{\geqslant 1}\right)g(x-z) &= \operatorname{Op}^{\mathrm{BW}}\left(\frac{\mathrm{J}^0(\eta;x)}{r^2}+V^1(\eta,g;x,z)\right)g(x-z),\\ \operatorname{Op}^{\mathrm{BW}}\left(\frac{g(x-z)}{r^4}\mathrm{J}_z'\left(\frac{\Delta_z\eta}{r^2}\right)\right)\Delta_z\eta &= \operatorname{Op}^{\mathrm{BW}}\left(\frac{g}{r^4}\mathrm{J}'^0(\eta;x)\right)\Delta_z\eta+\operatorname{Op}^{\mathrm{BW}}\left(V(\eta,g;x)+V^1(\eta,g;x,z)\right)\delta_z\eta,\\ \operatorname{Op}^{\mathrm{BW}}\left(V^0(\eta,g;x,z)\right)\eta &= \operatorname{Op}^{\mathrm{BW}}\left(V(\eta,g;x)+V^1(\eta,g;x,z)\right)\eta.\end{aligned} \tag{4.50}$$

We plug the results in Eqs. (4.44) and (4.47) to (4.50) into Eq. (4.43) and obtain, after an application of Proposition 3.28, that

$$\begin{aligned}\mathcal{D}_0(\eta)g = \operatorname{Op}^{\mathrm{BW}}\left(\frac{\mathrm{J}^0(\eta;x)}{r^2}\right)\int_{\mathbb{T}}\frac{g(x-z)}{2\sin(z/2)}\mathrm{d}z+\operatorname{Op}^{\mathrm{BW}}\left(\frac{g}{r^4}\mathrm{J}'^0(\eta;x)\right)\int_{\mathbb{T}}\frac{\delta_z\eta}{4\sin^2(z/2)}\mathrm{d}z\\ +\operatorname{Op}^{\mathrm{BW}}\left(A_{[0]}(\eta,g;x,\xi)\right)\eta+\operatorname{Op}^{\mathrm{BW}}\left(A_{[-2]}(\eta;x,\xi)\right)g+\boldsymbol{R}(\eta,g;z)\begin{bmatrix}\eta\\ g\end{bmatrix}\cdot\begin{bmatrix}1\\ 1\end{bmatrix},\end{aligned}$$

from which we conclude after standard symbolic manipulations the identity in (4.39).

Part 2 (Proof of Proposition 4.8, Item 2)**.** Arguing similarly as it was done in order to deduce (4.43) and using Eqs. (4.44) and (4.46) we have that

$$\mathcal{D}_0(\eta)[1] = r^{-2}+\int_{\mathbb{T}}\frac{1}{r^2}\mathrm{J}_z\left(\frac{\Delta_z\eta}{r^2}\right)_{\geqslant 1}\frac{\mathrm{d}z}{2\sin(z/2)}. \tag{4.51}$$

We apply Lemma 3.19 and obtain that

$$\mathcal{D}_0(\eta)[1] = r^{-2}+\int_{\mathbb{T}}\frac{1}{r^2}\left\{\operatorname{Op}^{\mathrm{BW}}\left(\mathrm{J}_z'\left(\frac{\Delta_z\eta}{r^2}\right)\right)\left[\frac{\Delta_z\eta}{r^2}\right]+R\left(\frac{\Delta_z\eta}{r^2}\right)\left[\frac{\Delta_z\eta}{r^2}\right]\right\}\frac{\mathrm{d}z}{2\sin(z/2)}. \tag{4.52}$$

Thus, computations similar to the ones that lead to (4.34) give us that

$$\int_{\mathbb{T}}\frac{1}{r^2}R\left(\frac{\Delta_z\eta}{r^2}\right)\left[\frac{\Delta_z\eta}{r^2}\right]\frac{\mathrm{d}z}{2\sin(z/2)} = R(\eta)\eta. \tag{4.53}$$

Next, similar computations to the ones that lead to (4.35) give us that

$$\begin{aligned}\int_{\mathbb{T}}\frac{1}{r^2}\operatorname{Op}^{\mathrm{BW}}\left(\mathrm{J}_z'\left(\frac{\Delta_z\eta}{r^2}\right)\right)\left[\frac{\Delta_z\eta}{r^2}\right]\frac{\mathrm{d}z}{2\sin(z/2)} &= \int_{\mathbb{T}}\operatorname{Op}^{\mathrm{BW}}\left(r^{-4}\mathrm{J}_z'\left(\frac{\Delta_z\eta}{r^2}\right)\right)\left[\frac{\delta_z\eta}{(2\sin(z/2))^2}\right]\mathrm{d}z\\ &\quad+\operatorname{Op}^{\mathrm{BW}}\left(V(\eta;x)\right)\eta+R(\eta)\eta,\end{aligned} \tag{4.54}$$

so that arguing as in order to deduce (4.36)

$$\begin{aligned}&\int_{\mathbb{T}}\operatorname{Op}^{\mathrm{BW}}\left(r^{-4}\mathrm{J}_z'\left(\frac{\Delta_z\eta}{r^2}\right)\right)\left[\frac{\delta_z\eta}{(2\sin(z/2))^2}\right]\mathrm{d}z\\ &= \operatorname{Op}^{\mathrm{BW}}\left(r^{-4}\mathrm{J}'^0(\eta;x)\right)|D|\eta+\operatorname{Op}^{\mathrm{BW}}\left(A_{[0]}(\eta;x,\xi)\right)\eta+R(\eta)\eta\\ &= \operatorname{Op}^{\mathrm{BW}}\left(r^{-4}\mathrm{J}'^0(\eta;x)|\xi|+A_{[0]}(\eta;x,\xi)\right)\eta+R(\eta)\eta.\end{aligned} \tag{4.55}$$

Thus, combining Eqs. (4.51) to (4.55) and the paralinearization $r^{-2} = 1+\operatorname{Op}^{\mathrm{BW}}\left(-\frac{2}{(1+2\eta)^2}\right)\eta+R(\eta)\eta$ derived using Lemma 3.19 we obtain Eq. (4.42).

□

4.3 Paralinearization of $\mathcal{H}(\eta)$

In view of (4.7), putting together the paralinearizations of Sections 4.1 and 4.2, we prove the following.

Lemma 4.10. *Let $N \in \mathbb{N}$, $\mathtt{b} \in \mathbb{R}$ and $\varrho \geqslant 0$, for any $K \in \mathbb{N}$ there exists $s_0 > 0$ and $\epsilon_0 > 0$ such that if $\eta, g \in B^K_{s_0,\mathbb{R}}(I;\epsilon_0)$, let $\mathcal{H}(\eta)$ be as in* (4.7) *then we have that*

1.

$$\mathcal{H}(\eta)\,g = \mathrm{Op}^{\mathrm{BW}}\left(-\mathrm{i}\ \mathrm{sgn}\,\xi + A_{[-2]}(\eta;x,\xi)\right)g + \mathrm{Op}^{\mathrm{BW}}\left(-\frac{g}{1+2\eta}\ \mathsf{J}^0(\eta;x)\ |\xi| + 2V_0(\eta,g;x)\,\mathrm{i}\xi + A_{[0]}(\eta,g;x,\xi)\right)\eta + \boldsymbol{R}(\eta,g)\begin{bmatrix}\eta\\ g\end{bmatrix}\cdot\begin{bmatrix}1\\ 1\end{bmatrix}, \tag{4.56}$$

where

- $\mathsf{J}^0(\eta;x) \in \mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]$ *and is explicitly defined in* (4.40)*;*
- $V_0(\eta,\partial_x^{-1}g;x) \triangleq \frac{1}{2}\mathcal{D}_0(\eta)\,g\ \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]$, *cf.* (1.5)*;*
- $A_{[m]}(\eta,g;x,\xi) \in \Sigma\Gamma^m_{K,0,1}[\epsilon_0,N]$*;*
- $R(\eta,g) \in \Sigma\mathcal{R}^{-\varrho}_{K,0,1}[\epsilon_0,N]$.

2.

$$\mathcal{H}(\eta)[1] = \mathrm{Op}^{\mathrm{BW}}\left(-\frac{\mathsf{J}^0(\eta;x)}{1+2\eta}\ |\xi| + \mathrm{i}\ \tilde{V}(\eta;x)\,\xi + A_{[0]}(\eta;x,\xi)\right)\eta + R(\eta)[\eta], \tag{4.57}$$

where

- $\mathsf{J}^0(\eta;x) \in \mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]$ *and is explicitly defined in* (4.40)*;*
- $\tilde{V}(\eta;x) \triangleq \mathcal{D}_0(\eta)[1]\ \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,0}[\epsilon_0,N]$ *with* $\tilde{V}(0;x)\equiv 1$*;*
- $A_{[0]}(\eta;x,\xi) \in \Sigma\Gamma^0_{K,0,1}[\epsilon_0,N]$*;*
- $R(\eta) \in \Sigma\mathcal{R}^{-\varrho}_{K,0,1}[\epsilon_0,N]$.

Proof. We use now the expression in Eq. (4.7) and Propositions 4.6 and 4.8 as well as Proposition 3.15 and obtain that

$$\mathcal{H}(\eta)\,g = \mathrm{Op}^{\mathrm{BW}}\left(-\mathrm{i}\left(1+\mathsf{K}^0(\eta;x)+\frac{\eta_x}{r^2}\ \mathsf{J}^0(\eta;x)\right)\mathrm{sgn}\,\xi + A_{[-2]}(\eta;x,\xi)\right)g + \mathrm{Op}^{\mathrm{BW}}\left(\frac{g}{r^2}\left(\mathsf{K}'^0(\eta;x)+\frac{\eta_x}{r^2}\ \mathsf{J}'^0(\eta;x)\right)|\xi| + \mathrm{i}\ V_0(\eta,\partial_x^{-1}g;x)\,\xi + A_{[0]}(\eta,g;x,\xi)\right)\eta + \boldsymbol{R}(\eta,g)\begin{bmatrix}\eta\\ g\end{bmatrix}\cdot\begin{bmatrix}1\\ 1\end{bmatrix}. \tag{4.58}$$

Notice that in order to deduce (4.58) we used the fact that $\mathcal{D}_0(\eta)\,g \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]$, which stems immediately from the paralinearization provided in Proposition 4.8, Item 1, next we relabeled $\frac{1}{2}\mathcal{D}_0(\eta)\,g = V_0(\eta,\partial_x^{-1}g;x)$. Next we use Equations (4.10), (4.40) and (4.41) and obtain that

$$\mathsf{K}^0(\eta;x)+\frac{\eta_x}{r^2}\ \mathsf{J}^0(\eta;x) = 0,$$
$$\mathsf{K}'^0(\eta;x)+\frac{\eta_x}{r^2}\ \mathsf{J}'^0(\eta;x) = -\mathsf{J}^0(\eta;x).$$

Thus, transforming (4.58) into (4.56). The proof of (4.57) is almost identical to the proof of (4.56) and is hence omitted. ☐

4.4 Proof of Theorem 4.2

We can finally paralinearize Eq. (1.9). We use Equations (4.56) and (4.57), we set $g = \partial_x\psi$ and Proposition 3.15 in order to obtain, from (1.9), that

$$\eta_t = \mathrm{Op}^{\mathrm{BW}}\left(-\frac{|\xi|}{2} + A_{[-2]}(\eta;x,\xi)\right)\psi + \mathrm{Op}^{\mathrm{BW}}\left(\frac{1}{2}B_{\mathsf{b}}(\eta,\psi;x)\,|\xi| - \mathrm{i}\ V_{\mathsf{b}}(\eta,\psi;x)\,\xi + A_{[0]}(\eta,\psi;x,\xi)\right)\eta + R(\eta,\psi)[\eta+\psi], \tag{4.59}$$

where

$$B_{\mathsf{b}}(\eta,\psi;x) \triangleq B_0(\eta,\psi;x) + \mathsf{b}\tilde{B}(\eta;x), \qquad B_0(\eta,\psi;x) \triangleq \frac{\psi_x}{1+2\eta}\mathsf{J}^0(\eta;x), \qquad \tilde{B}(\eta;x) \triangleq \frac{\mathsf{J}^0(\eta;x)}{1+2\eta} \tag{4.60}$$

and

$$V_{\mathtt{b}}(\eta,\psi;x) \triangleq \frac{1}{2}\mathcal{D}_0(\eta)[\mathtt{b}+\psi_x] - \frac{\mathtt{b}}{2}.$$

Let us now paralinearize the term

$$-\frac{\mathtt{b}+\psi_x}{2}\mathcal{D}_0(\eta)[\mathtt{b}+\psi_x].$$

We apply Lemma 3.16 and obtain that

$$\begin{aligned} -\frac{\psi_x}{2}\mathcal{D}_0(\eta)[\mathtt{b}+\partial_x\psi] = {} & \mathrm{Op}^{\mathrm{BW}}\left(-\frac{\psi_x}{2}\right)[\mathcal{D}_0(\eta)[\mathtt{b}+\partial_x\psi]] - \mathrm{Op}^{\mathrm{BW}}\left(\frac{\mathcal{D}_0(\eta)[\mathtt{b}+\partial_x\psi]}{2}\right)[\psi_x] \\ & + R_1\left(-\frac{\psi_x}{2}\right)[\mathcal{D}_0(\eta)[\mathtt{b}+\partial_x\psi]] + R_2(\mathcal{D}_0(\eta)[\mathtt{b}+\partial_x\psi])\left[-\frac{\psi_x}{2}\right]. \end{aligned} \tag{4.61}$$

Recall that $\frac{\mathcal{D}_0(\eta)[\mathtt{b}+\partial_x\psi]}{2} = V_{\mathtt{b}}(\eta,\psi;x)$, so that we can apply Propositions 3.15 and 3.17 and obtain that

$$\begin{aligned} \mathrm{Op}^{\mathrm{BW}}\left(-\frac{\mathcal{D}_0(\eta)[\mathtt{b}+\partial_x\psi]}{2}\right)[\psi_x] = {} & \mathrm{Op}^{\mathrm{BW}}(-\mathrm{i}\, V_{\mathtt{b}}(\eta,\psi;x)\xi)\psi + R(\eta,\psi)\psi, \\ R_1\left(-\frac{\psi_x}{2}\right)[\mathcal{D}_0(\eta)[\mathtt{b}+\partial_x\psi]] + R_2(\mathcal{D}_0(\eta)[\mathtt{b}+\partial_x\psi])\left[-\frac{\psi_x}{2}\right] = {} & \boldsymbol{R}(\eta,\psi)\begin{bmatrix}\eta\\ \psi\end{bmatrix}\cdot\begin{bmatrix}1\\1\end{bmatrix}. \end{aligned} \tag{4.62}$$

Next we apply the paralinearization stated in Proposition 4.8 and the composition theorems in Propositions 3.15 and 3.17 and obtain that

$$\begin{aligned} \mathrm{Op}^{\mathrm{BW}}\left(-\frac{\psi_x}{2}\right)[\mathcal{D}_0(\eta)[\mathtt{b}+\partial_x\psi]] = {} & \mathrm{Op}^{\mathrm{BW}}\left(-\frac{1}{2}\frac{\psi_x}{1+2\eta}\,\mathsf{J}^0(\eta;x)|\xi| + A_{[-2]}(\eta;x,\xi)\right)\psi \\ & + \mathrm{Op}^{\mathrm{BW}}\left(-\frac{1}{2}\frac{\psi_x(\mathtt{b}+\psi_x)}{1+2\eta}\mathsf{J}'^0(\eta;x)|\xi| + A_{[0]}(\eta,\psi;x,\xi)\right)\eta + \boldsymbol{R}(\eta,\psi)\begin{bmatrix}\eta\\ \psi\end{bmatrix}\cdot\begin{bmatrix}1\\1\end{bmatrix}, \end{aligned} \tag{4.63}$$

so, using (4.60), we obtain that

$$\begin{aligned} \mathrm{Op}^{\mathrm{BW}}\left(-\frac{\psi_x}{2}\right)[\mathcal{D}_0(\eta)[\mathtt{b}+\partial_x\psi]] = {} & \mathrm{Op}^{\mathrm{BW}}\left(-\frac{1}{2}B_0(\eta,\psi;x)|\xi| + A_{[-2]}(\eta;x,\xi)\right)\psi \\ & + \mathrm{Op}^{\mathrm{BW}}\left(-\frac{1}{2}\frac{\psi_x(\mathtt{b}+\psi_x)}{1+2\eta}\mathsf{J}'^0(\eta;x)|\xi| + A_{[0]}(\eta,\psi;x,\xi)\right)\eta + \boldsymbol{R}(\eta,\psi)\begin{bmatrix}\eta\\ \psi\end{bmatrix}\cdot\begin{bmatrix}1\\1\end{bmatrix}. \end{aligned} \tag{4.64}$$

We invoke now Proposition 4.8 and obtain, using the notation introduced in (4.60), that

$$\begin{aligned} -\frac{\mathtt{b}}{2}\mathcal{D}_0(\eta)[\mathtt{b}+\psi_x] = {} & \mathrm{Op}^{\mathrm{BW}}\left(-\frac{\mathtt{b}}{2}\tilde{B}(\eta;x)|\xi| + A_{[-2]}(\eta;x,\xi)\right)\psi \\ & -\frac{\mathtt{b}^2}{2} + \mathrm{Op}^{\mathrm{BW}}\left(\frac{\mathtt{b}^2}{(1+2\eta)} - \frac{\mathtt{b}}{2}\frac{\mathtt{b}+\psi_x}{(1+2\eta)^2}\mathsf{J}'^0(\eta;x)|\xi| + A_{[0]}(\eta,\psi;x,\xi)\right)\eta + \boldsymbol{R}(\eta,\psi)\begin{bmatrix}\eta\\ \psi\end{bmatrix}\cdot\begin{bmatrix}1\\1\end{bmatrix}. \end{aligned} \tag{4.65}$$

Notice that using Eqs. (4.61) and (4.62) we obtain that

$$\begin{aligned} -\frac{\mathtt{b}+\psi_x}{2}\mathcal{D}_0(\eta)[\mathtt{b}+\psi_x] = {} & -\frac{\mathtt{b}}{2}\mathcal{D}_0(\eta)[\mathtt{b}+\psi_x] + \mathrm{Op}^{\mathrm{BW}}\left(-\frac{\psi_x}{2}\right)[\mathcal{D}_0(\eta)[\mathtt{b}+\partial_x\psi]] \\ & + \mathrm{Op}^{\mathrm{BW}}(-\mathrm{i}\, V_{\mathtt{b}}(\eta,\psi;x)\xi)\psi + \boldsymbol{R}(\eta,\psi)\begin{bmatrix}\eta\\ \psi\end{bmatrix}\cdot\begin{bmatrix}1\\1\end{bmatrix}. \end{aligned} \tag{4.66}$$

From Equations (4.40) and (4.41) we obtain the relation

$$\mathsf{J}'^0(\eta;x) = \frac{2(1+2\eta)^4}{\left((1+2\eta)^2+\eta_x^2\right)^2} - \left(\mathsf{J}^0(\eta;x)\right)^2 \tag{4.67}$$

so that using Eqs. (4.60), (4.64), (4.65) and (4.67) and denoting with

$$W_{\mathtt{b}}(\eta,\psi;x) \triangleq W_0(\eta,\psi;x) + \mathtt{b}\tilde{W}(\eta;x) \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,0}[\epsilon_0,N], \tag{4.68}$$

where

$$W_0(\eta,\psi;x) \triangleq \frac{\psi_x(1+2\eta)}{(1+2\eta)^2+\eta_x^2} \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N], \qquad \tilde{W}(\eta;x) \triangleq \frac{1+2\eta}{(1+2\eta)^2+\eta_x^2} \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,0}[\epsilon_0,N], \tag{4.69}$$

we obtain that

$$\begin{aligned}-\frac{\mathtt{b}}{2}\mathcal{D}_0(\eta)[\mathtt{b}+\psi_x] + \mathrm{Op}^{\mathrm{BW}}\Big(-\frac{\psi_x}{2}\Big)[\mathcal{D}_0(\eta)[\mathtt{b}+\partial_x\psi]] &= -\frac{\mathtt{b}^2}{2} + \mathrm{Op}^{\mathrm{BW}}\Big(-\frac{1}{2}B_{\mathtt{b}}(\eta,\psi;x)|\xi| + A_{[-2]}(\eta;x,\xi)\Big)\psi \\ &+ \mathrm{Op}^{\mathrm{BW}}\Big(\frac{\mathtt{b}^2}{(1+2\eta)} - \frac{1}{2}\big(W_{\mathtt{b}}^2(\eta,\psi;x) - B_{\mathtt{b}}^2(\eta,\psi;x)\big)|\xi| + A_{[0]}(\eta,\psi;x,\xi)\Big)\eta + \boldsymbol{R}(\eta,\psi)\begin{bmatrix}\eta\\ \psi\end{bmatrix}\cdot\begin{bmatrix}1\\ 1\end{bmatrix}. \end{aligned} \tag{4.70}$$

We thus insert (4.70) in (4.66) and obtain that

$$\begin{aligned}-\frac{\mathtt{b}+\psi_x}{2}\mathcal{D}_0(\eta)[\mathtt{b}+\psi_x] = &-\frac{\mathtt{b}^2}{2} + \mathrm{Op}^{\mathrm{BW}}\Big(-\frac{1}{2}B_{\mathtt{b}}(\eta,\psi;x)|\xi| - \mathrm{i}\, V_{\mathtt{b}}(\eta,\psi;x)\,\xi + A_{[-2]}(\eta;x,\xi)\Big)\psi \\ &+ \mathrm{Op}^{\mathrm{BW}}\Big(\frac{\mathtt{b}^2}{(1+2\eta)} - \frac{1}{2}\big(W_{\mathtt{b}}^2(\eta,\psi;x) - B_{\mathtt{b}}^2(\eta,\psi;x)\big)|\xi| + A_{[0]}(\eta,\psi;x,\xi)\Big)\eta \\ &+ \boldsymbol{R}(\eta,\psi)\begin{bmatrix}\eta\\ \psi\end{bmatrix}\cdot\begin{bmatrix}1\\ 1\end{bmatrix}. \end{aligned} \tag{4.71}$$

An iterated application of Lemma 3.19 give us that

$$\begin{aligned} -\gamma\mathcal{K}(\eta) &= \mathrm{Op}^{\mathrm{BW}}\big(\gamma(1+\mathtt{f}(\eta;x))(|\xi|^2-1)\big)\eta + R(\eta)\eta, \\ \mathtt{f}(\eta;x) &\triangleq \left(\frac{1+2\eta}{(1+2\eta)^2+\eta_x^2}\right)^{\frac{3}{2}} - 1 \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]. \end{aligned} \tag{4.72}$$

We can thus now plug Eqs. (4.71) and (4.72) in the second equation of (1.9) (recall that Ω is set in (2.21) in order to cancel the 0-homogeneous components of the transport) and obtain that

$$\begin{aligned}\psi_t = \;&\mathrm{Op}^{\mathrm{BW}}\Big(-\frac{1}{2}B_{\mathtt{b}}(\eta,\psi;x)|\xi| - \mathrm{i}\, V_{\mathtt{b}}(\eta,\psi;x)\,\xi + A_{[-2]}(\eta;x,\xi)\Big)\psi \\ &+ \mathrm{Op}^{\mathrm{BW}}\Big(\gamma(1+\mathtt{f}(\eta;x))(|\xi|^2-1) + \frac{\mathtt{b}^2}{(1+2\eta)} - \frac{1}{2}\big(W_{\mathtt{b}}^2(\eta,\psi;x) - B_{\mathtt{b}}^2(\eta,\psi;x)\big)|\xi| + A_{[0]}(\eta,\psi;x,\xi)\Big)\eta \\ &+ \boldsymbol{R}(\eta,\psi)\begin{bmatrix}\eta\\ \psi\end{bmatrix}\cdot\begin{bmatrix}1\\ 1\end{bmatrix}. \end{aligned} \tag{4.73}$$

Finally, we combine Eqs. (4.59) and (4.73) to obtain (4.1) after a renaming, if needed.

5 Complex Hamiltonian formulation of the Kelvin-Helmholtz equations

We begin with the real Hamiltonian system (2.5), and introduce the following associated complex variables

$$\begin{bmatrix}\eta\\ \psi\end{bmatrix} \triangleq \mathcal{C}\begin{bmatrix}v\\ \bar{v}\end{bmatrix}, \qquad \mathcal{C} \triangleq \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -\mathrm{i} & \mathrm{i}\end{bmatrix}, \qquad \mathcal{C}^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & \mathrm{i}\\ 1 & -\mathrm{i}\end{bmatrix}. \tag{5.1}$$

Under this change of variables (2.5) is equivalent to

$$\begin{bmatrix}v_t\\ \bar{v}_t\end{bmatrix} = \mathcal{C}^{-1}\boldsymbol{J}\nabla_{(\eta,\psi)}H\left(\mathcal{C}\begin{bmatrix}v\\ \bar{v}\end{bmatrix}\right). \tag{5.2}$$

Defining $H_{\mathbb{C}} \triangleq H\circ\mathcal{C}$ and noting that

$$\begin{bmatrix}\partial_\eta\\ \partial_\psi\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ \mathrm{i} & -\mathrm{i}\end{bmatrix}\begin{bmatrix}\partial_v\\ \partial_{\bar{v}}\end{bmatrix},$$

we obtain that (5.2) can be written as an Hamiltonian system in the complex coordinates (5.1) as

$$V_t = \boldsymbol{J}_{\mathbb{C}} \nabla_U H_{\mathbb{C}}(V), \qquad V \triangleq \begin{bmatrix} v \\ \bar{v} \end{bmatrix}, \tag{5.3}$$

where the *complex Poisson tensor* (cf. (2.5)) is defined as

$$\boldsymbol{J}_{\mathbb{C}} \triangleq -\mathrm{i}\boldsymbol{J} = \begin{bmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{bmatrix}. \tag{5.4}$$

Next we define the Fourier symbol

$$\begin{aligned} \mathsf{m}_{\gamma,\mathsf{b}}(\xi) &\triangleq \sqrt{\frac{|\xi|}{2\omega_{\gamma,\mathsf{b}}(\xi)}} = \sqrt[4]{\frac{|\xi|}{2\left(\gamma\left(|\xi|^2-1\right)-\mathsf{b}^2\left(\frac{|\xi|}{2}-1\right)\right)}} \quad \in \tilde{\Gamma}_0^{-\frac{1}{4}}, \\ \mathsf{m}_{\gamma,\mathsf{b}}(\xi)^{-1} &= \sqrt[4]{\frac{2\left(\gamma\left(\xi^2-1\right)-\mathsf{b}^2\left(\frac{|\xi|}{2}-1\right)\right)}{|\xi|}} \quad \in \tilde{\Gamma}_0^{\frac{1}{4}}. \end{aligned} \tag{5.5}$$

Then, we consider the symplectic matrix

$$\boldsymbol{S}_{\gamma,\mathsf{b}}(D) \triangleq \begin{bmatrix} \mathsf{m}_{\gamma,\mathsf{b}}(D) & 0 \\ 0 & \mathsf{m}_{\gamma,\mathsf{b}}(D)^{-1} \end{bmatrix} \qquad \text{and} \qquad \mathsf{M}_{\gamma,\mathsf{b}}(D) \triangleq \boldsymbol{S}_{\gamma,\mathsf{b}}(D) \circ \mathcal{C}. \tag{5.6}$$

Notice that

$$\begin{aligned} \mathsf{M}_{\gamma,\mathsf{b}}(\xi) &\triangleq \frac{1}{\sqrt{2}} \begin{bmatrix} \mathsf{m}_{\gamma,\mathsf{b}}(\xi) & \mathsf{m}_{\gamma,\mathsf{b}}(\xi) \\ -\mathrm{i}\mathsf{m}_{\gamma,\mathsf{b}}(\xi)^{-1} & \mathrm{i}\mathsf{m}_{\gamma,\mathsf{b}}(\xi)^{-1} \end{bmatrix} \in \left(\tilde{\Gamma}_0^{1/4}\right)^{2\times 2}, \\ \mathsf{M}_{\gamma,\mathsf{b}}^{-1}(\xi) &\triangleq \frac{1}{\sqrt{2}} \begin{bmatrix} \mathsf{m}_{\gamma,\mathsf{b}}(\xi)^{-1} & \mathrm{i}\,\mathsf{m}_{\gamma,\mathsf{b}}(\xi) \\ \mathsf{m}_{\gamma,\mathsf{b}}(\xi)^{-1} & -\mathrm{i}\,\mathsf{m}_{\gamma,\mathsf{b}}(\xi) \end{bmatrix} \in \left(\tilde{\Gamma}_0^{1/4}\right)^{2\times 2}. \end{aligned} \tag{5.7}$$

We define the *complex coordinates* U as

$$U \triangleq \begin{bmatrix} u \\ \bar{u} \end{bmatrix} \triangleq \mathsf{M}_{\gamma,\mathsf{b}}^{-1}(D) \begin{bmatrix} \eta \\ \psi \end{bmatrix} = \mathcal{C}^{-1} \circ \boldsymbol{S}_{\gamma,\mathsf{b}}^{-1}(D) \begin{bmatrix} \eta \\ \psi \end{bmatrix}. \tag{5.8}$$

Notation 5.1. Let $K \in \mathbb{N}$, $\epsilon_0 > 0$, $m \in \mathbb{R}$, $N \in \mathbb{N}$ and $0 \leqslant K' \leqslant K$. We work on a time interval $I = [0, T]$ for some fixed $T > 0$ to be determined. This latter is a priori implicit but will correspond to $c\varepsilon^{-(N+1)}$. Moreover, from now on we denote with

- $A_{[m;K']}(U;x,\xi)$ any generic element in the space $\Sigma\Gamma^m_{K,K',1}[\epsilon_0,N]$, while $\boldsymbol{A}_{[m;K']}(U;x,\xi)$ is a generic element of $\left(\Sigma\Gamma^m_{K,K',1}[\epsilon_0,N]\right)^{2\times 2}$ such that $\boldsymbol{J}_{\mathbb{C}}\mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{A}_{[m;K']}(U;x,\xi)\right)$ is lineary Hamiltonian up to homogeneity N (cf. Definition A.2), whose explicit expression may vary from line to line. We use as well the simplified notation $A_{[m;0]}(U;x,\xi) \triangleq A_{[m]}(U;x,\xi)$ and $\boldsymbol{A}_{[m;0]}(U;x,\xi) \triangleq \boldsymbol{A}_{[m]}(U;x,\xi)$;
- $R_{[K']}(U)$ and $\boldsymbol{R}_{[K']}(U)$ any generic element in the space $\Sigma\mathcal{R}^{-\varrho}_{K,K',1}[\epsilon_0,N]$ and $\left(\Sigma\mathcal{R}^{-\varrho}_{K,K',1}[\epsilon_0,N]\right)^{2\times 2}$ respectively, whose explicit expression may vary from line to line. We denote as well $R_{[0]}(U) \triangleq R(U)$ and $\boldsymbol{R}_{[0]}(U) \triangleq \boldsymbol{R}(U)$.

We prove the following:

Proposition 5.2 (Kelvin-Helmholtz equations in complex Hamiltonian coordinates)**.** *Let* $N \in \mathbb{N}$, $\gamma > 0$, $\mathsf{b} \in \mathbb{R}$ *and* $\varrho \geqslant 0$, *for any* $K \in \mathbb{N}$ *there exists* $s_0 > 0$ *and* $\epsilon_0 > 0$ *such that if* $\eta, \psi \in B^K_{s_0,\mathbb{R}}(I;\epsilon_0)$ *is a solution of Eq.* (1.9), *then* U *defined in* (5.8) *solves the complex Hamiltonian system*

$$U_t = \boldsymbol{J}_{\mathbb{C}}\,\mathrm{Op}^{\mathrm{BW}}\Big(\boldsymbol{A}_{\frac{3}{2}}(U;x)\;\omega_{\gamma,\mathsf{b}}(\xi) + \boldsymbol{A}_1(U;x,\xi) + \boldsymbol{A}_{\frac{1}{2}}(U;x)\;|\xi|^{\frac{1}{2}} + \boldsymbol{A}_{[0]}(U;x,\xi)\Big)U + \boldsymbol{R}(U)\,U, \tag{5.9}$$

where

- $\boldsymbol{J}_{\mathbb{C}}$ *is defined in Eq.* (5.4)*;*

- $\boldsymbol{A}_{\frac{3}{2}}(U;x) \in \left(\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,0}[\epsilon_0,N]\right)^{2\times 2}$ *is defined as*

$$\boldsymbol{A}_{\frac{3}{2}}(U,x) \triangleq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{\mathtt{f}\left(\left(\mathsf{M}_{\gamma,\mathtt{b}}(D)\,U\right)_1;x\right)}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \tag{5.10}$$

where $\mathtt{f}(\eta;x)$ *is defined in Eq.* (4.72)*;*

- $\boldsymbol{A}_1(U;x,\xi) \in \left(\Sigma\Gamma^1_{K,0,1}[\epsilon_0,N]\right)^{2\times 2}$ *is defined as*

$$\boldsymbol{A}_1(U;x,\xi) \triangleq -V_{\mathtt{b}}\left(\mathsf{M}_{\gamma,\mathtt{b}}(D)\,U;x\right)\xi\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \frac{B_{\mathtt{b}}\left(\mathsf{M}_{\gamma,\mathtt{b}}(|D|)\,U;x\right)}{2}\,|\xi|\begin{bmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{bmatrix},$$

where $V_{\mathtt{b}}(\eta,\psi;x) \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]$ *and* $B_{\mathtt{b}}(\eta,\psi;\xi) \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]$ *are functions in defined in Eqs.* (4.5) *and* (4.60)*;*

- $\boldsymbol{A}_{\frac{1}{2}}(U;x) \in \left(\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]\right)^{2\times 2}$ *is defined as*

$$\boldsymbol{A}_{\frac{1}{2}}(U,x) \triangleq A_{\frac{1}{2}}(U;x)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \triangleq \frac{1}{2}\frac{1}{\sqrt{2\gamma}}\left(\frac{B^2_{\mathtt{b}}(\mathsf{M}U;x)}{2} + \frac{\mathtt{b}^2}{2}\mathtt{f}(\eta;x) - w_{\mathtt{b}}(\mathsf{M}U;x)\right)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \tag{5.11}$$

where $w_{\mathtt{b}}(\eta,\psi;x)$ *is defined in Eq.* (4.3)*;*

- $\boldsymbol{A}_{[0]}(U;x,\xi)$ *and* $\boldsymbol{R}(U)$ *are as in Notation 4.4.*

Proof.

Step 1 (Diagonalization of the linear part of (4.1))**.** We can diagonalize the matrix defined in (2.15) as

$$\boldsymbol{L}_{\gamma,\mathtt{b}}(\xi) = \mathsf{M}_{\gamma,\mathtt{b}}(\xi)\,\boldsymbol{D}_{\gamma,\mathtt{b}}(\xi)\,\mathsf{M}^{-1}_{\gamma,\mathtt{b}}(\xi), \qquad \boldsymbol{D}_{\gamma,\mathtt{b}}(\xi) \triangleq \mathrm{i}\omega_{\gamma,\mathtt{b}}(\xi)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \left(\tilde{\Gamma}^{\frac{3}{2}}_0\right)^{2\times 2}, \tag{5.12}$$

where $\omega_{\gamma,\mathtt{b}}(\xi) \in \tilde{\Gamma}^{\frac{3}{2}}_0$ are defined in Eq. (2.19). Defining U as in (5.8) where $\begin{bmatrix} \eta \\ \psi \end{bmatrix}$ solves (2.14), the vector-valued complex function U solves the diagonal linear system

$$U_t = \boldsymbol{D}_{\gamma,\mathtt{b}}(D)\,U.$$

In particular thanks to the relation (5.8) combined with (5.5), which implies that $\mathsf{m}_{\gamma,\mathtt{b}}(|D|)[1] = 0$, assure us that if $(\eta,\psi) \in H^{s+\frac{1}{4}}_0(\mathbb{T};\mathbb{R}) \times \dot{H}^{s-\frac{1}{4}}(\mathbb{T};\mathbb{R})$ then $u \in H^s_0(\mathbb{T};\mathbb{C}) \simeq \dot{H}^s(\mathbb{T};\mathbb{C})$ for $s > 0$.

Next, expanding (5.5) we have that

$$\mathsf{m}_{\gamma,\mathtt{b}}(\xi) = \frac{1}{(2\gamma|\xi|)^{\frac{1}{4}}} + \left(\frac{\mathtt{b}}{2}\right)^2\frac{1}{(2\gamma|\xi|)^{\frac{5}{4}}} + \mathsf{m}_{\gamma,\mathtt{b};-\frac{9}{4}}(|\xi|), \tag{5.13}$$

with $\mathsf{m}_{\gamma,\mathtt{b};-\frac{9}{4}}(|\xi|) \in \tilde{\Gamma}^{-\frac{9}{4}}_0$.

Notation 5.3. From now on we use the abbreviated notation $\mathsf{M} \triangleq \mathsf{M}_{\gamma,\mathtt{b}}(|D|)$ defined in (5.7) and $\mathsf{m} \triangleq \mathsf{m}_{\gamma,\mathtt{b}}(|D|)$ defined in (5.8) for the sake of brevity.

Step 2 (Reformulation of Eq. (4.1) in complex coordiantes)**.** Recall the matrix of Fourier symbols $\boldsymbol{L}_{\gamma,\mathtt{b}}$ in (2.14) and the matrix of symbols $\boldsymbol{Q}_{\gamma,\mathtt{b}}$ in (4.2); we define

$$\boldsymbol{Q}_{\gamma,\mathtt{b};\geqslant 1}(\eta,\psi;x,\xi) \triangleq \boldsymbol{Q}_{\gamma,\mathtt{b}}(\eta,\psi;x,\xi) - \boldsymbol{L}_{\gamma,\mathtt{b}}(\xi) = \begin{bmatrix} 0 & 0 \\ \mathsf{q}(\eta,\psi;x,\xi) & 0 \end{bmatrix} \qquad \in \Sigma\Gamma^2_{K,0,1}[\epsilon_0,N], \tag{5.14}$$

where, from Eqs. (4.2) and (5.20), $\mathsf{q}(U;x,\xi)$ is the real-valued symbol

$$
\begin{aligned}
\mathsf{q}\left(\eta,\psi;x,\xi\right) &= \gamma\, \mathtt{f}\left(\eta;x\right)\left(|\xi|^2-1\right) - w_{\mathsf{b}}\left(\eta,\psi;x\right)|\xi| + \mathsf{b}^2\left(\frac{1}{\left(1+2\eta\right)}-1\right) \\
&= \gamma\, \mathtt{f}\left(\eta;x\right)|\xi|^2 - w_{\mathsf{b}}\left(\eta,\psi;x\right)|\xi| + \mathsf{q}_0\left(\eta,\psi;x,\xi\right),
\end{aligned}
\tag{5.15}
$$

with

$$
\mathsf{q}_0\left(\eta,\psi;x,\xi\right) \triangleq -\gamma\, \mathtt{f}\left(\eta;x\right) + \mathsf{b}^2\left(\frac{1}{1+2\eta}-1\right) \in \Sigma\Gamma^0_{K,0,1}\left[\epsilon_0,N\right].
$$

Then, by (5.12), Eq. (4.1) becomes

$$
\begin{aligned}
&\begin{bmatrix}\eta\\ \psi\end{bmatrix}_t - \mathsf{M}\boldsymbol{D}_{\gamma,\mathsf{b}}(D)\,\mathsf{M}^{-1}\begin{bmatrix}\eta\\ \psi\end{bmatrix} \\
&= \mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{Q}_{\ \gamma,\mathsf{b};\geqslant 1}\left(\eta,\psi;x,\xi\right) + \boldsymbol{B}_{\mathsf{b}}\left(\eta,\psi;x\right)|\xi| - \mathrm{i}V_{\mathsf{b}}\left(\eta,\psi;x\right)\ \mathrm{Id}_{\mathbb{R}^2}\ \xi + \boldsymbol{A}_{[0]}\left(\eta,\psi;x,\xi\right)\right)\begin{bmatrix}\eta\\ \psi\end{bmatrix} \\
&\quad + \boldsymbol{R}\left(\eta,\psi\right)\begin{bmatrix}\eta\\ \psi\end{bmatrix}.
\end{aligned}
\tag{5.16}
$$

Since $U = \mathsf{M}^{-1}\begin{bmatrix}\eta\\ \psi\end{bmatrix}$ we can derive the evolution equation for U multiplying (5.16) from the left for M^{-1} obtaining

$$
\begin{aligned}
&U_t - \boldsymbol{D}_{\gamma,\mathsf{b}}(D)\,U \\
&= \mathsf{M}^{-1}\mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{Q}_{\ \gamma,\mathsf{b};\geqslant 1}(\mathsf{M}U;x,\xi) + \boldsymbol{B}_{\mathsf{b}}(\mathsf{M}U;x)|\xi| - \mathrm{i}V_{\mathsf{b}}(\mathsf{M}U;x)\ \mathrm{Id}_{\mathbb{R}^2}\ \xi + \boldsymbol{A}_{[0]}(\mathsf{M}U;x,\xi)\right)\mathsf{M}U \\
&\quad + \mathsf{M}^{-1}\boldsymbol{R}(\mathsf{M}U)\,\mathsf{M}U.
\end{aligned}
\tag{5.17}
$$

Recall that M^{-1} is explicitly defined in (5.7). We have the following relation

$$
\begin{aligned}
&\mathsf{M}^{-1}\begin{bmatrix}A_1 & A_2\\ A_3 & A_4\end{bmatrix}\mathsf{M} = \\
&\frac{1}{2}\begin{bmatrix}\mathsf{m}^{-1}A_1\mathsf{m}+\mathsf{m}A_4\mathsf{m}^{-1}+\mathrm{i}\mathsf{m}A_3\mathsf{m}-\mathrm{i}\mathsf{m}^{-1}A_2\mathsf{m}^{-1} & \mathsf{m}^{-1}A_1\mathsf{m}-\mathsf{m}A_4\mathsf{m}^{-1}+\mathrm{i}\mathsf{m}A_3\mathsf{m}+\mathrm{i}\mathsf{m}^{-1}A_2\mathsf{m}^{-1}\\ \mathsf{m}^{-1}A_1\mathsf{m}-\mathsf{m}A_4\mathsf{m}^{-1}-\mathrm{i}\mathsf{m}A_3\mathsf{m}-\mathrm{i}\mathsf{m}^{-1}A_2\mathsf{m}^{-1} & \mathsf{m}^{-1}A_1\mathsf{m}+\mathsf{m}A_4\mathsf{m}^{-1}-\mathrm{i}\mathsf{m}A_3\mathsf{m}+\mathrm{i}\mathsf{m}^{-1}A_2\mathsf{m}^{-1}\end{bmatrix}.
\end{aligned}
\tag{5.18}
$$

Notice now that given

$$
a(U;x,\xi)\in\Sigma\Gamma^{m_a}_{K,0,1}\left[\epsilon_0,N\right],\qquad \alpha(\xi)\in\tilde{\Gamma}^{m_\alpha}_0,\qquad \beta(\xi)\in\tilde{\Gamma}^{m_\beta}_0,
$$

applying Proposition 3.15 we have that

$$
\begin{aligned}
\alpha(D)\,\mathrm{Op}^{\mathrm{BW}}\left(a(U;x,\xi)\right)\beta(D) = \mathrm{Op}^{\mathrm{BW}}&\left(a(U;x,\xi)\,\alpha(\xi)\,\beta(\xi) + \frac{1}{2\mathrm{i}}\ \left(a(U;x,\xi)\right)_x\left(\alpha_\xi(\xi)\,\beta(\xi)-\alpha(\xi)\,\beta_\xi(\xi)\right)\right) \\
&+ \mathrm{Op}^{\mathrm{BW}}\left(\Sigma\Gamma^{m_a+m_\alpha+m_\beta-2}_{K,0,1}\left[\epsilon_0,N\right]\right) + \Sigma\mathcal{R}^{-\varrho}_{K,0,1}\left[\epsilon_0,N\right].
\end{aligned}
\tag{5.19}
$$

Then, applying (5.18), we have that

$$
\mathsf{M}^{-1}\mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{Q}_{\ \gamma,\mathsf{b};\geqslant 1}(\mathsf{M}U;x,\xi)\right)\mathsf{M} = \frac{\mathrm{i}}{2}\ \mathsf{m}\ \mathrm{Op}^{\mathrm{BW}}\left(\mathsf{q}(U;x,\xi)\right)\mathsf{m}\ \begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix},
\tag{5.20}
$$

where $\mathsf{q}(U;x,\xi)$ is the real valued symbol in (5.15) evaluated at $(\eta,\psi)=\mathsf{M}U$. We apply Equation (5.19) and obtain that

$$
\mathsf{m}\ \mathrm{Op}^{\mathrm{BW}}\left(\mathsf{q}(U;x,\xi)\right)\mathsf{m} = \mathrm{Op}^{\mathrm{BW}}\left(\mathsf{q}(U;x,\xi)\,\mathsf{m}^2_{\gamma,\mathsf{b}}(\xi) + A_{[-\frac{1}{2}]}(U;x,\xi)\right) + R(U).
\tag{5.21}
$$

Thus, combining Eqs. (5.20) and (5.21), we obtain

$$
\begin{aligned}
&\mathsf{M}^{-1}\mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{Q}_{\ \gamma,\mathsf{b};\geqslant 1}(\mathsf{M}U;x,\xi)\right)\mathsf{M} \\
&= \frac{\mathrm{i}}{2}\ \left(\mathrm{Op}^{\mathrm{BW}}\left(\mathsf{q}(U;x,\xi)\,\mathsf{m}^2_{\gamma,\mathsf{b}}(\xi) + A_{[-\frac{1}{2}]}(U;x,\xi)\right) + R(U)\right)\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}.
\end{aligned}
\tag{5.22}
$$

Next we apply Eqs. (5.18) and (5.19) to the matrix-valued symbol $\boldsymbol{B}_{\mathsf{b}}(\mathsf{M}U;x)|\xi|$ defined in (4.4) and obtain that

$$\begin{aligned}&\mathsf{M}^{-1}\mathrm{Op}^{\mathrm{BW}}(\boldsymbol{B}_{\mathsf{b}}(\mathsf{M}U;x)|\xi|)\mathsf{M}\\&=\frac{1}{2}\mathrm{Op}^{\mathrm{BW}}(B_{\mathsf{b}}(\mathsf{M}U;x)|\xi|)\begin{bmatrix}0&1\\1&0\end{bmatrix}+\frac{\mathrm{i}}{4}\mathrm{Op}^{\mathrm{BW}}\left(B_{\mathsf{b}}^2(\mathsf{M}U;x)|\xi|\,\mathsf{m}_{\gamma,\mathsf{b}}^2(\xi)\right)\begin{bmatrix}1&1\\-1&-1\end{bmatrix}\\&+\mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{A}_{[0]}(U;x,\xi)\right)+\boldsymbol{R}(U).\end{aligned}\tag{5.23}$$

The same procedure give us the conjugations

$$\begin{aligned}\mathsf{M}^{-1}\mathrm{Op}^{\mathrm{BW}}(\mathrm{i}V_{\mathsf{b}}(\mathsf{M}U;x)\xi)\mathrm{Id}_{\mathbb{R}^2}\mathsf{M}&=\mathrm{Op}^{\mathrm{BW}}(\mathrm{i}V_{\mathsf{b}}(\mathsf{M}U;x)\xi)\mathrm{Id}_{\mathbb{C}^2}+\mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{A}_{[0]}(U;x,\xi)\right)+\boldsymbol{R}(U),\\\mathsf{M}^{-1}\mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{A}_{[0]}(\mathsf{M}U;x,\xi)\right)\mathsf{M}&=\mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{A}_{[0]}(U;x,\xi)\right)+\boldsymbol{R}(U),\end{aligned}\tag{5.24}$$

while Proposition 3.17 implies that

$$\mathsf{M}^{-1}\boldsymbol{R}(\mathsf{M}U)\mathsf{M}\rightsquigarrow\boldsymbol{R}(U).\tag{5.25}$$

We plug Equations (5.22) to (5.25) in Equation (5.17) and obtain that

$$U_t-\boldsymbol{D}_{\gamma,\mathsf{b}}(D)U=\mathrm{Op}^{\mathrm{BW}}\left(\sum_{\mathrm{j}=0}^{2}\boldsymbol{A}^{\mathrm{NH}}_{\frac{3-\mathrm{j}}{2}}(U;x,\xi)+\boldsymbol{A}_{[0]}(U;x,\xi)\right)U+\boldsymbol{R}(U)U,\tag{5.26}$$

with

$$\boldsymbol{A}^{\mathrm{NH}}_{\frac{3-\mathrm{j}}{2}}(U;x,\xi)\in\left(\Sigma\Gamma^{\frac{3-\mathrm{j}}{2}}_{K,0,1}[\epsilon_0,N]\right)^{2\times2},$$

and are explicitly defined as

$$\begin{aligned}\boldsymbol{A}^{\mathrm{NH}}_{\frac{3}{2}}(U;x,\xi)&\triangleq\frac{\mathrm{i}}{2}\,\mathsf{q}(U;x,\xi)\,\mathsf{m}^2_{\gamma,\mathsf{b}}(\xi)\begin{bmatrix}1&1\\-1&-1\end{bmatrix},\\\boldsymbol{A}^{\mathrm{NH}}_{1}(U;x,\xi)&\triangleq-\mathrm{i}V_{\mathsf{b}}(\mathsf{M}U;x)\xi\begin{bmatrix}1&0\\0&1\end{bmatrix}+\frac{B_{\mathsf{b}}(\mathsf{M}U;x)}{2}|\xi|\begin{bmatrix}0&1\\1&0\end{bmatrix},\\\boldsymbol{A}^{\mathrm{NH}}_{\frac{1}{2}}(U;x,\xi)&\triangleq\frac{\mathrm{i}}{4}B_{\mathsf{b}}^2(\mathsf{M}U;x)|\xi|\,\mathsf{m}^2_{\gamma,\mathsf{b}}(\xi)\begin{bmatrix}1&1\\-1&-1\end{bmatrix}.\end{aligned}\tag{5.27}$$

We now expand the symbols in (5.27) in decreasing para-differential orders using Eqs. (5.13) and (5.15), thus obtaining that

$$\begin{aligned}\mathsf{q}(U;x,\xi)\,\mathsf{m}^2_{\gamma,\mathsf{b}}(\xi)&=\sqrt{\frac{\gamma}{2}}\,\mathtt{f}(\eta;x)|\xi|^{\frac{3}{2}}+\frac{1}{\sqrt{2\gamma}}\left(\left(\frac{\mathsf{b}}{2}\right)^2\mathtt{f}(\eta;x)-w_{\mathsf{b}}(\mathsf{M}U;x)\right)|\xi|^{\frac{1}{2}}+A_{[-\frac{1}{2}]}(U;x,\xi),\\B_{\mathsf{b}}^2(\mathsf{M}U;x)|\xi|\,\mathsf{m}^2_{\gamma,\mathsf{b}}(\xi)&=\frac{1}{\sqrt{2\gamma}}B^2_{\mathsf{b}}(\mathsf{M}U;x)|\xi|^{\frac{1}{2}}+A_{[-\frac{1}{2}]}(U;x,\xi).\end{aligned}\tag{5.28}$$

Thus, inserting (5.28) in (5.27), we obtain that

$$\sum_{\mathrm{j}=0}^{2}\boldsymbol{A}^{\mathrm{NH}}_{\frac{3-\mathrm{j}}{2}}(U;x,\xi)=\sum_{\mathrm{j}=0}^{2}\boldsymbol{A}^{\mathrm{H}}_{\frac{3-\mathrm{j}}{2}}(U;x,\xi)+\boldsymbol{A}_{[-\frac{1}{2}]}(U;x,\xi),$$

where

$$\begin{aligned}\boldsymbol{A}^{\mathrm{H}}_{\frac{3}{2}}(U;x,\xi)&\triangleq\frac{\mathrm{i}}{2}\sqrt{\frac{\gamma}{2}}\,\mathtt{f}(\eta;x)|\xi|^{\frac{3}{2}}\begin{bmatrix}1&1\\-1&-1\end{bmatrix},\\\boldsymbol{A}^{\mathrm{H}}_{1}(U;x,\xi)&\triangleq\boldsymbol{A}^{\mathrm{NH}}_{1}(U;x,\xi),\\\boldsymbol{A}^{\mathrm{H}}_{\frac{1}{2}}(U;x,\xi)&\triangleq\frac{\mathrm{i}}{2}\frac{1}{\sqrt{2\gamma}}\left(\frac{B^2_{\mathsf{b}}(\mathsf{M}U;x)}{2}+\left(\frac{\mathsf{b}}{2}\right)^2\mathtt{f}(\eta;x)-w_{\mathsf{b}}(\mathsf{M}U;x)\right)|\xi|^{\frac{1}{2}}\begin{bmatrix}1&1\\-1&-1\end{bmatrix}.\end{aligned}$$

Therefore, (5.26) becomes

$$U_t-\boldsymbol{D}_{\gamma,\mathsf{b}}(D)U=\mathrm{Op}^{\mathrm{BW}}\left(\sum_{\mathrm{j}=0}^{2}\boldsymbol{A}^{\mathrm{H}}_{\frac{3-\mathrm{j}}{2}}(U;x,\xi)+\boldsymbol{A}_{[0]}(U;x,\xi)\right)U+\boldsymbol{R}(U)U.\tag{5.29}$$

We now further refine the expression deduced in (5.29) by expressing the leading term as a quasilinear perturbation of the unperturbed frequencies $\omega_{\gamma,\mathtt{b}}(\xi)$, cf. (2.19). Using (2.20) we have that

$$\boldsymbol{A}^{\mathrm{H}}_{\frac{3}{2}}(U;x,\xi) = \mathrm{i}\left(\frac{\mathtt{f}(\eta;x)}{2}\omega_{\gamma,\mathtt{b}}(\xi) + \frac{1}{\sqrt{2\gamma}}\frac{\mathtt{f}(\eta;x)}{2}\left(\frac{\mathtt{b}}{2}\right)^2|\xi|^{\frac{1}{2}} + \boldsymbol{A}_{[-\frac{1}{2}]}(U;x,\xi)\right)\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix},$$

so that, defining

$$\begin{aligned}
\boldsymbol{J}_{\mathbb{C}}\boldsymbol{A}_{\frac{3}{2};\geqslant 1}(U,x) &\triangleq \frac{\mathtt{f}(\eta;x)}{2}\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}\mathrm{i}\,,\\
\boldsymbol{J}_{\mathbb{C}}\boldsymbol{A}_{\frac{1}{2}}(U,x) &\triangleq \frac{1}{2}\frac{1}{\sqrt{2\gamma}}\left(\frac{B^2_{\mathtt{b}}(\mathrm{M}U;x)}{2} + 2\left(\frac{\mathtt{b}}{2}\right)^2\mathtt{f}(\eta;x) - w_{\mathtt{b}}(\mathrm{M}U;x)\right)\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}\mathrm{i},\\
\boldsymbol{J}_{\mathbb{C}}\boldsymbol{A}_1(U;x,\xi) &\triangleq \boldsymbol{A}^{\mathrm{H}}_1(U;x,\xi)\,,
\end{aligned}$$

we transform (5.29) into

$$\begin{aligned}
&U_t - \boldsymbol{D}_{\gamma,\mathtt{b}}(D)\,U\\
&= \boldsymbol{J}_{\mathbb{C}}\,\mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{A}_{\frac{3}{2};\geqslant 1}(U;x)\ \omega_{\gamma,\mathtt{b}}(\xi) + \boldsymbol{A}_1(U;x,\xi) + \boldsymbol{A}_{\frac{1}{2}}(U;x)\ |\xi|^{\frac{1}{2}} + \boldsymbol{A}_{[0]}(U;x,\xi)\right)U + \boldsymbol{R}(U)\,U.
\end{aligned}\tag{5.30}$$

Next, an explicit computation shows that

$$\boldsymbol{D}_{\gamma,\mathtt{b}}(\xi) = \omega_{\gamma,\mathtt{b}}(\xi)\ \boldsymbol{J}_{\mathbb{C}}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}.$$

Thus, defining

$$\boldsymbol{A}_{\frac{3}{2}}(U;x) \triangleq \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + \boldsymbol{A}_{\frac{3}{2};\geqslant 1}(U,x)\,,$$

it remains only to show that (5.30) is a complex Hamiltonian system and that $\boldsymbol{J}_{\mathbb{C}}\boldsymbol{A}_{[0]}(U;x,\xi)$ is linearly symplectic. Indeed complex variables transformation $\mathsf{M}^{-1}_{\gamma,\mathtt{b}}$ in (5.8) is the composition of the complex transformation $\mathcal{C}^{-1}$ in (5.1) and the real symplectic map $\boldsymbol{S}^{-1}_{\gamma,\mathtt{b}}(D)$, see (5.6). Then the equation for U has the complex Hamiltonian form in (5.3). Then we apply Lemma A.14 to each homogeneous component in (5.30). As the explicit positive order symbols are linearly symplectic by direct inspections, we eventually replace $\boldsymbol{A}_{[0]}(U;x,\xi)$ as in (A.17) to get a linearly symplectic matrix of symbols. This conclude the proof.

□

6 Block diagonalization and reduction to constant coefficients

In this section, we perform spectrally localized, linearly symplectic, and bounded transformations to conjugate the complex Kelvin-Helmholtz system (5.9) into a diagonal constant-coefficient system, up to a smoothing remainder. Specifically, we prove the following.

Proposition 6.1. *Let $N \in \mathbb{N}$ and $\varrho > 3(N+1)$, there exists $\underline{K'} > 0$ such that for any $K > \underline{K'}$ there are $s_0 > 0$, $\epsilon_0 > 0$ such that, for any solution $U \in B^K_{s_0}(I;\epsilon_0)$ solution of* (5.9) *there exists a real-to-real invertible matrix of spectrally localized maps $\boldsymbol{B}(U;t)$ such that*

1. *$\boldsymbol{B}(U;t)$, $\boldsymbol{B}(U;t)^{-1} \in \left(\mathcal{S}^0_{K,\underline{K'}-1,0}[\epsilon_0,N]\right)^{2\times 2}$ are linearly symplectic up to homogeneity N, according to Definition A.4. Moreover, $\boldsymbol{B}(U;t) - \mathrm{Id} \in \left(\Sigma\mathcal{S}^{\frac{3}{2}(N+1)}_{K,\underline{K'}-1,1}[\epsilon_0,N]\right)^{2\times 2}$;*
2. *The variable $W \triangleq \boldsymbol{B}(U;t)\,U$ solves*

$$W_t = \mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}\left(\mathrm{i}\left[(1+v(U;t))\,\omega_{\gamma,\mathtt{b}}(\xi) + \mathcal{V}_{\mathtt{b}}(U;t)\,\xi + \mathscr{b}_{\frac{1}{2}}(U;t)\,|\xi|^{\frac{1}{2}} + \mathscr{b}_0(U;t,\xi)\right]\right)W + \mathsf{R}(U;t)\,W,\tag{6.1}$$

   where

- $\omega_{\gamma,\mathsf{b}}(\xi) \in \widetilde{\Gamma}_0^{\frac{3}{2}}$ *is defined in* (2.19);
- $v(U;t) \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,2}[\epsilon_0, N]$ *and is independent of* x;
- $\mathcal{V}_{\mathsf{b}}(U;t) \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,1,2}[\epsilon_0, N]$ *and is independent of* x;
- $\mathscr{b}_{\frac{1}{2}}(U;t) \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,2,2}[\epsilon_0, N]$ *and is independent of* x;
- $\mathscr{b}_0(U;t,\xi) \in \Sigma\Gamma^0_{K,\underline{K'},2}[\epsilon_0, N]$, *is independent of* x *and such that* $\operatorname{Im}\mathscr{b}_0(U;t,\xi) \in \Gamma^0_{K,\underline{K'},N+1}[\epsilon_0]$;
- $\mathsf{R}(U;t) \in \left(\Sigma\mathcal{R}^{-\varrho+3(N+1)}_{K,\underline{K'},1}[\epsilon_0, N]\right)^{2\times 2}$.

The rest of the section is devoted to the proof of Proposition 6.1. The procedure is now rather classical so we state the main steps, for the missing details, we refer the interested reader to [25, Section 6].

6.1 A complex Alinhac Good Unknown

We first perform a change of variable, known as *Alinhac good unknown* that cancels the anti-diagonal terms of order one in (5.9).

Lemma 6.2 (Alinhac Good Unknown for the Kelvin-Helmholtz equations in complex coordinates)**.** *Let* $N \in \mathbb{N}$, $\gamma > 0$, $\mathsf{b} \in \mathbb{R}$ *and* $\varrho \geqslant 0$, *for any* $K \in \mathbb{N}$ *there exists* $s_0 > 0$ *and* $\epsilon_0 > 0$ *such that if* $U \in B^K_{s_0}(I;\epsilon_0)$ *solves* (5.9) *there exists a real-to-real matrix of operators* $\mathbf{G}(U)$ *such that*

1. $\boldsymbol{G}(U)$, $\boldsymbol{G}(U)^{-1} \in \left(\mathcal{S}^0_{K,0,0}[\epsilon_0, N]\right)^{2\times 2}$ *are linearly symplectic, according to Definition A.3. Moreover,* $\boldsymbol{G}(U) - \mathrm{Id} \in \left(\Sigma\mathcal{S}^{-\frac{1}{2}}_{K,0,1}[\epsilon_0, N]\right)^{2\times 2}$;
2. *The variable* $U_0 \triangleq \boldsymbol{G}(U)\,U$ *solves*

$$
\begin{aligned}
\partial_t U_0 = \boldsymbol{J}_{\mathbb{C}}\operatorname{Op}^{\mathrm{BW}}\left(\boldsymbol{A}^{(0)}_{\frac{3}{2}}(U;x)\;\omega_{\gamma,\mathsf{b}}(\xi) + \boldsymbol{A}^{(0)}_{\frac{1}{2}}(U;x)\;|\xi|^{\frac{1}{2}} + \boldsymbol{A}_{[0;1]}(U;x,\xi)\right)U_0 \\
- \operatorname{Op}^{\mathrm{BW}}_{\mathrm{vec}}(\mathrm{i}V_{\mathsf{b}}(\mathsf{M}U;x)\,\xi)\,U_0 + \boldsymbol{R}(U)\,U_0,
\end{aligned}
\tag{6.2}
$$

where

- *the symbols* $\boldsymbol{A}^{(0)}_{\frac{3}{2}} \triangleq \boldsymbol{A}_{\frac{3}{2}}$, $\omega_{\gamma,\mathsf{b}}$ *and* V_{b} *are the same as in the statement of Proposition 5.2 and* $\boldsymbol{A}_{[0;1]}$, $\boldsymbol{R}$ *are as in Notation 5.1;*
- $\boldsymbol{A}^{(0)}_{\frac{1}{2}} \in \left(\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0, N]\right)^{2\times 2}$ *is defined as*

$$
\boldsymbol{A}^{(0)}_{\frac{1}{2}}(U;x) \triangleq A^{(0)}_{\frac{1}{2}}(U;x)\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} \triangleq \frac{1}{2}\frac{1}{\sqrt{2\gamma}}\left(\frac{\mathsf{b}^2}{2}\mathtt{f}(\eta;x) - w_{\mathsf{b}}(\mathsf{M}U;x)\right)\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix},
$$

where $w_{\mathsf{b}}(\eta,\psi;x)$ *is defined in Eq.* (4.3).

Proof. The procedure that we use here is the same as the one proved in [25, Section 6.1], thus we omit to perform detailed computations, in particular the Item 1 is true by direct computations and the fact that $\boldsymbol{G}(U) - \mathrm{Id} \in \left(\Sigma\mathcal{S}^0_{K,0,1}[\epsilon_0, N]\right)^{2\times 2}$. This latter fact is a consequence of Lemma 3.7. The only difference, compared to [25], is the symbol $\boldsymbol{A}_{\frac{1}{2}}(U;x)\,|\xi|^{\frac{1}{2}} \in \Sigma\Gamma^{\frac{1}{2}}_{K,0,1}[\epsilon_0, N]$ appearing in (5.9) and defined explicitly in (5.11), which shall be treated in detail. Let us define the transformation

$$
\boldsymbol{G}(U) \triangleq \mathrm{Id} - \frac{\mathrm{i}}{2}\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}\operatorname{Op}^{\mathrm{BW}}\left(B_{\mathsf{b}}(\mathsf{M}U;x)\,\mathsf{m}^2_{\gamma,\mathsf{b}}(\xi)\right).
\tag{6.3}
$$

The fact that $\boldsymbol{G}(U)$ is a nilpotent perturbation of the identity allows us to compute its inverse as

$$
\boldsymbol{G}(U)^{-1} \triangleq \mathrm{Id} + \frac{\mathrm{i}}{2}\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}\operatorname{Op}^{\mathrm{BW}}\left(B_{\mathsf{b}}(\mathsf{M}U;x)\,\mathsf{m}^2_{\gamma,\mathsf{b}}(\xi)\right).
$$

Thus, defining $U_0 \triangleq \boldsymbol{G}(U)\, U$, and using the conjugation rules

$$\begin{aligned}
\boldsymbol{G}(U)\,\boldsymbol{J}_{\mathbb{C}}\mathrm{Op}^{\mathrm{BW}}\Big(\boldsymbol{A}_{\frac{3}{2}}\,\omega_{\gamma,\mathtt{b}}(\xi)\Big)\boldsymbol{G}(U)^{-1} &= \boldsymbol{J}_{\mathbb{C}}\mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{A}_{\frac{3}{2}}\,\omega_{\gamma,\mathtt{b}}(\xi) - \mathrm{i}\frac{B_{\mathtt{b}}}{2}|\xi|\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} + \frac{B_{\mathtt{b}}^2}{4}\,\mathtt{m}_{\gamma,\mathtt{b}}^2(\xi)\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} + \boldsymbol{A}_{[0]}\right) + \boldsymbol{R}(U),\\
\boldsymbol{G}(U)\,\mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}(-\mathrm{i}V_{\mathtt{b}}\xi)\,\mathrm{Id}_{\mathbb{C}^2}\,\boldsymbol{G}(U)^{-1} &= \mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}(-\mathrm{i}V_{\mathtt{b}}\xi)\,\mathrm{Id}_{\mathbb{C}^2},\\
\boldsymbol{G}(U)\,\boldsymbol{J}_{\mathbb{C}}\mathrm{Op}^{\mathrm{BW}}\left(\mathrm{i}\frac{B_{\mathtt{b}}}{2}|\xi|\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right)\boldsymbol{G}(U)^{-1} &= \boldsymbol{J}_{\mathbb{C}}\mathrm{Op}^{\mathrm{BW}}\left(\mathrm{i}\frac{B_{\mathtt{b}}}{2}|\xi|\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} - \frac{B_{\mathtt{b}}^2}{2}\,\mathtt{m}_{\gamma,\mathtt{b}}^2(\xi)\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} + \boldsymbol{A}_{[0]}\right) + \boldsymbol{R}(U),
\end{aligned} \tag{6.4}$$

we get, using also Eqs. (5.13) and (6.4),

$$\begin{aligned}
\boldsymbol{G}(U)\,\boldsymbol{J}_{\mathbb{C}}\mathrm{Op}^{\mathrm{BW}}\Big(\boldsymbol{A}_{\frac{3}{2}}\,\omega_{\gamma,\mathtt{b}}(\xi) + \boldsymbol{A}_1\Big)\boldsymbol{G}(U)^{-1} = \boldsymbol{J}_{\mathbb{C}}\mathrm{Op}^{\mathrm{BW}}&\left(\boldsymbol{A}_{\frac{3}{2}}\,\omega_{\gamma,\mathtt{b}}(\xi) - \frac{1}{\sqrt{2\gamma}}\frac{B_{\mathtt{b}}^2}{4}|\xi|^{\frac{1}{2}}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} + \boldsymbol{A}_{[0]}\right)\\
&+ \mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}(-\mathrm{i}V_{\mathtt{b}}\xi) + \boldsymbol{R}(U).
\end{aligned} \tag{6.5}$$

Due to the particular nilpotent structure of $\boldsymbol{J}_{\mathbb{C}}\mathrm{Op}^{\mathrm{BW}}\Big(\boldsymbol{A}_{\frac{1}{2}}(U;x)\,|\xi|^{\frac{1}{2}}\Big)$ and $\boldsymbol{G}(U) - \mathrm{Id}$, the conjugation for the symbol of order $1/2$ explicitly given by

$$\boldsymbol{G}(U)\,\boldsymbol{J}_{\mathbb{C}}\mathrm{Op}^{\mathrm{BW}}\Big(\boldsymbol{A}_{\frac{1}{2}}(U;x)\,|\xi|^{\frac{1}{2}}\Big)\boldsymbol{G}(U)^{-1} = \boldsymbol{J}_{\mathbb{C}}\mathrm{Op}^{\mathrm{BW}}\Big(\boldsymbol{A}_{\frac{1}{2}}(U;x)\,|\xi|^{\frac{1}{2}}\Big). \tag{6.6}$$

Remark that there is a cancellation of the terms of the form $|\xi|^{\frac{1}{2}}$ between (5.11) and (6.5) leading the the expression of $\boldsymbol{A}^{(0)}_{\frac{1}{2}}$ in the statement. Moreover, we have

$$\boldsymbol{G}(U)\,U_t = \partial_t U_0 - (\partial_t \boldsymbol{G}(U))\,\boldsymbol{G}(U)^{-1}\,U_0. \tag{6.7}$$

Hence, from Item 1 and Proposition 3.15, we have that

$$(\partial_t \boldsymbol{G}(U))\,\boldsymbol{G}(U)^{-1} = \mathrm{Op}^{\mathrm{BW}}\Big(\boldsymbol{A}_{[-\frac{1}{2};1]}(U;x,\xi)\Big), \quad \boldsymbol{A}_{[-\frac{1}{2};1]} \triangleq -\frac{\mathrm{i}}{2}\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}\partial_t B_{\mathtt{b}}(\mathsf{M}U;x)\,\mathtt{m}_{\gamma,\mathtt{b}}^2(\xi). \tag{6.8}$$

The conjugations proved in Eqs. (6.5) to (6.8) and the computations performed in [25, Section 6.1] conclude thus the proof of Lemma 6.2. We remark that the zero-th order matrix $\boldsymbol{A}_{[0,1]}$ is the sum of the linearly Hamiltonian matrix $\boldsymbol{A}_{[-\frac{1}{2},1]}$ and the contributions coming from $\boldsymbol{A}_{[0]}$ in Eqs. (6.4) and (6.5). To prove that $\boldsymbol{J}_{\mathbb{C}}\boldsymbol{A}_{[0]}$ is linearly Hamiltonian up to homogeneity N we then apply Lemma A.15 to each homogeneous component of the operators in Eqs. (6.4) and (6.5) which are linearly Hamiltonian thanks to Lemma A.5. □

6.2 Diagonalization at the highest order

In the present section, we diagonalize the quasi-linear contribution

$$\boldsymbol{J}_{\mathbb{C}}\mathrm{Op}^{\mathrm{BW}}\Big(\boldsymbol{A}^{(0)}_{\frac{3}{2}}(U;x)\,\omega_{\gamma,\mathtt{b}}(\xi)\Big),$$

which reduces to the diagonalization of the matrix

$$\boldsymbol{J}_{\mathbb{C}}\boldsymbol{A}^{(0)}_{\frac{3}{2}}(U;x) = \mathrm{i}\begin{bmatrix}-(1+\mathtt{f}(U;x)) & -\mathtt{f}(U;x)\\ \mathtt{f}(U;x) & 1+\mathtt{f}(U;x)\end{bmatrix}, \tag{6.9}$$

where $\mathtt{f}(U;x)$ is defined in (4.72). The eigenvalues of (6.9) are given by $\pm\mathrm{i}\,\lambda(U;x)$ where

$$\lambda(U;x) \triangleq \sqrt{1+2\mathtt{f}(U;x)} \tag{6.10}$$

and we have that $\lambda(U;x) - 1 \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0, N]$. Defining

$$\mathtt{h} \triangleq \frac{1+\mathtt{f}+\lambda}{\sqrt{(1+\mathtt{f}+\lambda)^2 - \mathtt{f}^2}}, \qquad \mathtt{g} \triangleq \frac{-\mathtt{f}}{\sqrt{(1+\mathtt{f}+\lambda)^2 - \mathtt{f}^2}},$$

the symmetric and symplectic matrix

$$\boldsymbol{F} \triangleq \begin{bmatrix} \mathtt{h} & \mathtt{g} \\ \mathtt{g} & \mathtt{h} \end{bmatrix} \in \left(\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,0}[\epsilon_0, N]\right)^{2\times 2}, \qquad \boldsymbol{F}^{-1} \triangleq \begin{bmatrix} \mathtt{h} & -\mathtt{g} \\ -\mathtt{g} & \mathtt{h} \end{bmatrix} \in \left(\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,0}[\epsilon_0, N]\right)^{2\times 2}, \tag{6.11}$$

which are such that

$$\boldsymbol{F} - \mathrm{Id},\ \boldsymbol{F}^{-1} - \mathrm{Id} \in \left(\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0, N]\right)^{2\times 2}, \tag{6.12}$$

diagonalize (6.9) in the sense that

$$\boldsymbol{F}^{-1}\boldsymbol{J}_{\mathbb{C}}\boldsymbol{A}^{(0)}_{\frac{3}{2}}\boldsymbol{F} = \begin{bmatrix} -\mathrm{i}\lambda & 0 \\ 0 & \mathrm{i}\lambda \end{bmatrix}.$$

We prove the following result.

Lemma 6.3. *Let $N \in \mathbb{N}$ and $\varrho > 0$, for any $K \in \mathbb{N}^*$ there are $s_0 > 0$, $\epsilon_0 > 0$ such that, for any solution $U \in B^K_{s_0}(I;\epsilon_0)$ of* (6.2) *there exists a real-to-real invertible matrix of spectrally localized maps $\boldsymbol{\Psi}_1(U)$ such that*

1. *$\boldsymbol{\Psi}_1(U)$, $\boldsymbol{\Psi}_1(U)^{-1} \in \left(\mathcal{S}^0_{K,0,0}[\epsilon_0]\right)^{2\times 2}$ are linearly symplectic as per Definition A.3. Moreover, $\boldsymbol{\Psi}_1(U) - \mathrm{Id} \in \left(\Sigma\mathcal{S}^0_{K,0,1}[\epsilon_0, N]\right)^{2\times 2}$.*
2. *The variable $U_1 \triangleq \boldsymbol{\Psi}_1(U)U_0$ solves*

$$\begin{aligned} \partial_t U_1 = {} & \mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}\left(\mathrm{i}\,\lambda(U;x)\,\omega_{\gamma,\mathtt{b}}(\xi) - \mathrm{i}\,V^{(1)}_{\mathtt{b}}(U;x)\,\xi\right)U_1 \\ & + \boldsymbol{J}_{\mathbb{C}}\mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{A}^{(1)}_{\frac{1}{2}}(U;x)\,|\xi|^{\frac{1}{2}} + \boldsymbol{A}_{[0;1]}(U;x,\xi)\right)U_1 + \boldsymbol{R}_{[1]}(U)\,U_1, \end{aligned} \tag{6.13}$$

where

- *$\omega_{\gamma,\mathtt{b}}(\xi) \in \Gamma^{\frac{3}{2}}_0$ is defined in* (2.19)*;*
- *$\lambda(U;x)$ is defined in* (6.10)*;*
- *$V^{(1)}_{\mathtt{b}}(U;x) \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0, N]$;*
- *$\boldsymbol{A}^{(1)}_{\frac{1}{2}}(U;x) \triangleq \left(\mathtt{h}(U;x) + \mathtt{g}(U;x)\right)^2 \boldsymbol{A}^{(0)}_{\frac{1}{2}}(U;x) \in \left(\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0, N]\right)^{2\times 2}$.*

Proof. Arguing as in [25, Lemma 6.4] the 1-flow $\boldsymbol{\Psi}_1(U) \triangleq \boldsymbol{\Psi}^\tau(U)|_{\tau=0}$ of

$$\begin{cases} \partial_\tau \boldsymbol{\Psi}^\tau(U) = \boldsymbol{J}_{\mathbb{C}}\mathrm{Op}^{\mathrm{BW}}\left(\begin{bmatrix} \mathrm{i}\,\log\left(\mathtt{h}(U;x) + \mathtt{g}(U;x)\right) & 0 \\ 0 & -\mathrm{i}\,\log\left(\mathtt{h}(U;x) + \mathtt{g}(U;x)\right) \end{bmatrix}\right)\boldsymbol{\Psi}^\tau, \\ \boldsymbol{\Psi}^0(U) = \mathrm{Id} \end{cases} \tag{6.14}$$

is such that

$$\boldsymbol{\Psi}_1(U) = \mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{F}^{-1}(U;x)\right) + \boldsymbol{R}(U), \qquad \boldsymbol{\Psi}_1(U)^{-1} = \mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{F}(U;x)\right) + \boldsymbol{R}(U). \tag{6.15}$$

Since the generator in (6.14) is linearly Hamiltonian, Lemma A.6 ensures that $\boldsymbol{\Psi}_1(U)$ is a linearly symplectic, spectrally localized map in $\left(\Sigma\mathcal{S}^0_{K,K',1}[r, N]\right)^{2\times 2}$. Thus, the new variable $U_1 \triangleq \boldsymbol{\Psi}_1(U)\,U_0$ solves (cf. (6.2))

$$\begin{aligned} \partial_t U_1 = {} & \boldsymbol{\Psi}_1(U)\,\boldsymbol{J}_{\mathbb{C}}\,\mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{A}^{(0)}_{\frac{3}{2}}(U;x)\ \omega_{\gamma,\mathtt{b}}(\xi) + \boldsymbol{A}^{(0)}_{\frac{1}{2}}(U;x)\ |\xi|^{\frac{1}{2}} + \boldsymbol{A}_{[0;1]}(U;x,\xi)\right)\boldsymbol{\Psi}_1(U)^{-1}\,U_1 \\ & + \partial_t\boldsymbol{\Psi}_1(U)\,\boldsymbol{\Psi}_1(U)^{-1}\,U_1 - \boldsymbol{\Psi}_1(U)\,\mathrm{Op}^{\mathrm{BW}}\left(\mathrm{i}V_{\mathtt{b}}(\mathsf{M}U;x)\,\xi\right)\boldsymbol{\Psi}_1(U)^{-1}\,U_1 + \boldsymbol{\Psi}_1(U)\,\boldsymbol{R}(U)\,\boldsymbol{\Psi}_1(U)^{-1}\,U_1. \end{aligned} \tag{6.16}$$

Following the same computations as in [25, Eq. (6.3)] we have that

$$\begin{aligned} & \boldsymbol{\Psi}_1(U)\,\boldsymbol{J}_{\mathbb{C}}\,\mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{A}^{(0)}_{\frac{3}{2}}(U;x)\ \omega_{\gamma,\mathtt{b}}(\xi)\right)\boldsymbol{\Psi}_1(U)^{-1} \\ & = \mathrm{Op}^{\mathrm{BW}}\left(\mathrm{i}\lambda(U;x)\,\omega_{\gamma,\mathtt{b}}(\xi)\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + A_{[-\frac{1}{2};0]}(U;x,\xi)\right) + \boldsymbol{R}(U). \end{aligned} \tag{6.17}$$

Similar computations give us the conjugation

$$\begin{aligned}&\boldsymbol{\Psi}_1(U)\boldsymbol{J}_{\mathbb{C}}\operatorname{Op}^{\mathrm{BW}}\left(\boldsymbol{A}^{(0)}_{\frac12}(U;x)\ |\xi|^{\frac12}\right)\boldsymbol{\Psi}_1(U)^{-1}\\&=\boldsymbol{J}_{\mathbb{C}}\operatorname{Op}^{\mathrm{BW}}\left(\left(\mathtt{h}(U;x)+\mathtt{g}(U;x)\right)^2\boldsymbol{A}^{(0)}_{\frac12}(U;x)\ |\xi|^{\frac12}+A_{[-\frac12;0]}(U;x,\xi)\right)+\boldsymbol{R}(U)\end{aligned}\tag{6.18}$$

and

$$\boldsymbol{\Psi}_1(U)\operatorname{Op}^{\mathrm{BW}}\left(\boldsymbol{A}_{[0;1]}(U;x,\xi)\right)\boldsymbol{\Psi}_1(U)^{-1}=\operatorname{Op}^{\mathrm{BW}}\left(\boldsymbol{A}_{[0;1]}(U;x,\xi)\right)+\boldsymbol{R}(U),\tag{6.19}$$

while from Eqs. (6.12) and (6.15) we obtain that

$$\partial_t\boldsymbol{\Psi}_1(U)\boldsymbol{\Psi}_1(U)^{-1}=\operatorname{Op}^{\mathrm{BW}}\left(\boldsymbol{A}_{[0;1]}(U;x,\xi)\right)+\boldsymbol{R}_{[1]}(U),\tag{6.20}$$

and standard symbolic computations (cf. Proposition 3.15 and Eq. (6.15)) give that

$$\boldsymbol{\Psi}_1(U)\operatorname{Op}^{\mathrm{BW}}_{\mathrm{vec}}\left(\mathrm{i}V_{\mathtt{b}}(U;x)\xi\right)\boldsymbol{\Psi}_1(U)^{-1}=\operatorname{Op}^{\mathrm{BW}}_{\mathrm{vec}}\left(\mathrm{i}V^{(1)}_{\mathtt{b}}(U;x)\xi\right)+\boldsymbol{R}(U),\qquad V^{(1)}_{\mathtt{b}}(U;x)\in\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N].\tag{6.21}$$

We conclude plugging Eqs. (6.17) to (6.21) in Eq. (6.16). We remark that the zero-th order matrix $\boldsymbol{A}_{[0,1]}$ is the sum of $\boldsymbol{A}_{[-\frac12;0]}$ form Eqs. (6.17) and (6.18) and $\boldsymbol{A}_{[0;1]}$ from Eqs. (6.19) and (6.20). To prove that $\boldsymbol{J}_{\mathbb{C}}\boldsymbol{A}_{[0;1]}$ is linearly symplectic we then apply Lemma A.15 to each homogeneous components of the spectrally localized operators in Eqs. (6.17) to (6.20), which are linearly Hamiltonian thanks to Lemma A.5. ☐

6.3 Reduction to constant coefficients at the highest order

Lemma 6.4 (Reduction of the highest order). *Let $N\in\mathbb{N}$ and $\varrho>2(N+1)$. Then for any $K\in\mathbb{N}^*$ there are $s_0>0$, $\epsilon_0>0$ such that for any solution $U\in B^K_{s_0}(I;\epsilon_0)$ of* (6.13)*, there exists a real-to-real invertible matrix of spectrally localized maps $\boldsymbol{\Psi}_2(U)$ satisfying*

1. *$\boldsymbol{\Psi}_2(U)$, $\boldsymbol{\Psi}_2(U)\in\left(\mathcal{S}^0_{K,0,0}[\epsilon_0]\right)^{2\times2}$ are linearly symplectic as per Definition A.3. Moreover, $\boldsymbol{\Psi}_2(U)-\mathrm{Id}\in\left(\Sigma\mathcal{S}^{N+1}_{K,0,2}[\epsilon_0,N]\right)^{2\times2}$;*
2. *The variable $U_2\triangleq\boldsymbol{\Psi}_2(U)U_1$ solves the system*

$$\begin{aligned}\partial_tU_2=&\operatorname{Op}^{\mathrm{BW}}_{\mathrm{vec}}\left(\mathrm{i}\left[(1+v(U))\,\omega_{\gamma,\mathtt{b}}(\xi)-V^{(2)}_{\mathtt{b}}(U;x)\xi\right]\right)U_2\\&+\boldsymbol{J}_{\mathbb{C}}\operatorname{Op}^{\mathrm{BW}}\left(\boldsymbol{A}^{(2)}_{\frac12}(U;x)|\xi|^{\frac12}+\boldsymbol{A}_{[0;1]}(U;x,\xi)\right)U_2+\mathbf{R}(U)U_2,\end{aligned}\tag{6.22}$$

where

- *$v(U)$ is a x-independent function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,2}[\epsilon_0,N]$ and $\omega_{\gamma,\mathtt{b}}(\xi)$ is defined in* (2.19)*;*
- *$V^{(2)}_{\mathtt{b}}(U;x)$ is a real valued function in $\left(\Sigma\mathcal{F}^{\mathbb{R}}_{K,1,1}[\epsilon_0,N]\right)^2$;*
- *$\boldsymbol{A}^{(2)}_{\frac12}(U;x)\triangleq A^{(2)}_{\frac12}(U;x)\begin{bmatrix}1&1\\1&1\end{bmatrix}$ where $A^{(2)}_{\frac12}(U;x)\in\left(\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]\right)^{2\times2}$;*
- *$\mathbf{R}(U)\in\Sigma\mathcal{R}^{-\varrho+2(N+1)}_{K,1,1}[\epsilon_0,N]$.*

Proof. We refer the interested reader to [25, Lemma 6.7], the only difference being the conjugation of the term of order 1/2, which is achieved by standard paracomposition theorems, such as [18, Theorem 3.27]. Notice that the conjugation worsens the regularizing properties of the smoothing operator in Eq. (6.22). ☐

6.4 Diagonalization up to smoothing remainders

In this section we block-diagonalize the Kelvin-Helmholtz system up to a smoothing remainder.

Lemma 6.5 (Diagonalization to arbitrary order)**.** *Let $N \in \mathbb{N}$ and $\varrho \gg N$. Then for any $n \in \mathbb{N} \cup \{-1\}$, there exists $K' \triangleq K'(n) \geqslant 0$ such that for all $K \geqslant K'+1$, there exist $s_0 > 0$ and $\epsilon_0 > 0$ such that for any solution $U \in B^K_{s_0,\mathbb{R}}(I;\epsilon_0)$ of Eq.* (6.22)*, there exists a real-to-real invertible matrix of spectrally localized maps $\mathbf{\Phi}_n(U)$ satisfying*

1. *$\mathbf{\Phi}_n(U), \mathbf{\Phi}_n(U)^{-1} \in \left(\mathcal{S}^0_{K,0,0}[\epsilon_0]\right)^{2\times 2}$ are linearly symplectic up to homogeneity N as per Definition A.4. Moreover, $\mathbf{\Phi}_n(U) - \mathrm{Id} \in \left(\Sigma\mathcal{S}^0_{K,K',1}[r,N]\right)^{2\times 2}$;*
2. *The variable $U_{n+3} \triangleq \mathbf{\Phi}_n(U)U_2$ solves*

$$\begin{aligned}\partial_t U_{n+4} = {} & \mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}\left(\mathrm{i}\left[(1+v(U))\,\omega_{\gamma,\mathrm{b}}(\xi) - V^{(2)}_{\mathrm{b}}(U;x)\xi + a^{(n)}_{1/2}(U;x)\,|\xi|^{\frac12} + a^{(n)}_0(U;x,\xi)\right]\right)U_{n+3} \\ & + \boldsymbol{J}_{\mathbb{C}}\mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{A}_{[-n;K'+1]}(U;x,\xi)\right)U_{n+3} + \mathbf{R}(U)\,U_{n+3},\end{aligned} \tag{6.23}$$

where

- $a^{(n)}_{\frac12}(U;x) \in \mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N]$,
- *$a^{(n)}_0(U;x,\xi)$ is a symbol in $\Sigma\Gamma^0_{K,K',1}[\epsilon_0,N]$ with $\mathrm{Im}\,a^{(n)}_0(U;x,\xi) \in \Gamma^0_{K,K',N+1}[\epsilon_0]$,*
- $\mathbf{R}(U) \in \Sigma\mathcal{R}^{-\varrho+2(N+1)}_{K,1,1}[\epsilon_0,N]$.

Proof.

Step 1 (Reduction of the term of order $1/2$)**.** Notice that the term of order $1/2$ can be written as

$$\boldsymbol{J}_{\mathbb{C}}\boldsymbol{A}^{(2)}_{\frac12}(U;x)\,|\xi|^{1/2} = -\mathrm{i}a^{(-1)}_{1/2}(U;x)\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}|\xi|^{\frac12}, \qquad a^{(-1)}_{1/2}(U;x) \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[\epsilon_0,N].$$

Let now

$$\boldsymbol{F}^{(-1)}(U;x,\xi) \triangleq \begin{bmatrix}0 & F^{(-1)}(U;x,\xi)\\ F^{(-1)}(U;x,\xi) & 0\end{bmatrix}, \qquad F^{(-1)}(U;x,\xi) \triangleq \frac{a^{(-1)}_{1/2}(U;x)\,|\xi|^{1/2}}{2\,(1+v(U))\,\omega_{\gamma,\mathrm{b}}(\xi)} \in \Sigma\Gamma^{-1}_{K,0,1}[\epsilon_0,N]$$

and let $\left(\mathbf{\Phi}^{\tau}_{\boldsymbol{F}^{(-1)}}(U)\right)_{|\tau|\leqslant 1}$ be the flow generated by $\mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{F}^{(-1)}(U;x,\xi)\right)$ with initial condition $\mathbf{\Phi}^0_{\boldsymbol{F}^{(-1)}}(U) = \mathrm{Id}$ as per Lemma A.6. We denote with $\mathbf{\Phi}(U) \triangleq \mathbf{\Phi}^1_{\boldsymbol{F}^{(-1)}}(U)$. Since $\boldsymbol{F}^{(-1)}$ is linearly Hamiltonian, Lemma A.6 ensures that $\mathbf{\Phi}(U)$ is a linearly symplectic, spectrally localized map in $\left(\Sigma\mathcal{S}^0_{K,K',1}[r,N]\right)^{2\times 2}$. Let us define

$$U_3 \triangleq \mathbf{\Phi}^1_{\boldsymbol{F}^{(-1)}}(U)\,U_2 = \mathbf{\Phi}(U)\,U_2,$$

which is the solution of

$$\begin{aligned}\partial_t U_3 = {} & \mathbf{\Phi}(U)\,\mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}\left(\mathrm{i}\left[(1+v(U))\,\omega_{\gamma,\mathrm{b}}(\xi) - V^{(2)}_{\mathrm{b}}(U;x)\xi\right]\right)\mathbf{\Phi}(U)^{-1}\,U_3 \\ & + (\partial_t\mathbf{\Phi}(U))\,\mathbf{\Phi}(U)^{-1}\,U_3 + \mathbf{\Phi}(U)\,\boldsymbol{J}_{\mathbb{C}}\mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{A}^{(2)}_{\frac12}(U;x)\,|\xi|^{\frac12} + \boldsymbol{A}_{[0;1]}(U;x,\xi)\right)\mathbf{\Phi}(U)^{-1}\,U_3 \\ & + \mathbf{\Phi}(U)\,\mathbf{R}(U)\mathbf{\Phi}(U)^{-1}\,U_3.\end{aligned} \tag{6.24}$$

The following conjugation rule apply (cf. [19, Lemma A.1]) setting $\mathbf{F} \triangleq \mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{F}^{(-1)}(U;x,\xi)\right)$

$$\mathbf{\Phi}(U)\,\boldsymbol{M}(U)\,\mathbf{\Phi}(U)^{-1} = \boldsymbol{M} + \sum_{q=1}^{L}\frac{1}{q!}\mathrm{Ad}^q_{\mathbf{F}}[\boldsymbol{M}] + \frac{1}{L!}\int_0^1 (1-\tau)^L\,\mathbf{\Phi}^{\tau}_{\boldsymbol{F}^{(-1)}}(U)\,\mathrm{Ad}^{L+1}_{\mathbf{F}}[\boldsymbol{M}]\,\mathbf{\Phi}^{\tau}_{\boldsymbol{F}^{(-1)}}(U)^{-1}\,\mathrm{d}\tau, \tag{6.25}$$

$$(\partial_t\mathbf{\Phi}(U))\,\mathbf{\Phi}(U)^{-1} = \boldsymbol{J}_{\mathbb{C}}\mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{A}_{[-1;1]}(U;x,\xi)\right) + \boldsymbol{R}_{[1]}(U), \tag{6.26}$$

An application of Eq. (6.25) for $L > \frac{3+2\varrho}{4} - 1$ combined with symbolic calculus considerations give us that

$$\begin{aligned} &\boldsymbol{\Phi}(U)\left(\mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}\left(\mathrm{i}\,(1+v(U))\,\omega_{\gamma,\mathtt{b}}(\xi)\right) + \boldsymbol{J}_{\mathbb{C}}\mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{A}^{(2)}_{\frac12}(U;x)\,|\xi|^{\frac12} + \boldsymbol{A}_{[0;1]}(U;x,\xi)\right)\right)\boldsymbol{\Phi}(U)^{-1} \\ &= \mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}\left(\mathrm{i}\left[(1+v(U))\,\omega_{\gamma,\mathtt{b}}(\xi) + a^{(-1)}_{1/2}(U;x)\,|\xi|^{\frac12}\right]\right) + \boldsymbol{J}_{\mathbb{C}}\mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{A}_{[0;1]}(U;x,\xi)\right) + \boldsymbol{R}(U), \end{aligned} \tag{6.27}$$

in which the off-diagonal terms of order 1/2 have been canceled. Applying Eqs. (6.25) to (6.27) to (6.24) (and renaming $\boldsymbol{\Phi}(U)\,\mathbf{R}(U)\boldsymbol{\Phi}(U)^{-1} \rightsquigarrow \mathbf{R}(U)$ in light of Proposition 3.17, Item 2 we obtain that

$$\begin{aligned} \partial_t U_3 = \mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}&\left(\mathrm{i}\left[(1+v(U))\,\omega_{\gamma,\mathtt{b}}(\xi) - V^{(2)}_{\mathtt{b}}(U;x)\xi + a^{(2)}_{1/2}(U;x)\,|\xi|^{\frac12}\right]\right)U_3 \\ &+ \boldsymbol{J}_{\mathbb{C}}\mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{A}_{[0;1]}(U;x,\xi)\right)U_3 + \boldsymbol{R}(U)U_3. \end{aligned} \tag{6.28}$$

Notice that the conjugation rule in Eq. (6.25) does not modify the principal symbol of the transport term. Moreover the zero-th order matrix $\boldsymbol{A}_{[0,1]}$ is the sum of $\boldsymbol{A}_{[-1;1]}$ form Eq. (6.26) and $\boldsymbol{A}_{[0;1]}$ from Eq. (6.27). To prove that $\boldsymbol{J}_{\mathbb{C}}\boldsymbol{A}_{[0;1]}$ is linearly symplectic we then apply Lemma A.15 to each homogeneous components of the spectrally localized operators in Eqs. (6.26) and (6.27), which are linearly Hamiltonian thanks to Lemma A.5.

Step 2 (Reduction at non-positive order)**.** We sketch the proof which is very similar to the one outlined in Step 1, we refer the interested reader to [25, Lemma 6.8]. The proof is performed by induction on $n \in \mathbb{N}$, the case $n = 0$ is proven in Step 1, so that we can consider the case $n \rightsquigarrow n+1$. Suppose Eq. (6.23) holds, we have to find a bounded, lineary-symplectic transformation that pushes the off diagonal terms of $\boldsymbol{J}_{\mathbb{C}}\mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{A}_{[-n;K'+1]}\right)$ to lower order, similarly as in was done in Step 1. Since the matrix $\boldsymbol{A}_{[-n;K'+1]}$ is of the form

$$\boldsymbol{J}_{\mathbb{C}}\boldsymbol{A}_{[-n;K'+1]} = \boldsymbol{J}_{\mathbb{C}}\begin{bmatrix} -\mathrm{i}\bar{b}^{\vee}_{[-n]} & -\bar{a}^{\vee}_{[-n]} \\ -a_{[-n]} & \mathrm{i}\bar{b}_{[-n]} \end{bmatrix}, \qquad a_{[-n]},\ \bar{b}_{[-n]} \in \Sigma\Gamma^{-n}_{K,K'+1,1}[\epsilon_0, N], \tag{6.29}$$

such cancellation is achieved via conjugation by the flow

$$\begin{cases} \partial_\tau \boldsymbol{\Phi}^{\tau}_{\boldsymbol{F}^{(n)}}(U) = \mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{F}^{(n)}(U)\right)\boldsymbol{\Phi}^{\tau}_{\boldsymbol{F}^{(n)}}(U), \\ \boldsymbol{\Phi}^{0}_{\boldsymbol{F}^{(n)}}(U) = \mathrm{Id}, \end{cases} \qquad \boldsymbol{F}^{(n)}(U) \triangleq \begin{bmatrix} 0 & f_{-n-\frac32} \\ f^{\vee}_{-n-\frac32} & 0 \end{bmatrix},$$

$$f_{-n-\frac32}(U;t,x,\xi) \triangleq -\frac{b_{-n}(U;x,\xi)}{2\mathrm{i}\omega(\xi)(1+v(U))} \in \Sigma\Gamma^{-n-\frac32}_{K,K'+1,1}[\epsilon_0,N],$$

and defining the variable $U_{n+4} \triangleq \boldsymbol{\Phi}^{1}_{\boldsymbol{F}^{(n)}}(U)\,U_{n+3}$ and we refer the reader to [25, Lemma 6.8] for further details. As the matrix in (6.29) is linearly Hamiltonian up to homogeneity N by inductive hypothesis, the explicitly defined generator $\boldsymbol{F}^{(n)}$ is linearly Hamiltonian up to homogeneity N as well. Thus Lemma A.6 ensures that $\boldsymbol{\Phi}^{\tau}_{\boldsymbol{F}^{(n)}}(U)$ is a linearly symplectic, spectrally localized map in $\left(\Sigma\mathcal{S}^{0}_{K,K',1}[r,N]\right)^{2\times 2}$. The bounded, linearly symplectic transformation is thus defined as

$$\boldsymbol{\Phi}_n(U) \triangleq \prod_{\mathrm{j}=-1}^{n} \boldsymbol{\Phi}^{1}_{\boldsymbol{F}^{(\mathrm{j})}}(U).$$

□

We can thus apply Lemma 6.5 setting $n \triangleq n_1(\varrho) \geqslant -\varrho + 2(N+1)$ so that $\boldsymbol{J}_{\mathbb{C}}\mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{A}_{[-n;K'+1]}\right)$ can be incorporated in the smoothing reminder $\mathbf{R}(U)$, thus setting $Z \triangleq U_{n_1+4}$ we obtain that Z solves the evolution equation

$$\partial_t Z = \mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}\left(\mathrm{i}\left[(1+v(U))\,\omega_{\gamma,\mathtt{b}}(\xi) - V^{(2)}_{\mathtt{b}}(U;x)\xi + a^{(n_1)}_{1/2}(U;x)\,|\xi|^{\frac12} + a^{(n_1)}_{0}(U;x,\xi)\right]\right)Z + \mathbf{R}(U)\,Z. \tag{6.30}$$

6.5 Reduction to constant coefficients up to smoothing remainders

Notation 6.6. From now on we shall denote functions and symbols that are x-independent with calligraphic lower- and upper-case letters.

Lemma 6.7. *Let $N \in \mathbb{N}$ and $\varrho > 3(N+1)$. Then for any $n \in \mathbb{N}$ there is $K'' \triangleq K''(\varrho, n) > 0$ such that for all $K \geqslant K''+1$ there are $s_0 > 0$, $\epsilon_0 > 0$ such that for any solution $U \in B^K_{s_0,\mathbb{R}}(I;\epsilon_0)$ of (5.9), there exists a real-to-real invertible matrix of spectrally localized maps $\mathbf{\Theta}_n(U)$ such that*

1. *$\mathbf{\Theta}_n(U), \mathbf{\Theta}_n(U)^{-1} \in \left(\mathcal{S}^0_{K,K'',0}[\epsilon_0]\right)^{2\times 2}$ are linearly symplectic up to homogeneity N according to Definition A.4. Moreover, $\mathbf{\Theta}_n(U) - \mathrm{Id} \in \left(\Sigma\mathcal{S}^{\frac{N+1}{2}}_{K,K'',1}[\epsilon_0, N]\right)^{2\times 2}$.*

2. *If Z solves (6.30) then the variable $Z_n \triangleq \mathbf{\Theta}_n(U)Z$ solves*

$$\partial_t Z_n = \mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}\left(\mathrm{i}\,\mathcal{d}^{(n)}_{3/2}(U;t,\xi) + \mathrm{i}\,a_{-\frac{n}{2}}(U;t,x,\xi)\right) Z_n + \mathsf{R}(U;t) Z_n, \tag{6.31}$$

with the x–independent symbol

$$\mathcal{d}^{(n)}_{3/2}(U;t,\xi) \triangleq (1+v(U;t))\omega(\xi) + \mathcal{V}_{\mathfrak{b}}(U;t)\xi + \mathcal{b}_{\frac{1}{2}}(U;t)|\xi|^{\frac{1}{2}} + \mathcal{b}^{(n)}_0(U;t,\xi), \tag{6.32}$$

where

- *$v(U) \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,2}[\epsilon_0, N]$;*
- *the function $\mathcal{V}(U;t) \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,1,2}[\epsilon_0, N]$ is x-independent;*
- *the function $\mathcal{b}_{\frac{1}{2}}(U;t) \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,2,2}[\epsilon_0, N]$ is x-independent;*
- *the symbol $\mathcal{b}^{(n)}_0(U;t,\xi) \in \Sigma\Gamma^0_{K,K'',2}[\epsilon_0, N]$ is x–independent and its imaginary part $\mathrm{Im}\,\mathcal{b}^{(n)}_0(U;t,\xi)$ is in $\Gamma^0_{K,K'',N+1}[\epsilon_0]$;*
- *the symbol $a_{-\frac{n}{2}}(U;t,x,\xi)$ belongs to $\Sigma\Gamma^{-\frac{n}{2}}_{K,K''+1,1}[\epsilon_0, N]$ and its imaginary part $\mathrm{Im}\,a_{-\frac{n}{2}}(U;t,x,\xi)$ is in $\Gamma^{-\frac{n}{2}}_{K,K''+1,N+1}[\epsilon_0]$;*
- *$\mathsf{R}(U;t)$ is a real-to-real matrix of smoothing operators in $\left(\Sigma\mathcal{R}^{-\varrho+3(N+1)}_{K,K''+1,1}[\epsilon_0, N]\right)^{2\times 2}$.*

Proof. The proof consist of a minor modification of the proof of [25, lemma 6.9], so we refer the interested reader to [25, lemma 6.9] for details. □

Finally we prove Proposition 6.1.

Proof of Proposition 6.1 In Lemma 6.7 we set

$$n_2 \triangleq n_2(\varrho, N) \triangleq n \geqslant 2\left(\varrho + 3(N+1)\right), \qquad \underline{K'} \triangleq K''+1$$

and we define

$$W \triangleq Z_{n_2} = \boldsymbol{B}(U;t)\,U,$$

where

$$\boldsymbol{B}(U;t) \triangleq \mathbf{\Theta}_{n_2}(U;t) \circ \mathbf{\Phi}_{n_1}(U;t) \circ \mathbf{\Psi}_2(U;t) \circ \mathbf{\Psi}_1(U;t) \circ \boldsymbol{G}(U;t).$$

At last we set $\mathcal{b}^{(n)}_0 \triangleq \mathcal{b}_0$ and we conclude. □

7 Hamiltonian Birkhoff normal form and energy estimate

In this section we finally prove the almost global existence Theorem 1.1. The previous reduction broke the Hamiltonian structure of the system. Therefore, a first step is to recover the complex Hamiltonian formulation through a symplectic correction. The corresponding result is stated in Proposition 7.1 and serves as the foundation for the subsequent normal form analysis. Then, in the technical subsection 7.1, we introduce the class of super-action preserving Hamiltonians that play a central role in the forthcoming analysis, as they do not contribute to the energy estimate. Next, in Section 7.2, we perform a Birkhoff normal form procedure, which extracts the Hamiltonian super-action preserving property up to homogeneity $N+1$, for a given fixed integer N. Finally, we implement in Section 7.3 the energy estimate allowing to conclude the desired Theorem.

The transformation $\boldsymbol{B}(U)$ in Proposition 6.1 destroyed the Hamiltonian property of the system. However $\boldsymbol{B}(U)$ was still linearly symplectic and therefore, we can recover the complex Hamiltonian structure by applying the abstract Darboux Theorem of Berti-Maspero-Murgante [25, Theorem 7.1], see also Theorem A.16. The corresponding result states as follows.

Proposition 7.1 (**Hamiltonian reduction up to smoothing operators).** *Let $N\in\mathbb{N}$ and $\varrho>\varrho(N)\triangleq 3(N+1)+\frac{3}{2}(N+1)^3$. Then, for any $K\geqslant \underline{K}'$ (fixed in Proposition 6.1) there is $s_0>0,\epsilon_0>0$, such that for any solution $U\in B^K_{s_0,\mathbb{R}}(I;\epsilon_0)$ of* (5.9)*, there exists a real-to-real matrix of pluri–homogeneous smoothing operators $\boldsymbol{R}(U)$ in $\left(\Sigma_1^N\widetilde{\mathcal{R}}_q^{-\varrho'}\right)^{2\times 2}$ for any $\varrho'\geqslant 0$, such that defining*

$$Z_0\triangleq\big(\mathrm{Id}+\boldsymbol{R}(\Phi(U))\big)\Phi(U),\qquad \Phi(U)\triangleq\boldsymbol{B}(U;t)U,\tag{7.1}$$

where $\boldsymbol{B}(U;t)$ is defined in Proposition 6.1, the following holds true:

- **i Symplecticity:** *The non-linear map $\big(\mathrm{Id}+\boldsymbol{R}(\cdot)\big)\circ\Phi$ in* (7.1) *is symplectic up to homogeneity N according to Definition A.11.*
- **ii Conjugation:** *The variable Z_0 solves the* Hamiltonian system up to homogeneity N *(cfr. Definition A.10).*

$$\partial_t Z_0=\mathrm{i}\boldsymbol{\omega}_{\gamma,\mathtt{b}}(D)Z_0+\mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}\Big(\mathrm{i}(d_{\frac32})_{\leqslant N}(Z_0;\xi)+\mathrm{i}(d_{\frac32})_{>N}(U;t,\xi)\Big)Z_0+\boldsymbol{R}_{\leqslant N}(Z_0)Z_0+\boldsymbol{R}_{>N}(U;t)U,\tag{7.2}$$

where

- $\boldsymbol{\omega}_{\gamma,\mathtt{b}}(D)\triangleq\mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}\big(\omega_{\gamma,\mathtt{b}}(\xi)\big)$ *is a matrix in the space $\big(\Gamma_0^{3/2}\big)^{2\times 2}$;*
- $(d_{\frac32})_{\leqslant N}$ *is a pluri-homogeneous, real valued symbol, independent of x, in $\Sigma_2^N\widetilde{\Gamma}_q^{\frac32}$;*
- $(d_{\frac32})_{>N}$ *is a non–homogeneous symbol, independent of x, in $\Gamma^{\frac32}_{K,\underline{K}',N+1}[\epsilon_0]$ with imaginary part $\mathrm{Im}(d_{\frac32})_{>N}$ in $\Gamma^0_{K,\underline{K}',N+1}[\epsilon_0]$;*
- $\boldsymbol{R}_{\leqslant N}(Z_0)$ *is a real-to-real matrix of smoothing operators in $\left(\Sigma_1^N\widetilde{\mathcal{R}}_q^{-\varrho+\varrho(N)}\right)^{2\times 2}$;*
- $\boldsymbol{R}_{>N}(U;t)$ *is a real-to-real matrix of non–homogeneous smoothing operators in $\left(\mathcal{R}^{-\varrho+\varrho(N)}_{K,\underline{K}',N+1}[\epsilon_0]\right)^{2\times 2}$.*

iii Boundedness: *The variable $Z_0=\mathbf{M}_0(U;t)U$ with $\mathbf{M}_0(U;t)\in\left(\mathcal{M}^0_{K,\underline{K}'-1,0}[\epsilon_0]\right)^{2\times 2}$ and for any $s\geqslant s_0$, there is $0<\epsilon_0(s)<\epsilon_0$, such that for any $U\in B^K_{s_0}(I;\epsilon_0)\cap C^K_{*\mathbb{R}}(I;\dot{H}^s(\mathbb{T},\mathbb{C}^2))$, there is a constant $C\triangleq C_{s,K}>0$ such that, for all $k=0,\ldots,K-\underline{K}'$,*

$$C^{-1}\|U\|_{k,s}\leqslant\|Z_0\|_{k,s}\leqslant C\|U\|_{k,s}.\tag{7.3}$$

7.1 Super-action preserving symbols and Hamiltonians

In this section we define the special class of “super–action preserving" SAP homogeneous symbols and Hamiltonians which will appear in the Birkhoff normal form reduction of the Section 7.2.

Definition 7.2. (SAP **monomial**) Let $p \in \mathbb{N}^*$. Given $(\vec{\jmath},\vec{\sigma}) = (j_a,\sigma_a)_{a=1,\dots,p} \in (\mathbb{Z}^*)^p \times \{\pm\}^p$ we define the multi-index $(\alpha,\beta) \in \mathbb{N}^{\mathbb{Z}^*} \times \mathbb{N}^{\mathbb{Z}^*}$ with components, for any $k \in \mathbb{Z}^*$,

$$
\begin{aligned}
\alpha_k(\vec{\jmath},\vec{\sigma}) &\triangleq \#\big\{a = 1,\dots,p : (j_a,\sigma_a) = (k,+)\big\}, \\
\beta_k(\vec{\jmath},\vec{\sigma}) &\triangleq \#\big\{a = 1,\dots,p : (j_a,\sigma_a) = (k,-)\big\}.
\end{aligned}
\tag{7.4}
$$

We say that a monomial of the form $z_{\vec{\jmath}}^{\vec{\sigma}} = z_{j_1}^{\sigma_1}\dots z_{j_p}^{\sigma_p}$ is super-action preserving if the associated multi-index $(\alpha,\beta) = (\alpha(\vec{\jmath},\vec{\sigma}),\beta(\vec{\jmath},\vec{\sigma}))$ is super-action preserving according to Definition 2.5.

We now introduce the subset $\mathfrak{S}_p$ of the indexes of $\mathfrak{T}_p$ composed by super-action preserving indexes

$$
\mathfrak{S}_p \triangleq \Big\{(\vec{\jmath},\vec{\sigma}) \in \mathfrak{T}_p \quad \text{s.t.} \quad (\alpha(\vec{\jmath},\vec{\sigma}),\beta(\vec{\jmath},\vec{\sigma})) \in \mathbb{N}^{\mathbb{Z}^*} \times \mathbb{N}^{\mathbb{Z}^*} \text{ in (7.4) are super action preserving}\Big\}. \tag{7.5}
$$

We remark that the multi-index (α,β) associated to $(\vec{\jmath},\vec{\sigma}) \in (\mathbb{Z}^* \times \{\pm\})^p$ as in (7.4) satisfies $|\alpha+\beta| = p$ and

$$
z_{\vec{\jmath}}^{\vec{\sigma}} = z^\alpha \bar{z}^\beta \triangleq \prod_{j\in\mathbb{Z}\setminus\{0\}} z_j^{\alpha_j}\overline{z_j}^{\beta_j} = \prod_{n\in\mathbb{N}} z_n^{\alpha_n} z_{-n}^{\alpha_{-n}}\overline{z_n}^{\beta_n}\overline{z_{-n}}^{\beta_{-n}} . \tag{7.6}
$$

It turns out

$$
\vec{\sigma}\cdot\vec{\omega}_{\gamma,\mathtt{b}}(\vec{\jmath}) = \sigma_1\omega_{\gamma,\mathtt{b}}(j_1) + \dots + \sigma_p\omega_{\gamma,\mathtt{b}}(j_p) = (\alpha-\beta)\cdot\vec{\omega}_{\gamma,\mathtt{b}} = \sum_{k\in\mathbb{Z}^*}(\alpha_k-\beta_k)\omega_{\gamma,\mathtt{b}}(k), \tag{7.7}
$$

where we denote

$$
\vec{\omega}_{\gamma,\mathtt{b}}(\vec{\jmath}) \triangleq (\omega_{\gamma,\mathtt{b}}(j_1),\dots,\omega_{\gamma,\mathtt{b}}(j_p)), \qquad \vec{\omega}_{\gamma,\mathtt{b}} \triangleq \{\omega_{\gamma,\mathtt{b}}(j)\}_{j\in\mathbb{Z}^*}. \tag{7.8}
$$

Remark 7.3. If the monomial $z_{\vec{\jmath}}^{\vec{\sigma}}$ is super–action preserving then, for any $j \in \mathbb{Z}^*$, the monomial $z_{\vec{\jmath}}^{\vec{\sigma}} z_j \bar{z}_j$ is super-action preserving as well.

For any $n \in \mathbb{N}^*$, we define the *super-action*

$$
J_n \triangleq |z_n|^2 + |z_{-n}|^2. \tag{7.9}
$$

The following is Lemma 7.7 in [25].

Lemma 7.4. *The Poisson bracket between a monomial $z_{\vec{\jmath}}^{\vec{\sigma}}$ and a super-action J_n, $n \in \mathbb{N}$, defined in* (7.9)*, is*

$$
\Big\{z_{\vec{\jmath}}^{\vec{\sigma}}, J_n\Big\} = \mathrm{i}\big(\beta_n + \beta_{-n} - \alpha_n - \alpha_{-n}\big)\, z_{\vec{\jmath}}^{\vec{\sigma}}, \tag{7.10}
$$

where $(\alpha,\beta) = (\alpha(\vec{\jmath},\vec{\sigma}),\beta(\vec{\jmath},\vec{\sigma}))$ is the multi-index defined in (7.4)*. In particular a super-action preserving monomial $z_{\vec{\jmath}}^{\vec{\sigma}}$ (according to Definition 7.2) Poisson commutes with any super-action J_n, $n \in \mathbb{N}$.*

We now define a super-action preserving Hamiltonian.

Definition 7.5. (SAP **Hamiltonian**) Let $p \in \mathbb{N}$. A $(p+2)$–homogeneous super-action preserving Hamiltonian $H_{p+2}^{(\mathtt{SAP})}(Z)$ is a real function of the form

$$
H_{p+2}^{(\mathtt{SAP})}(Z) = \frac{1}{p+2}\sum_{(\vec{\jmath}_{p+2},\vec{\sigma}_{p+2})\in\mathfrak{S}_{p+2}} H_{\vec{\jmath}_{p+2}}^{\vec{\sigma}_{p+2}} z_{\vec{\jmath}_{p+2}}^{\vec{\sigma}_{p+2}}
$$

where $\mathfrak{S}_{p+2}$ is defined as in (7.5). A pluri-homogeneous super-action preserving Hamiltonian is a finite sum of homogeneous super-action preserving Hamiltonians. A Hamiltonian vector field is super-action preserving if it is generated by a super-action preserving Hamiltonian.

We now define a super-action preserving symbol.

Definition 7.6. (SAP **symbol**) Let $p \in \mathbb{N}$ and $m \in \mathbb{R}$. For $p \geqslant 1$ a real valued, p–homogeneous *super-action preserving symbol* of order m is a symbol $\mathtt{m}_p^{(\mathtt{SAP})}(Z;\xi)$ in $\widetilde{\Gamma}_p^m$, independent of x, of the form

$$\mathtt{m}_p^{(\mathtt{SAP})}(Z;\xi) = \sum_{(\vec{\jmath}_p,\vec{\sigma}_p)\in\mathfrak{S}_p} M_{\vec{\jmath}_p}^{\vec{\sigma}_p}(\xi) z_{\vec{\jmath}_p}^{\vec{\sigma}_p}.$$

For $p = 0$ we say that any symbol in $\widetilde{\Gamma}_0^m$ is super-action preserving. A pluri-homogeneous super-action preserving symbol is a finite sum of homogeneous super-action preserving symbols.

Remark 7.7. A super-action preserving symbol has even degree p of homogeneity. Indeed, if $z_{\vec{\jmath}_p}^{\vec{\sigma}_p}$ is super-action preserving then (α,β) defined in (7.4) satisfies $|\alpha| = |\beta|$ and $p = |\alpha+\beta| = 2|\alpha|$ is even.

Given a super-action preserving symbol we associate a super-action preserving Hamiltonian according to the following lemma (see Lemma 7.11 in [25]).

Lemma 7.8. *Let $p \in \mathbb{N}$, $m \in \mathbb{R}$. If $(\mathtt{m}^{(\mathtt{SAP})})_p(Z;\xi)$ is a p–homogeneous super-action preserving symbol in $\widetilde{\Gamma}_p^m$ according to Definition 7.6 then*

$$H_{p+2}^{(\mathtt{SAP})}(Z) \triangleq \mathrm{Re}\left\langle \mathrm{Op}^{\mathrm{BW}}\left((\mathtt{m}^{(\mathtt{SAP})})_p(Z;\xi)\right)z \;\middle|\; \bar{z}\right\rangle_{\mathbb{R}}$$

is a $(p+2)$–homogeneous super-action preserving Hamiltonian according to Definition 7.5.

7.2 Birkhoff normal form reduction

In this section we finally transform the system (7.2) into its Hamiltonian Birkhoff normal form, up to homogeneity N.

Proposition 7.9. (Hamiltonian Birkhoff normal form) *Let $N \in \mathbb{N}$ and $0 < \beta_1 < \beta_2 < 4\left(2+\sqrt{3}\right)$. Assume that the parameter $\beta = \frac{\mathtt{b}^2}{\gamma} \in [\beta_1,\beta_2]$ is outside the zero measure set $\mathcal{B} \subset [\beta_1,\beta_2]$ defined in Proposition 2.6. Then, there exists $\underline{\varrho} = \underline{\varrho}(N) > 0$ such that, for any $\varrho \geqslant \underline{\varrho}$, for any $K \geqslant K' \triangleq \underline{K'}(\varrho)$ (defined in Proposition 6.1), there exists $\underline{s_0} > 0$ such that, for any $s \geqslant \underline{s_0}$ there is $\underline{\epsilon_0} \triangleq \underline{\epsilon_0}(s) > 0$ such that for all $0 < \epsilon_0 < \underline{\epsilon_0}(s)$ small enough, and any solution $U \in B_{\underline{s_0}}^K(I;\epsilon_0) \cap C_{*\mathbb{R}}^K(I;\dot{H}^s(\mathbb{T};\mathbb{C}^2))$ of the complex Kelvin-Helmholtz system* (5.9)*, there exists a non–linear map $\mathcal{F}_{\mathtt{nf}}(Z_0)$ such that:*

- **i Symplecticity:** *$\mathcal{F}_{\mathtt{nf}}(Z_0)$ is symplectic up to homogeneity N (Definition A.11);*
- **ii Conjugation:** *If Z_0 solves the system* (7.2) *then the variable $Z \triangleq \mathcal{F}_{\mathtt{nf}}(Z_0)$ solves the* Hamiltonian system up to homogeneity N *(cfr. Definition A.10)*

$$\partial_t Z = \mathrm{i}\boldsymbol{\omega}_{\gamma,\mathtt{b}}(D)Z + \boldsymbol{J}_{\mathbb{C}}\nabla H_{\frac{3}{2}}^{(\mathtt{SAP})}(Z) + \boldsymbol{J}_{\mathbb{C}}\nabla H_{-\varrho}^{(\mathtt{SAP})}(Z) + \mathrm{Op}_{\mathrm{vec}}^{\mathrm{BW}}\left(\mathrm{i}(d_{\frac{3}{2}})_{>N}(U;t,\xi)\right)Z + \boldsymbol{R}_{>N}(U;t)U, \tag{7.11}$$

  where

  - *$H_{\frac{3}{2}}^{(\mathtt{SAP})}(Z)$ is the super-action preserving Hamiltonian*

$$\mathrm{Re}\left\langle \mathrm{Op}^{\mathrm{BW}}\left((d_{\frac{3}{2}}^{(\mathtt{SAP})})_{\leqslant N}(Z;\xi)\right)z \;\middle|\; \bar{z}\right\rangle_{\mathbb{R}},$$

    with a pluri-homogeneous super-action preserving symbol $(d_{\frac{3}{2}}^{(\mathtt{SAP})})_{\leqslant N}(Z;\xi)$ in $\Sigma_2^N\widetilde{\Gamma}_q^{\frac{3}{2}}$, according to Definition 7.6;
  - *$\boldsymbol{J}_{\mathbb{C}}\nabla H_{-\varrho}^{(\mathtt{SAP})}(Z)$ is a super-action preserving, Hamiltonian, smoothing vector field in $\Sigma_3^{N+1}\widetilde{\mathfrak{X}}_q^{-\varrho+\underline{\varrho}}$ (see 7.5);*
  - *$(d_{\frac{3}{2}})_{>N}(U;t,\xi)$ is a non–homogeneous symbol in $\Gamma_{K,\underline{K'},N+1}^{\frac{3}{2}}[\epsilon_0]$ with imaginary part $\mathrm{Im}(d_{\frac{3}{2}})_{>N}(U;t,\xi)$ in $\Gamma_{K,\underline{K'},N+1}^{0}[\epsilon_0]$;*

- $\boldsymbol{R}_{>N}(U;t)$ *is a real-to-real matrix of non–homogeneous smoothing operators in* $\left(\mathcal{R}^{-\varrho+\underline{\varrho}}_{K,\underline{K'},N+1}[\epsilon_0]\right)^{2\times 2}$.

iii Boundedness: *There exists a constant* $C \triangleq C_{s,K} > 0$ *such that for all* $0 \leqslant k \leqslant K$ *and any* $Z_0 \in B^K_{\underline{s}_0}(I;\epsilon_0) \cap C^K_{*\mathbb{R}}(I;\dot{H}^s(\mathbb{T},\mathbb{C}^2))$ *one has*

$$C^{-1}\|Z_0\|_{k,s} \leqslant \|\mathcal{F}_{\mathtt{nf}}(Z_0)\|_{k,s} \leqslant C\|Z_0\|_{k,s} \tag{7.12}$$

and

$$C^{-1}\|U(t)\|_{\dot{H}^s} \leqslant \|Z(t)\|_{\dot{H}^s} \leqslant C\|U(t)\|_{\dot{H}^s}, \quad \forall t \in I. \tag{7.13}$$

Notation 7.10. From now on we denote with $\boldsymbol{R}^{\mathtt{H}}_p$ any smoothing remainder in $\widetilde{\mathcal{R}}^{-\varrho}_p$ such that $\boldsymbol{R}^{\mathtt{H}}_p(Z)Z$ is a $p+1$-homogeneous Hamiltonian vector field, namely $\boldsymbol{R}^{\mathtt{H}}_p(Z)Z = \boldsymbol{J}_{\mathbb{C}}\nabla H_{p+2}(Z)$ for some $p+2$ homogeneous Hamiltonian as per Definition A.8.

Proof of Proposition 7.9. The proof consists in N steps and is analogous to the proof of Proposition 7.12 in [25]. For completeness, we only sketch the main steps and we refer to [25] for a detailed proof.

We first reduce the quadratic terms of the vector field in (7.2).

Step 1: Elimination of the quadratic smoothing remainder in equation (7.2)**.**

The x-independent symbol $(d_{\frac{3}{2}})_{\leqslant N}(Z_0;\xi)$ in (7.2) belongs to $\Sigma_2^N\widetilde{\Gamma}^{\frac{3}{2}}_q$ and the only quadratic component of the vector field in (7.2) is $\boldsymbol{R}^{\mathtt{H}}_1(Z_0)Z_0$ where (recall the notation in (3.18))

$$\boldsymbol{R}^{\mathtt{H}}_1(Z_0) \triangleq \mathcal{P}_1[\boldsymbol{R}_{\leqslant N}(Z_0)] \in \left(\widetilde{\mathcal{R}}^{-\varrho+\varrho(N)}_1\right)^{2\times 2}. \tag{7.14}$$

Since the system (7.2) is Hamiltonian up to homogeneity N, $\boldsymbol{R}^{\mathtt{H}}_1(Z_0)Z_0$ is a Hamiltonian vector field that we expand in Fourier coordinates as (recall (3.19))

$$\left(\boldsymbol{R}^{\mathtt{H}}_1(Z_0)Z_0\right)^{\sigma}_k = \sum_{(j_1,j_2,k,\sigma_1,\sigma_2,-\sigma)\in\mathfrak{T}_3} X^{\sigma_1,\sigma_2,\sigma}_{j_1,j_2,k}(z_0)^{\sigma_1}_{j_1}(z_0)^{\sigma_2}_{j_2}. \tag{7.15}$$

In order to remove $\boldsymbol{R}^{\mathtt{H}}_1(Z_0)Z_0$ from (7.2), we perform the change of variable $Z_1 = \mathtt{F}^{(1)}_{\leqslant N}(Z_0)$ where $\mathtt{F}^{(1)}_{\leqslant N}(Z_0)$ is the map given by Lemma A.17 relative to a Hamiltonian smoothing vector field

$$\left(\boldsymbol{G}^{\mathtt{H}}_1(Z_0)Z_0\right)^{\sigma}_k = \sum_{(j_1,j_2,k,\sigma_1,\sigma_2,-\sigma)\in\mathfrak{T}_3} G^{\sigma_1,\sigma_2,\sigma}_{j_1,j_2,k}(z_0)^{\sigma_1}_{j_1}(z_0)^{\sigma_2}_{j_2}, \quad \boldsymbol{G}^{\mathtt{H}}_1(Z_0) \in \left(\mathcal{R}^{-\varrho'}_1\right)^{2\times 2} \tag{7.16}$$

for some $\varrho' > 0$ to be determined. Since $\boldsymbol{G}^{\mathtt{H}}_1(Z)Z$ is a Hamiltonian vector field, Lemma A.17 gives by construction that $\mathtt{F}^{(1)}_{\leqslant N}$ is symplectic up to homogeneity N and it has the form

$$Z_1 \triangleq \mathtt{F}^{(1)}_{\leqslant N}(Z_0) = Z_0 + \boldsymbol{G}^{\mathtt{H}}_1(Z_0)Z_0 + \boldsymbol{F}_{\geqslant 2}(Z_0)Z_0, \qquad \boldsymbol{F}_{\geqslant 2}(Z_0) \in \left(\Sigma^N_2\widetilde{\mathcal{R}}^{-\varrho'}_q\right)^{2\times 2}. \tag{7.17}$$

Applying Lemma A.12, we obtain that the variable Z_1 solves a system which is Hamiltonian up to homogeneity N. We compute it using Lemma B.2. Its assumption **(A)** at page 65 holds since Z_0 solves (7.2). Then Lemma B.2 implies that the variable Z_1 solves

$$\begin{aligned}\partial_t Z_1 = \mathrm{i}\boldsymbol{\omega}_{\gamma,\mathtt{b}}(D)Z_1 + \mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}\left(\mathrm{i}(d_{\frac{3}{2}})^+_{\leqslant N}(Z_1;\xi) + \mathrm{i}(d_{\frac{3}{2}})^+_{>N}(U;t,\xi)\right)Z_1 \\ + [\boldsymbol{R}^{\mathtt{H}}_1(Z_1) + \boldsymbol{G}^+_1(Z_1)]Z_1 + R^+_{\geqslant 2}(Z_1)Z_1 + R^+_{>N}(U;t)U,\end{aligned} \tag{7.18}$$

where

- $(d_{\frac{3}{2}})^+_{\leqslant N}(Z_1;\xi)$ is a real valued symbol, independent of x, in $\Sigma^N_2\widetilde{\Gamma}^{\frac{3}{2}}_q$;
- $(d_{\frac{3}{2}})^+_{>N}(U;t,\xi)$ is a non-homogeneous real valued symbol, independent of x, in $\Gamma^{\frac{3}{2}}_{K,\underline{K'},N+1}[\epsilon_0]$ with imaginary part $\mathrm{Im}(d_{\frac{3}{2}})_{>N}(U;t,\xi)$ in $\Gamma^0_{K,\underline{K'},N+1}[\epsilon_0]$;
- $\boldsymbol{R}^{\mathtt{H}}_1(Z_1)$ is defined in (7.14) and $\boldsymbol{G}^+_1(Z_1)Z_1 \in \widetilde{\mathfrak{X}}^{-\varrho'+\frac{3}{2}}_2$ has Fourier expansion, by (B.15) and (7.16),

$$(\boldsymbol{G}^+_1(Z_1)Z_1)^{\sigma}_k = \sum_{(j_1,j_2,k,\sigma_1,\sigma_2,-\sigma)\in\mathfrak{T}_3} \mathrm{i}\big(\sigma_1\omega_{\gamma,\mathtt{b}}(j_1) + \sigma_2\omega_{\gamma,\mathtt{b}}(j_2) - \sigma\omega_{\gamma,\mathtt{b}}(k)\big)G^{\sigma_1,\sigma_2,\sigma}_{j_1,j_2,k}(z_1)^{\sigma_1}_{j_1}(z_1)^{\sigma_2}_{j_2}; \tag{7.19}$$

• $\boldsymbol{R}^{+}_{\geqslant 2}(Z_1)$ is a matrix of pluri-homogeneous smoothing operators in $\left(\Sigma_2^N \tilde{\mathcal{R}}_q^{-\varrho+\underline{\varrho}(2)}\right)^{2\times 2}$ where

$$-\varrho+\underline{\varrho}(2) \triangleq -\varrho' + \frac{3}{2}\,; \tag{7.20}$$

• $\boldsymbol{R}^{+}_{>N}(U;t)$ is a matrix of non–homogeneous smoothing operators in $\left(\mathcal{R}^{-\varrho+\underline{\varrho}(2)}_{K,\underline{K'},N+1}[\epsilon_0]\right)^{2\times 2}$.

By (7.15), (7.19) we solve

$$\boldsymbol{R}^{\mathrm{H}}_1(Z_1)Z_1 + \boldsymbol{G}^{+}_1(Z_1)Z_1 = 0 \tag{7.21}$$

by taking

$$G^{\sigma_1,\sigma_2,\sigma}_{j_1,j_2,k} \triangleq \begin{cases} 0, & \text{if } (j_1,j_2,k,\sigma_1,\sigma_2,-\sigma)\notin \mathfrak{T}_3, \\ -\dfrac{X^{\sigma_1,\sigma_2,\sigma}_{j_1,j_2,k}}{\mathrm{i}\big(\sigma_1\omega_{\gamma,\mathtt{b}}(j_1)+\sigma_2\omega_{\gamma,\mathtt{b}}(j_2)-\sigma\omega_{\gamma,\mathtt{b}}(k)\big)}, & \text{if } (j_1,j_2,k,\sigma_1,\sigma_2,-\sigma)\in \mathfrak{T}_3\,. \end{cases} \tag{7.22}$$

The previous expression and the fact that the vector field $\boldsymbol{R}^{\mathrm{H}}_1(Z_0)Z_0$ is Hamiltonian justifies a posteriori that indeed $\boldsymbol{G}^{\mathrm{H}}_1(Z_0)Z_0$ is Hamiltonian. Moreover, combining (7.14) and Proposition 2.6, taking $\beta\in[\beta_1,\beta_2]\setminus\mathcal{B}$, we get that $\boldsymbol{G}^{\mathrm{H}}_1(Z_0)Z_0 \in \widetilde{\mathfrak{X}}^{-\varrho'}_2$ with $\varrho' \triangleq \varrho-\varrho(N)-\tau$. Thus, by Eq. (7.20), we get $\underline{\varrho}(2)=\varrho(N)+\tau+\frac{3}{2}$.

Step $p\geqslant 2$**:** We now assume that the system is in normal form up to degree $p-1$ of homogeneity. Next, we illustrate how to perform the normal form transformation on the terms of degree p. Specifically, we assume that Z_{p-1} solves

$$\begin{aligned}\partial_t Z_{p-1} &= \mathrm{i}\boldsymbol{\omega}_{\gamma,\mathtt{b}}(D)Z_{p-1} + \boldsymbol{J}_{\mathbb{C}}\nabla\big(H^{(\mathtt{SAP})}_{\frac{3}{2}}\big)_{\leqslant p+1}(Z_{p-1}) + \boldsymbol{J}_{\mathbb{C}}\nabla\big(H^{(\mathtt{SAP})}_{-\varrho}\big)_{\leqslant p+1}(Z_{p-1}) \\ &\quad + \mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}\Big(\mathrm{i}(d_{\frac{3}{2}})_p(Z_{p-1};\xi) + \mathrm{i}(d_{\frac{3}{2}})_{\geqslant p+1}(Z_{p-1};\xi)\Big)Z_{p-1} + \boldsymbol{R}^{\mathrm{H}}_{\geqslant p}(Z_{p-1})Z_{p-1} \\ &\quad + \mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}\Big(-\mathrm{i}(d_{\frac{3}{2}})_{>N}(U;t,\xi)\Big)Z_{p-1} + \boldsymbol{R}_{>N}(U;t)U,\end{aligned} \tag{7.23}$$

where $\big(H^{(\mathtt{SAP})}_{\frac{3}{2}}\big)_{\leqslant p+1}$, $\big(H^{(\mathtt{SAP})}_{-\varrho}\big)_{\leqslant p+1}$ are pluri-homogeneous super-action preserving Hamiltonians, $(d_{\frac{3}{2}})_p\in\widetilde{\Gamma}^{\frac{3}{2}}_p$ and expands as

$$(d_{\frac{3}{2}})_p(Z_{p-1};\xi) = \sum_{(\vec{\jmath}_p,\vec{\sigma}_p)\in\mathfrak{T}_p} \mathtt{D}^{\vec{\sigma}_p}_{\vec{\jmath}_p}(\xi)\,(z_{p-1})^{\vec{\sigma}_p}_{\vec{\jmath}_p}, \qquad \overline{\mathtt{D}^{-\vec{\sigma}_p}_{\vec{\jmath}_p}}(\xi) = \mathtt{D}^{\vec{\sigma}_p}_{\vec{\jmath}_p}(\xi). \tag{7.24}$$

Moreover $(d_{\frac{3}{2}})_{\geqslant(p+1)}\in\Sigma^N_{p+1}\Gamma^{\frac{3}{2}}_q$ and $\boldsymbol{R}^{\mathrm{H}}_{\geqslant p}\in\Sigma^N_p\mathcal{R}^{-\varrho+\underline{\varrho}(p)}_q$. We reduce the homogeneous component $\mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}\Big(-\mathrm{i}(d_{\frac{3}{2}})_p\Big)+R_p$ into its resonant normal form. First of all we define the intermediate variable

$$W \triangleq \Phi_p(Z_{p-1}) \triangleq \boldsymbol{\mathcal{G}}^1_{g_p}(Z_{p-1})Z_{p-1}, \tag{7.25}$$

where $\boldsymbol{\mathcal{G}}^1_{g_p}(Z_{p-1})$ is the time 1-linear flow generated by $\mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}(\mathrm{i}g_p)$, where g_p is the Fourier multiplier

$$g_p(Z_{p-1};\xi) \triangleq \sum_{(\vec{\jmath}_p,\vec{\sigma}_p)\in\mathfrak{T}_p} \mathtt{G}^{\vec{\sigma}_p}_{\vec{\jmath}_p}(\xi)(z_{p-1})^{\vec{\sigma}_p}_{\vec{\jmath}_p} \in \widetilde{\Gamma}^{\frac{3}{2}}_p. \tag{7.26}$$

Then Lemma B.1 implies that the variable W defined in (7.25) solves

$$\begin{aligned}\partial_t W &= \mathrm{i}\boldsymbol{\omega}_{\gamma,\mathtt{b}}(D)W + \boldsymbol{J}_{\mathbb{C}}\nabla\big(H^{(\mathtt{SAP})}_{\frac{3}{2}}\big)_{\leqslant p+1}(W) + \boldsymbol{J}_{\mathbb{C}}\nabla\big(H^{(\mathtt{SAP})}_{-\varrho}\big)_{\leqslant p+1}(W) \\ &\quad + \mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}\Big(\mathrm{i}[(d_{\frac{3}{2}})_p(W;\xi)+g^{+}_p(W;\xi)] + \mathrm{i}(d_{\frac{3}{2}})^{+}_{\geqslant p+1}(W;\xi)\Big)W + \boldsymbol{R}_{\geqslant p}(W)W \\ &\quad + \mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}\Big(\mathrm{i}(d_{\frac{3}{2}})^{+}_{>N}(U;t,\xi)\Big)W + \boldsymbol{R}_{>N}(U;t)U,\end{aligned} \tag{7.27}$$

where g^{+}_p expands as

$$g^{+}_p(W;\xi) = \sum_{(\vec{\jmath}_p,\vec{\sigma}_p)\in\mathfrak{T}_p} \mathrm{i}\vec{\sigma}_p\cdot\vec{\omega}_{\gamma,\mathtt{b}}(\vec{\jmath}_p)\mathtt{G}^{\vec{\sigma}_p}_{\vec{\jmath}_p}(\xi)\,w^{\vec{\sigma}_p}_{\vec{\jmath}_p}, \tag{7.28}$$

while $\boldsymbol{R}_{\geqslant p} \in \left(\Sigma_1^N \widetilde{\mathcal{R}}_q^{-\varrho+\underline{\varrho}(p)+c(N,p)}\right)^{2\times 2}$, $\boldsymbol{R}_{>N} \in \left(\mathcal{R}_{K,K',N+1}^{-\varrho+\underline{\varrho}(p)+c(N,p)}[\epsilon_0]\right)^{2\times 2}$ for some $c(N,p)>0$. In order to get a p-homogeneous super-action preserving normal form symbol, we solve the homological equation

$$(d_{\frac{3}{2}})_p(W;\xi) + g_p^+(W;\xi) = (d_{\frac{3}{2}}^{(\mathtt{SAP})})_p(W;\xi) \triangleq \sum_{(\vec{\jmath}_p,\vec{\sigma}_p)\in\mathfrak{S}_p} \mathtt{D}_{\vec{\jmath}_p}^{\vec{\sigma}_p}(\xi)\, w_{\vec{\jmath}_p}^{\vec{\sigma}_p},$$

where $\mathfrak{S}_p$ has been introduced in (7.5). One solution is obtained by choosing

$$\mathtt{G}_{\vec{\jmath}_p}^{\vec{\sigma}_p}(\xi) \triangleq \begin{cases} 0, & \text{if } (\vec{\jmath}_p,\vec{\sigma}_p)\in\mathfrak{S}_p, \\ \dfrac{\mathtt{D}_{\vec{\jmath}_p}^{\vec{\sigma}_p}(\xi)}{\mathrm{i}\vec{\sigma}_p\cdot\vec{\omega}_{\gamma,\mathtt{b}}(\vec{\jmath}_p)}, & \text{if } (\vec{\jmath}_p,\vec{\sigma}_p)\notin\mathfrak{S}_p. \end{cases} \tag{7.29}$$

The previous expression (7.29) and the fact that $(d_{\frac{3}{2}})_p$ is real valued justifies a posteriori that indeed g_p is real valued. Moreover, combining estimate (3.5) for $(d_{\frac{3}{2}})_p$ and Proposition 2.6, taking $\beta\in[\beta_1,\beta_2]\setminus\mathcal{B}$, we get that g_p satisfies the corresponding estimate (3.5) with $\mu\rightsquigarrow\mu+\tau$. Now, we observe that, by Lemma A.13,

$$\mathrm{Op}_{\mathrm{vec}}^{\mathrm{BW}}\left(\mathrm{i}(d_{\frac{3}{2}}^{(\mathtt{SAP})})_p(W;\xi)\right)W = \boldsymbol{J}_{\mathbb{C}}\nabla\big(H_{\frac{3}{2}}^{(\mathtt{SAP})}\big)_{p+2}(W) + \boldsymbol{R}_p(W)W, \tag{7.30}$$

with Hamiltonian

$$\big(H_{\frac{3}{2}}^{(\mathtt{SAP})}\big)_{p+2}(W) \triangleq \mathrm{Re}\Big\langle \mathrm{Op}^{\mathrm{BW}}\left((d_{\frac{3}{2}}^{(\mathtt{SAP})})_p(W;\xi)\right)w,\bar{w}\Big\rangle_{\mathbb{R}}, \tag{7.31}$$

which is super-action preserving by Lemma 7.8, and a matrix of smoothing operators $\boldsymbol{R}_p(W)$ in $\left(\widetilde{\mathcal{R}}_p^{-\varrho'}\right)^{2\times 2}$ for any $\varrho'\geqslant 0$. Therefore (7.27) becomes

$$\begin{aligned} \partial_t W = & \mathrm{i}\boldsymbol{\omega}_{\gamma,\mathtt{b}}(D)W + \boldsymbol{J}_{\mathbb{C}}\nabla\big(H_{\frac{3}{2}}^{(\mathtt{SAP})}\big)_{\leqslant p+2}(W) + \boldsymbol{J}_{\mathbb{C}}\nabla\big(H_{-\varrho}^{(\mathtt{SAP})}\big)_{\leqslant p+1}(W) \\ & + \mathrm{Op}_{\mathrm{vec}}^{\mathrm{BW}}\left(\mathrm{i}(d_{\frac{3}{2}})_{\geqslant p+1}^{+}(W;\xi)\right)W + [\boldsymbol{R}_{\geqslant p}(W)]W \\ & + \mathrm{Op}_{\mathrm{vec}}^{\mathrm{BW}}\left(\mathrm{i}(d_{\frac{3}{2}})_{>N}^{+}(U;t,\xi)\right)W + \boldsymbol{R}_{>N}(U;t)U, \end{aligned} \tag{7.32}$$

where (see (7.23),(7.31))

$$\big(H_{\frac{3}{2}}^{(\mathtt{SAP})}\big)_{\leqslant p+2} \triangleq \big(H_{\frac{3}{2}}^{(\mathtt{SAP})}\big)_{\leqslant p+1} + \big(H_{\frac{3}{2}}^{(\mathtt{SAP})}\big)_{p+2}, \tag{7.33}$$

while $\boldsymbol{R}_{\geqslant p}$ is a new smoothing remainder in $\left(\Sigma_1^N\widetilde{\mathcal{R}}_q^{-\varrho+\underline{\varrho}(p)+c(N,p)}\right)^{2\times 2}$. Note that the new system (7.32) is not Hamiltonian up to homogeneity N (unlike system (7.23) for Z_{p-1}), since the map $\Phi_p(Z_{p-1}) = \boldsymbol{\mathcal{G}}_{g_p}^1(Z_{p-1})Z_{p-1}$ in (7.25) is not symplectic up to homogeneity N. By Lemma A.7 we only know that $\boldsymbol{\mathcal{G}}_{g_p}^1(Z_{p-1})$ is linearly symplectic. We now apply Theorem A.16 to find a correction of $\Phi_p(Z_{p-1})$ which is symplectic up to homogeneity N. The assumptions of Theorem A.16 are verified applying Lemma A.7, therefore there exists $\boldsymbol{R}_{\leqslant N}^{(p)} \in \left(\Sigma_p^N\widetilde{\mathcal{R}}_q^{-\varrho-\frac{3}{2}}\right)^{2\times 2}$ such that the variable

$$V \triangleq \mathrm{C}_N^{(p)}(W) \triangleq \big(\mathrm{Id} + \boldsymbol{R}_{\leqslant N}^{(p)}(W)\big)W = \big(\mathrm{Id} + \boldsymbol{R}_{\leqslant N}^{(p)}(\Phi_p(Z_{p-1}))\big)\Phi_p(Z_{p-1}) \tag{7.34}$$

is symplectic up to homogeneity N, thus solves a system which is Hamiltonian up to homogeneity N. Since W solves (7.32) then Lemma B.2 implies that $V = W + \boldsymbol{R}_{\leqslant N}^{(p)}(W)W$ solves

$$\begin{aligned} \partial_t V = & \mathrm{i}\boldsymbol{\omega}_{\gamma,\mathtt{b}}(D)V + \boldsymbol{J}_{\mathbb{C}}\nabla\big(H_{\frac{3}{2}}^{(\mathtt{SAP})}\big)_{\leqslant p+2}(V) + \boldsymbol{J}_{\mathbb{C}}\nabla\big(H_{-\varrho}^{(\mathtt{SAP})}\big)_{\leqslant p+1}(V) \\ & + \mathrm{Op}_{\mathrm{vec}}^{\mathrm{BW}}\left(\mathrm{i}\widetilde{(d_{\frac{3}{2}})}_{\geqslant p+1}(V;\xi)\right)V + \boldsymbol{J}_{\mathbb{C}}\nabla\left(H_{-\varrho}\right)_{p+2}(V) + \boldsymbol{R}_{\geqslant p+1}(V)V \\ & + \mathrm{Op}_{\mathrm{vec}}^{\mathrm{BW}}\left(\mathrm{i}\widetilde{(d_{\frac{3}{2}})}_{>N}(U;t,\xi)\right)V + \boldsymbol{R}_{>N}(U;t)U, \end{aligned} \tag{7.35}$$

where $\widetilde{(d_{\frac{3}{2}})}_{\geqslant p+1} \in \Sigma_{p+1}^N\widetilde{\Gamma}_q^{\frac{3}{2}}$, $\boldsymbol{J}_{\mathbb{C}}\nabla\left(H_{-\varrho}\right)_{p+2} \in \mathfrak{X}_p^{-\varrho+\underline{\varrho}(p)+C(N,p)}$, $\boldsymbol{R}_{\geqslant p+1} \in \Sigma_p^N\widetilde{\mathcal{R}}^{-\varrho+\underline{\varrho}(p)+C(N,p)}$. Notice that the Hamiltonian structure of $\boldsymbol{J}_{\mathbb{C}}\nabla\left(H_{-\varrho}\right)_p(V)$ is justified a posteriori by the fact that the p-homogeneous component of the vector field in (7.35) is Hamiltonian and the term in its first line is Hamiltonian, thus we deduce the Hamiltonianity by difference. Thus we can proceed as in Step 1. First we Fourier expand $\boldsymbol{J}_{\mathbb{C}}\nabla\left(H_{-\varrho}\right)_{p+2}$ as

$$\left(\boldsymbol{J}_{\mathbb{C}}\nabla\left(H_{-\varrho}\right)_{p+2}(V)\right)_k^{\sigma}=\sum_{(\vec{\jmath}_{p+1},k,\vec{\sigma}_{p+1},-\sigma)\in\mathfrak{T}_{p+2}} X_{\vec{\jmath}_{p+1},k}^{\vec{\sigma}_{p+1},\sigma}\, v_{\vec{\jmath}_{p+1}}^{\vec{\sigma}_{p+1}}. \tag{7.36}$$

Then we use Lemma A.17 to generate a symplectic up to homogeneity N map $\mathcal{F}_{\leqslant N}^{(p)}$ associate to the Hamiltonian smoothing vector field

$$\left(\boldsymbol{G}_p(V)V\right)_k^{\sigma}=\sum_{(\vec{\jmath}_{p+1},k,\vec{\sigma}_{p+1},-\sigma)\in\mathfrak{T}_{p+2}} G_{\vec{\jmath}_{p+1},k}^{\vec{\sigma}_{p+1},\sigma}\, v_{\vec{\jmath}_{p+1}}^{\vec{\sigma}_{p+1}}. \tag{7.37}$$

Applying Lemma B.2 the analogue of the homological equation (7.21) is

$$\boldsymbol{J}_{\mathbb{C}}\nabla\left(H_{-\varrho}\right)_{p+2}(Z_p)+\boldsymbol{G}_p^{+}(Z_p)Z_p=\boldsymbol{J}_{\mathbb{C}}\nabla\left(H_{-\varrho}^{(\mathsf{SAP})}\right)_p(Z_p)\triangleq\sum_{(\vec{\jmath}_{p+1},k,\vec{\sigma}_{p+1},-\sigma)\in\mathfrak{S}_{p+2}} X_{\vec{\jmath}_{p+1},k}^{\vec{\sigma}_{p+1},\sigma}\,(z_p)_{\vec{\jmath}_{p+1}}^{\vec{\sigma}_{p+1}} \tag{7.38}$$

where the vector field $\boldsymbol{G}_p^{+}(Z_p)Z_p$ is explicitly given by

$$(\boldsymbol{G}_p^{+}(Z_p)Z_p)_k^{\sigma}\triangleq\sum_{(\vec{\jmath}_{p+1},k,\vec{\sigma}_{p+1},-\sigma)\in\mathfrak{T}_{p+2}}\mathrm{i}\big(\vec{\sigma}_{p+1}\cdot\vec{\omega}_{\gamma,\mathtt{b}}(\vec{\jmath}_{p+1})-\sigma\omega_{\gamma,\mathtt{b}}(k)\big)G_{\vec{\jmath}_{p+1},k}^{\vec{\sigma}_{p+1},\sigma}\,(z_p)_{\vec{\jmath}_{p+1}}^{\vec{\sigma}_{p+1}}.$$

Then a solution of (7.38) is given by

$$G_{\vec{\jmath}_{p+1},k}^{\vec{\sigma}_{p+1},\sigma}\triangleq\begin{cases}0, & \text{if } (\vec{\jmath}_{p+1},k,\vec{\sigma}_{p+1},-\sigma)\in\mathfrak{S}_{p+2},\\ -\dfrac{X_{\vec{\jmath}_{p+1},k}^{\vec{\sigma}_{p+1},\sigma}}{\mathrm{i}(\vec{\sigma}_{p+1}\cdot\vec{\omega}_{\gamma,\mathtt{b}}(\vec{\jmath}_{p+1})-\sigma\omega_{\gamma,\mathtt{b}}(k))}, & \text{if } (\vec{\jmath}_{p+1},k,\vec{\sigma}_{p+1},-\sigma)\notin\mathfrak{S}_{p+2}.\end{cases} \tag{7.39}$$

The previous expression and the fact that the vector field $\boldsymbol{J}_{\mathbb{C}}\nabla\left(H_{-\varrho}\right)_{p+2}$ is Hamiltonian justifies a posteriori that indeed $G_p(Z_p)Z_p$ is Hamiltonian. Moreover, since $\boldsymbol{J}_{\mathbb{C}}\nabla\left(H_{-\varrho}\right)_{p+2}$ is ϱ_p-smoothing, we apply Proposition 2.6, taking $\beta\in[\beta_1,\beta_2]\setminus\mathcal{B}$ and we get that $\boldsymbol{G}_p(Z_p)Z_p\in\widetilde{\mathfrak{X}}_2^{-\varrho_{p+1}}$ with $\varrho_{p+1}\triangleq\varrho_p-c(N)-\tau$. □

7.3 Energy estimate and proof of the main Theorem

Remark 7.11. We can derive local existence for (5.9) following the computations in the main result of [23] and exploiting the fact that the local existence result of [23] uses the very same paradifferential functional setting as in the present manuscript. Minor modifications in the works [3, 51] can also provide local existence for the setting of (1.4).

The following result, analogous to [19, Lemma 6.3], enables to bound the norms $\|\partial_t^k U(t)\|_{s-\frac{3}{2}k}$ of the time derivatives of a solution $U(t)$ of (5.9) by $\|U(t)\|_s$.

Lemma 7.12. *Let $K\in\mathbb{N}$. There exists $s_0>0$ such that for any $s\geqslant s_0$, any $\epsilon\in\left(0,\overline{\epsilon_0}(s)\right)$ small, if U belongs to $B_{s_0,\mathbb{R}}^0(I;\epsilon)\cap C_*^0\left(I;\dot{H}^s(\mathbb{T};\mathbb{C}^2)\right)$ and solves* (5.2) *then $U\in C_{*\mathbb{R}}^K\left(I;\dot{H}^s(\mathbb{T};\mathbb{C}^2)\right)$ and there exists $C_1\triangleq C_1(s,K)\geqslant 1$ such that*

$$\|U(t)\|_s\leqslant\|U(t)\|_{K,s}\leqslant C_1\|U(t)\|_s,\qquad\forall t\in I.$$

Proof. The proof is inductive and follows the same line of [19, Lemma 6.3], so we sketch only the first step. As $\|U(t)\|_s\leqslant\|U(t)\|_{K,s}$ is obvious in view of the definition (3.3), it remains to prove the second inequality. We start by estimating $\|\partial_t U(t)\|_{s-\frac{3}{2}}$. Using equation (5.2) for $U(t)$ and then (3.13) with $k=K'=p=0$, $m=\frac{3}{2}$ and estimate (3.16) with $m\rightsquigarrow-\varrho$ and $k=0$, we get

$$\begin{aligned}\|\partial_t U(t)\|_{s-\frac{3}{2}}&\leqslant\left\|\mathrm{Op}^{\mathrm{BW}}\Big(\boldsymbol{A}_{\frac{3}{2}}(U;x)\ \omega_{\gamma,\mathtt{b}}(\xi)+\boldsymbol{A}_1(U;x,\xi)+\boldsymbol{A}_{\frac{1}{2}}(U;x)\ |\xi|^{\frac{1}{2}}+\boldsymbol{A}_{[0]}(U;x,\xi)\Big)U\right\|_{s-\frac{3}{2}}+\|\boldsymbol{R}(U)U\|_{s-\frac{3}{2}}\\&\lesssim\|U(t)\|_s.\end{aligned}$$

The estimates for $k\geqslant 2$ follow in a similar manner, additionally using estimates (3.13) and (3.16) with $k\geqslant 1$ to handle higher-order derivatives.

□

We fix now the parameters appearing in Proposition 7.9. In its statement, we set $\varrho \triangleq \underline{\varrho}(N)$ and $K \triangleq \underline{K'}(\varrho)$, which implies the existence of the constant $\underline{s_0} > 0$ which we increase, if necessary, to fit the range of Lemma 7.12, such that for any $s \geqslant \underline{s_0}$, and any fixed $0 < \epsilon_0 \leqslant \min\{\overline{\epsilon_0}(s), \underline{\epsilon_0}(s)\}$, where $\underline{\epsilon_0}(s)$ is defined in Proposition 7.9, and $\overline{\epsilon_0}(s)$ in Lemma 7.12, the conclusions of Proposition 7.9 and Lemma 7.12 hold. Therefore, one can obtain the following energy estimate. Notice that the time-reversibility of the Kelvin-Helmholtz system allows us to restrict the discussion to positive times $t > 0$.

Lemma 7.13 (Energy estimate)**.** *Let $U(t)$ be a solution of equation* (5.9) *in* $B^K_{\underline{s_0}}(I;\epsilon_0) \cap C^K_*\left(I; \dot{H}^s\left(\mathbb{T};\mathbb{C}^2\right)\right)$. *Then there exists $\bar{C}_2(s) > 1$ such that*

$$\|U(t)\|_s^2 \leqslant \bar{C}_2(s)\left(\|U(0)\|_s^2 + \int_0^t \|U(\tau)\|_{s_0}^{N+1} \|U(\tau)\|_s^2 \,\mathrm{d}\tau\right), \quad \forall 0 < t < T. \tag{7.40}$$

Proof. The variable Z defined in Proposition 7.9 solves the normal form Equation (7.11). Then, denoting by $H^{(\mathtt{SAP})} \triangleq H^{(\mathtt{SAP})}_{\frac{3}{2}} + H^{(\mathtt{SAP})}_{-\varrho}$ we have that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|Z(t)\|_s^2 &= \mathrm{d}_Z\|Z\|_s^2\left[\mathrm{i}\boldsymbol{\omega}_{\gamma,\mathtt{b}}(D)Z + \boldsymbol{J}_{\mathbb{C}}\nabla H^{(\mathtt{SAP})} + \mathrm{Op}^{\mathrm{BW}}\left(\mathrm{i}(d_{\frac{3}{2}})_{>N}(U;t,\xi)\right)Z + \boldsymbol{R}_{>N}(U;t)U\right] \\
&= \sum_{n\in\mathbb{N}} |n|^{2s}\left\{J_n,\ H^{(\mathtt{SAP})}\right\} \\
&\quad + \left\langle |D|^{2s}\,\mathrm{Op}^{\mathrm{BW}}\left(-\mathrm{Im}\left(\left(d_{\frac{3}{2}}\right)_{>N}(U;t,\xi)\right)\right)Z \,\middle|\, Z\right\rangle + 2\mathrm{Re}\left\langle |D|^{2s}\boldsymbol{R}_{>N}(U;t)\,U \,\middle|\, Z\right\rangle \\
&\lesssim \|U\|_{K',s_0}^{N+1}\left(\|Z\|_s + \|U\|_s\right)\|Z\|_s,
\end{aligned}$$

where we used Lemma 7.4 and the fact that $-\mathrm{Im}\left(\left(d_{\frac{3}{2}}\right)_{>N}(U;t,\xi)\right)$ is in $\Gamma^0_{K,K',N+1}[\epsilon_0]$ and $\boldsymbol{R}_{>N}(U;t)$ is in $\mathcal{M}^0_{K,K',N+1}[\epsilon_0]$. Integrating in time the above inequality, then using Eq. (7.13) and Lemma 7.12, we obtain (7.40). ☐

The energy estimate (7.40) and the equivalence

$$\left\|\eta(t)\right\|_{H_0^{s+\frac{1}{4}}} + \left\|\psi(t)\right\|_{\dot{H}^{s-\frac{1}{4}}} \sim \|U(t)\|_s, \tag{7.41}$$

which is a consequence of (5.8), imply, by a standard bootstrap argument, Theorem 1.1. For detailed computations we refer the interested reader to [18, 19, 25].

A Hamiltonian formalism

In this appendix we set the Hamiltonian formalism for the problem at hand, following [25, Section 3].

A.1 Linearly Hamiltonian systems

In this section we provide the linear Hamiltonian framework needed in our analysis. The linear Hamiltonian structure is the natural algebraic property that emerges from the paralinearization of an Hamiltonian system.

Definition A.1 (Linearly Hamiltonian paradifferential operator)**.** A real-to-real matrix of paradifferential complex operators is *linearly Hamiltonian* if it is of the form

$$\boldsymbol{J}_{\mathbb{C}}\mathrm{Op}^{\mathrm{BW}}(\boldsymbol{B}), \qquad \boldsymbol{B} \triangleq \begin{bmatrix} b_1(U;t,x,\xi) & b_2(U;t,x,\xi) \\ \overline{b_2(U;t,x,-\xi)} & \overline{b_1(U;t,x,-\xi)} \end{bmatrix}, \qquad \begin{cases} b_1(U;t,x,-\xi) = b_1(U;t,x,\xi), \\ b_2(U;t,x,\xi) \in \mathbb{R}. \end{cases} \tag{A.1}$$

In other words, b_1 is even in ξ and b_2 is real valued. This is equivalent to require that the matrix valued paradifferential operator $\mathrm{Op}^{\mathrm{BW}}(\boldsymbol{B})$ is symmetric.

Definition A.2 (Linearly Hamiltonian operator up to homogeneity N)**.** A real-to-real matrix of spectrally localized maps $\boldsymbol{J}_{\mathbb{C}}\mathbf{B}(U;t)$ in $\left(\Sigma\mathcal{S}_{K,K',p}[\epsilon_0,N]\right)^{2\times2}$ is *linearly Hamiltonian up to homogeneity N* if its pluri-homogeneous component $\mathcal{P}_{\leqslant N}(\mathbf{B}(U;t))$ (defined through (3.18)) is symmetric, namely

$$\mathcal{P}_{\leqslant N}(\mathbf{B}(U;t)) = \mathcal{P}_{\leqslant N}\left(\mathbf{B}(U;t)^{\intercal}\right).$$

In particular, a real-to-real matrix of paradifferential operators is linearly Hamiltonian up to homogeneity N if it has the form (cfr. (A.1))

$$\boldsymbol{J}_{\mathbb{C}}\mathrm{Op}^{\mathrm{BW}}\begin{pmatrix} b_1(U;t,x,\xi) & b_2(U;t,x,\xi) \\ b_2(U;t,x,-\xi) & b_1(U;t,x,-\xi)\end{pmatrix}, \qquad \begin{cases} b_1(U;t,x,-\xi)-b_1(U;t,x,\xi)\in\Gamma^m_{K,K',N+1}[\epsilon_0], \\ \mathrm{Im}\, b_2(U;t,x,\xi)\in\Gamma^{m'}_{K,K',N+1}[\epsilon_0]\end{cases} \tag{A.2}$$

for some m, m' in $\mathbb{R}$.

Definition A.3 (Linearly symplectic map)**.** A real-to-real linear transformation $\mathcal{A}$ is linearly symplectic if $\mathcal{A}^*\Omega_{\mathbb{C}} = \Omega_{\mathbb{C}}$, where $\Omega_{\mathbb{C}}$ is defined as

$$\Omega_{\mathbb{C}}\left(\begin{bmatrix} u_1 \\ \bar{u}_1\end{bmatrix}, \begin{bmatrix} u_2 \\ \bar{u}_2\end{bmatrix}\right) \triangleq \left\langle \boldsymbol{E}_{\mathbb{C}}\begin{bmatrix} u_1 \\ \bar{u}_1\end{bmatrix} \,\middle|\, \begin{bmatrix} u_2 \\ \bar{u}_2\end{bmatrix}\right\rangle_{\mathbb{R}}, \qquad \boldsymbol{E}_{\mathbb{C}} \triangleq \boldsymbol{J}_{\mathbb{C}}^{-1}, \tag{A.3}$$

namely

$$\mathcal{A}^\top \boldsymbol{E}_{\mathbb{C}}\mathcal{A} = \boldsymbol{E}_{\mathbb{C}}.$$

Definition A.4. (Linearly symplectic map up to homogeneity N**)** A real-to-real matrix of spectrally localized maps $\mathbf{S}(U;t)$ in $\left(\Sigma\mathcal{S}_{K,K',0}[r,N]\right)^{2\times 2}$ is *linearly symplectic up to homogeneity N* if

$$\mathbf{S}(U;t)^\top \boldsymbol{E}_{\mathbb{C}}\,\mathbf{S}(U;t) = \boldsymbol{E}_{\mathbb{C}} + S_{>N}(U;t), \tag{A.4}$$

where $\boldsymbol{E}_{\mathbf{C}}$ is the symplectic operator defined in (A.3) and $S_{>N}(U;t)$ is a matrix of spectrally localized maps in $\left(\mathcal{S}_{K,K',N+1}[\epsilon_0]\right)^{2\times 2}$.

Linearly symplectic maps up to homogeneity N preserve the linear Hamiltonian structure up to homogeneity N. The following result is borrowed from [25, Lemma 3.9].

Lemma A.5. *Let $\boldsymbol{J}_{\mathbb{C}}\boldsymbol{B}(U;t)$ be a linearly Hamiltonian operator up to homogeneity N (Definition A.2) and $\boldsymbol{S}(U;t)$ be an invertible map, linearly symplectic to homogeneity N (Definition A.4). Then the operators $\boldsymbol{S}(U;t)\boldsymbol{J}_{\mathbb{C}}\boldsymbol{B}(U;t)\boldsymbol{S}(U;t)^{-1}$ and $(\partial_t\boldsymbol{S}(U;t))\boldsymbol{S}^{-1}(U;t)$ are linearly Hamiltonian up to homogeneity N.*

We consider the flow of a linearly Hamiltonian up to homogeneity N paradifferential operator. The following is Lemma 3.16 of [25].

Lemma A.6. (Linear symplectic flow) *Let $p\in\mathbb{N}$, $N,K,K'\in\mathbb{N}$ with $K'\leqslant K$, $m\leqslant 1$, $r>0$. Let*

$$\boldsymbol{J}_{\mathbb{C}}\mathrm{Op}^{\mathrm{BW}}(\boldsymbol{B}) = \mathrm{Op}^{\mathrm{BW}}\left(\begin{bmatrix} \overline{\mathrm{i}b_2^\vee} & \overline{\mathrm{i}b_1^\vee} \\ \mathrm{i}b_1 & \mathrm{i}b_2\end{bmatrix}\right)$$

be a linearly Hamiltonian operator up to homogeneity N (Definition A.2) where $\boldsymbol{B}$ is a matrix of symbols

$$\boldsymbol{B}\triangleq\boldsymbol{B}(\tau,U;t,x,\xi)\triangleq\begin{pmatrix} b_1(\tau,U;t,x,\xi) & b_2(\tau,U;t,x,\xi) \\ b_2(\tau,U;t,x,-\xi) & b_1(\tau,U;t,x,-\xi)\end{pmatrix}, \qquad \begin{cases} b_1\in\Sigma\Gamma^0_{K,K',p}[r,N], \\ b_2\in\Sigma\Gamma^m_{K,K',p}[r,N],\end{cases}$$

with $b_1^\vee - b_1$ in $\Gamma^0_{K,K',N+1}[\epsilon_0]$ and the imaginary part $\mathrm{Im}\ b_2$ *in $\Gamma^0_{K,K',N+1}[\epsilon_0]$ (cfr.* (A.2)*) uniformly in $|\tau|\leqslant 1$. Then there exists $s_0>0$ such that, for any $U\in B^K_{s_0,\mathbb{R}}(I;r)$, the system*

$$\partial_\tau\mathcal{G}^\tau_{\boldsymbol{B}}(U;t) = \boldsymbol{J}_{\mathbb{C}}\,\mathrm{Op}^{\mathrm{BW}}(\boldsymbol{B}(\tau,U;t,x,\xi))\mathcal{G}^\tau_{\boldsymbol{B}}(U;t), \qquad \mathcal{G}^0_{\boldsymbol{B}}(U;t)=\mathrm{Id}$$

has a unique solution $\mathcal{G}^\tau_{\boldsymbol{B}}(U)$ defined for all $|\tau|\leqslant 1$, satisfying the following properties:

(i) **Boundedness:** *For any $s\in\mathbb{R}$ the linear map $\mathcal{G}^\tau_{\boldsymbol{B}}(U;t)$ is invertible and $\mathcal{G}^\tau_{\boldsymbol{B}}(U;t)$ and $\mathcal{G}^\tau_{\boldsymbol{B}}(U;t)^{-1}$ are non–homogeneous spectrally localized maps in $\left(\mathcal{S}^0_{K,K',0}[\epsilon_0]\right)^{2\times 2}$ according to Definition 3.20.*

(ii) **Linear symplecticity:** *The map $\mathcal{G}^\tau_{\boldsymbol{B}}(U;t)$ is linearly symplectic up to homogeneity N (Definition A.4). If $\boldsymbol{J}_{\mathbb{C}}\mathrm{Op}^{\mathrm{BW}}(\mathbf{B})$ is linearly Hamiltonian (Definition A.1), then $\mathcal{G}^\tau_{\boldsymbol{B}}(U;t)$ is linearly symplectic (Definition A.3).*

(iii) **Homogeneous expansion:** $\mathcal{G}^{\tau}_{\boldsymbol{B}}(U;t)$ *and its inverse are spectrally localized maps and* $\mathcal{G}^{\tau}_{\boldsymbol{B}}(U;t)^{\pm}-\mathrm{Id}$ *belong to* $\left(\Sigma\mathcal{S}^{(N+1)m_0}_{K,K',p}[r,N]\right)^{2\times 2}$ *with* $m_0 \triangleq \max(m,0)$*, uniformly in* $|\tau|\leqslant 1$.

The flow generated by a Fourier multiplier satisfies similar properties. The following is Lemma 3.17 of [25].

Lemma A.7. (Flow of a Fourier multiplier) *Let* $p\in\mathbb{N}$ *and* $g_p(Z;\xi)$ *be a* p*–homogeneous,* x*-independent, real symbol in* $\widetilde{\Gamma}^{\frac{3}{2}}_p$*. Then, the flow* $\mathcal{G}^{\tau}_{g_p}(Z)$ *defined by*

$$\partial_\tau \mathcal{G}^{\tau}_{g_p}(Z) = \mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}\left(\mathrm{i}g_p(Z;\xi)\right)\mathcal{G}^{\tau}_{g_p}(Z), \qquad \mathcal{G}^{0}_{g_p}(Z)=\mathrm{Id}, \tag{A.5}$$

is well defined for any $|\tau|\leqslant 1$ *and satisfies the following properties:*

(i) **Boundedness:** *For any* $K\in\mathbb{N}$ *and* $r>0$ *the flow* $\mathcal{G}^{\tau}_{g_p}(Z)$ *and its inverse* $\mathcal{G}^{-\tau}_{g_p}(Z)$ *are real-to-real diagonal matrix of spectrally localized maps in* $\left(\mathcal{S}^0_{K,0,0}[\epsilon_0]\right)^{2\times 2}$.

(ii) **Linear symplecticity:** *The flow map* $\mathcal{G}^{\tau}_{g_p}(Z)$ *is linearly symplectic (Definition A.3).*

(iii) **Homogeneous expansion:** *The flow map* $\mathcal{G}^{\tau}_{g_p}(Z)$ *and its inverse* $\mathcal{G}^{-\tau}_{g_p}(Z)$ *are matrices of spectrally localized maps such that* $\mathcal{G}^{\pm\tau}_{g_p}(Z)-\mathrm{Id}$ *belong to* $\left(\Sigma\mathcal{S}^{\frac{3}{2}(N+1)}_{K,0,p}[r,N]\right)^{2\times 2}$*, uniformly in* $|\tau|\leqslant 1$.

A.2 Non-linear Hamiltonian systems

We first give the definition of p-homogeneous Hamiltonian functions.

Definition A.8 (Homogeneous Hamiltonian)**.** Let $p\in\mathbb{N}$, a $p+2$-homogeneous Hamiltonian is a function defined on $\dot{H}^\infty(\mathbb{T};\mathbb{C}^2)$ with values in $\mathbb{C}$ which is real valued for any $U\in\dot{H}^\infty_{\mathbb{R}}(\mathbb{T};\mathbb{C}^2)$ of the form, c.f. (3.19)

$$H_{p+2}(U)=\langle M_p(U)U,U\rangle=\sum_{(\vec{\jmath},\vec{\sigma})\in\mathfrak{T}_{p+2}} H^{\vec{\sigma}}_{\vec{\jmath}}\, u^{\vec{\sigma}}_{\vec{\jmath}}, \qquad M_p(U)\in\left(\widetilde{\mathcal{M}}^m_p\right)^{2\times 2} \tag{A.6}$$

for some $m\in\mathbb{R}$ and complex valued coefficients $H^{\vec{\sigma}}_{\vec{\jmath}}=H^{\sigma_1,\ldots,\sigma_{p+2}}_{j_1,\ldots,j_{p+2}}$. A pluri-homogeneous Hamiltonian is a polynomial of the form

$$H(U)=\sum_{p=0}^{N} H_{p+2}(U), \tag{A.7}$$

where, for $p=0,\ldots,N$, H_{p+2} is a $p+2$-homogeneous Hamiltonian.

From the definition the coefficients in (A.6) satisfy the following:

1. **Symmetry restrictions:** for any $\vec{\sigma}=(\sigma_1,\ldots,\sigma_{p+2})\in\{\pm\}^{p+2}$ and $\vec{\jmath}=(j_1,\ldots,j_{p+2})\in(\mathbb{Z}^*)^{p+2}$, one has

$$\textbf{Reality condition: } \overline{H^{\vec{\sigma}}_{\vec{\jmath}}}=H^{-\vec{\sigma}}_{\vec{\jmath}}, \qquad \textbf{Momentum condition: } H^{\vec{\sigma}}_{\vec{\jmath}}\neq 0 \implies \vec{\sigma}\cdot\vec{\jmath}=0. \tag{A.8}$$

2. **Polynomial bound:** There are $\mu, C>0$ and $m\in\mathbb{R}$ such that

$$\left|H^{\vec{\sigma}}_{\vec{\jmath}}\right|\leqslant C\max{}_3(|j_1|,\ldots,|j_{p+2}|)^{\mu}\max{}_2(|j_1|,\ldots,|j_{p+2}|)^m.$$

Remark A.9. In view of the momentum condition $\vec{\sigma}\cdot\vec{\jmath}=0$ in (A.8), one has the equivalence

$$\max(|j_1|,\ldots,|j_{p+2}|)\sim\max{}_2(|j_1|,\ldots,|j_{p+2}|).$$

We now define the class of Hamiltonian systems up to homogeneity N that we shall use in Section 7. Let $K,K'\in\mathbb{N}$ with $K'\leqslant K$, $r>0$ and $U\in B^K_{s_0}(I;r)$. Let

$$Z\triangleq \mathbf{M}_0(U;t)U \quad \text{with} \quad \mathbf{M}_0(U;t)\in\left(\mathcal{M}^0_{K,K',0}[\epsilon_0]\right)^{2\times 2}. \tag{A.9}$$

Definition A.10. (Hamiltonian system up to homogeneity N**)** Let $N,K,K'\in\mathbb{N}$ with $K\geqslant K'+1$ and assume (A.9). A U–dependent system

$$\partial_t Z = \boldsymbol{J}_{\mathbb{C}}\nabla H(Z) + \boldsymbol{M}_{>N}(U;t)[U] \tag{A.10}$$

is *Hamiltonian up to homogeneity* N if

- $H(Z)$ is a pluri-homogeneous Hamiltonian as in (A.7);
- $\boldsymbol{M}_{>N}(U;t)$ is a matrix of non-homogeneous operators in $\big(\mathcal{M}_{K,K'+1,N+1}[\epsilon_0]\big)^{2\times 2}$.

We shall perform nonlinear changes of variables which are symplectic up to homogeneity N according to the following definition.

Definition A.11. (Symplectic map up to homogeneity N**)** Let $p,N\in\mathbb{N}$ with $p\leqslant N$. We say that

$$\mathrm{D}(Z;t) = \boldsymbol{M}(Z;t)Z \qquad \text{with} \qquad \boldsymbol{M}(Z;t)-\mathrm{Id}\in\big(\Sigma\mathcal{M}_{K,K',p}[r,N]\big)^{2\times 2}, \tag{A.11}$$

is *symplectic up to homogeneity* N, if its pluri-homogeneous component $\mathcal{D}_{\leqslant N}(Z)\triangleq\big(\mathcal{P}_{\leqslant N}M(Z;t)\big)Z$ satisfies

$$\big[\lceil_Z\mathcal{D}_{\leqslant N}(Z)\big]^\top \boldsymbol{E}_{\mathbb{C}}\lceil_Z\mathcal{D}_{\leqslant N}(Z) = \boldsymbol{E}_{\mathbb{C}} + \boldsymbol{E}_{>N}(Z) \qquad \text{with} \qquad \boldsymbol{E}_{>N}(Z)\in\Big(\Sigma_{N+1}\widetilde{\mathcal{M}}_q\Big)^{2\times 2}. \tag{A.12}$$

A symplectic map up to homogeneity N transforms a Hamiltonian system up to homogeneity N into another Hamiltonian system up to homogeneity N.

Lemma A.12. *Let $p,N\in\mathbb{N}$ with $p\leqslant N$, $K,K'\in\mathbb{N}$ with $K\geqslant K'+1$. Let $Z\triangleq\mathbf{M}_0(U;t)U$ as in* (A.9)*. Assume $\mathcal{D}(Z;t)=M(Z;t)Z$ is a symplectic map up to homogeneity N (Definition A.11) such that*

$$M(Z;t)-\mathrm{Id}\in\begin{cases}\big(\Sigma\mathcal{M}_{K,K',p}[r,N]\big)^{2\times 2}, & \text{if}\quad \mathbf{M}_0(U;t)=\mathrm{Id},\\ \big(\Sigma\mathcal{M}_{K,0,p}[\check{r},N]\big)^{2\times 2}, & \forall\,\check{r}>0\quad \text{otherwise}.\end{cases} \tag{A.13}$$

If $Z(t)$ solves a U-dependent Hamiltonian system up to homogeneity N (Definition A.10), then the variable $W\triangleq\mathcal{D}(Z;t)$ solves another U-dependent Hamiltonian system up to homogeneity N (generated by the transformed Hamiltonian).

The following is Lemma 3.19 in [25].

Lemma A.13. *Let $p\in\mathbb{N}$, $m\in\mathbb{R}$ and $a(U;x,\xi)$ a real valued homogeneous symbol in $\widetilde{\Gamma}^m_p$. Then the Hamiltonian vector field generated by the Hamiltonian*

$$H(U)\triangleq\mathrm{Re}\,\langle\boldsymbol{A}(U)u\mid\bar{u}\rangle_{\mathbb{R}}, \qquad \boldsymbol{A}(U)\triangleq\mathrm{Op}^{\mathrm{BW}}(a(U;x,\xi)),$$

is

$$\boldsymbol{J}_{\mathbb{C}}\nabla H(U) = \mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}(\mathrm{i}a(U;x,\xi))\,U + \boldsymbol{R}(U)U,$$

where $\boldsymbol{R}(U)$ is a real-to-real matrix of homogeneous smoothing operators in $\big(\widetilde{\mathcal{R}}^{-\varrho}_p\big)^{2\times 2}$ for any $\varrho\geqslant 0$.

The following is Lemma 3.20 in [25].

Lemma A.14. *Let $p\in\mathbb{N}$, $m\in\mathbb{R}$ and $\varrho\geqslant 0$. Let*

$$X(U) = \boldsymbol{J}_{\mathbb{C}}\mathrm{Op}^{\mathrm{BW}}(\boldsymbol{A}(U;x,\xi))U + \boldsymbol{R}(U)U = \boldsymbol{J}_{\mathbb{C}}\nabla H(U) \tag{A.14}$$

be a $(p+1)$-homogeneous Hamiltonian vector field, where

$$\boldsymbol{A}(U;x,\xi) = \begin{pmatrix}\dfrac{a(U;x,\xi)}{b(U;x,-\xi)} & \dfrac{b(U;x,\xi)}{a(U;x,-\xi)}\end{pmatrix} \tag{A.15}$$

is a matrix of symbols in $\Big(\widetilde{\Gamma}^m_p\Big)^{2\times 2}$ and $\boldsymbol{R}(U)$ is a real-to-real matrix of smoothing operators in $\big(\widetilde{\mathcal{R}}^{-\varrho}_p\big)^{2\times 2}$. Then, we may write

$$X(U) = \boldsymbol{J}_{\mathbb{C}}\mathrm{Op}^{\mathrm{BW}}(\boldsymbol{A}_1(U;x,\xi))U + \boldsymbol{R}_1(U)U, \tag{A.16}$$

where the matrix of paradifferential operators $\mathrm{Op}^{\mathrm{BW}}(\boldsymbol{A}_1(U;x,\xi))$ is symmetric, with matrix of symbols

$$\boldsymbol{A}_1(U;x,\xi) = \frac{1}{2}\begin{pmatrix}\dfrac{a+a^\vee}{\overline{b}^\vee+b^\vee} & \dfrac{b+\overline{b}}{a+a^\vee}\end{pmatrix} \tag{A.17}$$

and $\boldsymbol{R}_1(U)$ is another real-to-real matrix of smoothing operators in $\big(\widetilde{\mathcal{R}}^{-\varrho}_p\big)^{2\times 2}$.

The following is Lemma 3.21 in [25].

Lemma A.15. *Let $p \in \mathbb{N}$, $m \in \mathbb{R}$ and $\varrho \geqslant 0$. Let $\boldsymbol{S}(U)$ be a matrix of spectrally localized homogeneous maps in $\big(\widetilde{\mathcal{S}}_p\big)^{2\times 2}$ which is linearly Hamiltonian of the form*

$$\boldsymbol{S}(U) = \boldsymbol{J}_{\mathbb{C}} \mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{A}(U;x,\xi)\right) + \boldsymbol{R}(U), \tag{A.18}$$

where $\boldsymbol{A}(U;x,\xi)$ is a real-to-real matrix of symbols in $\left(\widetilde{\Gamma}^m_p\right)^{2\times 2}$ as in (A.15), and $\boldsymbol{R}(U)$ is a real-to-real matrix of smoothing operators in $\left(\widetilde{\mathcal{R}}^{-\varrho}_p\right)^{2\times 2}$. Then, we may write

$$\boldsymbol{S}(U) = \boldsymbol{J}_{\mathbb{C}} \mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{A}_1(U;x,\xi)\right) + \boldsymbol{R}_1(U),$$

where the matrix of symbols $\boldsymbol{A}_1(U;x,\xi)$ in $\left(\widetilde{\Gamma}^m_p\right)^{2\times 2}$ has the form (A.17) and $\boldsymbol{R}_1(U)$ is another matrix of real-to-real smoothing operators in $\left(\widetilde{\mathcal{R}}^{-\varrho}_p\right)^{2\times 2}$. In particular the homogeneous operator $\boldsymbol{J}_{\mathbb{C}}\mathrm{Op}^{\mathrm{BW}}\left(\boldsymbol{A}_1(U)\right)$ is linearly Hamiltonian.

A.3 Symplectic corrections

In this section we provide a symplectic correction to two different class of maps: linearly symplectic spectrally localized maps and smoothing perturbations of the identity. The main result is Theorem 7.1 in [25] that we state below.

Theorem A.16. *Let $p, N \in \mathbb{N}$ with $p \leqslant N$, $K, K' \in \mathbb{N}$ with $K'+1 \leqslant K$, $r > 0$. Let $Z = \mathbf{M}_0(U;t)U$ with $\mathbf{M}_0(U;t) \in \left(\mathcal{M}^0_{K,K',0}[\epsilon_0]\right)^{2x2}$. Assume that $Z(t)$ solves a Hamiltonian system up to homogeneity N. Consider*

$$\Phi(Z) \triangleq \boldsymbol{B}(Z;t)Z, \tag{A.19}$$

where

- *$\boldsymbol{B}(Z;t) - \mathrm{Id}$ is a matrix of spectrally localized maps in*

$$\boldsymbol{B}(Z;t) - \mathrm{Id} \in \begin{cases} \left(\Sigma \mathbf{S}_{K,K',p}[r,N]\right)^{2x2}, & \text{if } \quad \mathbf{M}_0(U;t) = \mathrm{Id}, \\ \left(\Sigma \mathcal{S}_{K,0,p}[\check{r},N]\right)^{2x2}, \ \forall \check{r} > 0 & \text{otherwise.} \end{cases} \tag{A.20}$$

- *$\boldsymbol{B}(Z;t)$ is linearly symplectic up to homogeneity N, according to Definition A.4.*

Then, there exists a real-to-real matrix of pluri–homogeneous smoothing operators $\boldsymbol{R}_{\leqslant N}(\cdot)$ in $\left(\Sigma_p^N \widetilde{\mathcal{R}}^{-\varrho}_q\right)^{2x2}$, for any $\varrho > 0$, such that the non-linear map

$$Z_+ \triangleq \big(\mathrm{Id} + \boldsymbol{R}_{\leqslant N}(\Phi(Z))\big)\Phi(Z)$$

is symplectic up to homogeneity N and thus Z_+ solves a system which is Hamiltonian up to homogeneity N.

Given a map of the form $U \mapsto U + \boldsymbol{J}_{\mathbb{C}}\nabla H_{p+2}(U)$ where $\boldsymbol{J}_{\mathbb{C}}\nabla H_{p+2}(U)$ is a Hamiltonian, smoothing vector field, we find a correction which is symplectic up to homogeneity N.

Lemma A.17. *Let $p, N \in \mathbb{N}$ with $p \leqslant N$. Let $Y_{p+1}(U) = \boldsymbol{J}_{\mathbb{C}}\nabla H_{p+2}(U)$ be a homogeneous Hamiltonian smoothing vector field in $\widetilde{\mathfrak{X}}^{-\varrho}_{p+1}$ for some $\varrho \geqslant 0$. Then there is a map of the form*

$$\mathfrak{F}_{\leqslant N}(U) = U + Y_{p+1}(U) + F_{\geqslant (p+2)}(U), \qquad F_{\geqslant(p+2)}(U) \in \Sigma^N_{p+2}\widetilde{\mathfrak{X}}^{-\varrho}_q, \tag{A.21}$$

which is symplectic up to homogeneity N (Definition A.11).

Proof. It is a direct consequence of Lemmata 2.27 and 3.14 in [24]. A careful examination of the proofs reveals that $F_{\geqslant(p+2)}(U)$ actually belongs to $\Sigma^N_{2p+1}\widetilde{\mathfrak{X}}^{-\varrho}_q$. However, since this stronger result is not required for our purposes, we prefer to state the weaker conclusion $F_{\geqslant(p+2)}(U) \in \Sigma^N_{p+2}\widetilde{\mathfrak{X}}^{-\varrho}_q$. □

Remark A.18. The map $\mathfrak{F}_{\leqslant N}(U)$ in (A.21) is indeed the truncation up to homogeneity N of the time-one flow generated by the Hamiltonian vector field $\boldsymbol{J}_{\mathbb{C}}\nabla H_{p+2}(U)$.

B Auxiliary flows and conjugations

The following conjugation Lemmata B.1 and B.2 are used in the nonlinear Hamiltonian Birkhoff normal form reduction performed in Section 7. Their proof can be found in Appendix A of [25].

The following hypothesis shall be assumed in both Lemmata B.1 and B.2:

Assumption (A): *Assume* $Z \triangleq \mathbf{M}_0(U;t)U$ *where* $\mathbf{M}_0(U;t) \in \left(\mathcal{M}^0_{K,K',0}[\epsilon_0]\right)^{2\times 2}$, $U \in B^K_{s_0,\mathbb{R}}(I;\epsilon_0)$ *for some* $\epsilon_0, s_0 > 0$ *and* $0 \leqslant K' \leqslant K$. *Let* $N \in \mathbb{N}$ *and assume that* Z *solves the system*

$$\partial_t Z = \mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}\left(\mathrm{i}\omega_{\gamma,\mathtt{b}}(\xi) + \mathrm{i}a_{\leqslant N}(Z;\xi) + \mathrm{i}a_{>N}(U;t,\xi)\right) Z + \boldsymbol{R}_{\leqslant N}(Z)Z + \boldsymbol{R}_{>N}(U;t)U, \tag{B.1}$$

where:

- $a_{\leqslant N}(Z;\xi)$ *is a real valued pluri-homogeneous symbol, independent of* x, *in* $\Sigma_2^N \widetilde{\Gamma}^{\frac{3}{2}}_q$;
- $a_{>N}(U;t,\xi)$ *is a non-homogeneous symbol, independent of* x, *in* $\Gamma^{\frac{3}{2}}_{K,K',N+1}[\epsilon_0]$ *with imaginary part* $\mathrm{Im}\, a_{>N}(U;t,\xi)$ *in* $\Gamma^0_{K,K',N+1}[\epsilon_0]$;
- $\boldsymbol{R}_{\leqslant N}(Z)$ *is a real-to-real matrix of pluri-homogeneous smoothing operators in* $\left(\Sigma_1^N \widetilde{\mathcal{R}}^{-\varrho}_q\right)^{2\times 2}$;
- $\boldsymbol{R}_{>N}(U;t)$ *is a real-to-real matrix of non-homogeneous smoothing operators in* $\left(\mathcal{R}^{-\varrho}_{K,K',N+1}[\epsilon_0]\right)^{2\times 2}$.

Lemma B.1 (Conjugation under the flow of a Fourier multiplier). *Assume* **(A)** *at page 65. Let* $g_p(Z;\xi)$ *be a* p*–homogeneous real symbol independent of* x *in* $\widetilde{\Gamma}^{\frac{3}{2}}_p$, $p \geqslant 2$, *that we expand as*

$$g_p(Z;\xi) = \sum_{(\vec{\jmath}_p,\vec{\sigma}_p)\in\mathfrak{T}_p} G^{\vec{\sigma}_p}_{\vec{\jmath}_p}(\xi) z^{\vec{\sigma}_p}_{\vec{\jmath}_p}, \qquad \overline{G^{-\vec{\sigma}_p}_{\vec{\jmath}_p}}(\xi) = G^{\vec{\sigma}_p}_{\vec{\jmath}_p}(\xi) \in \mathbb{C} \tag{B.2}$$

and denote by $\boldsymbol{\mathcal{G}}_{g_p}(Z) \triangleq \boldsymbol{\mathcal{G}}^1_{g_p}(Z)$ *the time* 1*-flow defined in* (A.5) *generated by* $\mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}\left(\mathrm{i}g_p(Z;\xi)\right)$. *If* $Z(t)$ *solves system* (B.1), *then the variable*

$$W \triangleq \boldsymbol{\mathcal{G}}_{g_p}(Z)Z \tag{B.3}$$

solves the system

$$\partial_t W = \mathrm{i}\boldsymbol{\omega}_{\gamma,\mathtt{b}}(D)W + \mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}\left(\mathrm{i}a^+_{\leqslant N}(W;\xi) + \mathrm{i}a^+_{>N}(U;t,\xi)\right) W + \boldsymbol{R}^+_{\leqslant N}(W)W + \boldsymbol{R}^+_{>N}(U;t)U, \tag{B.4}$$

where

- $a^+_{\leqslant N}(W;\xi)$ *is a real valued pluri-homogeneous symbol, independent of* x, *in* $\Sigma_2^N \widetilde{\Gamma}^{\frac{3}{2}}_q$, *with components*
$$\begin{aligned} \mathcal{P}_{\leqslant p-1}[a^+_{\leqslant N}(W;\xi)] &= \mathcal{P}_{\leqslant p-1}[a_{\leqslant N}(W;\xi)], \\ \mathcal{P}_p\left[a^+_{\leqslant N}(W;\xi)\right] &= \mathcal{P}_p\left[a_{\leqslant N}(W;\xi)\right] + g^+_p(W;\xi), \end{aligned} \tag{B.5}$$
where $g^+_p(W;\xi) \in \widetilde{\Gamma}^{\frac{3}{2}}_p$ *is the real,* x*-independent symbol*
$$g^+_p(W;\xi) \triangleq \sum_{(\vec{\jmath}_p,\vec{\sigma}_p)\in\mathfrak{T}_p} \mathrm{i}\left(\vec{\sigma}_p \cdot \vec{\omega}_{\gamma,\mathtt{b}}(\vec{\jmath}_p)\right) G^{\vec{\sigma}_p}_{\vec{\jmath}_p}(\xi) w^{\vec{\sigma}_p}_{\vec{\jmath}_p}; \tag{B.6}$$
- $a^+_{>N}(U;t,\xi)$ *is a non-homogeneous symbol, independent of* x, *in* $\Gamma^{\frac{3}{2}}_{K,K',N+1}[\epsilon_0]$ *with imaginary part* $\mathrm{Im}\, a^+_{>N}(U;t,\xi)$ *belonging to* $\Gamma^0_{K,K',N+1}[\epsilon_0]$;
- $\boldsymbol{R}^+_{\leqslant N}(W)$ *is a real-to-real matrix of pluri–homogeneous smoothing operators in* $\left(\Sigma_1^N \widetilde{\mathcal{R}}^{-\varrho+c(N,p)}_q\right)^{2\times 2}$ *for some* $c(N,p) > 0$ *(depending only on* N,p*) and fulfilling*
$$\mathcal{P}_{\leqslant p}[\boldsymbol{R}^+_{\leqslant N}(W)] = \mathcal{P}_{\leqslant p}[\boldsymbol{R}_{\leqslant N}(W)]; \tag{B.7}$$

- $\boldsymbol{R}^+_{>N}(U;t)$ *is a real-to-real matrix of non–homogeneous smoothing operators in* $\Big(\mathcal{R}^{-\varrho+c(N,p)}_{K,K',N+1}[\epsilon_0]\Big)^{2\times 2}$.

The following lemma describes how a system is conjugated under a smoothing perturbation of the identity.

Lemma B.2 (Conjugation under a smoothing perturbation of the identity). *Assume* **(A)** *at page 65. Let* $F_{\leqslant N}(Z)$ *be a real-to-real matrix of pluri-homogeneous smoothing operators in* $\Big(\Sigma_p^N \widetilde{\mathcal{R}}_q^{-\varrho'}\Big)^{2\times 2}$ *for some* $\varrho' \geqslant 0$. *If* $Z(t)$ *solves* (B.1) *then the variable*

$$W \triangleq \mathcal{F}_{\leqslant N}(Z) \triangleq Z + F_{\leqslant N}(Z)Z \tag{B.8}$$

solves

$$\partial_t W = \mathrm{i}\boldsymbol{\omega}_{\gamma,\mathtt{b}}(D)W + \mathrm{Op}^{\mathrm{BW}}_{\mathrm{vec}}\Big(\mathrm{i}a^+_{\leqslant N}(W;\xi) + \mathrm{i}a^+_{>N}(U;t,\xi)\Big)W + \boldsymbol{R}^+_{\leqslant N}(W)W + \boldsymbol{R}^+_{>N}(U;t)U, \tag{B.9}$$

where

- $a^+_{\leqslant N}(W;\xi)$ *is a real valued pluri-homogeneous symbol, independent of* x, *in* $\Sigma_2^N \widetilde{\Gamma}_q^{\frac{3}{2}}$, *with components*

$$\mathcal{P}_{\leqslant p+1}[a^+_{\leqslant N}(W;\xi)] = \mathcal{P}_{\leqslant p+1}[a_{\leqslant N}(W;\xi)]\,; \tag{B.10}$$

- $a^+_{>N}(U;t,\xi)$ *is a non-homogeneous symbol, independent of* x, *in* $\Gamma^{\frac{3}{2}}_{K,K',N+1}[\epsilon_0]$ *with imaginary part* $\mathrm{Im}\, a^+_{>N}(U;t,\xi)$ *belonging to* $\Gamma^0_{K,K',N+1}[\epsilon_0]$;

- $\boldsymbol{R}^+_{\leqslant N}(W)$ *is a real-to-real matrix of pluri–homogeneous smoothing operators in* $\Big(\Sigma_1^N \widetilde{\mathcal{R}}_q^{-\varrho_*}\Big)^{2\times 2}$, $\varrho_* \triangleq \min(\varrho, \varrho' - \frac{3}{2})$ ($\varrho \geq 0$ *is the smoothing order in Assumption* **(A)** *at page 65), with components*

$$\mathcal{P}_{\leqslant p-1}[\boldsymbol{R}^+_{\leqslant N}(W)] = \mathcal{P}_{\leqslant p-1}[\boldsymbol{R}_{\leqslant N}(W)]\,, \tag{B.11}$$

  and, denoting $\boldsymbol{F}_p(W) \triangleq \mathcal{P}_p(\boldsymbol{F}_{\leqslant N}(W))$ *in* $\Big(\widetilde{\mathcal{R}}_p^{-\varrho'}\Big)^{2\times 2}$, *one has*

$$\mathcal{P}_p[\boldsymbol{R}^+_{\leqslant N}(W)] = \mathcal{P}_p[\boldsymbol{R}_{\leqslant N}(W)] + d_W\big(\boldsymbol{F}_p(W)W\big)\big[i\boldsymbol{\omega}_{\gamma,\mathtt{b}}(D)\big] - i\boldsymbol{\omega}_{\gamma,\mathtt{b}}(D)\boldsymbol{F}_p(W); \tag{B.12}$$

- $\boldsymbol{R}^+_{>N}(U;t)$ *is a real-to-real matrix of non–homogeneous smoothing operators in* $\Big(\mathcal{R}^{-\varrho_*}_{K,K',N+1}[\epsilon_0]\Big)^{2\times 2}$.

In addition, if $\mathcal{F}_{\leqslant N}(Z)$ *in* (B.8) *is the symplectic up to homogeneity* N *map associated to a Hamiltonian vector field* $\boldsymbol{G}_p(Z)Z = \boldsymbol{J}_{\mathbb{C}}\nabla H_{p+2}(Z)$ *as per Lemma A.17, where* $\Big(\boldsymbol{G}_p(Z) \in \widetilde{\mathcal{R}}_p^{-\varrho'}\Big)^{2\times 2}$ *has Fourier expansion*

$$\big(\boldsymbol{G}_p(Z)Z\big)^\sigma_k = \sum_{(\vec{\jmath}_{p+1},k,\vec{\sigma}_{p+1},-\sigma)\in\mathfrak{T}_{p+2}} G^{\vec{\sigma}_{p+1},\sigma}_{\vec{\jmath}_{p+1},k}\, z^{\vec{\sigma}_{p+1}}_{\vec{\jmath}_{p+1}}\,, \tag{B.13}$$

then (B.12) *reduces to*

$$\mathcal{P}_p[\boldsymbol{R}^+_{\leqslant N}(W)] = \mathcal{P}_p[\boldsymbol{R}_{\leqslant N}(W)] + \boldsymbol{G}^+_p(W), \tag{B.14}$$

where $\boldsymbol{G}^+_p(W) \in \Big(\widetilde{\mathcal{R}}_p^{-\varrho'+\frac{3}{2}}\Big)^{2\times 2}$ *is the smoothing operator with Fourier expansion*

$$(\boldsymbol{G}^+_p(W)W)^\sigma_k = \sum_{(\vec{\jmath}_{p+1},k,\vec{\sigma}_{p+1},-\sigma)\in\mathfrak{T}_{p+2}} \mathrm{i}\big(\vec{\sigma}_{p+1}\cdot\vec{\omega}_{\gamma,\mathtt{b}}(\vec{\jmath}_{p+1}) - \sigma\omega_{\gamma,\mathtt{b}}(k)\big)G^{\vec{\sigma}_{p+1},\sigma}_{\vec{\jmath}_{p+1},k}\, w^{\vec{\sigma}_{p+1}}_{\vec{\jmath}_{p+1}}\,. \tag{B.15}$$

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