

Constructing Four-Body Ballistic Lunar Transfers via Analytical Energy Conditions

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This paper derives and summarizes the analytical conditions for lunar ballistic capture and constructs ballistic lunar transfers based on these conditions. We adopt the Sun-Earth/Moon planar bicircular restricted four-body problem as the dynamical model to construct lunar transfers. First, the analytical conditions for ballistic capture are derived based on the relationship between the Keplerian energy with respect to the Moon and the angular momentum with respect to the Moon, summarized in form of exact ranges of the Jacobi energy at the lunar insertion point. Both sufficient and necessary condition and necessary condition are developed. Then, an optimization method combined with the analytical energy conditions is proposed to construct ballistic lunar transfers. Simulations shows that a high ballistic capture ratio is achieved by our proposed method (100% for direct insertion and 99.15% for retrograde insertion). Examining the obtained ballistic lunar transfers, the effectiveness of the analytical energy conditions is verified. Samples of our obtained lunar transfers achieves a lower impulse and shorter time of flight compared to two conventional methods, further strengthening the advantage of our proposed method.

I. Introduction

LUNAR transfers have attracted considerable attention due to renewed global interest in lunar exploration. To construct lunar transfers satisfying mission requirements, scholars have proposed several types of transfers, such as direct transfers and low-energy transfers [1]. Compared to direct transfers, low-energy transfers are characterized by low fuel consumption and flexible launch windows [2, 3], and has been successfully applied to practical missions (e.g., *Hiten* [4] and *Danuri* [5]). To construct low-energy transfers, selecting appropriate dynamical models is important. Trajectories constructed using the Hohmann transfer and the patched-conic method in the two-body dynamics [6, 7] typically require high fuel consumption (about 3.9 km/s from 167 km Earth parking orbit to 100 km lunar insertion orbit). Consequently, the two-body model is unsuitable for low-energy transfer construction. Belbruno and Miller [4] pioneered

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low-energy transfers in the multi-body dynamics, verifying their solutions in the *Lunar Observer* and *Hiten* missions. Their work highlighted the role of multi-body gravity perturbation in constructing low-energy transfers. Sweetser [8] theoretically demonstrated potential fuel savings when using the Earth-Moon planar circular restricted three-body problem (PCR3BP). Furthermore, Topputo [9] and Oshima et al. [3] pointed out that using the Sun-Earth/Moon planar bicircular restricted four-body problem (PBCR4BP) can further reduce impulses. Therefore, this Note focuses on the construction of low-energy lunar transfers in the Sun-Earth/Moon PBCR4BP.

Since closed-form solutions do not exist for the Earth-Moon PCR3BP and Sun-Earth/Moon PBCR4BP, the construction of low-energy transfers depends on numerical methods, typically categorized into construction methods based on prior knowledge and methods based on direct optimization [10]. Construction methods based on prior knowledge typically use weak stability boundary (WSB) theory [4, 11–13] and invariant manifolds of the libration periodic orbits [14, 15]. Among these, the method using the WSB theory constructs low-energy transfers via numerical calculation of lunar capture sets, utilizing forward and backward propagation strategies [16]. These low-energy transfers yield ballistic capture [4]; therefore they can also be called ballistic lunar transfers. In the following texts, lunar ballistic capture refers to ballistic capture. Meanwhile, the theory of invariant manifold has also been developed to provide prior knowledge for constructing low-energy transfers. Typical methods involve the patched invariant manifold method [14] and patched Lagrangian coherent structure (LCS) method [15] (LCSs are analogs of invariant manifolds in time-varying systems). The trajectories constructed from these methods also achieved ballistic capture. Construction methods based on prior knowledge effectively use natural dynamics to assist design. However, they typically yield fewer solutions than direct optimization methods, which provide a more comprehensive perspective of the solution space.

Compared to the methods based on prior knowledge, methods based on direct optimization globally search the solution space of bi-impulsive lunar transfers. Yagasaki [17, 18] transformed the construction of lunar transfers into a nonlinear boundary value problem, obtaining various low-energy solutions. Based on his work, Topputo [9] and Oshima et al. [3] employed direct optimization method to obtain the global map of impulse and time of flight (TOF) for lunar transfers within 100 and 200 days, further exploring the transfer characterizations. In particular, Oshima et al. [3] pointed out that most of low-energy transfers end up with ballistic capture. Although direct optimization method can provide a comprehensive perspective to analyze the transfer characterizations, its application often involves extensive computational effort and large-scale searches [16]. Furthermore, the Sun-Earth/Moon PBCR4BP introduces additional complexity due to increased parameter dimensionality caused by time-varying Sun-perturbed dynamics [9]. To address these challenges and construct low-energy transfers in a target way, several optimization methods combined with prior knowledge about multi-body dynamics have been proposed. Mengali and Quarta [19] derived analytical expression about impulses with respect to Jacobi energy, facilitating the optimization of lunar transfers. Moore et al. [20] combined invariant manifold with the discrete mechanics and optimal control to design low-energy transfers, while Fu et al. [21, 22] investigated the characterizations of the transit orbits in the Sun-Earth/Moon PBCR4BP and constructed

low-energy transfers based on these characterizations. Optimization methods combined with prior knowledge about multi-body dynamics not only provide a global map of the solutions space for low-energy transfers but also construct low-energy transfers in a target way, eliminating the need to select low-energy solutions through large numbers of trajectories with high fuel consumption. Therefore, this Note investigates the optimization method combined with prior knowledge. Based on the aforementioned discussion, we find that ballistic capture plays an important role in low-energy transfers. Ballistic capture can reduce the Moon insertion impulse, thereby lowering the total impulse required [4]. Therefore, we focus on prior knowledge about ballistic capture, and propose a method to construct lunar transfers with ballistic capture, i.e., ballistic lunar transfers.

For ballistic capture around the secondary body in the PCR3BP, several scholars developed the corresponding theory, both numerically and analytically. Belbruno [11] presented the approximation value of the Jacobi energy when the trajectory with respect to the Moon is a lunar parabolic trajectory. This approximation value was used to determine the energy-match relationship between the WSBs with respect to the Earth and the Moon. Li et al. [23] further derived the analytical expression of the eccentricity of the spacecraft at periapsis under the specific values of the Jacobi energy, providing a valuable theoretical insight into stable sets and capture trajectories around the Mars. Their main focus was on direct capture (i.e., positive angular momentum with respect to the secondary body). Furthermore, way of capture (i.e., direct and retrograde capture) can have an effect on the fuel consumption of low-energy transfers [3, 24]. The numerical results obtained from Qi and Xu [25] and Anoè et al. [26] implied that there are specific ranges of the Jacobi energy for direct and retrograde capture around the Moon. Motivated by their results, we analytically derived energy conditions for direct and retrograde capture at the lunar insertion point. Differing from the approximation value presented by Belbruno [11], we obtain the close-form expressions of the energy conditions, as constructing ballistic lunar transfers though optimization method necessitates the exact ranges of the Jacobi energy. The obtained sufficient and necessary condition for direct and retrograde capture is expressed in form of the specific ranges of the Jacobi energy at the lunar insertion point. The lower boundaries of the Jacobi energy for direct and retrograde capture are equivalent to the expressions of the Jacobi energy developed by Fantino et al. [27] when the eccentricity with respect to the Moon is set to 1. However, the monotonicity between the Jacobi energy and eccentricity (equivalent to the Keplerian energy with respect to the Moon) remains unanalyzed, and the energy conditions for ballistic capture have not been comprehensively investigated. Therefore, we provide this theoretical supplementation by detailed analysis. Then, to use this energy condition for construction of ballistic lunar transfers, we weaken the sufficient and necessary condition to the necessary condition, and propose an optimization method combined with this necessary condition. Since the necessary condition is expressed in form of the states of the insertion point, backward time propagation is employed to construct lunar transfers. The transfer trajectories obtained from our method achieve a high ratio of ballistic capture, verifying the effectiveness of the developed energy conditions and proposed method. Obtained solutions also achieve lower fuel impulse than those obtained from conventional methods using prior knowledge about multi-body dynamics,

strengthening the advantage of using the optimization method combined with prior knowledge about ballistic capture. Therefore, main contributions of this Note can be summarized as follows:

- 1) We derive and summarize analytical energy conditions for ballistic capture, providing a theoretical supplementation to pervious works [3, 11, 26, 27].
- 2) We propose an optimization method combined with the energy conditions to construct ballistic lunar transfers in a target way. Simulations show that our solutions achieve a high ratio of ballistic capture.
- 3) Compared to solutions obtained from conventional methods, samples of our obtained solutions achieve a lower impulse and shorter TOF, strengthening the advantage of our proposed method.

The rest of this Note is organized as follows. Section II presents the mathematical background about this work. Section III derives the analytical analytical energy conditions for ballistic capture. Section IV proposes the method to construct the ballistic lunar transfers and verifies the effectiveness of the developed energy conditions and proposed method. Finally, conclusions are drawn in Section V.

II. Mathematical Background

This section presents the mathematical background of this Note, including the Sun-Earth/Moon PBCR4BP, ballistic capture, and ballistic lunar transfers.

A. Sun-Earth/Moon PBCR4BP

Compared to the Earth-Moon PCR3BP, the Sun-Earth/Moon PBCR4BP provides higher fidelity when describing spacecraft trajectories, particularly when for exterior lunar transfers [9, 28]. Therefore, we adopt the Sun-Earth/Moon PBCR4BP to construct ballistic lunar transfers. In this model, the Sun, Earth, Moon, and spacecraft are assumed to move in the same plane. The Earth and Moon are assumed to move in the circular orbit around their barycenter, while the Earth-Moon barycenter is in the circular orbit around the Sun. The spacecraft is treated as a massless point that does not affect the motion of the Sun, Earth, and Moon. The dimensionless units are selected as follows: the length unit (LU) is set as the Earth-Moon distance, the mass unit (MU) is set as the combined mass of the Earth and Moon, and the time unit (TU) is set as $TU = T_{EM}/2\pi$, where T_{EM} is the orbital period of the Earth and Moon about their barycenter. With these dimensionless units, the Earth-Moon rotating frame [9] is adopted to describe the dynamical equations:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} u \\ v \\ 2v + \frac{\partial \Omega_4}{\partial x} \\ -2u + \frac{\partial \Omega_4}{\partial y} \end{bmatrix} \quad (1)$$

$$\Omega_4 = \frac{1}{2} [x^2 + y^2 + \mu(1 - \mu)] + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2} + \frac{\mu_S}{r_3} - \frac{\mu_S}{\rho^2} (x \cos \theta_S + y \sin \theta_S) \quad (2)$$

where $\mathbf{X} = [x, y, u, v]^T$ denotes the orbital state, Ω_4 denotes the effective potential of the PBCR4BP, and the parameter μ_S denotes the dimensionless mass of the Sun. The distances between the spacecraft and the Earth (r_1), Moon (r_2), and Sun (r_3) are expressed as:

$$r_1 = \sqrt{(x + \mu)^2 + y^2} \quad r_2 = \sqrt{(x + \mu - 1)^2 + y^2} \quad r_3 = \sqrt{(x - \rho \cos \theta_S)^2 + (y - \rho \sin \theta_S)^2} \quad (3)$$

where ρ denotes the distance between the Earth-Moon barycenter and the Sun, and the solar phase angle θ_S is calculated by $\theta_S = \theta_{S0} + \omega_S T$, where θ_{S0} denotes the initial solar phase angle expressed as $\theta_{S0} = \omega_S t_0$ (note that when $t_0 = 0$, the Sun is located at $(\rho, 0)$ in the Earth-Moon rotating frame) and T denotes the propagation time. In the Sun-Earth/Moon PBCR4BP, the Jacobi energy is time-varying because of Sun-perturbed dynamics. Therefore, the instantaneous Jacobi energy is defined as:

$$C = -\left(u^2 + v^2\right) + \left(x^2 + y^2\right) + \frac{2(1 - \mu)}{r_1} + \frac{2\mu}{r_2} + \mu(1 - \mu) \quad (4)$$

For numerical propagation of trajectories in the Sun-Earth/Moon PBCR4BP, we use the variable step-size, variable order (VSVO) Adams-Bashforth-Moulton algorithm with absolute and relative tolerances set to 1×10^{-13} [29], performed by MATLAB®'s ode113 command. Specific values of the parameters used in simulations can be found in Ref. [21]. Subsequently, the concepts of the bi-impulsive lunar transfers, ballistic capture, and ballistic lunar transfers are introduced.

B. Lunar Transfers And Ballistic Lunar Transfers

In this Note, we focus on bi-impulsive lunar transfers, particularly those with ballistic capture (i.e., ballistic lunar transfers). In such transfers, the spacecraft departs a circular Earth parking orbit with an Earth injection impulse (Δv_i) and inserts into a circular lunar insertion orbit after performing a Moon insertion impulse (Δv_f). The impulses should be tangential to the orbital velocity to maximize energy variations [30]. Therefore, the constraints of lunar transfers can be expressed as [9]:

$$\boldsymbol{\psi}_i = \begin{bmatrix} (x_i + \mu)^2 + y_i^2 - (R_E + h_i)^2 \\ (x_i + \mu)(u_i - y_i) + y_i(v_i + x_i + \mu) \end{bmatrix} = \mathbf{0} \quad (5)$$

$$\boldsymbol{\psi}_f = \begin{bmatrix} (x_f + \mu - 1)^2 + y_f^2 - (R_M + h_f)^2 \\ (x_f + \mu - 1)(u_f - y_f) + y_f(v_f + x_f + \mu - 1) \end{bmatrix} = \mathbf{0} \quad (6)$$

where h_i and h_f denote altitudes of the Earth parking orbit and lunar insertion orbit. The subscript 'i' and 'f' denote quantities associated with the departure and insertion points. With these constraints, we present the definition of

bi-impulsive lunar transfers:

Definition II.1 (Bi-impulsive Lunar Transfer). Bi-impulsive lunar transfer is the transfer trajectory that satisfies Eqs. (5)-(6), where $X_f = \phi_{t_i}^{t_f}(X_i)$ and $\phi_{t_i}^{t_f} : \mathbb{R} \times \mathbb{R} \times \mathbb{D} \rightarrow \mathbb{D}$; $(t_i, t_f, X_i) \rightarrow \phi_{t_i}^{t_f}(X_i)$ is the flow map of the Sun-Earth/Moon PBCR4BP. The parameters t_i and t_f denote the departure and insertion epochs.

In this Note, we set h_i and h_f to 167 km and 100 km [3, 9], respectively. In the following text, we denote $R_E + h_i$ and $R_M + h_f$ as r_i and r_f . Then, the impulses of the transfer trajectory can be calculated by:

$$\Delta v_i = \sqrt{(u_i - y_i)^2 + (v_i + x_i + \mu)^2} - \sqrt{\frac{1 - \mu}{r_i}} \quad (7)$$

$$\Delta v_f = \sqrt{(u_f - y_f)^2 + (v_f + x_f + \mu - 1)^2} - \sqrt{\frac{\mu}{r_f}} \quad (8)$$

$$\Delta v = \Delta v_i + \Delta v_f \quad (9)$$

where Δv denotes the total impulse. In bi-impulsive lunar transfers, ballistic capture (i.e., negative Keplerian energy with respect to the Moon) at the insertion point plays an important role in reducing Δv_f , which consequently leads to the reduction of Δv [4, 31]. Moreover, way of ballistic capture (i.e., direct and retrograde capture) can further affect Δv_f [3, 21, 24]. Therefore, both direct and retrograde capture are investigated. To define direct and retrograde capture, two parameters at the insertion point are introduced, namely, the Keplerian energy with respect to the Moon (E_f) and the angular momentum with respect to the Moon (M_f). The Keplerian energy with respect to the Moon at the insertion point is expressed as:

$$E_f = \frac{1}{2} \left[(u_f - y_f)^2 + (v_f + x_f + \mu - 1)^2 \right] - \frac{\mu}{r_f} \quad (10)$$

With the expression of the Keplerian energy with respect to the Moon at the insertion point, we present the definition of the ballistic capture at the insertion point [9]:

Definition II.2 (Ballistic Capture). Ballistic capture at the insertion point is the state that satisfies $E_f \leq 0$.

Therefore, E_f is the key parameter to identify transfer trajectories with ballistic capture. Then, the angular momentum with respect to the Moon at the insertion point is expressed as:

$$M_f = (x_f + \mu - 1)(v_f + x_f + \mu - 1) - y_f(u_f - y_f) \quad (11)$$

According to the signs of M_f , direct and retrograde capture can be defined as:

Definition II.3 (Direct Capture). Direct capture at the insertion point is ballistic capture that satisfies $M_f > 0$.

Definition II.4 (Retrograde Capture). Retrograde capture at the insertion point is ballistic capture that satisfies $M_f < 0$.

Then, with the definitions of bi-impulsive lunar transfers and ballistic capture we present the definition of ballistic lunar transfers:

Definition II.5 (Ballistic Lunar Transfer). Ballistic lunar transfer is the transfer trajectory that satisfies Definition II.1 and ends up with ballistic capture defined in Definition II.2.

The parameters E_f and M_f have a relationship shown as follows:

$$C_f = -2E_f + 2M_f + 2(1 - \mu)x_f + (1 - \mu)(2\mu - 1) + \frac{2(1 - \mu)}{r_{1f}} \quad (12)$$

where C_f denotes the Jacobi energy at the insertion point. This relationship has been previously developed by Anoè et al. [26] using the curvilinear frame. Subsequently, based on this relationship, we derive analytical energy conditions for ballistic capture.

III. Analytical Energy Conditions for Ballistic Capture

In this section, we analytical energy conditions for ballistic capture at the insertion point based on Eq. (12). We parameterize the states at the insertion point, and use this parameterization to derive analytical energy conditions.

A. Parameterization of The Insertion Point

Starting from the constraints presented in Eq. (6), we simplify them into:

$$\begin{cases} (x_f + \mu - 1)^2 + y_f^2 - r_f^2 = 0 \\ (x_f + \mu - 1)(u_f - y_f) + y_f(v_f + x_f + \mu - 1) = (x_f + \mu - 1)u_f + y_f v_f = 0 \end{cases} \quad (13)$$

Using this relationship, we parameterize the states of the insertion point under the specific value of Jacobi energy. The states of the direct insertion point (i.e., $M_f > 0$) can be parameterized as:

$$\begin{cases} x_{fD} = r_f \cos \alpha_{fD} + 1 - \mu, y_{fD} = r_f \sin \alpha_{fD}, u_{fD} = -V_{fD} \sin \alpha_{fD}, v_{fD} = V_{fD} \cos \alpha_{fD} \\ V_{fD} = \sqrt{-C_{fD} + (x_{fD}^2 + y_{fD}^2) + \frac{2(1 - \mu)}{r_{1fD}} + \frac{2\mu}{r_f} + \mu(1 - \mu)} \end{cases} \quad (14)$$

where α_f denotes the phase angle, and the subscript 'D' denotes quantities associated with direct insertion (including direct capture). The states of the retrograde insertion point (i.e., $M_f < 0$) can be parameterized as:

$$\begin{cases} x_{fR} = r_f \cos \alpha_{fR} + 1 - \mu, y_{fR} = r_f \sin \alpha_{fR}, u_{fR} = V_{fR} \sin \alpha_{fR}, v_{fR} = -V_{fR} \cos \alpha_{fR} \\ V_{fR} = \sqrt{-C_{fR} + (x_{fR}^2 + y_{fR}^2) + \frac{2(1 - \mu)}{r_{1fR}} + \frac{2\mu}{r_f} + \mu(1 - \mu)} \end{cases} \quad (15)$$

where the subscript ‘ R ’ denotes quantities associated with retrograde insertion (including retrograde capture). Subsequently, based on these two sets of parameters, we explore analytical energy conditions for ballistic capture.

B. Analytical Conditions for Ballistic Capture

For direct insertion, the angular momentum with respect to the Moon at the insertion point can be expressed as:

$$M_{fD} = (x_{fD} + \mu - 1)(v_{fD} + x_{fD} + \mu - 1) - y_{fD}(u_{fD} - y_{fD}) = r_f^2 + r_f V_{fD} = r_f^2 + r_f \sqrt{-C_{fD} + W(\alpha_{fD})} \quad (16)$$

where

$$W(\alpha_f) = (x_f^2 + y_f^2) + \frac{2(1-\mu)}{r_{1f}} + \frac{2\mu}{r_f} + \mu(1-\mu) \quad (17)$$

According to the relationship shown in Eq. (12), we consider the critical case, i.e., $E_{fD} = 0$, and reconstruct Eq. (12) as:

$$C_{fD}^* = 2r_f^2 + 2r_f \sqrt{-C_{fD}^* + W(\alpha_{fD})} + 2(1-\mu)(r_f \cos \alpha_{fD} - \mu + 1) + (1-\mu)(2\mu - 1) + \frac{2(1-\mu)}{r_{1fD}} \quad (18)$$

where C_{fD}^* denotes the critical Jacobi energy level satisfying $E_{fD} = 0$ for direct insertion. Rewriting Eq. (18) as:

$$C_{fD}^* = G(\alpha_{fD}) + 2r_f \sqrt{-C_{fD}^* + W(\alpha_{fD})} \quad (19)$$

where

$$G(\alpha_f) = 2r_f^2 + 2(1-\mu)(r_f \cos \alpha_f - \mu + 1) + (1-\mu)(2\mu - 1) + \frac{2(1-\mu)}{r_{1f}} \quad (20)$$

Therefore, the equations that C_{fD}^* should satisfy can be expressed as:

$$\begin{cases} C_{fD}^{*2} - 2(G(\alpha_{fD}) - 2r_f^2)C_{fD}^* + (G^2(\alpha_{fD}) - 4r_f^2W(\alpha_{fD})) = 0 \\ 2r_f \sqrt{-C_{fD}^* + W(\alpha_{fD})} = C_{fD}^* - G(\alpha_{fD}) \geq 0 \Rightarrow C_{fD}^* \geq G(\alpha_{fD}) \end{cases} \quad (21)$$

where the equation $C_{fD}^{*2} - 2(G(\alpha_{fD}) - 2r_f^2)C_{fD}^* + (G^2(\alpha_{fD}) - 4r_f^2W(\alpha_{fD})) = 0$ is a quadratic equation. The discriminant of this equation can be calculated by:

$$\Delta = 4(G(\alpha_{fD}) - 2r_f^2)^2 - 4(G^2(\alpha_{fD}) - 4r_f^2W(\alpha_{fD})) = 16r_f^2(r_f^2 - G(\alpha_{fD}) + W(\alpha_{fD})) = 32\mu r_f > 0 \quad (22)$$

Since $\Delta > 0$, there exist two solutions to the equation. Due to $C_{fD}^* \geq G(\alpha_{fD})$, the value of C_{fD}^* is solved by:

$$C_{fD}^* = C_{fD}^*(\alpha_{fD}) = \frac{2(G(\alpha_{fD}) - 2r_f^2) + \sqrt{\Delta}}{2} = G(\alpha_{fD}) - 2r_f^2 + 2\sqrt{2\mu r_f} \quad (23)$$

According to Eq. (23), the value of C_{fD}^* depends on the value of α_{fD} . Then, we illustrate that at each α_{fD} , when the Jacobi energy level of the trajectory satisfies $C_{fD} < C_{fD}^*(\alpha_{fD})$, the Keplerian energy with respect to the Moon at the insertion point satisfies $E_{fD} > 0$. The expression of E_{fD} can be expressed as:

$$E_{fD}(C_{fD}, \alpha_{fD}) = \frac{1}{2} \left[(u_{fD} - y_{fD})^2 + (v_{fD} + x_{fD} + \mu - 1)^2 \right] - \frac{\mu}{r_f} = \frac{1}{2} \left(\sqrt{-C_{fD} + W(\alpha_{fD})} + r_f \right)^2 - \frac{\mu}{r_f} \quad (24)$$

In this expression, we can observe that E_{fD} is a quadratic function with respect to $V_{fD} = \sqrt{-C_{fD} + W(\alpha_{fD})}$. At each α_{fD} , when $C_{fD} = C_{fD}^*$, $V_{fD} = V_{fD}^*$ and the value of E_{fD} satisfies:

$$E_{fD}(C_{fD}^*(\alpha_{fD}), \alpha_{fD}) = 0 \quad (25)$$

Therefore, the relationship of the quadratic function can be presented in Fig. 1 (note that $V_{fD} \geq 0$).

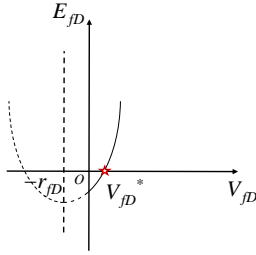


Fig. 1 The schematic of E_{fD} as a quadratic function with respect to V_{fD} .

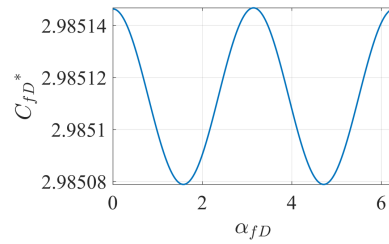


Fig. 2 The variation in C_{fD}^* with respect to α_{fD} .

As shown in Fig. 1, when $C_{fD} < C_{fD}^*$, i.e., $V_{fD} > V_{fD}^*$, E_{fD} satisfies $E_{fD} > 0$ when the value of α_{fD} is presented. Therefore, at each α_{fD} , there exists a lower boundary of C_{fD} to achieve direct capture, i.e., C_{fD}^* . Then, the variation of C_{fD}^* with respect to α_{fD} is analyzed. The derivative of C_{fD}^* with respect to α_{fD} is expressed as:

$$\frac{dC_{fD}^*(\alpha_{fD})}{d\alpha_{fD}} = \frac{dG(\alpha_{fD})}{d\alpha_{fD}} = -2(1-\mu)r_f \sin \alpha_{fD} + \frac{2(1-\mu)r_f \sin \alpha_{fD}}{r_{1fD}^3} \quad (26)$$

The derivative of C_{fD}^* with respect to α_{fD} has five zero points, expressed as follows:

$$\alpha_{fD1} = 0, \quad \alpha_{fD2} = \arccos\left(-\frac{r_f}{2}\right), \quad \alpha_{fD3} = \pi, \quad \alpha_{fD4} = 2\pi - \arccos\left(-\frac{r_f}{2}\right), \quad \alpha_{fD5} = 2\pi \quad (27)$$

With these five zero points, the variation in C_{fD}^* with respect to α_{fD} is presented in Fig. 2. Then, the minimum value of C_{fD}^* with respect to α_{fD} ($\alpha_{fD} = \alpha_{fD2}$) can be calculated by:

$$C_{fD}^*(\alpha_{fD})_{\min} = 3(1-\mu) - (1-\mu)r_f^2 + 2\sqrt{2\mu r_f} \quad (28)$$

Subsequently, we investigate the case of retrograde insertion. For retrograde insertion, the angular momentum with respect to the Moon at the insertion point can be expressed as:

$$M_{fR} = (x_{fR} + \mu - 1) (v_{fR} + x_{fR} + \mu - 1) - y_{fR} (u_{fR} - y_{fR}) = r_f^2 - r_f V_{fR} = r_f^2 - r_f \sqrt{-C_{fR} + W(\alpha_{fR})} \quad (29)$$

Similarly, considering the critical case $E_{fR} = 0$ first, the equations requiring C_{fR}^* to satisfy are presented as follows:

$$C_{fR}^* = G(\alpha_{fR}) - 2r_f \sqrt{-C_{fR}^* + W(\alpha_{fR})} \quad (30)$$

$$\begin{cases} C_{fR}^{*2} - 2(G(\alpha_{fR}) - 2r_f^2) C_{fR}^* + (G^2(\alpha_{fR}) - 4r_f^2 W(\alpha_{fR})) = 0 \\ 2r_f \sqrt{-C_{fR}^* + W(\alpha_{fR})} = -C_{fR}^* + G(\alpha_{fR}) \geq 0 \Rightarrow C_{fR}^* \leq G(\alpha_{fR}) \end{cases} \quad (31)$$

The equation $C_{fR}^{*2} - 2(G(\alpha_{fR}) - 2r_f^2) C_{fR}^* + (G^2(\alpha_{fR}) - 4r_f^2 W(\alpha_{fR})) = 0$ has the same form as Eq. (21) but with different inequalities. Therefore, C_{fR}^* is solved by:

$$C_{fR}^* = C_{fR}^*(\alpha_{fR}) = \frac{2(G(\alpha_{fR}) - 2r_f^2) - \sqrt{\Delta}}{2} = G(\alpha_{fR}) - 2r_f^2 - 2\sqrt{2\mu r_f} \quad (32)$$

Similarly, E_{fR} is a quadratic function with respect to $V_{fR} = \sqrt{-C_{fR} + W(\alpha_{fR})}$:

$$E_{fR}(C_{fR}, \alpha_{fR}) = \frac{1}{2} \left[(u_{fR} - y_{fR})^2 + (v_{fR} + x_{fR} + \mu - 1)^2 \right] - \frac{\mu}{r_f} = \frac{1}{2} \left(\sqrt{-C_{fR} + W(\alpha_{fR})} - r_f \right)^2 - \frac{\mu}{r_f} \quad (33)$$

At each α_{fR} , when $C_{fR} = C_{fR}^*$, $V_{fR} = V_{fR}^*$ and the value of E_{fR} satisfies:

$$E_{fR}(C_{fR}^*(\alpha_{fR}), \alpha_{fR}) = 0 \quad (34)$$

When $C_{fR} < C_{fR}^*$, i.e., $V_{fR} > V_{fR}^*$, E_{fR} satisfies $E_{fR} > 0$, illustrating that retrograde capture cannot be achieved, as shown in Fig. 3.

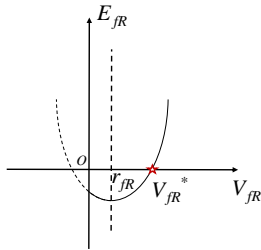


Fig. 3 The schematic of E_{fR} as a quadratic function with respect to V_{fR} .

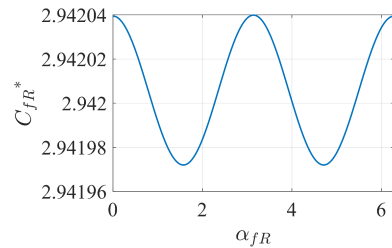


Fig. 4 The variation in C_{fR}^* with respect to α_{fR} .

Since C_{fR}^* has the same monotonicity as C_{fD}^* with respect to α_{fR} (as shown in Fig. 4), the minimum value of C_{fR}^* within $\alpha_{fR} \in [0, 2\pi)$ is presented as follows:

$$C_{fR}^*(\alpha_{fR})_{\min} = 3(1 - \mu) - (1 - \mu)r_f^2 - 2\sqrt{2\mu r_f} \quad (35)$$

Meanwhile, V_f for direct and retrograde capture should satisfy $V_f \geq 0$. Then, we summarize the **sufficient and necessary condition** for ballistic capture at the insertion point as the following theorem:

Theorem III.1 (Sufficient And Necessary Condition for Ballistic Capture). Ballistic capture at the insertion point (x_f, y_f) takes place if the Jacobi energy at the insertion point C_f satisfies $C_f^*(\alpha_f) \leq C_f \leq W(\alpha_f)$ (the converse is also true), where:

- 1) $\alpha_f = \text{atan2}(y_f, x_f + \mu - 1)$;
- 2) $C_f^*(\alpha_f) = (1 - \mu) + 2(1 - \mu)r_f \cos \alpha_f + \frac{2(1-\mu)}{r_{1f}} \pm 2\sqrt{2\mu r_f}$ (+ for direct capture and - for retrograde capture);
- 3) $W(\alpha_f) = (x_f^2 + y_f^2) + \frac{2(1-\mu)}{r_{1f}} + \frac{2\mu}{r_f} + \mu(1 - \mu)$.

Here we present the proof of this theorem (taking the direct capture for example):

Proof of Theorem III.1. (Proof of Sufficiency) Sufficiency of Theorem III.1 can be proved by Eqs. (16)-(25). When $C_{fD} \geq C_{fD}^*$, i.e., $V_{fD} \leq V_{fD}^*$, E_{fD} satisfies $E_{fD} \leq 0$, as shown in Fig. 1. Meanwhile $V_{fD} \geq 0$, thus $C_{fD} \leq W(\alpha_{fD})$. Therefore, we have:

$$C_{fD}^*(\alpha_{fD}) \leq C_{fD} \leq W(\alpha_{fD}) \Rightarrow E_{fD} \leq 0 \quad (36)$$

(Proof of Necessity) As shown in Fig. 1, when $E_{fD} \leq 0$, we have $0 \leq V_{fD} \leq V_{fD}^*$. Since V_{fD} and C_{fD} have a relationship $V_{fD} = \sqrt{-C_{fD} + W(\alpha_{fD})}$. Therefore, the Jacobi energy C_{fD} should satisfy $C_{fD}^*(\alpha_{fD}) \leq C_{fD} \leq W(\alpha_{fD})$. Therefore, we have:

$$E_{fD} \leq 0 \Rightarrow C_{fD}^*(\alpha_{fD}) \leq C_{fD} \quad (37)$$

□

To apply the analytical energy conditions to constructing ballistic lunar transfers in the Sun–Earth/Moon PBCR4BP and to facilitate a grid search at each α_f , we weaken the aforementioned **necessary conditions** for direct and retrograde capture to the following **necessary condition**:

Theorem III.2 (Necessary Condition for Ballistic Capture). The Jacobi energy at the insertion point C_f satisfies $C_f^*(\alpha_f)_{\min} \leq C_f \leq W(\alpha_f)_{\max}$ if ballistic capture at the insertion point (x_f, y_f) takes place, where $C_f^*(\alpha_f)_{\min} = 3(1 - \mu) - (1 - \mu)r_f^2 \pm 2\sqrt{2\mu r_f}$ (+ for direct capture and - for retrograde capture).

The proof of the necessary condition is apparent.

Remark (1). According to Eq. (12), the difference between the angular momentum with respect to the Moon for direct and retrograde insertion leads directly to the difference between $C_{fD}^*(\alpha_{fD})$ and $C_{fR}^*(\alpha_{fR})$. Although these values can also be found by solving $E_{fD} = 0$ and $E_{fR} = 0$, our derivation clearly reveals the cause of the difference. This derivation result agrees with the numerical findings obtained from Anoè et al. [26].

Remark (2). The obtained analytical energy conditions can further provide insights into constructing ballistic lunar transfers. To achieve ballistic capture, only the initial guesses satisfying the analytical energy condition Theorem III.2 are selected to construct transfers by differential correction [9], detailed in Section IV.

Remark (3). In 2004, Belbruno [11] presented the approximation value of the Jacobi energy when the trajectory with respect to the Moon is a parabolic trajectory in the Earth-Moon PCR3BP. He presented this approximation value as $C_f^* \approx 3$ under the assumptions $r_f \approx 0$ and $\mu \approx 0$. However, to determine the optimization parameters for constructing ballistic lunar transfers, more exact ranges of the Jacobi energy at the insertion point are required than those provided by the previous approximation. Therefore, we derive the close-form expressions of $C_{fD}^*(\alpha_{fD})$ and $C_{fR}^*(\alpha_{fR})$ without approximation.

Remark (4). In 2010, Fantino et al. [27] developed the close-form expressions of the Jacobi energy as functions of r_f , α_f , and eccentricity with respect to the Moon e_M . These expressions are equivalent to the expressions of $C_{fD}^*(\alpha_{fD})$ and $C_{fR}^*(\alpha_{fR})$ when e_M is set to 1. However, they originally developed them only to calculate WSB points and did not analyze the monotonicity relationship between the Jacobi energy and e_M (e_M is equivalent to E_f). Although this monotonicity is not necessary for WSB calculation, it is important for the analysis of the energy conditions. To our best knowledge, these conditions have not been comprehensively analyzed, and our work provides this theoretical complement. Meanwhile, we weaken the obtained sufficient and necessary condition to the necessary condition, further distinguishing our contributions from Ref. [27].

Based on the aforementioned discussion, our first contribution is the derivation and summary of analytical energy conditions for ballistic capture, including the sufficient and necessary condition Theorem III.1 and necessary condition Theorem III.2.

IV. Application to Constructing Ballistic Lunar Transfers

This section proposes construction method of ballistic lunar transfers. The ranges of the construction parameters and optimization variables are selected based on the analytical energy conditions, and the initial guess trajectories are generated from backward time propagation. Then, the initial guesses are corrected by differential correction, and the results are discussed.

A. Optimization Method Combined with Prior Knowledge about Ballistic Capture

When constructing lunar transfers in the Sun-Earth/Moon PBCR4BP, the optimization method [3, 9] is adopted. In particular, the method developed in this Note uses prior knowledge about ballistic capture, i.e., analytical energy condition Theorem III.2. This condition define feasible ranges for construction parameters and optimization variables. Since Theorem III.2 focuses on the Jacobi energy at the insertion point, we propose a backward strategy, i.e., we select the states at the insertion point that satisfy Theorem III.2, and perform backward time propagation to search the transfer trajectories satisfying the constraints Eqs. (5)-(6). The specific procedure is summarized as follows, including selecting ranges of construction parameters, generating initial guesses, and performing differential correction.

1. Selecting Ranges of Construction Parameters

The construction parameters are selected as α_f , C_f , and $\theta_{Sf} = \omega_{Stf}$. Differing from the construction parameters used by Topputo [9] and Oshima et al. [3] as the states of the departure point, this setting effectively uses the developed prior knowledge about ballistic capture. For lunar transfers considered in this Note ($h_f = 100$ km), the exact values of $C_{fD}^* (\alpha_{fD})_{\min}$ and $C_{fR}^* (\alpha_{fR})_{\min}$ are presented as follows (five significant figures retained):

$$\begin{cases} C_{fD}^* (\alpha_{fD})_{\min} = 2.9851 \\ C_{fR}^* (\alpha_{fR})_{\min} = 2.9420 \end{cases} \quad (38)$$

To construct lunar transfers with direct capture, the ranges of parameters are set as $\alpha_{fD} \in [0, 2\pi)$ with a step-size of $\pi/360$, $C_{fD} \in [2.9851, 3.2003]$ with a step-size of 0.0001, and $\theta_{SfD} \in [0, 2\pi)$ with a step-size of $\pi/360$. Among these parameters, the lower boundary of C_{fD} is selected as $C_{fD}^* (\alpha_{fD})_{\min}$ according to Theorem III.2. When determining the upper boundary of C_{fD} , we refer to the results shown in Fig. 4 (b) of Ref. [3] and select it as the Jacobi energy at the L1 libration point $C_{fD} = 3.2003$. Similarly, the ranges of parameters to construct lunar transfers with retrograde capture are set as $\alpha_{fR} \in [0, 2\pi)$ with a step-size of $\pi/360$, $C_{fR} \in [2.9420, 3.2003]$ with a step-size of 0.0001, and $\theta_{SfR} \in [0, 2\pi)$ with a step-size of $\pi/360$.

2. Generating Initial Guesses

With the construction parameters selected in Section IV.A.1, the states of insertion point can be calculated by Eqs. (14) or (15). Then, we perform backward time propagation of these states, and the propagation time is set to 200 days. Since the states calculated by Eqs. (14) or (15) satisfy the constraint Eq. (6) rigorously, the residual of Eq. (5) is focused on. When the states during the propagation satisfy:

$$\|\psi_i\| < 1 \times 10^{-4} \quad (39)$$

the corresponding construction parameters are recorded as the initial guesses of the states at the insertion point, and the epoch of sates satisfying Eq. (39) is recorded as an initial guess of the departure epoch t_i . Based on the aforementioned discussion, the parameters determining an initial guess trajectory can be expressed as:

$$\mathbf{y} = [\alpha_f, C_f, \theta_{Sf}, t_i]^T \quad (40)$$

With these parameters, initial guesses are generated using backward time propagation. Subsequently, these initial guesses are corrected to satisfy Eq. (5). Notably, the Earth/Moon collision trajectories [3, 9] are excluded during the generation of initial guesses.

3. Performing Differential Correction

When performing differential correction, the problem is transformed into a nonlinear programming (NLP) problem [9, 32]. The optimization variables of the NLP problem are set as \mathbf{y} defined in Eq. (40), and the constraints of the NLP problem are set as Eq. (5), and the function to minimize is set as $\|\boldsymbol{\psi}_i\|$. In simulations, the NLP problem is solved by MATLAB®'s `fmincon` command with the sequential quadratic programming method. The parameters of the `fmincon` command are selected through trial and error to ensure efficiency and accuracy, as presented in Table 1. The lower and upper boundaries of the optimization variables during the correction are presented in Table 2 (T_0

Table 1 Parameter settings for `fmincon` command.

Parameter	Value
TolX	1×10^{-13}
TolFun	1×10^{-8}
TolCon	1×10^{-8}
MaxIter	1000
MaxFunEvals	1000

denotes the dimensionless time of 200 days and θ_{Sf} is selected as the value of initial guess parameter). Notably, the lower boundaries of C_f are selected based on Theorem III.2. After differential correction, lunar transfers satisfying the

Table 2 Lower and upper boundaries of optimization variables (direct capture).

Optimization Variables	Lower Boundary	Upper Boundary
α_{fD}, α_{fR}	0, 0	$2\pi, 2\pi$
C_{fD}, C_{fR}	2.9851, 2.9420	3.2003, 3.2003
$\theta_{SfD}, \theta_{SfR}$	0, 0	$2\pi, 2\pi$
t_{iD}, t_{iR}	$\theta_{SfD}/\omega_S - T_0, \theta_{SfR}/\omega_S - T_0$	$\theta_{SfD}/\omega_S - \pi/10, \theta_{SfR}/\omega_S - \pi/10$

tolerance of the constraints (i.e., $\|\psi_i\| < 5 \times 10^{-8}$) are recorded. For practical consideration, only the lunar transfers from the prograde Earth parking orbit are recorded. During the correction, the Earth/Moon collision trajectories are also excluded. Notably, the solutions obtained from the aforementioned method are only preliminary lunar transfers. Whether they achieve ballistic capture must be determined further, following Definition II.5.

Remark (5). The solutions satisfying Eqs. (5)-(6) can be obtained from the aforementioned procedure. To obtain more feasible solutions, the continuation method [3, 33, 34] can be used.

Subsequently, the results are presented and analyzed.

B. Results and Discussion

Following the method proposed in Section IV.A, we obtain 828 solutions, including 711 solutions with direct insertion and 117 solutions with retrograde insertion. Investigating the values of E_f of these solutions, it is found that the ratio of ballistic capture reaches 711/711 (100%) for direct insertion and 116/117 (99.15%) for retrograde insertion, respectively. These results confirm a high ballistic capture ratio. The values of α_{fR} and C_{fR} of the transfer trajectory without ballistic capture is shown in Fig. 5. The red curve shows the variation of C_{fR}^* with respect to α_{fR} , and the blue line shows the value of $C_{fR}^*(\alpha_{fR})_{\min}$, which is selected as the lower boundary of C_{fR} when performing the differential correction. It is observed that C_{fR} of this trajectory is higher than $C_{fR}^*(\alpha_{fR})_{\min}$ but lower than $C_{fR}^*(\alpha_{fR})$, explaining why it does not achieve ballistic capture. In the following texts, we focus on the obtained ballistic lunar transfers, and the effectiveness of the developed analytical energy conditions Theorem III.1 is verified.

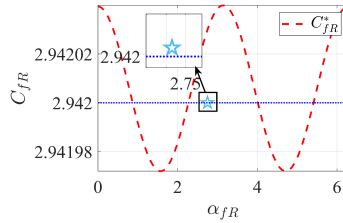


Fig. 5 The values of α_{fR} and C_{fR} of the transfer trajectory without ballistic capture.

The distributions of C_f with respect to α_f of the obtained transfers with direct and retrograde capture are shown in Fig. 6. From this figure we can observe that all of the values of C_f satisfy $C_f(\alpha_f) \geq C_f^*(\alpha_f)$ for direct and retrograde capture, which illustrates that $C_f^*(\alpha_f) \leq C_f(\alpha_f) \leq W(\alpha_f)$ ($W(\alpha_f) \approx 8.0486$). Therefore, the effectiveness of Theorem III.1 is verified. Subsequently, the transfer characterizations of these ballistic lunar transfers are analyzed.

Figure 7 presents the (TOF, Δv) map of the obtained solutions. We select the samples with the minimum Δv with direct and retrograde capture. The trajectories are presented in Fig. 8 (denoted as Sample I, II).

From Fig. 8, it is observed that the four samples are all exterior transfers, which effectively utilizes the Sun-perturbed dynamics. Meanwhile, the trajectories with minimum Δv are both trajectories with lunar gravity assist [3], which helps

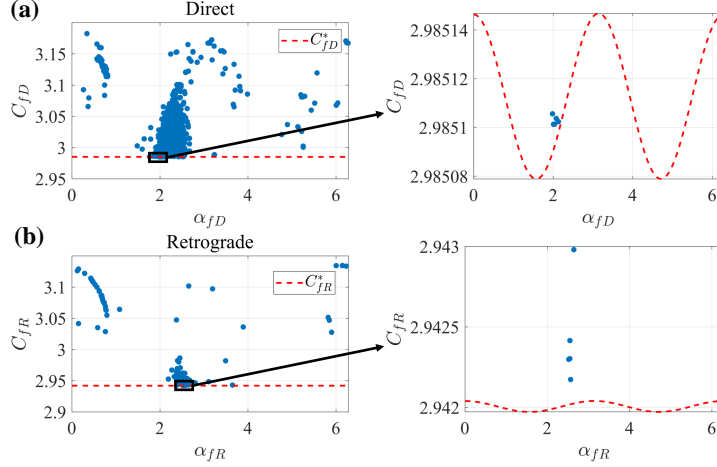


Fig. 6 The distributions of C_f with respect to α_f of the obtained transfers with direct and retrograde capture. (a) Direct capture; (b) Retrograde capture.

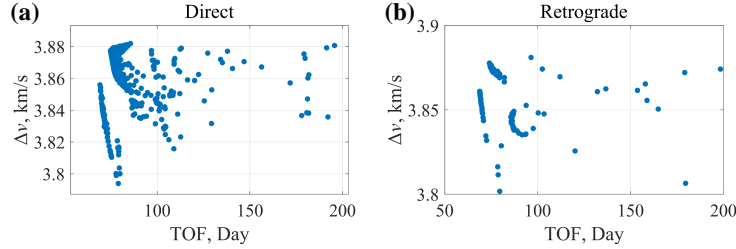


Fig. 7 The $(TOF, \Delta v)$ and $(TOF, \Delta v_f)$ maps. (a) Lunar transfers with direct capture; (b) Lunar transfers with retrograde capture.

reduce Δv_i . Since our proposed method in this Note is an optimization method combined with prior knowledge about ballistic capture, to illustrate the advantage of our proposed method, we perform a comparison with the conventional construction methods only using prior knowledge about multi-body dynamics (i.e., the WSB method and patched LCS method). The comparison results are shown in Table 3. From Table 3, it can be observed that our obtained solutions achieve a lower Δv and shorter TOF compared to the solutions obtained from the WSB method [4] and patched LCS method in the Sun-Earth/Moon PBCR4BP [15]. The maximum Δv savings compared to the two conventional methods are 44 m/s and 86 m/s, respectively. This comparison further highlights the advantage of our proposed method in reduction of Δv over the methods only using prior knowledge about multi-body dynamics.

Table 3 Comparison between the obtained solutions in this Note and previous works.

Solution	Δv , km/s	TOF, Day
Sample I	3.794	79
Sample II	3.802	80
WSB [4]	3.838	160
[15]	3.880	100

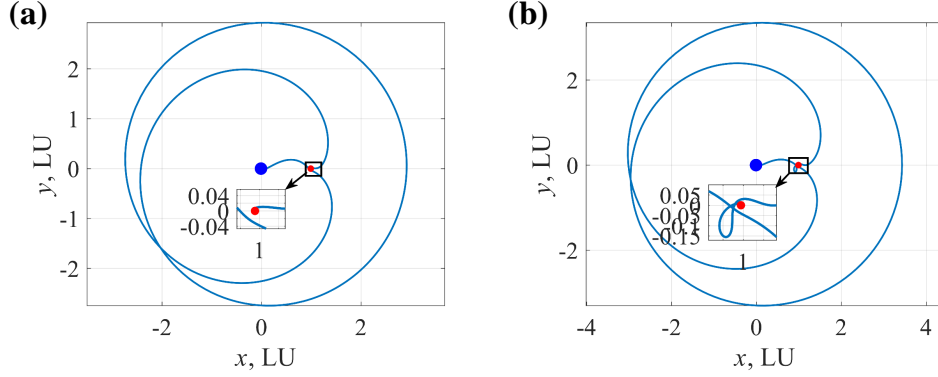


Fig. 8 The trajectories of samples with the minimum Δv and Δv_f for two cases. (a) Sample with minimum Δv and direct capture (Sample I); (b) Sample with minimum Δv and retrograde capture (Sample II).

V. Conclusion

This Note is devoted to deriving analytical energy conditions for lunar ballistic capture and proposing an optimization method combined with these conditions to construct ballistic lunar transfers in a target way. Considering the role of Sun-perturbed dynamics in ballistic capture and low-energy transfers, we adopt the Sun-Earth/Moon planar bicircular restricted four-body problem to construct lunar transfers. A theoretical supplementation to ballistic capture has been provided, and a target way to construct ballistic lunar transfers has been developed. First, analytical energy conditions for ballistic capture are derived based on the relationship between the Keplerian energy and angular momentum with respect to the Moon. Then, based on these conditions, optimization method is proposed and a backward strategy is employed to construct ballistic lunar transfers. Analytical conditions are used to determine feasible ranges of construction parameters and optimization variables. For solutions obtained from our method, a high ratio of ballistic capture is achieved: 711/711 (100%) and 116/117 (99.15%) for direct and retrograde insertion. This result verifies the effectiveness of the developed energy conditions and proposed method. Furthermore, several samples are selected to perform a comparison with the solutions obtained from the conventional methods only based on prior knowledge about multi-body dynamics (the weak stability boundary method and patched Lagrangian coherent structure method). Comparison results illustrate that utilizing the optimization method combined with prior knowledge about ballistic capture achieves a low fuel consumption and shorter time of flight compared to methods only using prior knowledge. The maximum impulse savings compared to these two methods reach 44 m/s and 86 m/s for transfers from 167 km Earth parking orbit to 100 km lunar insertion orbit, revealing the advantage of our proposed method in finding solutions with lower fuel consumption over the conventional methods only using prior knowledge about multi-body dynamics.

Funding Sources

The authors acknowledge the financial support from the National Natural Science Foundation of China (Grant No. 12372044), the National Natural Science Foundation of China (No. U23B6002), the National Natural Science Foundation

of China (Grant No. 12302058), and the Postdoctoral Science Foundation of China (Grant No. 2024T170480).

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