AN INTERSECTION PRODUCT FOR THE POLYTOPE ALGEBRA

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Abstract. We introduce a new multiplication for the polytope algebra, defined via the intersection of polytopes. After establishing the foundational properties of this intersection product, we investigate finite-dimensional subalgebras that arise naturally from this construction. These subalgebras can be regarded as volumetric analogues of the graded Möbius algebra, which appears in the context of the Dowling–Wilson conjecture. We conjecture that they also satisfy the injective hard Lefschetz property and the Hodge–Riemann relations, and we prove these in degree one.

1. Introduction

The polytope algebra, introduced in the late 1980s by McMullen, is a remarkable object at the crossroads of convex geometry, combinatorics, and algebraic geometry [14, 20, 27]. In a landmark paper, McMullen [28] used this framework to show that the number of faces of simple polytopes is characterized by a short list of properties. The sufficiency of these properties was already known from an ingenious construction due to Billera and Lee [11], while their necessity was deduced by Stanley [35] from deep results in algebraic geometry. The contribution of McMullen was to provide a convex geometric proof. The proofs of Stanley and McMullen have served as prototypes and sources of inspiration for recent decisive progress in algebraic combinatorics, see, e.g., [1, 13, 21, 22].

The polytope algebra of an n-dimensional real vector space V

$$\Pi_*(V) = \bigoplus_{k=0}^{n} \Pi_k(V)$$

can be defined as the group of polytopal chains, i.e. integer linear combinations

$$\sum_{i=1}^{m} a_i \mathbf{1}_{P_i}, \quad a_1, \dots, a_m \in \mathbb{Z},$$

of indicator functions of polytopes in V, modulo translations. In particular, if $[P] := [\mathbf{1}_P]$ denotes the class of a polytope, then $[x + P] = [P]$ for all translations $x \in V$. The multiplication in the polytope algebra is uniquely determined by the equation

$$[P] * [Q] = [P + Q], \tag{1}$$

where $P + Q$ is the Minkowski sum.

In convex geometry, intersection is often viewed as an operation dual to Minkowski addition. At the level of polytopal chains, intersection behaves in a straightforward way: $\mathbf{1}_{P \cap Q} = \mathbf{1}_P \cdot \mathbf{1}_Q$. However, because intersection depends on the relative position of P and Q, it does not descend to a well-defined operation on the polytope algebra. One way to address this defect is to average over all relative positions of P and Q. Setting aside questions about the existence of the following integral, one is led to consider the definition

$$[P] \cdot [Q] = \int_V [P \cap (x + Q)]\, dx. \tag{2}$$

The starting point for this work was the realization that this definition does, in fact, extend to a new multiplicative structure on the polytope algebra—one that captures nontrivial geometric and combinatorial phenomena, as illustrated by the results discussed below.

1.1. **Main results.** One, albeit minor, issue with the preliminary definition (2) is the absence of a canonical choice of a Lebesgue measure dx on V. To address this, we denote by $\mathrm{Dens}(V)$ the one-dimensional vector space of densities on V, consisting of all (including negative) translation-invariant Radon measures on V. Moreover, it will be convenient to introduce the following notation. Throughout this paper, all tensor products are over $\mathbb{Z}$.

Definition 1.1. We define $\Pi^k(V) = \Pi_{n-k}(V) \otimes \mathrm{Dens}(V)$ for every integer $k \in \{0, \ldots, n\}$ and

$$\Pi^*(V) = \bigoplus_{k=0}^{n} \Pi^k(V).$$

Our first theorem asserts that the integral in (2) is indeed well defined and briefly summarizes the main properties of the resulting multiplication, which we call the intersection product in the polytope algebra. For more precise statements, the reader is directed to Theorems 4.4 and 4.9.

Theorem 1.2. *There exists a multiplication, uniquely determined by the equation*

$$([P] \otimes \mu) \cdot ([P'] \otimes \mu') = \int_V [P \cap (x + P')] \otimes \mu'\, d\mu(x), \tag{3}$$

that endows $\Pi^*(V)$ with the structure of a unital graded commutative algebra satisfying Poincaré duality.

The intersection product is compatible with an operation

$$f^*\colon \Pi^*(V) \to \Pi^*(W)$$

that we call the pullback along a linear map $f\colon W \to V$. If f is injective, then the pullback of $[P] \otimes \mu$ can interpreted as an average of the fibers of P under the canonical projection $V \to V/f(W)$. An analogous construction—based on a different addition—leads to the notion of the fiber polytope, introduced by Billera and Sturmfels [12]. We elaborate on this connection in Remark 5.5 below. It turns out that the pullback can be defined for all linear maps, not just injective ones, but the definition in the general case is somewhat less straightforward.

The following theorem summarizes the main properties of the pullback.

Theorem 1.3. *The pullback along a linear map $f\colon W \to V$ satisfies the following properties:*

(a) *It is is a morphism of algebras when $\Pi^*(V)$ and $\Pi^*(W)$ are equipped with the intersection product.*

(b) *The pullback is compatible with the grading,*

$$f^*(\Pi^k(V)) \subseteq \Pi^k(W).$$

(c) *If* $g\colon U \to W$ *is another linear map, then* $(f \circ g)^* = g^* \circ f^*$.

Remark 1.4. For continuous translation-invariant valuations, Alesker introduced in [5] a notion of pushforward along linear maps that can be regarded as dual to the pullback in the polytope algebra; see Remarks 5.6 and 7.4 below. There is another notable analogy with the theory of valuations on convex bodies. Bernig and Faifman [8] show that, in a precise sense, McMullen's multiplication (1) is closely related to the convolution of smooth translation-invariant valuations, introduced by Bernig and Fu [9]. We expect a similar relationship between intersection and Alesker product.

It seems that the only significant formal difference between these two settings is the existence of the Fourier transform within Alesker's theory of smooth translation-invariant valuations. A key property of this operation is that it intertwines the Alesker product and the Bernig–Fu convolution. In light of the recent explicit description of the Fourier transform obtained by Faifman and Wannerer [17], it appears unlikely that this powerful operation exists also within the context of the polytope algebra.

If we fix a positive density vol on V, as we will do from now on, we abbreviate our notation to $\alpha[P] := [P] \otimes \alpha \operatorname{vol} \in \Pi^*(V)$ for $\alpha \in \mathbb{R}$. Moreover, when our discussion involves the notion of positivity, such as in (4) below, we will tacitly identify $\Pi^n(V) \cong \mathbb{R}$ via $\alpha[\{0\}] \mapsto \alpha$.

Definition 1.5. For each polytope $P \subseteq V$, we define $\ell_P := [P]_{n-1} \in \Pi^1(V)$.

The Euler–Verdier involution σ is an algebra automorphism of $\Pi^*(V)$ and compatible with the pullback. It differs from McMullen's Euler map by the sign $(-1)^n$. For every polytope P, $\sigma(\ell_P) = -\ell_{-P}$.

The intersection product in the polytope algebra is capable of expressing deep geometric and combinatorial properties, as the theorems that follow will demonstrate. Our first result is an Alexandrov–Fenchel-type inequality.

Theorem 1.6. *Let* $C_1, \ldots, C_{n-2}$ *be centrally symmetric polytopes in* V *and define* $\ell_{\mathbf{C}} = \ell_{C_1} \cdots \ell_{C_{n-2}}$. *Then, for every* $x \in \Pi^1(V)$ *and every polytope* $Q \subseteq \mathbb{R}^n$, *the following inequality holds:*

$$(\sigma(x) \cdot \ell_Q \cdot \ell_{\mathbf{C}})^2 \geq (\sigma(x) \cdot x \cdot \ell_{\mathbf{C}})\,(\sigma(\ell_Q) \cdot \ell_Q \cdot \ell_{\mathbf{C}})\,. \tag{4}$$

Notice that inequality (4) can be interpreted as a statement about the symmetric bilinear form

$$q_{\mathbf{C}}(x, y) = \sigma(x) \cdot y \cdot \ell_{\mathbf{C}}, \quad x, y \in \Pi^1(V).$$

Indeed, since σ is an algebra automorphism and the polytopes C_i are centrally symmetric, for all $x, y \in \Pi^1(V)$

$$\sigma(x) \cdot y \cdot \ell_{\mathbf{C}} = \sigma(y) \cdot x \cdot \ell_{\mathbf{C}}.$$

For every simple polytope with nonempty interior, McMullen [28] defined $\mathsf{M}_*(P) \subseteq \Pi_*(V)$ as the $\mathbb{Z}$-span of all weak Minkowski summands of P. This subset is, in fact, a finite-dimensional subalgebra with truly remarkable algebraic properties. These imply restrictions for the dimensions of the graded components $\mathsf{M}_k(P)$, which in turn imposes necessary conditions on the number of faces of P.

Also $\Pi^*(V)$ contains natural subalgebras, and these too are connected to combinatorics.

Definition 1.7. Let E be a finite set of lines in V^* that pass through the origin and are not contained in a single hyperplane.

(a) We denote by $\mathsf{A}^*(E) = \bigoplus_{k=0}^n \mathsf{A}^k(E)$ the subalgebra of $\Pi^*(V)$ generated by ℓ_P for all polytopes P such that each facet conormal of P belongs to a line in E.

(b) We denote by $\mathsf{A}_+^*(E)$ the subalgebra generated only by those elements $\ell_P \in \mathsf{A}^*(E)$ such that P is centrally symmetric.

(c) $\mathsf{K}(E) \subseteq \mathsf{A}_+^1(E)$ is the open convex cone of all $\ell_P \in \mathsf{A}_+^1(E)$ with the property that each line in E contains a facet conormal of P.

It is not difficult to see that $\mathsf{A}^*(E)$ and $\mathsf{A}_+^*(E)$ are finite-dimensional algebras for the intersection product. Moreover, the dimension of $\mathsf{A}_+^k(E)$ admits a straightforward combinatorial interpretation. Indeed, let $\mathscr{L}_k(E)$ denote the set of k-dimensional linear subspaces that can be obtained as sums $L_1 + \cdots + L_k$ of the lines in E. Then,

$$\dim \mathsf{A}_+^k(E) = |\mathscr{L}_k(E)|.$$

The Dowling–Wilson conjecture [16] asserts that for every nonnegative integer $k \leq n/2$

$$|\mathscr{L}_k(E)| \leq |\mathscr{L}_{n-k}(E)|.$$

After several partial results, starting from the 1940s with papers by de Bruijn–Erdős [15] and Motzkin [31], this conjecture was finally resolved in the affirmative by Huh–Wang [23]. Shortly afterwards, the matroid version of the Dowling–Wilson conjecture was established in the landmark paper [13]. In both papers, the strategy is to establish an injective hard Lefschetz theorem for the graded Möbius algebra.

Poincaré duality, the hard Lefschetz theorem, and the Hodge–Riemann relations are fundamental properties on the cohomology ring of a compact Kähler manifold. This algebraic structure—colloquially described as a Kähler package—has been observed to arise in different areas of mathematics. In algebraic combinatorics in particular, the central importance of this concept has recently become apparent, see, e.g., [1, 13, 21–23].

As we will discuss in detail in Section 10, the graded Möbius algebra of E and $\mathsf{A}_+^*(E)$ are closely related. Based on Theorem 1.6 and analogous results for smooth translation-invariant valuations on convex bodies [10, 25] and the graded Möbius algebra [13, 23], we propose the following conjecture. If true, it would directly imply the Dowling–Wilson conjecture.

Conjecture 1.8. *Let k be a nonnegative integer satisfying $k \leq n/2$. Suppose that $\ell_{C_0}, \dots, \ell_{C_{n-2k}} \in \mathsf{K}(E)$, and define $\ell_{\mathbf{C}} = \ell_{C_1} \cdots \ell_{C_{n-2k}}$. Then, the following statements hold:*

(a) Injective hard Lefschetz property. *The linear map*

$$\mathsf{A}^k(E) \to \mathsf{A}^{n-2k}(E), \quad x \mapsto x \cdot \ell_{\mathbf{C}},$$

is injective.

(b) Hodge–Riemann relations. *If $x \in \mathsf{A}^k(E)$ satisfies $x \cdot \ell_{C_0} \cdot \ell_{\mathbf{C}} = 0$, then*

$$\sigma(x) \cdot x \cdot \ell_{\mathbf{C}} \geq 0 \tag{5}$$

Moreover, equality holds if and only if $x = 0$.

For $k = 0$, the above conjecture reduces to the statement

$$\ell_{\mathbf{C}} = \ell_{C_1} \cdots \ell_{C_n} > 0,$$

which we will prove in Proposition 9.3. With considerably more effort, based on Theorem 1.6 and a characterization of the equality case in (5), we are able to show the following.

Theorem 1.9. *Conjecture 1.8 holds for $k = 1$.*

The above theorem directly implies the Dowling–Wilson conjecture for $k = 1$. A notable feature of our proof of Theorem 1.9 is that it uses a lemma by Motzkin on points in the real projective plane that appears already in his work [31] on a special case of the Dowling–Wilson conjecture.

2. PRELIMINARIES ON CONVEX GEOMETRY

We collect here for later use important definitions and results from convex geometry. For more information we refer the reader to the monograph by Schneider [32].

2.1. **Convex bodies.** In this paper, a convex body is a nonempty compact convex subset of $\mathbb{R}^n$. We denote the set all convex bodies in $\mathbb{R}^n$ by $\mathscr{K}(\mathbb{R}^n)$. The Hausdorff distance $\delta(K, L)$ turns this set into a locally compact metric spaces. The intersection of convex bodies is in general not continuous. However, under the assumption that K and L cannot be separated by a hyperplane, one has the following result. Recall that two convex bodies can be separated by a hyperplane if there exists a linear functional $f\colon \mathbb{R}^n \to \mathbb{R}$ and a number α such that $K \subseteq \{x\colon f(x) \leq \alpha\}$ and $L \subseteq \{x\colon f(x) \geq \alpha\}$.

Lemma 2.1. *Let K, L be convex bodies in $\mathbb{R}^n$ that cannot be separated by a hyperplane. If K_i, L_i ($i \in \mathbb{N}$) are convex bodies with $K_i \to K$ and $L_i \to L$ for $i \to \infty$, then $K_i \cap L_i \neq \emptyset$ for almost all i and $K_i \cap L_i \to K \cap L$ for $i \to \infty$.*

A convex body $K \subseteq \mathbb{R}^n$ is uniquely determined by its support function $h_K(x) = \sup\{\langle x, y\rangle\colon y \in K\}$, $x \in \mathbb{R}^n$. Here $\langle x, y\rangle$ denotes the euclidean inner product.

For any convex body $K \subseteq \mathbb{R}^n$ there exists a convex body $\mathsf{P}K \subseteq \mathbb{R}^n$ such that

$$h_{\mathsf{P}K}(u) = \operatorname{vol}(P_{u^\perp} K), \quad u \in S^{n-1}.$$

Here vol denotes the Lebesgue measure in $u^\perp$, $S^{n-1} \subseteq \mathbb{R}^n$ is the euclidean unit sphere, and $P_{u^\perp}$ is the orthogonal projection onto $u^\perp$. $\mathsf{P}K$ is called the projection body of K.

The surface area measure S_K of a convex body $K \subseteq \mathbb{R}^n$ is a finite Borel measure on the unit sphere such that $S_K(\omega)$ is for any Borel set $\omega \subseteq S^{n-1}$ the $(n-1)$-dimensional Hausdorff measure of the set of all boundary points of K at which there exists a normal vector of K belonging to ω. If P is a polytope, then

$$S_P = \sum_{i=1}^{m} \operatorname{vol}_{n-1}(F_i)\delta_{u_i}$$

where $F_1, \dots, F_m$ are the facets of P, $u_1, \dots, u_m$ are the corresponding facet normals, and δ_u denotes the Dirac measure.

Necessary and sufficient conditions for a finite Borel measure to be the surface area measure of a convex body are given by Minkowski's existence theorem:

Theorem 2.2. *A finite Borel measure μ on S^{n-1} is the surface area measure of a convex body with non-empty interior if and only if*

$$\int_{S^{n-1}} u \, d\mu(u) = 0$$

and μ is not concentrated on an equator. Moreover, the equation $\mu = S_K$ determines convex bodies with nonempty interior uniquely up to translations.

The Blaschke sum of two convex bodies K, L with nonempty interior is defined as the convex body $K\#L$ with centroid at the origin such that $S_{K\#L} = S_K + S_L$. The existence and uniqueness are a consequence of Theorem 2.2. The Blaschke sum is continuous with respect to the Hausdorff metric:

Lemma 2.3. *Let $K, L \subseteq \mathbb{R}^n$ be convex bodies with nonempty interior. If K_i, L_i, $i \in \mathbb{N}$, are convex bodies such that $K_i \to K$ and $L_i \to L$ for $i \to \infty$, then $K_i\#L_i$ exists for almost all i and $K_i\#L_i \to K\#L$ for $i \to \infty$.*

The support function of a projection body satisfies

$$h_{\mathsf{P}K}(x) = \frac{1}{2} \int_{S^{n-1}} |\langle u, x \rangle| \, dS_K(u). \tag{6}$$

One consequence is this formula is that $\mathsf{P}(K\#L) = \mathsf{P}K + \mathsf{P}L$ for convex bodies with nonempty interior.

The mixed volume of n convex bodies $K_1, \dots K_n$ in $\mathbb{R}^n$ is denoted by $\mathsf{V}(K_1, \dots, K_n)$. The volume of a Minkowski linear combination of m convex bodies is a homogeneous polynomial of degree n,

$$\operatorname{vol}(x_1 K_1 + \cdots + x_m K_m) = \sum_{\substack{\alpha \in \mathbb{N}^m \\ \alpha_1 + \cdots + \alpha_m = n}} \binom{n}{\alpha} \mathsf{V}(K_1[\alpha_1], \dots, K_m[\alpha_m]) x^\alpha,$$

where $x_1, \dots, x_m$ are nonnegative numbers, $\binom{n}{\alpha}$ is the multinomial coefficient,

$$\mathsf{V}(K_1[\alpha_1], \dots, K_m[\alpha_m]) = \mathsf{V}(\underbrace{K_1, \dots, K_1}_{\alpha_1 \text{ times}}, \dots, \underbrace{K_m, \dots, K_m}_{\alpha_m \text{ times}}),$$

and $x^\alpha = x_1^{\alpha_1} \cdots x_m^{\alpha_m}$.

2.2. **Polytopes.**

A polytope in $\mathbb{R}^n$ or, more generally, in a finite-dimensional real vector space V is the convex hull of finitely many points. We denote by $\mathscr{P}(V)$ the set of all polytopes in V.

Let $P \in \mathscr{P}(V)$ be a polytope. A subset $F \subseteq P$ is called a face of the polytope P if there exists a nonzero linear functional $f \colon V \to \mathbb{R}$ such that

$$P \subseteq \{x \in V \colon f(x) \leq \alpha\} \text{ and } F = P \cap \{x \in V \colon f(x) = \alpha\}.$$

If F is a facet, then f is called a facet conormal. Moreover, P itself is considered to be a (non-proper) face of P. If F is a face, then $\overline{F}$ denotes a linear subspace, namely the direction of the affine hull of F.

The conormal cone of a polytope $P \subseteq V$ at a nonempty face F is

$$N(F,P) = \{\xi \in V^*\colon \langle \xi, v\rangle = h_P(\xi) \text{ for every } v \in F\}.$$

Here $h_P\colon V^* \to \mathbb{R}$ denotes the invariant version of the support function of P. The conormal cone of P at a point $x \in P$ is

$$N(x,P) = \{\xi \in V^*\colon \langle \xi, x\rangle = h_P(\xi)\}.$$

We denote by $\operatorname{relint} A$ the relative interior of a convex set A, i.e., the interior of A relative to affine hull of A.

The following lemma describes the faces and conormal cones of the intersection of two polytopes.

Lemma 2.4. *Let $P, P' \subseteq V$ be polytopes and let G be a nonempty face of $P \cap P'$. Then the following properties hold:*

(a) *If F is a face of P and F' is a face of P', then $F \cap F'$ is a face of $P \cap P'$.*

(b) *If G is a nonempty face of $P \cap P'$, then there exist faces $F \subseteq P$ and $F' \subseteq P'$ such that $G = F \cap F'$.*

(c) *Let F be a face of P and F' be a face of P'. If $\operatorname{relint} F \cap \operatorname{relint} F' \neq \emptyset$, then conormal cones satisfy*

$$N(F \cap F', P \cap P') = N(F,P) + N(F',P').$$

We will also need an analogous result for the intersection of a polytope with a linear subspace.

Lemma 2.5. *Let V and W be finite-dimensional vector spaces and let $f\colon W \to V$ be a linear injection. Let $P \subseteq V$ be a polytope. Then the following properties hold:*

(a) *If F is a face of P, then $f^{-1}(F)$ is a face of $f^{-1}(P)$.*

(b) *If G is a nonempty face of $f^{-1}(P)$, then there exists a face F of P such that $G = f^{-1}(F)$.*

(c) *Let F be a face of P. If $f^{-1}(\operatorname{relint} F) \neq \emptyset$, then the conormal cones satisfy*

$$N(f^{-1}(F), f^{-1}(P)) = f^*(N(F,P)).$$

3. Preliminaries on the polytope algebra

Throughout this paper V denotes an n-dimensional real vector space.

3.1. Definitions and basic properties.

The polytope algebra $\Pi_*(V)$, which was introduced by McMullen in [27], is the free abelian group formally generated by the polytopes in V quotiented by the ideal generated by the relations

$$P \cup Q + P \cap Q - P - Q, \tag{7}$$

whenever $P, Q \in \mathscr{P}(V)$ are so that $P \cup Q$ is convex, and

$$(x + P) - P, \quad x \in V. \tag{8}$$

The equivalence class of a polytope P in $\Pi_*(V)$ denoted by $[P]$. Moreover, we define $[\emptyset] = 0$.

The polytope algebra enjoys a universal property: it linearizes valuations. Here a valuation with values in an abelian group A is by definition a function $\phi\colon \mathscr{P}(V) \to A$ such that for all $P, Q \in \mathscr{P}(V)$

$$\phi(P \cup Q) = \phi(P) + \phi(Q) - \phi(P \cap Q),$$

provided $P \cup Q$ is convex. A valuation is called translation-invariant if $\phi(x + P) = \phi(P)$ holds for all $x \in V$ and all polytopes P.

The universal property of the polytope algebra is an immediate consequence of a classical result of Groemer concerning the extension of valuations to convex chains (see, e.g., [24, Theorem 2.2.1]).

Proposition 3.1 (Universal property of $\Pi_*(V)$)**.** *Let $\phi\colon \mathscr{P}(V) \to A$ be a translation-invariant valuation with values in an abelian group A. Then there exists a unique morphism of abelian groups $\overline{\phi}\colon \Pi_*(V) \to A$ such that the following diagram commutes:*

$$\begin{tikzcd}
\mathscr{P}(V) \arrow[r, "\phi"] \arrow[d] & A \\
\Pi_*(V) \arrow[ur, "\overline{\phi}"'] &
\end{tikzcd}$$

Minkowski addition endows $\Pi_*(V)$ with the structure of a commutative ring

$$[P] * [Q] = [P + Q]. \tag{9}$$

To distinguish this multiplication from the one we introduce in this paper, we will refer to (9) as the *convolution* in the polytope algebra.

Dilation of polytopes by nonzero numbers $\alpha \in \mathbb{R}$ descends to a ring endomorphism of $\Pi_*(V)$ that is uniquely determined by

$$\Delta(\alpha)[P] = [\alpha P].$$

The polytope algebra admits a natural grading that is compatible with dilations. We summarize the fundamental properties of polytope algebra in the following theorem.

Theorem 3.2 ([27, Theorem 1])**.** *As an abelian group, the polytope algebra admits a decomposition*

$$\Pi_*(V) = \bigoplus_{k=0}^{n} \Pi_k(V)$$

that satisfies the following properties:

(a) *The grading is compatible with convolution:*

$$\Pi_k(V) * \Pi_l(V) \subseteq \Pi_{k+l}(V),$$

where $\Pi_k(V) = \{0\}$ for $k > n$.

(b) $\Pi_0(V) \cong \mathbb{Z}$ *is generated by the points $\{x\}$, $x \in V$. For $k \in \{1, \dots n\}$, $\Pi_k(V)$ is naturally a vector space over $\mathbb{R}$.*

(c) *For every positive real number* α *and* $x \in \Pi_k(V)$

$$\Delta(\alpha)x = \alpha^k x.$$

(d) *If* $x, y \in \bigoplus_{k=1}^n \Pi_k(V)$ *and* $\alpha \in \mathbb{R}$*, then* $(\alpha x) * y = x * (\alpha y) = \alpha(x * y)$*.*

Remark 3.3. Given the natural vector space structure on $\bigoplus_{k=1}^n \Pi_k(V)$, one may seek to characterize its linear functionals. A translation-invariant valuation $\phi\colon \mathscr{P}(V) \to \mathbb{R}$ is called dilation continuous if for any polytope P the function $\lambda \mapsto \phi(\lambda P)$, $\lambda \geq 0$, is continuous. Given a translation-invariant valuation ϕ, let $\overline{\phi}$ denote its extension to $\Pi_*(V)$. As observed by McMullen in [29, Theorem 6.2], one can show that ϕ is dilation continuous if and only if the restriction of $\overline{\phi}$ to $\bigoplus_{k=1}^n \Pi_k(V)$ is a linear functional.

3.2. **The normal cycle embedding.** The polytope algebra can be embedded into a vector space that seems easier to visualize than $\Pi_*(V)$. This embedding was first discovered by McMullen [27, Theorem 5], and below we give an invariant description of it. Under this embedding, the image of $[P]$ is essentially the normal cycle of P, a concept from geometric measure theory that plays an important role in Alesker's theory of smooth valuations [2–4, 6, 7]. We therefore propose to call this embedding the normal cycle embedding.

We call a subset $C \subseteq V$ a polyhedral cone if it is the intersection of finitely many halfspaces of the form $\{v \in V\colon f(v) \leq 0\}$, where $f\colon V \to \mathbb{R}$ is linear functional. The cone group $\widehat{\Sigma}(V)$ the free abelian group formally generated by polyhedral cones in V quotiented by the ideal generated by the relations

$$C \cup C' + C \cap C' - C - C',$$

whenever $C \cup C'$ is convex, and

$$C, \quad \text{if} \dim C < \dim V.$$

The equivalence class of a polyhedral cone C in $\widehat{\Sigma}(V)$ denoted by $[C]$.

Recall that a left Haar measure on a locally compact group is unique up to normalization. The one-dimensional vector space $\mathrm{Dens}(V)$ of densities on V consists of all (including negative) Haar measures on V. Alternatively, $\mathrm{Dens}(V)$ can be defined as the vector space of all functions $\mu\colon \Lambda^n V \to \mathbb{R}$ on the n-th exterior power of V satisfying

$$\mu(\alpha w) = |\alpha|\mu(w) \text{ for all } \alpha \in \mathbb{R} \text{ and } w \in \Lambda^n V.$$

A density is called positive if $\mu(w) > 0$ for all nonzero w.

Recall that if F is a nonempty face of a polytope P, then $\overline{F}$ denotes the direction of the affine hull of F. We denote by ε_F the element of $\mathrm{Dens}(\overline{F})^*$ defined by $\mu \mapsto \mu(F)$.

Let us take the opportunity to comment on a canonical isomorphism $\mathrm{Dens}(V^*) \to \mathrm{Dens}(V)^*$. Since $V \times V^*$ carries a canonical symplectic structure, there is a canonical density μ_ω on $V \times V^*$ called the Liouville measure. For any densities μ on V and ν on V^*, the product measure $\mu \times \nu$ is proportional to μ_ω. This factor of proportionality defines a canonical non-degenerate pairing $\mathrm{Dens}(V) \times \mathrm{Dens}(V^*) \to \mathbb{R}$, which in turn defines the isomorphism

$$\mathrm{Dens}(V^*) \to \mathrm{Dens}(V)^*. \tag{10}$$

We denote by

$$L^\perp = \{\xi \in V^*\colon \langle \xi, x\rangle = 0 \text{ for all } x \in L\}$$

the annihilator of a linear subspace $L \subseteq V$.

Definition 3.4. We define

$$\Sigma(V) = \bigoplus_{k=0}^{n} \bigoplus_{L \in \mathrm{Gr}_k(V)} \mathrm{Dens}(L)^* \otimes \widehat{\Sigma}(L^\perp),$$

where the inner sum extends over all k-dimensional linear subspaces of V. For every nonempty polytope P, we call

$$\mathrm{nc}(P) = \sum_F \varepsilon_F \otimes [N(F,P)] \in \Sigma(V),$$

where the summation extends over all nonempty faces of P, the normal cycle of P.

Unless specified otherwise, all tensor products in this paper are over $\mathbb{Z}$. If A is an abelian group and U is a real vector space, then $U \otimes A$ is vector space by declaring $\lambda \cdot (v \otimes a) = (\lambda v) \otimes a$ for $\lambda \in \mathbb{R}$. In this paper, A will sometimes happen to be a real vector space as well, hence one can define $\lambda \cdot (v \otimes a) = v \otimes (\lambda a)$. Note that these two—a priori different—vector space structures coincide.

Remark 3.5. For every polytope, $\mathrm{nc}(P)$ can be regarded as a linear functional on translation-invariant smooth $(n-1)$-forms on the sphere bundle of $\mathbb{R}^n$. Indeed, assuming $V = \mathbb{R}^n$ in what follows, translation-invariant smooth $(n-1)$-forms on $\mathbb{R}^n \times S^{n-1}$ can be identified with the elements of

$$\bigoplus_{k=0}^{n-1} \Lambda^k(\mathbb{R}^n)^* \otimes \Omega^{n-1-k}(S^{n-1}),$$

where $\Omega^*(S^{n-1})$ denotes the space of smooth differential forms on the euclidean unit sphere. For any oriented linear subspace $L \subseteq \mathbb{R}^n$, the elements of $\mathrm{Dens}(L)^*$ pair with forms on L and therefore, after restriction to L, also with forms on V. Consequently, for every face F of P and every translation-invariant $(n-1)$-form ω, one obtains a form $\langle \varepsilon_F, \omega \rangle \in \Omega^{n-1-\dim F}(S^{n-1})$. Integration of these forms defines the linear functional

$$\mathrm{nc}(P)(\omega) = \sum_F \int_{N(F,P) \cap S^{n-1}} \langle \varepsilon_F, \omega \rangle,$$

where the sum extends over all proper faces of F and the orientation of $\overline{F}^\perp$ is chosen so that $\overline{F} \oplus \overline{F}^\perp = \mathbb{R}^n$ has the standard orientation. It follows that $\omega \mapsto \mathrm{nc}(P)(\omega)$ coincides with the normal cycle of P introduced by Fu [18, 19] when the latter is integrated against translation invariant forms.

Theorem 3.6 ([27, Theorem 5])**.** *The map* $\mathrm{nc}\colon \mathscr{P}(V) \to \Sigma(V)$ *extends to an injective map of abelian groups* $\mathrm{nc}\colon \Pi_*(V) \to \Sigma(V)$*. Moreover, for* $k > 0$ *the restriction of* nc *to* $\Pi_k(V)$ *is linear.*

As an important corollary we obtain:

Corollary 3.7. $\Pi_n(V)$ *is canonically isomorphic to* $\mathrm{Dens}(V)^*$.

4. CONSTRUCTION OF THE INTERSECTION PRODUCT

At first glance, the definition of the intersection product via the integral in (3) might seem problematic, as it appears to require a topology on the infinite-dimensional vector space $\Pi^*(V)$. However, since every finite-dimensional real vector space carries a unique topology that makes it into a topological vector space, we need not concern ourselves with the topology on $\Pi^*(V)$—as long as all functions we consider take values in finite-dimensional subspaces of $\Pi^*(V)$.

The first few lemmas of this section take care of these technical issues. Recall that throughout this paper V denotes an n-dimensional real vector space.

Lemma 4.1. *Let $P, P' \subseteq V$ be polytopes. There exists an open set $U \subseteq V$ whose complement has measure zero and such that for each $x \in U$ and each pair of nonempty faces F of P and F' of P' the following property holds: If $F \cap (x + F') \neq \emptyset$, then*

$$\operatorname{relint} F \cap (x + \operatorname{relint} F') \neq \emptyset,$$

and

$$\dim F + \dim F' \geq n.$$

Proof. This follows from a standard reasoning, and we omit the proof for brevity. □

Lemma 4.2. *Let $E = \{\xi_1, \dots, \xi_m\} \subseteq V^*$ be a finite set of nonzero linear functionals. Let $S_E \subseteq \Pi^*(V)$ denote the subspace spanned by all elements $[P] \otimes \mu$ with polytopes of the form*

$$P = \{x \in V \colon \langle \xi_i, x\rangle \leq c_i \text{ for } i = 1, \dots, m\}, \tag{11}$$

where $c_1, \dots, c_m$ are certain real numbers. Then S_E is finite-dimensional.

Proof. If P is of the form (11), then each normal cone $N(F, P)$ is generated by a subset of E. Consequently, the normal cycle of $[P]$ is contained in a finite-dimensional subspace of $\Sigma(V)$. Since $\operatorname{nc} \colon \Pi^*(V) \to \Sigma(V) \otimes \operatorname{Dens}(V)$ is injective by Theorem 3.6, the claim follows. □

Lemma 4.3. *Let $E = \{\xi_1, \dots, \xi_m\} \subseteq V^*$ be a finite set of nonzero linear functionals and let S_E be defined as in Lemma 4.2. For all polytopes $P = \{x \in V \colon \langle \xi_i, x\rangle \leq c_i$ for $i = 1, \dots, m\}$ and $P' = \{x \in V \colon \langle \xi_i, x\rangle \leq c'_i$ for $i = 1, \dots, m\}$ and all densities $\mu \in \operatorname{Dens}(V)$, the following properties hold:*

(a) *$[P \cap (x + P')] \otimes \mu \in S_E$ for all $x \in V$.*

(b) *The function $V \to S_E$, $x \mapsto [P \cap (x + P')] \otimes \mu$, is measurable.*

(c) *For each $\mu' \in \operatorname{Dens}(V)$, the integral*

$$\int_V [P \cap (x + P')] \otimes \mu \, d\mu'(x) \in S_E$$

is well defined.

Proof. (a) is clear from the definition of S_E.

(b) Let U be as in Lemma 4.1 and let $x \in U$. By Lemma 2.4, the faces of $P \cap (x + P')$ can be represented in the form $F \cap (x + F')$, where F is a face of P and F' is a face of

P'. There is an open neighborhood of $U' \subseteq U$ of x such that for all $y \in U'$ and all faces F and F' the following property holds:

$$F \cap (y + F') \neq \emptyset \quad \Leftrightarrow \quad F \cap (x + F') \neq \emptyset.$$

By Lemma 2.4, the conormal cones $N(F \cap (y + F'), P \cap (y + P'))$ coincide for all $y \in U'$. Since by Lemma 2.1 the map $y \mapsto F \cap (y + F')$ is continuous in the Hausdorff metric, we conclude that

$$U' \ni y \mapsto \operatorname{nc}(P \cap (y + P')) \in \operatorname{nc}(S_E) \subseteq \Sigma(V) \otimes \operatorname{Dens}(V)$$

is continuous. Consequently, as the complement of U has measure zero, the map $V \to S_E$, $x \mapsto \operatorname{nc}(P \cap (x + P'))$, is measurable. Using that $\operatorname{nc}\colon \Pi^*(V) \to \Sigma(V) \otimes \operatorname{Dens}(V)$ is a linear injection, this finishes the proof of (b).

(c) is an immediate consequence of (a) and (b).

□

Before stating the main theorem of this section, we first describe the action of $\operatorname{GL}(V)$ on the polytope algebra. This action is the standard one, where the general linear group acts on polytopes, extended to the polytope algebra. More precisely, by the universal property of the polytope algebra (Proposition 3.1), there exists for every $g \in \operatorname{GL}(V)$ a unique linear transformation $L_g \in \operatorname{GL}(\Pi^*(V))$ such that

$$L_g([P] \otimes \mu) = [gP] \otimes g_* \mu$$

holds for every $[P] \otimes \mu \in \Pi^*(V)$. Here $g_*\mu$ denotes the pushforward of the measure μ under the linear transformation g. For nonzero α, we define dilation on $\Pi^*(V)$ by $\Delta(\alpha) = L_{\alpha \operatorname{id}}$.

Theorem 4.4. *There exists a multiplication that endows $\Pi^*(V)$ with the structure of a commutative algebra, uniquely determined by the equation* (3)*. This operation is called the* intersection product *and satisfies the following additional properties:*

(a) *It is compatible with the grading:*

$$\Pi^k(V) \cdot \Pi^l(V) \subseteq \Pi^{k+l}(V).$$

(b) *The canonical element $\mu^* \otimes \mu \in \operatorname{Dens}(V)^* \otimes \operatorname{Dens}(V) = \Pi^0(V)$, which we denote by e_V, is the identity.*

(c) *It is equivariant under the natural action of the general linear group.*

Proof. We first establish existence. Fix $\mu' \in \operatorname{Dens}(V)$ and a polytope P'. By Lemma 4.3, the map $\mathscr{P}(V) \times \operatorname{Dens}(V) \to \Pi^*(V)$,

$$\phi_{P',\mu'}(P, \mu) = \int_V [P \cap (x + P')] \otimes \mu' \, d\mu(x)$$

is well defined. Since it is a translation-invariant valuation as a function of P, by the universal property of the polytope algebra (Proposition 3.1), we obtain a map $\phi_{P',\mu'}\colon \Pi_*(V) \times \operatorname{Dens}(V) \to \Pi^*(V)$. Consequently, as the latter map is $\mathbb{Z}$-balanced, there is an extension

$$\phi_{P',\mu'}\colon \Pi^*(V) \to \Pi^*(V).$$

Let $x = \sum_{i=1}^m [P_i] \otimes \mu_i$ be fixed. One immediately verifies that

$$P' \mapsto \phi_{P',\mu'}(x) = \sum_{i=1}^m \phi_{P',\mu'}(P_i, \mu_i)$$

is a translation-invariant valuation. Again by the universal properties of the polytope algebra and the tensor product, we obtain an extension to $\Pi^*(V)$. We thus have constructed a map

$$\Pi^*(V) \otimes \Pi^*(V) \to \Pi^*(V)$$

that extends (3). This finishes the construction of the intersection product. Notice that equation (3) implies that the intersection product is commutative.

We claim that for any $\lambda \neq 0$ and $x, y \in \Pi^*(V)$

$$\Delta(\lambda)x \cdot \Delta(\lambda)y = \Delta(\lambda)(x \cdot y). \tag{12}$$

Indeed, it suffices to prove this for generators, where it is straightforward to verify using (3). Let $x \in \Pi^k(V)$ and $y \in \Pi^l(V)$ and $\lambda > 0$. Then $\Delta(\lambda)x = \lambda^{-k}x$ and $\Delta(\lambda)y = \lambda^{-l}y$. In combination with equation (12), we obtain

$$\Delta(\lambda)(x \cdot y) = \lambda^{-(k+l)} \cdot (x \cdot y).$$

Hence $x \cdot y \in \Pi^{k+l}(V)$, as desired. This proves (a).

For (b), observe that $[\lambda P] \otimes \mu$, $\lambda > 0$, lies in a finite-dimensional subspace of $\Pi^*(V)$. Hence we can obtain $[P]_n \otimes \mu$ as the limit of $\lambda^{-n}[\lambda P] \otimes \mu$ as $\lambda \to \infty$. A straightforward computation using (3) shows that

$$([P]_n \otimes \mu) \cdot ([P'] \otimes \mu') = \mu(P) \cdot ([P'] \otimes \mu')$$

Choosing P and μ so that $\mu(P) = 1$, we obtain $e_V \cdot ([P'] \otimes \mu') = [P'] \otimes \mu'$.

Finally, we consider the natural action of $\mathrm{GL}(V)$ on $\Pi^*(V)$. It suffices to verify $(L_g x) \cdot (L_g y) = L_g(x \cdot y)$ on the generators. In this case, the desired equation is an immediate consequence of (3). ☐

Definition 4.5. The Euler–Verdier involution on $\Pi^*(V)$ is defined by

$$\sigma|_{\Pi^k(V)} = (-1)^k \Delta(-1).$$

Remark 4.6. The Euler–Verdier involution was introduced for smooth valuations by Alesker [3], and it is closely related to the Verdier duality of constructible functions. In the context of the polytope algebra, the Euler–Verdier involution coincides with McMullen's Euler map up to the factor $(-1)^n$. As a consequence of this modification, the Euler–Verdier involution commutes with the pullback. Moreover, this choice of sign is convenient for the formulation of the Hodge–Riemann relations in Conjecture 1.8.

Corollary 4.7. *σ is an algebra automorphism of $\Pi^*(V)$.*

Proof. This follows immediately from Theorem 4.4(a) and (c). ☐

If x and y have complementary degrees, intersection product and McMullen convolution are closely related.

Lemma 4.8. *Fix a euclidean inner product on V to identify $V \cong \mathbb{R}^n$ and* $\operatorname{Dens}(V) \cong \operatorname{Dens}(V)^* \cong \mathbb{R}$. *Then*

$$(x \cdot y)_0 = (x * \Delta(-1)y)_n$$

holds for all $x, y \in \Pi^*(V)$.

Proof. Again, it suffices to prove the desired identity on the generators. Let $\chi\colon \Pi_0(V) \to \mathbb{R}$ denote the Euler characteristic and let $\operatorname{vol}\colon \Pi_n(V) \to \mathbb{R}$ be the restriction of the Lebesgue measure to polytopes. With this notation, the lemma follows from

$$\begin{aligned}\chi([P] \cdot [Q]) &= \int_V \chi(P \cap (x+Q))dx = \operatorname{vol}(P + (-Q)) \\ &= \operatorname{vol}([P] * \Delta(-1)[Q]).\end{aligned}$$

□

Theorem 4.9 (Poincaré duality)**.** *For every nonzero $x \in \Pi^k(V)$ there exists $y \in \Pi^{n-k}(V)$ such that $x \cdot y \neq 0$.*

Proof. Theorem 11 of [27] asserts that for every $x \in \Pi_k(V)$ with $k \in \{0, \dots, n-1\}$, there exists $y_1 \in \Pi_1(V)$ such that $x * y_1 \neq 0$. Applying this result repeatedly shows the existence of an element $y \in \Pi_{n-k}(V)$ such that $x * y \neq 0$. From Lemma 4.8 we deduce that $(x \otimes \mu) \cdot \Delta(-1)(y \otimes \mu) \neq 0$ for any $\mu \neq 0$.

□

5. CONSTRUCTION OF THE PULLBACK ALONG LINEAR INJECTIONS

We construct the pullback along a linear map $f\colon W \to V$ in two stages. In this section, we consider the special case where f is a linear injection. We address the general case in Section 7, once we have established the fundamental properties of the exterior product. Throughout this section, W denotes a finite-dimensional real vector space.

Lemma 5.1. *Let $f\colon W \to V$ be a linear injection and let $P \subseteq V$ be a polytope. There exists an open set $U \subseteq V/f(W)$ whose complement has measure zero such that for each face of F of P and each $[x] \in U$ the following properties hold: If $f^{-1}(x+F) \neq \emptyset$, then*

$$f^{-1}(x + \operatorname{relint} F) \neq \emptyset$$

and

$$\dim F \geq \dim V - \dim W.$$

Proof. As was the case for Lemma 4.1, the proof is straightforward and is therefore omitted. □

Lemma 5.2. *Let $E = \{\xi_1, \dots, \xi_m\} \subseteq V^*$ be a finite set of nonzero linear functionals and let S_{f^*E} be defined as in Lemma 4.2 with*

$$f^*E = \{f^*\xi_i\colon i = 1, \dots, m\} \setminus \{0\}.$$

For all polytopes of the form $P = \{x \in V\colon \langle \xi_i, x\rangle \leq c_i$ for $i = 1, \dots, m\}$ and every $\mu_W \in \operatorname{Dens}(W)$ the following properties hold:

(a) $[f^{-1}(x+P)] \otimes \mu_W \in S_{f^*E}$ *for all $x \in V$.*

(b) *The function $V/f(W) \to S_{f^*E}$, $[x] \mapsto [f^{-1}(x+P)] \otimes \mu_W$, is measurable.*

(c) *For each density* $\mu_{V/f(W)} \in \mathrm{Dens}(V/f(W))$*, the integral*

$$\int_{V/f(W)} [f^{-1}(x+P)] \otimes \mu_W \, d\mu_{V/f(W)}([x]) \in S_{f^*E}$$

is well-defined.

Proof. We omit the proof, because it is parallel to the proof of Lemma 4.3. □

Before we state the main result of this section, we remind the reader of a useful isomorphism for densities.

Lemma 5.3. *Let* $0 \to X \xrightarrow{i} Y \xrightarrow{\pi} Z \to 0$ *be an exact sequence of finite-dimensional real vector spaces. There exists a canonical isomorphism*

$$\mathrm{Dens}(Y) \to \mathrm{Dens}(X) \otimes \mathrm{Dens}(Z)$$

so that μ *is mapped to an element* $\mu_X \otimes \mu_Z$ *satisfying*

$$\int_Y h \, d\mu = \int_Z \int_X h(i(x)+y) d\mu_X(x) d\mu_Z(z), \quad y \in \pi^{-1}(z), \tag{13}$$

for compactly supported continuous functions $h\colon Y \to \mathbb{R}$*.*

Proof. To see the existence of an isomorphism with the desired properties just note that

$$\int_Z \int_X h(i(x)+y) d\sigma(x) d\tau(z), \quad y \in \pi^{-1}(z),$$

defines for all densities $\sigma \in \mathrm{Dens}(X)$ and $\tau \in \mathrm{Dens}(Z)$ a density on $\mathrm{Dens}(Y)$. □

The following result collects the fundamental properties of the pullback along a linear injection.

Theorem 5.4. *Let* $f\colon W \to V$ *be a linear injection. There exists a linear map* $f^*\colon \Pi^*(V) \to \Pi^*(W)$*, called the* pullback *along* f*, that is uniquely determined by the equation*

$$f^*([P] \otimes \mu) = \int_{V/f(W)} [f^{-1}(x+P)] \otimes \mu_W \, d\mu_{V/f(W)}([x]), \tag{14}$$

where $\mu = \mu_W \otimes \mu_{V/f(W)}$ *under the canonical isomorphism of Lemma 5.3. The pullback satisfies the following additional properties:*

(a) f^* *is a morphism of algebras when* $\Pi^*(V)$ *and* $\Pi^*(W)$ *are equipped with the intersection product.*

(b) $f^*(\Pi^k(V)) \subseteq \Pi^k(W)$*.*

(c) *If* $g\colon U \to W$ *is another linear injection, then* $(f \circ g)^* = g^* \circ f^*$*.*

(d) f^* *commutes with the Euler–Verdier involution.*

Proof of Theorem 5.4. Since uniqueness is clear, it suffices to prove existence. For any $\mu = \mu_W \otimes \mu_{V/f(W)} \in \mathrm{Dens}(V)$ and any polytope $P \subseteq V$ define

$$\phi(P, \mu) = \int_{V/f(W)} [f^{-1}(x+P)] \otimes \mu_W d\mu_{V/f(W)}([x]).$$

Lemma 5.2 guarantees that this is well defined. Observe that the map $\mathscr{P}(V) \to \Pi^*(W)$, $P \mapsto \phi(P,\mu)$ is a valuation. By the universal property of polytope algebra, there exists a unique extension $\Pi_*(V) \to \Pi^*(W)$. Since $(x,\mu) \mapsto \phi(x,\mu)$ is $\mathbb{Z}$-balanced, we have constructed a group homomorphism $f^* \colon \Pi^*(V) \to \Pi^*(W)$ with the desired property (14). Moreover, it is clear that f^* depends linearly on the density μ.

To prove (a) we verify $f^*(x \cdot y) = f^*x \cdot f^*y$ for generators and then show that the identity is mapped to the the idenity. Using Lemma 5.3 applied to μ' and $0 \to W \to V \to V/W \to 0$ in the fourth equality, we compute

$$\begin{aligned}
&f^*(([P]\otimes\mu)\cdot([P']\otimes\mu'))\\
&= f^*\left(\int_V [P\cap(x+P')]\otimes\mu\, d\mu'(x)\right)\\
&= \int_V f^*([P\cap(x+P')]\otimes\mu)d\mu'(x)\\
&= \int_V \left(\int_{V/f(W)} [f^{-1}(y+P)\cap f^{-1}(x+y+P')]\otimes\mu_W\, d\mu_{V/f(W)}([y])\right) d\mu'(x)\\
&= \int_W \int_{V/f(W)} \left(\int_{V/f(W)} [f^{-1}(y+P)\cap f^{-1}(f(w)+y'+P')]\otimes\mu_W\, d\mu_{V/f(W)}([y])\right)\\
&\quad d\mu'_{V/f(W)}([y'])d\mu'_W(w)\\
&= \int_{V/f(W)} \int_{V/f(W)} \left(\int_W [f^{-1}(y+P)\cap(w+f^{-1}(y'+P'))]\otimes\mu_W\, d\mu'_W(w)\right)\\
&\quad d\mu_{V/f(W)}([y])d\mu'_{V/f(W)}([y'])\\
&= \int_{V/f(W)} \int_{V/f(W)} ([f^{-1}(y+P)\otimes\mu_W)\cdot([f^{-1}(y'+P')]\otimes\mu'_W)\\
&\quad d\mu_{V/f(W)}([y])d\mu'_{V/f(W)}([y'])\\
&= f^*([P]\otimes\mu)\cdot f^*([P']\otimes\mu').
\end{aligned}$$

Let us write $n = \dim V$ and $m = \dim W$. Recall that the identity element may be expressed as $e_V = [P]_n \otimes \mu$ with $\mu(P) = 1$. Also recall that $[P]_n = \lim_{\lambda\to\infty} \lambda^{-n}[\lambda P]$. Using this, we obtain

$$\begin{aligned}
f^*(e_V) &= \lim_{\lambda\to\infty} \lambda^{-n} \int_{V/f(W)} [f^{-1}(y+\lambda P)]\otimes\mu_W\, d\mu_{V/f(W)}([y])\\
&= \lim_{\lambda\to\infty} \lambda^{-m} \int_{V/f(W)} [\lambda f^{-1}(z+P)]\otimes\mu_W\, d\mu_{V/f(W)}([z])\\
&= \int_{V/f(W)} [f^{-1}(z+P)]_m\otimes\mu_W\, d\mu_{V/f(W)}([z]).
\end{aligned}$$

Observe that

$$\int_{V/f(W)} \mu_W(f^{-1}(z+P))d\mu_{V/f(W)}([z]) = \mu(P) = 1$$

by (13). Hence $f^*(e_V) = e_U$ is the identity in $\Pi^*(W)$.

To prove (b), observe that for any $\lambda > 0$ and any $x \in \Pi^*(V)$

$$f^*(\Delta(\lambda)x) = \Delta(\lambda)f^*(x).$$

Indeed, this identity is obvious in the case of generators $x = [P] \otimes \mu$. If $x \in \Pi^k(V)$, it follows that $\Delta(\lambda)f^*(x) = \lambda^{-k}f^*(x)$. Thus $f^*(x) \in \Pi^k(W)$, as required.

To prove (c) observe that by the definition of the pullback

$$g^*(f^*([P]\otimes\mu))) = \int_{V/f(W)}\int_{W/g(U)}[g^{-1}(f^{-1}(P + x + f(y))] \otimes \mu_U \, d\mu_{W/g(U)}([y])d\mu_{V/f(W)}([x]).$$

Let $h\colon V \to \mathbb{R}$ be a compactly supported continuous function, and define

$$h_U([v]) = \int_U h(f(g(u)) + v)d\mu_U(u), \quad [v] \in V/(f(g(U)).$$

On the one hand, applying Lemma 5.3 to $0 \to W/g(U) \to V/f(g(U)) \to V/f(W) \to 0$ yields a density $\widetilde{\mu}_{V/f(g(U))}$ such that

$$\begin{aligned}\int_{V/f(W)}\int_{W/g(U)} & h_U([f(y) + x])d\mu_{W/g(U)}([y])d\mu_{V/f(W)}([x]) \\ &= \int_{V/f(g(U))} h_U([v])d\widetilde{\mu}_{V/f(g(U))}([v]).\end{aligned}$$

On the other hand, applying Lemma 5.3 to $0 \to U \to V \to V/f(g(U)) \to 0$ yields

$$\begin{aligned}\int_{V/f(W)}\int_{W/g(U)} & h_U([f(y) + x])d\mu_{W/g(U)}([y])d\mu_{V/f(W)}([x]) \\ &= \int_V h d\mu \\ &= \int_{V/f(g(U))} h_U([v])d\mu_{V/f(g(U))}([v]).\end{aligned}$$

Hence $\widetilde{\mu}_{V/f(g(U))} = \mu_{V/f(g(U))}$ and we conclude that

$$\begin{aligned}g^*(f^*([P] \otimes \mu))) &= \int_{V/f(g(U))}\int_U [(f \circ g)^{-1}(P + v)] \otimes \mu_U \, d\mu_{V/f(g(U))}([v]) \\ &= (f \circ g)^*([P] \otimes \mu).\end{aligned}$$

This finishes the proof of (c).

Finally, that the pullback commutes with the Euler–Verdier involution is an immediate consequence of (b) and (c). □

Remark 5.5. There is a similarly looking, but essentially different construction known as the fiber polytope. For comparison, let W be a linear subspace of V. Let $f\colon W \to V$ denote the inclusion and let $\pi\colon V \to V/W$ denote the canonical projection. Let P be a polytope in V and put $Q = \pi(P)$. In this situation, the fiber polytope of $\pi\colon P \to Q$ introduced in [12] is the polytope in V defined by

$$\Sigma_\pi(P, Q) = \int_Q P \cap \pi^{-1}(y) \, dy,$$

where dy is some positive density on V/W. This integral can be interpreted as the Minkowski sum of the fibers $P \cap \pi^{-1}(y)$, see [12, Proposition 1.2]. For every continuous choice $x \in \pi^{-1}(y)$, the integral

$$\int_Q f^{-1}(x+P)\, dy,$$

is a translate of $\Sigma_\pi(P,Q)$. While this integral may resemble (14), the two are fundamentally different: the latter represents an average of the classes $[f^{-1}(x+P)]$ taken with respect to the addition in the polytope algebra.

Remark 5.6. In the paper [5], Alesker introduced a pushforward of continuous translation-invariant valuations along linear injections. More precisely, if $\mathrm{Val}(V)$ denotes the Banach space of translation-invariant continuous valuations on convex bodies in V, then the pushforward along a linear injection $f\colon W \to V$ is the continuous linear map

$$f_*\colon\ \mathrm{Val}(W) \otimes \mathrm{Dens}(W)^* \to \mathrm{Val}(V) \otimes \mathrm{Dens}(V)^*$$

defined by

$$f_*(\phi \otimes \varepsilon)(K) = \int_{V/f(W)} \phi(f^{-1}(x+K)) \otimes \varepsilon_V \, d\nu_{V/f(W)}(x),$$

where $K \subseteq V$ is a convex body and $\varepsilon \in \mathrm{Dens}(W)^*$, $\varepsilon_V \in \mathrm{Dens}(V)^*$, and $\nu_{V/f(W)} \in \mathrm{Dens}(V/f(W))$ are related via canonical isomorphism of Lemma 5.3. If P is a polytope in V, then

$$\langle f_*(\phi \otimes \varepsilon), [P] \otimes \mu\rangle = \langle \phi \otimes \varepsilon, f^*([P] \otimes \mu)\rangle.$$

In this sense, the pushforward f_* of translation-invariant continuous valuations can be regarded as dual to the pullback f^* in the polytope algebra.

6. Exterior product

The exterior product, which we introduce in this section, will play an important role in the construction of the pullback along general linear maps. Moreover, it will facilitate the evaluation of the intersection product in certain special situations, see Theorem 8.2 and Proposition 9.3 below.

Theorem 6.1. *There exists a bilinear map*

$$\boxtimes\colon \Pi^*(V) \times \Pi^*(W) \to \Pi^*(V \times W),$$

called the exterior product*, that is uniquely determined by*

$$([P] \otimes \mu) \boxtimes ([Q] \otimes \nu) = [P \times Q] \otimes (\mu \times \nu).$$

Moreover, it has the following additional properties:

(a) *It is compatible with the grading:*

$$\Pi^k(V) \boxtimes \Pi^l(W) \subseteq \Pi^{k+l}(V \times W).$$

(b) *If U is another vector space and $z \in \Pi^*(U)$, then*

$$(x \boxtimes y) \boxtimes z = x \boxtimes (y \boxtimes z).$$

(c) *If $f\colon U_1 \times U_2 \to V \times W$, $f = f_1 \times f_2$, is a linear injection, then*

$$f^*(x \boxtimes y) = f_1^* x \boxtimes f_2^* y$$

(d) *The identity elements for the intersection product satisfy* $e_V \boxtimes e_W = e_{V\times W}$.

(e) *Let* $\operatorname{diag}\colon V \to V \times V$ *denote the diagonal embedding. For all* $x, y \in \Pi^*(V)$

$$x \cdot y = \operatorname{diag}^*(x \boxtimes y).$$

(f) *For all* $x, y \in \Pi^*(V)$ *and* $x', y' \in \Pi^*(W)$,

$$(x \cdot y) \boxtimes (x' \cdot y') = (x \boxtimes x') \cdot (y \boxtimes y').$$

Proof. The existence part follows immediately from the universal property of the polytope algebra. Item (a) follows from $\Delta(\lambda)(x \boxtimes y) = \Delta(\lambda)x \boxtimes \Delta(\lambda)y$, $\lambda > 0$, which is straightforward to verify for generators. Likewise, (b) is satisfied for generators and hence true in general. By the same token, (c) follows from

$$f^{-1}((v, w) + P \times Q) = f_1^{-1}(v + P) \times f_2^{-1}(w + Q).$$

Property (d) concerning the identity elements is also straightforward.

To prove (e) note that

$$\begin{aligned}
\operatorname{diag}^*([P] \otimes \mu) \boxtimes ([P'] \otimes \mu')) &= \int_{V^2/\operatorname{diag}(V)} [P \cap (x' - x + P')] \otimes \mu_V \, d\mu_{V^2/\operatorname{diag}(V)}([(x, x')]) \\
&= \int_V [P \cap (y + P')] \otimes \mu_V \, d(g_*\mu_{V^2/\operatorname{diag}(V)})(y)
\end{aligned}$$

where $\mu \times \mu' = \mu_V \otimes \mu_{V^2/\operatorname{diag}(V)}$ under the isomorphism of Lemma 5.3 and $g\colon V^2/\operatorname{diag}(V) \to V$ denotes the isomorphism $g([(x, x')]) = x' - x$.

Put $T\colon V^2 \to V^2$, $T(x, y) = (x, y - x)$. On the one hand, since $\det T = 1$,

$$\begin{aligned}
&\int_V \int_{V^2/\operatorname{diag}(V)} (h \circ T)(\operatorname{diag}(u) + (x, x')) d\mu_V(u) d\mu_{V^2/\operatorname{diag}(V)}([(x, x')]) \\
&\qquad = \int_{V\times V} h(u, v) d(\mu \times \mu')(u, v).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_V \int_{V^2/\operatorname{diag}(V)} (h \circ T)(\operatorname{diag}(u) + (x, x')) d\mu_V(u) d\mu_{V^2/\operatorname{diag}(V)}([(x, x')]) \\
&\qquad = \int_{V\times V} h(u, v) d(\mu_V \times g_*\mu_{V^2/\operatorname{diag}(V)})(u, v).
\end{aligned}$$

We conclude that $\mu \times \mu' = \mu_V \times g_*\mu_{V^2/\operatorname{diag}(V)}$. This shows that (e) holds for generators, which suffices to finish the proof.

Finally, item (f) is a formal consequence of (c) and (e). Indeed, if diag_V, diag_W, and $\operatorname{diag}_{V\times W}$ denote the diagonal embeddings and $\rho\colon (V\times W)^2 \to V^2 \times W^2$, $\rho(v_1, w_1, v_2, w_2) = (v_1, v_2, w_1, w_2)$, then using $\rho \circ \operatorname{diag}_{V\times W} = \operatorname{diag}_V \times \operatorname{diag}_W$, we obtain

$$\begin{aligned}
(x \cdot y) \boxtimes (x' \cdot y') &= \operatorname{diag}_V^*(x \boxtimes y) \boxtimes \operatorname{diag}_W^*(x' \boxtimes y') \\
&= (\operatorname{diag}_V \times \operatorname{diag}_W)^*(x \boxtimes y \boxtimes x' \boxtimes y') \\
&= (\operatorname{diag}_{V\times W})^* \rho^*(x \boxtimes y \boxtimes x' \boxtimes y') \\
&= (x \boxtimes x') \cdot (y \boxtimes y')
\end{aligned}$$

□

Corollary 6.2. *Let $m \geq 2$ be an integer, let $x_1, \ldots, x_m \in \Pi^*(V)$, and let $\mathrm{diag}_m \colon V \to V^m$ denote the diagonal embedding. Then*

$$x_1 \cdots x_m = (\mathrm{diag}_m)^*(x_1 \boxtimes \cdots \boxtimes x_m).$$

Proof. For $m = 2$, the statement follows directly from Theorem 6.1(e). The case $m > 2$ follows by induction, using $(\mathrm{diag}_{m-1} \times \mathrm{id}_V) \circ \mathrm{diag}_2 = \mathrm{diag}_m$. □

7. CONSTRUCTION OF THE PULLBACK IN THE GENERAL CASE

Equipped with the properties of the exterior product, we are now ready to complete the construction of the pullback.

Definition 7.1. Let $f \colon V \to W$ be a linear map. Let X be a complement of $\ker f$ in V. Let $e_{\ker f}$ denote the identity element in $\Pi^*(\ker f)$. The pullback along f is defined by

$$f^*(x) = e_{\ker f} \boxtimes (f|_X)^* x, \quad x \in \Pi^*(W).$$

Theorem 7.2. *Let $f \colon V \to W$ be a linear map. The definition of the pull-back f^* does not depend on the choice of complement to $\ker f$. Moreover, the pullback has the following additional properties:*

(a) *It is a morphism of algebras when $\Pi^*(V)$ and $\Pi^*(W)$ are equipped with the intersection product.*

(b) *It is compatible with the grading, $f^*(\Pi^k(W)) \subseteq \Pi^k(V)$.*

(c) *If $g \colon U \to V$ is another linear map, then $(f \circ g)^* = g^* \circ f^*$.*

(d) *The pullback commutes with the Euler–Verdier involution.*

For the proof, we will use the following lemma.

Lemma 7.3. *Let $f \colon V \to W$ be a linear surjection, and let X be a complement to $\ker f$. For all densities $\mu \in \mathrm{Dens}(\ker f)$ and $\nu \in \mathrm{Dens}(W)$,*

$$\rho = \mu \times (f|_X)_*^{-1}\nu \in \mathrm{Dens}(V)$$

is independent of the choice of complement X.

Proof. Let X' be another complement to $\ker f$, and define $\rho' = \mu \times (f|_{X'})_*^{-1}\nu$. Let $g_1 \colon X \to \ker f$ and $g_2 \colon X \to X'$ be the linear maps, uniquely determined by $v + x = v + g_1(x) + g_2(x)$ for all $v \in \ker f$ and $x \in X$. For all bounded Borel sets $A \subseteq \ker f$ and $B \subseteq X$, one has

$$\begin{aligned}
\rho'(A \times B) &= \int_{X'} \int_{\ker f} \mathbf{1}_A(v - g_1(g_2^{-1}(x'))) \mathbf{1}_B(g_2^{-1}(x')) dv dx' \\
&= \mu(A)\nu((f|_{X'} \circ g_2(B)) \\
&= \mu(A)\nu((f|_X(B)) \\
&= \rho(A \times B).
\end{aligned}$$

This implies $\rho = \rho'$, as claimed. □

Proof. To see that the definition of the pullback is independent of the choice of complement, define $m = \dim \ker f$ and choose a polytope $Q \subseteq \ker f$ and a density $\mu \in \mathrm{Dens}(\ker f)$ such that $e_{\ker f} = [Q]_m \otimes \mu$. Since $f\colon V \to W$ can be factored as $V \to \mathrm{im}(f) \hookrightarrow W$, we may assume in what follows that f is surjective. Choose a complement X of $\ker f$ in V. For $x = [P] \otimes \nu \in \Pi^*(W)$, one has

$$\mathrm{nc}(e_{\ker f} \boxtimes (f|_X)^* x) = \sum_F \varepsilon_{Q+(f|_X)^{-1}F} \otimes (\mathrm{pr}_X)^*(f|_X)^* N(F,P) \otimes (\mu \times (f|_X)_*^{-1}\nu),$$

where the sum extends over all faces of P and $\mathrm{pr}_X\colon V \to X$ denotes the projection onto X. Let X' be another complement to $\ker f$, and let $\mathrm{pr}_{X'}\colon V \to X'$ denote the projection onto X'. We claim that each factor in

$$\varepsilon_{Q+(f|_X)^{-1}F} \otimes (\mathrm{pr}_X)^*(f|_X)^* N(F,P) \otimes (\mu \times (f|_X)_*^{-1}\nu)$$

is independent of the choice of complement X to $\ker f$. Indeed, for the second factor this is a consequence of $f \circ \mathrm{pr}_X = f \circ \mathrm{pr}_{X'}$ and for the third one this follows from Lemma 7.3.

For each face F, we will show that $\varepsilon_{Q+(f|_X)^{-1}F}$ and $\varepsilon_{Q+(f|_{X'})^{-1}F}$ define the same dual density on $f^{-1}(\overline{F})$. Consider the linear subspaces $L = (f|_X)^{-1}\overline{F}$ and $L' = (f|_{X'})^{-1}\overline{F}$ and the density μ on $\ker f + L = f^{-1}(\overline{F}) = \ker f + L'$ defined by

$$\int_L \int_{\ker f} h(v,x)\,dv\,dx,$$

for some positive densities dv and dx. Let $g_1\colon L \to \ker f$ and $g_2\colon L \to L'$ be the linear maps uniquely determined by $v + x = v + g_1(x) + g_2(x)$ for all $v \in \ker f$ and $x \in L$. Then, since $f|_X = f|_{X'} \circ g_2$,

$$\begin{aligned}\mu(Q + (f|_{X'})^{-1}F) &= \int_L \int_{\ker f} \mathbf{1}_Q(v + g_1(x))\mathbf{1}_{(f|_{X'})^{-1}F}(g_2(x))\,dv\,dx \\ &= \mu(Q + (f|_X)^{-1}F)\end{aligned}$$

and, in turn, $\varepsilon_{Q+(f|_X)^{-1}F} = \varepsilon_{Q+(f|_{X'})^{-1}F}$.

We conclude that

$$\mathrm{nc}(e_{\ker f} \boxtimes (f|_X)^* x) = \mathrm{nc}(e_{\ker f} \boxtimes (f|_{X'})^* x).$$

Since nc is injective, this shows that the pullback does not depend on the choice of complement.

To prove (a), observe that

$$f^*(x \cdot y) = e_{\ker f} \boxtimes ((f|_X)^* x \cdot (f|_X)^* y) = (e_{\ker f} \cdot e_{\ker f}) \boxtimes ((f|_X)^* x \cdot (f|_X)^* y)$$

and use Theorem 6.1(f). The equation $f^*(e_V) = e_W$ follows immediately from $(f|_X)^* e_V = e_X$ and $e_{\ker f} \boxtimes e_X = e_W$, see Theorem 6.1(d).

(b) is a straightforward consequence of Theorem 6.1(a) and the corresponding property of the pullback along linear injections.

For the proof of (c) choose a complement $\ker g \oplus X = U$. Put $X_1 = \ker(f \circ g|_X)$ and choose complements $X_1 \oplus X_2 = X$ and $\ker f \oplus Y = V$ with $Y \supseteq g(X_2)$. Since $g(X_1) \subseteq \ker f$ and $g(X_2) \subseteq Y$ by construction, we may write $g|_X = g_1 \times g_2\colon X_1 \times X_2 \to \ker f \times Y$. Using the properties of the pullback along linear injections and Theorem 6.1, we compute

$$g^*(f^*(x)) = e_{\ker g} \boxtimes (g|_X)^*(e_{\ker f} \boxtimes (f|_Y)^* x)$$

$$\begin{aligned}
&= e_{\ker g} \boxtimes (g_1^*(e_{\ker f}) \boxtimes g_2^*(f|_Y)^* x) \\
&= e_{\ker g} \boxtimes (e_{X_1} \boxtimes (f \circ g_2)^* x) \\
&= (e_{\ker g} \boxtimes e_{X_1}) \boxtimes (f \circ g_2)^* x \\
&= e_{\ker(f\circ g)} \boxtimes (f \circ g|_{X_2})^* x \\
&= (f \circ g)^* x.
\end{aligned}$$

Finally, (d) follows immediately from (b) and (c).

□

Remark 7.4. Alesker introduced in [5] the pushforward of continuous translation-invariant valuations along general linear maps. If $f\colon W \to V$ is a linear surjection, then this operation can be described as follows. We keep the notation of Remark 5.6. Choose a complement $\ker f \oplus X = W$ to the kernel of f. Under the canonical isomorphism $\mathrm{Dens}(W)^* \cong \mathrm{Dens}(\ker f)^* \otimes \mathrm{Dens}(X)^*$, any dual density $\varepsilon \in \mathrm{Dens}(W)^*$ is represented as $\varepsilon_{\ker f} \otimes \varepsilon_X$. Choose a convex body $E \subseteq \ker f$ so that $\varepsilon_{\ker f}(\mu) = \mu(E)$ for all densities $\mu \in \mathrm{Dens}(\ker f)$. The pushforward

$$f_*\colon \mathrm{Val}(W) \otimes \mathrm{Dens}(W)^* \to \mathrm{Val}(V) \otimes \mathrm{Dens}(V)^*$$

is defined by

$$f_*(\phi \otimes \varepsilon)(K) = \frac{1}{m!} \frac{d^m}{dt^m}\bigg|_{t=0} \phi(tE \times (f|_X)^{-1}(K)) \otimes (f|_X)_* \varepsilon_X,$$

where $m = \dim X$ and K is a convex body in V. If P is a polytope in V, then one readily verifies the identity

$$\langle f_*(\phi \otimes \varepsilon), [P] \otimes \mu) \rangle = \langle \phi \otimes \varepsilon, f^*([P] \otimes \mu) \rangle.$$

Hence also in the case of linear surjections, the pushforward of continuous translation-invariant valuations can be regarded as dual to the pullback in the polytope algebra. Since every linear map f can be expressed as $f = j \circ p$, where j is injective and p is surjective, this relationship extends to all linear maps.

8. Identities for special elements of the polytope algebra

In this section, we are concerned with special elements of $\Pi^*(V)$, namely those proportional to $[P]_{n-k} \otimes \mu$ for $(n-k)$-dimensional polytopes P. Viewed appropriately, the operations of intersection product, pullback, and exterior product of these elements translate into linear algebraic operations on linear subspaces.

Let $L \subseteq V^*$ be a linear subspace. Inclusion and restriction yield an exact sequence $0 \to L^\perp \to V \to L^* \to 0$. Hence, given elements of $\mathrm{Dens}(V)$ and $\mathrm{Dens}(L^\perp)^*$, Lemma 5.3 yields a density on L^*. Combined with (10), this construction describes an isomorphism

$$\mathrm{Dens}(L^\perp)^* \otimes \mathrm{Dens}(V) \cong \mathrm{Dens}(L)^* \tag{15}$$

that we will frequently use in the following.

Definition 8.1. Let $L \subseteq V^*$ be a k-dimensional linear subspace and let $\varepsilon \in \mathrm{Dens}(L)^*$ be a dual density. We denote by $x_{L,\varepsilon}$ the unique element in $\Pi^k(V)$ such that

$$\mathrm{nc}(x_{L,\varepsilon}) = \varepsilon \otimes [L] \in \Sigma(V) \otimes \mathrm{Dens}(V)$$

under the isomorphism (15).

Theorem 8.2. *Let $L \subseteq V^*$ and $L' \subseteq V'^*$ be linear subspaces and let $\varepsilon \in \mathrm{Dens}(L)^*$ and $\varepsilon' \in \mathrm{Dens}(L')^*$ be dual densities. The following statements hold:*

(a) *The exterior product satisfies*

$$x_{L,\varepsilon} \boxtimes x_{L',\varepsilon'} = x_{L\times L',\varepsilon\otimes\varepsilon'}.$$

(b) *For any linear map $f\colon W \to V$,*

$$f^* x_{L,\varepsilon} = \begin{cases} x_{f^*L,(f^*)_*\varepsilon} & \textit{if } f(W) + L^\perp = V, \\ 0 & \textit{otherwise.} \end{cases}$$

In the first case, $f^*L = \{f^*\xi \colon \xi \in L\}$ and $f^*\colon L \to f^*L$ is an isomorphism.

(c) *If $V = V'$, then*

$$x_{L,\varepsilon} \cdot x_{L',\varepsilon'} = \begin{cases} x_{L+L',a_*(\varepsilon\otimes\varepsilon')} & \textit{if } L \cap L' = \{0\}, \\ 0 & \textit{otherwise,} \end{cases}$$

where $a\colon V^* \times V^* \to V^*$ denotes the vector space addition.

For the proof of the theorem, the following description of the isomorphism (15) will be helpful. Recall that μ_ω denotes the Liouville measure on $V \times V^*$. If $K \subseteq V$ is a convex body containing the origin in its interior, then $K^\circ \subseteq V^*$ denotes the polar body of K.

Lemma 8.3. *Let $L \subseteq V^*$ be a linear subspace and let $X \subseteq V$ be a complement to $L^\perp$. Suppose the following:*

- *$\nu \in \mathrm{Dens}(V)$ and $\varepsilon \in \mathrm{Dens}(L)^*$;*
- *$P \subseteq L^\perp$ is a convex body with nonempty interior and $\varepsilon_P \in \mathrm{Dens}(L^\perp)^*$ denotes the corresponding dual density;*
- *$Q \subseteq X$ is a convex body containing the origin in its interior such that $\varepsilon = \frac{1}{\mu_\omega(\pi(Q)^\circ \times \pi(Q))}\varepsilon_{\pi(Q)^\circ}$, where $\pi\colon V \to L^*$ is the canonical map.*

In this situation,

$$\varepsilon_P \otimes \nu \mapsto \varepsilon \quad \textit{under the isomorphism } (15)$$

if and only if

$$\nu(P+Q) = 1.$$

Proof. Lemma 5.3 applied to $0 \to L^\perp \to V \to L^* \to 0$ yields $\nu = \nu_{L^\perp} \otimes \nu_{L^*}$ and

$$\nu(P+Q) = \nu_{L^\perp}(P)\nu_{L^*}(\pi(Q)).$$

Consequently, $\nu(P+Q) = 1$ if and only if $\nu_{L^*}(\pi(Q)) = 1$. By the definition of the isomorphism (10), the latter is equivalent to the statement that the dual density corresponding to ν_{L^*} is $\frac{1}{\mu_\omega(\pi(Q)^\circ \times \pi(Q))}\varepsilon_{\pi(Q)^\circ}$. □

Proof of Theorem 8.2. Choose polytopes $P \subseteq L^\perp$, $P' \subseteq (L')^\perp$ of dimensions $n-k$ and $n-k'$, containing the origin in their relative interior, together with suitable densities $\nu \in \mathrm{Dens}(V)$, $\nu' \in \mathrm{Dens}(V')$ such that $\varepsilon_P \otimes \nu = \varepsilon$ and $\varepsilon_{P'} \otimes \nu' = \varepsilon'$ under the isomorphism (15). Using that $\lambda^{-(n-k)}[\lambda P] \to [P]_{n-k}$ as $\lambda \to \infty$, we obtain

$$\mathrm{nc}(x_{L,\varepsilon} \boxtimes x_{L',\varepsilon'}) = \lim_{\lambda\to\infty} \lim_{\lambda'\to\infty} \lambda^{-(n-k)} \lambda'^{-(n-k')} \mathrm{nc}([\lambda P \times \lambda' P'] \otimes \nu \times \nu')$$

$$
\begin{aligned}
&= \varepsilon_{P\times P'} \otimes \nu \times \nu' \otimes [L \times L'] \\
&= \varepsilon \otimes \varepsilon' \otimes [L \times L'].
\end{aligned}
$$

This proves (a).

To show (b), let as before $P \subseteq L^\perp$ be an $(n-k)$-dimensional polytope and $\nu \in \mathrm{Dens}(V)$ a density.

We first consider the case where f is a linear injection. It follows from the definition of the pullback that $f^* x_{L,\varepsilon} = 0$, if $f(W) + L^\perp \neq V$. We therefore assume from now on that $f(W) + L^\perp = V$. Under this assumption, $f^* \colon L \to f^*L$ is clearly injective. Unraveling the definitions and using Lemma 2.5, we obtain

$$
\begin{aligned}
\mathrm{nc}(f^* x_{L,\varepsilon}) &= \lim_{\lambda\to\infty} \lambda^{-(n-k)} \int_{V/f(W)} \mathrm{nc}([f^{-1}(x+\lambda P)]) \otimes \nu_W \, d\nu_{V/f(W)}([x]) \\
&= \lim_{\lambda\to\infty} \lambda^{-(m-k)} \int_{V/f(W)} \mathrm{nc}([\lambda f^{-1}(y+P)]) \otimes \nu_W \, d\nu_{V/f(W)}([y]) \\
&= \int_{V/f(W)} \varepsilon_{f^{-1}(y+P)} d\nu_{V/f(W)}[y] \otimes \nu_W \otimes [f^*L] \\
&=: \varepsilon' \otimes \nu_W \otimes [f^*L].
\end{aligned}
$$

Choose a complement $L^\perp \oplus X = V$ such that $X \subseteq f(W)$. This is possible since $f(W) + L^\perp = V$ by assumption. Choose a convex body $Q \subseteq X$ containing the origin in its interior such that $\nu(P+Q) = 1$. Note that by Lemma 8.3,

$$
\varepsilon = \frac{1}{\mu_\omega(\pi(Q)^\circ \times \pi(Q))} \varepsilon_{\pi(Q)^\circ}. \tag{16}
$$

As $(f^*L)^\perp \oplus f^{-1}(X) = W$ and $f^{-1}(y+P) + f^{-1}(Q) = f^{-1}(y+P+Q)$ for $y \in L^\perp$, we have

$$
\begin{aligned}
\langle \varepsilon' \otimes \varepsilon_{f^{-1}(Q)}, \nu_W \rangle &= \int_{V/f(W)} \nu_W(f^{-1}(y+P) + f^{-1}(Q)) d\nu_{V/f(W)}([y]) \\
&= \nu(P+Q) = 1.
\end{aligned}
$$

Consequently, Lemma 8.3 implies that $\varepsilon' \otimes \nu_W$ is mapped to

$$
\frac{1}{\mu_\omega(\pi(Q)^\circ \times \pi(Q)))} \varepsilon_{f^*(\pi(Q)^\circ)} \tag{17}
$$

under the isomorphism (15). Comparing this expression with (16), shows that (17) equals $(f^*)_*\varepsilon$, as claimed.

Assume next that $f\colon W \to V$ is a linear surjection. Choose a complement $X \subseteq W$ to the kernel of f and put $m = \dim \ker f$. Put $P' = (f|_X)^{-1}(P)$ and $\nu' = ((f|_X)^{-1})_*\nu$. Represent $e_{\ker f}$ as $[R]_m \otimes \rho$. One has

$$
\begin{aligned}
\mathrm{nc}(f^*x) &= \mathrm{nc}(([R]_m \otimes \rho) \boxtimes (f|_X)^*([P]_{n-k} \otimes \nu)) \\
&= \mathrm{nc}([R \times P']_{m+n-k} \otimes \rho \times \nu') \\
&= \varepsilon_{R\times P'} \otimes \rho \times \nu' \otimes [f^*L].
\end{aligned}
$$

Choose a complement $L^\perp \oplus Y = V$ and a convex body $Q \subseteq Y$ containing the origin in its interior such that $\nu(P+Q) = 1$. Put $Q' = (f|_X)^{-1}Q$ and $Y' = (f|_X)^{-1}(Y)$. Notice that $(f^*L)^\perp \oplus Y' = W$ and

$$\rho \times \nu'(R \times P' + Q') = 1.$$

If $\pi_W \colon W \to (f^*L)^*$ denotes the canonical map, then Lemma 8.3 implies that $\varepsilon_{R\times P'} \otimes \rho \times \nu'$ is mapped to

$$\frac{1}{\mu_\omega(\pi_W(Q')^\circ \times \pi_W(Q')))} \varepsilon_{\pi_W(Q')^\circ} = (f^*)_* \varepsilon.$$

under the isomorphism (15). This finishes the proof of (b), since every linear map f can be expressed as the composition of a surjection and an injection.

Finally, we prove (c). We will apply the description of the pullback to the diagonal embedding $\operatorname{diag} \colon V \to V \times V$. Note that $(L \times L')^\perp = L^\perp \times L'^\perp$ and that $\operatorname{diag}(V) + L^\perp \times L'^\perp = V \times V$ is equivalent to the statement that the image of $L^\perp \times L'^\perp$ under the canonical projection $\pi \colon V \times V \to V \times V/\operatorname{diag}(V)$ equals $V \times V/\operatorname{diag}(V)$. Observe that $V \times V \to V$, $(x,y) \mapsto x - y$, descends to an isomorphism $T \colon V \times V/\operatorname{diag}(V) \to V$. Since $T(\pi(L^\perp \times L'^\perp)) = L^\perp + L'^\perp$, we conclude that $\operatorname{diag}(V) + L^\perp \times L'^\perp = V \times V$ is equivalent to $V = L^\perp + L'^\perp = (L \cap L')^\perp$.

Let $a \colon V^* \times V^* \to V^*$ denote the vector space addition, $a(\xi, \eta) = \xi + \eta$. It is straightforward to check that $a = \operatorname{diag}^*$. We conclude that $\operatorname{diag}^*(L \times L') = L + L'$ and $(\operatorname{diag}^*)_* = a_*$. $\square$

If $V = \mathbb{R}^n$, then the above expression for the product can be made even more explicit. Since there is a canonical positive density on $\mathbb{R}^n$ and all of its subspaces, namely the Lebesgue measure vol, we introduce the following notation.

Definition 8.4. If $V = \mathbb{R}^n$, then we define $x_L := x_{L,\mathrm{vol}} \in \Pi^*(\mathbb{R}^n)$ for each linear subspace $L \subseteq \mathbb{R}^n \cong (\mathbb{R}^n)^*$. Moreover, we abbreviate our notation to $\alpha[P] := [P] \otimes \alpha \operatorname{vol} \in \Pi^*(\mathbb{R}^n)$ for polytopes P and real numbers α.

If $L, L' \subseteq \mathbb{R}^n$ are such that $L \cap L' = \{0\}$, define

$$\sin(L, L') = \frac{\operatorname{vol}(B + B')}{\operatorname{vol}(B)\operatorname{vol}(B')}, \tag{18}$$

where B, B' are the euclidean unit balls in L and L'. In terms of the principal angles $0 \le \theta_1 \le \cdots \le \theta_m \le \pi/2$, $m = \min(\dim L, \dim L')$, between L and L', one has

$$\sin(L, L') = \sin\theta_1 \cdots \sin\theta_m,$$

see, e.g., [30]. We call $\sin(L, L')$ the sine of the principal angles between L and L'. Sometimes this quantity is also called the subspace determinant of L and L', see [33, Section 14.1].

With the above definitions, the description of the product immediately translates into the following

Corollary 8.5. *Let L, L' be linear subspaces in $\mathbb{R}^n$. Then*

$$x_L \cdot x_{L'} = \begin{cases} \sin(L, L')\, x_{L+L'} & \text{if } L \cap L' = \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

There is a particularly simple description of the elements x_L in terms of the pullback along orthogonal projections.

Corollary 8.6. *Let $L \subseteq \mathbb{R}^n$ be a linear subspace. Let $p\colon \mathbb{R}^n \to L$ denote the orthogonal projection. Then $x_L = p^*[\{0\}]$.*

We return now to the general case of an n-dimensional real vector space. The following proposition shows that multiplication by the element $x_{L,\varepsilon}$ is essentially the same as the pullback to along the inclusion $L^\perp \to V$. This fact will be important for us as it allows us prove to certain statements by induction on the dimension of V.

Proposition 8.7. *Let $L \subseteq V^*$ be a line through the origin. Denote $H = L^\perp$ and write $i\colon H \to V$ for the inclusion. Choose a complement X to the hyperplane $H \subseteq V$ and let $\mu_X \in \operatorname{Dens}(X)$ correspond to ε under the isomorphism $X \to L^*$. For every $y \in \Pi^*(V)$*

$$x_{L,\varepsilon} \cdot y = ([\{0\}] \otimes \mu_X) \boxtimes i^* y.$$

Proof. Let us write $x_{L,\varepsilon} = [P]_{n-1} \otimes \nu$ for a suitable $(n-1)$-dimensional polytope P and a density $\nu \in \operatorname{Dens}(V)$. It suffices to prove the statement for elements of the form $y = [Q] \otimes \rho$. In this case, we find

$$\begin{aligned} x_{L,\varepsilon} \cdot y &= \lim_{\lambda \to \infty} \lambda^{-(n-1)} ([\lambda P] \otimes \nu) \cdot ([Q] \otimes \rho) \\ &= \lim_{\lambda \to \infty} \lambda^{-(n-1)} \int_V [\lambda P \cap (x + Q)] \otimes \nu \, d\rho(x) \\ &= \lim_{\lambda \to \infty} \int_{V/H} \left(\lambda^{-(n-1)} \int_H [(x + \lambda P) \cap (t + Q)] \otimes \nu \, d\rho_H(x) \right) d\rho_{V/H}([t]). \end{aligned}$$

After replacing integration over H by integration over $(-\lambda P) + (t + Q)$ and the change of variables $\lambda^{-1} x = z$, we find that the inner integral satisfies

$$\begin{aligned} \lambda^{-(n-1)} &\int_H [(x + \lambda P) \cap (t + Q)] \otimes \nu \, d\rho_H(x) \\ &= \lambda^{-(n-1)} \int_{(-\lambda P) + (t+Q)} [(x + \lambda P) \cap (t + Q)] \otimes \nu \, d\rho_H(x) \\ &= \int_{(-P) + \lambda^{-1}(t+Q)} [\lambda(z + P) \cap (t + Q)] \otimes \nu \, d\rho_H(z) \end{aligned}$$

For this last integral, it is clear that one can interchange limit and integral and so

$$\lim_{\lambda \to \infty} \lambda^{-(n-1)} \int_H [(x + \lambda P) \cap (t + Q)] \otimes \nu \, d\rho_H(x) = [H \cap (t + Q)] \otimes \rho_H(P) \nu.$$

We conclude that

$$x_{L,\varepsilon} \cdot y = \int_{V/H} [H \cap (t + Q)] \otimes \rho_H(P) \nu \, d\rho_{V/H}([t]).$$

This last expression resembles the definition of the pullback. The difference is that here the bracket $[H \cap (t + Q)]$ is understood inside $\Pi(V)$ and not inside $\Pi(H)$. Denoting the latter bracket by $[H \cap (t + Q)]_{\Pi(H)}$, we have

$$([\{0\}] \otimes \mu_X) \boxtimes ([H \cap (t + Q)]_{\Pi(H)} \otimes \rho_H) = [H \cap (t + Q)] \otimes (\mu_X \times \rho_H).$$

Thus

$$([\{0\}]\otimes \mu_X)\boxtimes i^*y=\int_{V/H}[H\cap(t+Q)]\otimes(\mu_X\times\rho_H)\,d\rho_{V/H}([t]).$$

Since $\mu_X\times\rho_H=\rho_H(P)\nu$, this finishes the proof of the proposition. ☐

9. An Alexandrov–Fenchel inequality

The main goal of this section is to prove Theorem 1.6. In fact, we will establish a version of (4) for general convex bodies. The following quantities are a particular instance of a larger family of higher-rank mixed volumes, introduced in [25].

Definition 9.1. Let $K_1,\dots,K_n$ be convex bodies in $\mathbb{R}^n$. Let ι_i denote the inclusion of $\mathbb{R}^n$ into $(\mathbb{R}^n)^n=\mathbb{R}^n\oplus\cdots\oplus\mathbb{R}^n$ as the i-th summand and let $\pi\colon(\mathbb{R}^n)^n\to\Delta^\perp$ denote the orthogonal projection onto the orthogonal complement of the diagonal $\Delta=\{(x,\dots,x)\colon x\in\mathbb{R}^n\}$ in $(\mathbb{R}^n)^n$. We define

$$\widetilde{\mathsf{V}}(K_1,\dots,K_n)=\frac{(n(n-1))!}{(n-1)!^n}\mathsf{V}(\pi\circ\iota_1K_1[n-1],\dots,\pi\circ\iota_nK_n[n-1]) \tag{19}$$

where the mixed volume on the right-hand side is to be understood inside $\Delta^\perp$.

Note that the normalization in (19) is different from [25], but more convenient for the purposes of the present paper.

We will deduce Theorem 1.6 from the following Alexandrov–Fenchel-type inequality for the higher-rank mixed volume.

Theorem 9.2. *Let $\mathbf{C}=(C_1,\dots C_{n-2})$ be a tuple of centrally symmetric convex bodies in $\mathbb{R}^n$. For any convex bodies K and L*

$$\widetilde{\mathsf{V}}(K,-L,\mathbf{C})^2\geq\widetilde{\mathsf{V}}(K,-K,\mathbf{C})\widetilde{\mathsf{V}}(L,-L,\mathbf{C}).$$

With the added assumption of central symmetry of L, Theorem 9.2 was proved in [25, Theorem 1.4]. The main idea in that paper was to apply the Fourier transform of smooth translation-invariant valuations (see [5,17]) to deduce the desired inequality from the classical Alexandrov–Fenchel inequality. This approach required the bodies to have a C^∞-smooth and strictly positively curved boundary and L to be centrally symmetric. It remains to show that the assumption of centrally symmetry of L can be omitted.

Recall from Section 2.1 that we denote by $K\#L$ the Blaschke sum of convex bodies in $\mathbb{R}^n$ with nonempty interior and by $\mathsf{P}K$ the projection body of K.

Proposition 9.3. *Let $C_1,\dots,C_n$ be convex bodies in $\mathbb{R}^n$. The following properties hold:*

(a) *If the bodies $C_1,\dots,C_n$ are polytopes, then*

$$\ell_{C_1}\cdots\ell_{C_n}=\widetilde{\mathsf{V}}(C_1,\dots,C_n).$$

(b) *$\widetilde{\mathsf{V}}$ is a continuous function on $\mathscr{K}^n\times\cdots\times\mathscr{K}^n$.*

(c) *For all convex bodies $K,L\subseteq\mathbb{R}^n$ with nonempty interior,*

$$\widetilde{\mathsf{V}}(K\#L,C_1,\dots,C_{n-1})=\widetilde{\mathsf{V}}(K,C_1,\dots,C_{n-1})+\widetilde{\mathsf{V}}(L,C_1,\dots,C_{n-1}).$$

(d) *If the bodies $C_1,\dots,C_{n-1}$ are centrally symmetric, then*

$$2^n\widetilde{\mathsf{V}}(C_1,\dots,C_n)=\mathsf{V}(\mathsf{P}C_1,\dots,\mathsf{P}C_n).$$

With a different proof, item (d) has appeared in already in [25, Proposition 5.6]. For the proof of Proposition 9.3, we need the following simple lemma.

Lemma 9.4. *Let $P, P' \subseteq \mathbb{R}^n$ be polytopes with nonempty interior. Then*

$$\ell_P + \ell_{P'} = \ell_{P\#P'}$$

Proof. On the one hand, the area measure of a polytope is the discrete measure

$$S_P = \sum_{i=1}^m \mathrm{vol}_{n-1}(F_i)\delta_{u_i}$$

where $F_1, \dots, F_m$ are the facets of P with facet normals $u_1, \dots u_m$. On the other hand, applying the normal cycle embedding yields

$$\mathrm{nc}(\ell_P) = \sum_{i=1}^m \mathrm{vol}_{n-1}(F_i)[N(F_i, P)].$$

Since by definition $S_{P\#P'} = S_P + S_{P'}$, the claim follows from the injectivity of nc. □

Proof of Proposition 9.3. To prove (a) let $\chi\colon \Pi^n(\mathbb{R}^n) \to \mathbb{R}$ denote the Euler characteristic, $\mathrm{diag}_n\colon \mathbb{R}^n \to (\mathbb{R}^n)^n$ the diagonal embedding, and ι_i the inclusion of $\mathbb{R}^n$ into $(\mathbb{R}^n)^n$. Let $\Delta = \mathrm{diag}_n(\mathbb{R}^n)$ denote the diagonal. By Corollary 6.2, for any positive numbers $\lambda_1, \dots, \lambda_n$ we have

$$\begin{aligned}\chi([\lambda_1 P_1] \cdots [\lambda_n P_n]) &= \chi(\mathrm{diag}_n^*([\lambda_1 \iota_1 P_1 + \cdots + \lambda_n \iota_n P_n]) \\ &= \int_{\Delta^\perp} \chi([(\lambda_1 \iota_1 P_1 + \cdots + \lambda_n \iota_n P_n) \cap (x + \Delta)])dx \\ &= \mathrm{vol}(\lambda_1 \pi \circ \iota_1 P_1 + \cdots + \lambda_n \pi \circ \iota_n P_n)\end{aligned}$$

Expanding both the first and the last expression into a polynomial in the λ_i and comparing the coefficients of the monomial $\lambda_1^{n-1} \cdots \lambda_n^{n-1}$, the claim follows.

(b) is an immediate consequence of the definition of $\widetilde{\mathsf{V}}$ and the continuity of the mixed volume.

Since $\widetilde{\mathsf{V}}$ and, by Lemma 2.3, Blaschke addition are continuous in the Hausdorff metric, (c) follows via approximation by polytopes from (a) and Lemma 9.4.

To prove (d) assume first that the bodies $C_1, \dots, C_n$ are centrally symmetric polytopes. If C is a centrally symmetric polytope, then

$$\ell_C = \sum_{i=1}^m \mathrm{vol}_{n-1}(F_i) x_{\mathbb{R}u_i}.$$

Therefore, using Corollary 8.5 we obtain

$$\ell_{C_1} \cdots \ell_{C_n} = \sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} |\det(u_{i_1}, \dots, u_{i_n})| \, \mathrm{vol}_{n-1}(F_{i_1}) \cdots \mathrm{vol}_{n-1}(F_{i_n})$$

For centrally symmetric bodies C, equation (6) implies

$$\mathsf{P}C = \sum_{i=1}^m \mathrm{vol}_{n-1}(F_i)[-u_i, u_i],$$

and hence $\mathsf{V}(\mathsf{P}C_1,\dots,\mathsf{P}C_n)$ equals

$$\sum_{i_1=1}^{m_1}\cdots\sum_{i_n=1}^{m_n}\mathsf{V}([u_{i_1},-u_{i_1}],\dots,[u_{i_n},-u_{i_n}])\operatorname{vol}_{n-1}(F_{i_1})\cdots\operatorname{vol}_{n-1}(F_{i_n})$$

Since $\mathsf{V}([u_{i_1},-u_{i_1}],\dots,[u_{i_n},-u_{i_n}])=2^n|\det(u_{i_1},\dots,u_{i_n})|$, this proves (d) for centrally symmetric polytopes and by approximation for all centrally symmetric convex bodies. It suffices to prove the general case for convex bodies with nonempty interior. If all bodies except perhaps the last one are centrally symmetric, then

$$\begin{aligned}2\widetilde{\mathsf{V}}(C_1,\dots,C_n)&=\widetilde{\mathsf{V}}(C_1,\dots,C_{n-1},C_n\#(-C_n))\\&=2^{-n}\mathsf{V}(\mathsf{P}C_1,\dots,\mathsf{P}C_{n-1},\mathsf{P}(C_n\#(-C_n))).\end{aligned}$$

Since $\mathsf{P}(K\#L)=\mathsf{P}K+\mathsf{P}L$, the claim follows. ☐

Lemma 9.5. *For every finite signed Borel measure μ on S^{n-1} with centroid at the origin, there exist convex bodies K and L such that*

$$\mu=S_K-S_L.$$

Moreover, if μ is discrete, then there exist polytopes K and L with this property.

Proof. There exist nonnegative finite Borel measures such that $\mu=\mu_+-\mu_-$ and both measures have their centroids at the origin. Adding a measure to both if necessary, we can assume that μ_+ and μ_- are not concentrated on an equator. Thus μ_+ and μ_- satisfy the hypothesis of Minkowski's existence theorem and hence there exist convex bodies K,L as claimed. ☐

Definition 9.6. Let $C_1,\dots,C_{n-1}$ be convex bodies in $\mathbb{R}^n$ and let μ and ν be finite signed Borel measures on the unit sphere with centroid at the origin. We define

$$\widetilde{\mathsf{V}}(\mu,C_1,\dots,C_{n-1})=\widetilde{\mathsf{V}}(K,C_1,\dots,C_{n-1})-\widetilde{\mathsf{V}}(L,C_1,\dots,C_{n-1}),$$

where K,L any convex bodies satisfying $\mu=S_K-S_L$. The expression $\widetilde{\mathsf{V}}(\mu,\nu,C_1,\dots,C_{n-2})$ is defined analogously.

Lemma 9.7. *The definition of $\widetilde{\mathsf{V}}(\mu,C_1,\dots,C_{n-1})$ is independent of the choice of decomposition $\mu=S_K-S_L$. An analogous statement holds for $\widetilde{\mathsf{V}}(\mu,\nu,C_1,\dots,C_{n-2})$.*

Proof. We present a short proof based on McMullen's characterization [26] of $(n-1)$-homogeneous continuous translation-invariant valuations on convex bodies in $\mathbb{R}^n$. Since the function $\phi(K)=\widetilde{\mathsf{V}}(K,C_1,\dots,C_{n-1})$ is such a valuation, there exists by McMullen's theorem a continuous function $f\colon S^{n-1}\to\mathbb{R}$ on the unit sphere such that for every convex body K

$$\phi(K)=\int_{S^{n-1}}f(u)dS_K(u).\tag{20}$$

Suppose that $\mu=S_K-S_L=S_{K'}-S_{L'}$. To prove the lemma, it suffices to show that

$$\phi(K)-\phi(L)=\phi(K')-\phi(L')$$

This follows immediately from (20). ☐

The classical Alexandrov–Fenchel inequality has three equivalent formulations, see [34, Lemma 3.11]. Analogous equivalent formulations exist in the context of Theorem 9.2. Let $a(u) = -u$, $u \in S^{n-1}$, denote the antipodal map.

Lemma 9.8. *Let* $\mathbf{C} = (C_1, \dots C_{n-2})$ *be a tuple of centrally symmetric convex bodies in* $\mathbb{R}^n$. *Then the following are equivalent:*

(a) *For all convex bodies* K *and* L

$$\widetilde{\mathsf{V}}(K, -L, \mathbf{C})^2 \geq \widetilde{\mathsf{V}}(K, -K, \mathbf{C})\widetilde{\mathsf{V}}(L, -L, \mathbf{C}).$$

(b) *For all finite signed Borel measures* μ *with centroid at the origin and all convex bodies* L

$$\widetilde{\mathsf{V}}(\mu, -L, \mathbf{C})^2 \geq \widetilde{\mathsf{V}}(\mu, a_*\mu, \mathbf{C})\widetilde{\mathsf{V}}(L, -L, \mathbf{C}).$$

(c) *For all finite signed Borel measures* μ *with centroid at the origin and all convex bodies* L *with* $\widetilde{\mathsf{V}}(L, -L, \mathbf{C}) > 0$,

$$\widetilde{\mathsf{V}}(\mu, -L, \mathbf{C}) = 0 \textit{ implies } \widetilde{\mathsf{V}}(\mu, a_*\mu, \mathbf{C}) \leq 0.$$

Moreover, if L *is throughout assumed to be centrally symmetric, then these statements are also equivalent.*

Proof. The implications (b)⇒(c) and (b)⇒(a) are trivial. Assume that (a) holds. Suppose $\mu = S_{K'} - S_{K''}$. Since $\widetilde{\mathsf{V}}$ is continuous by Proposition 9.3, approximating K', K'', and L by convex bodies with smooth and strictly positively curved boundary, we may assume that μ and S_L have continuous densities with respect to the spherical Lebesgue measure, $d\mu = f(u)du$ and $dS_L = f_L(u)du$, and that f_L is strictly positive. Hence there exists a number $\alpha > 0$ such that $(f + \alpha f_L) - \alpha f_L$ is a decomposition into strictly positive functions. Since

$$\int_{S^{n-1}} u(f + \alpha f_L)(u)du = 0,$$

the hypothesis of Minkowski's existence theorem is satisfied and, consequently, there exists a convex body K such that $\mu = S_K - \alpha S_L$. Plugging this decomposition into (b) and expanding both sides, we see that the terms containing α cancel. This shows (a)⇒(b).

Suppose that (c) holds. To show (b) we may assume by approximation that $\widetilde{\mathsf{V}}(L, -L, \mathbf{C}) > 0$. Since

$$\widetilde{\mathsf{V}}(\mu - \alpha S_L, -L, \mathbf{C}) = 0$$

for $\alpha = \widetilde{\mathsf{V}}(\mu, -L, \mathbf{C})/\widetilde{\mathsf{V}}(L, -L, \mathbf{C})$, (c) yields

$$0 \geq \widetilde{\mathsf{V}}(\mu - \alpha S_L, a_*(\mu - \alpha S_L), \mathbf{C}) = \widetilde{\mathsf{V}}(\mu, a_*\mu, \mathbf{C}) - \frac{\widetilde{\mathsf{V}}(\mu, -L, \mathbf{C})^2}{\widetilde{\mathsf{V}}(L, -L, \mathbf{C})}.$$

This finishes the proof of (c)⇒(b).

Finally, the above proof works without change if one assumes that L is centrally symmetric. □

Proof of Theorem 9.2. For centrally symmetric convex bodies L, the statement was proved in [25, Theorem 1.4]. Therefore, all the statements of Lemma 9.8 hold for centrally symmetric convex bodies L.

Let L be a convex body with nonempty interior, possibly not centrally symmetric, and suppose $\widetilde{\mathsf{V}}(L,-L,\mathbf{C})>0$. Put $L':=L\#(-L)$. Then L' is centrally symmetric and

$$\widetilde{\mathsf{V}}(L,L',\mathbf{C})=\widetilde{\mathsf{V}}(L,L,\mathbf{C})+\widetilde{\mathsf{V}}(L,-L,\mathbf{C})>0.$$

Suppose $\widetilde{\mathsf{V}}(\mu,-L,\mathbf{C})=0$. Since $\widetilde{\mathsf{V}}(\mu-\alpha S_L,L',\mathbf{C})=0$ for

$$\alpha=\widetilde{\mathsf{V}}(\mu,L',\mathbf{C})/\widetilde{\mathsf{V}}(L,L',\mathbf{C})$$

and since L' is centrally symmetric,

$$0\geq\widetilde{\mathsf{V}}(\mu-\alpha S_L,a_*(\mu-\alpha S_L),\mathbf{C})=\widetilde{\mathsf{V}}(\mu,a_*\mu,\mathbf{C})+\alpha^2\widetilde{\mathsf{V}}(L,-L,\mathbf{C}).$$

We conclude that $0\geq\widetilde{\mathsf{V}}(\mu,a_*\mu,\mathbf{C})$. Thus we proved that Lemma 9.8(c) holds for convex bodies L with nonempty interior. The proof of (c)⇒(b) applies without change and shows that Lemma 9.8(b) holds for convex bodies L with non-empty interior. A standard approximation argument now concludes the proof. ☐

Proof of Theorem 1.6. Choose a euclidean inner product to identify V with $\mathbb{R}^n$ such that vol is the Lebesgue measure. Let $x\in\Pi^1(\mathbb{R}^n)$ be given. By Lemma 9.5, we may write $x=\ell_P-\ell_{P'}$ for certain polytopes P and P'. Hence

$$-\sigma(x)\cdot\ell_Q\cdot\ell_{\mathbf{C}}=(\ell_P-\ell_P')\cdot\ell_{-Q}\cdot\ell_{\mathbf{C}}=\widetilde{\mathsf{V}}(\mu,-Q,\mathbf{C})$$

with $\mu=S_P-S_{P'}$. The assertion follows now from immediately from Theorem 9.2 in the formulation of Lemma 9.8(b). ☐

10. FINITE-DIMENSIONAL SUBALGEBRAS

In this section, we take a closer look at the subalgebras $\mathsf{A}^*(E)$ and $\mathsf{A}^*_+(E)$, defined in the introduction, and their connection with algebraic combinatorics. Throughout this section, we work with a fixed positive density vol on V. Let E be a finite set of lines in V^* that pass through the origin and are not contained in a single hyperplane.

First of all, observe that

$$\operatorname{nc}(\mathsf{A}^1(E))\subseteq\operatorname{span}\{[L^+],[L^-]\colon L\in E\},$$

where L^+ and L^- denote the two half lines corresponding to L. Hence $\mathsf{A}^1(E)$ and, consequently, $\mathsf{A}^*(E)$ are indeed finite-dimensional.

Recall that $\sin(L,L')$ denotes the sine of the principal angles between L and L'.

Proposition 10.1. *Let E be a finite set of lines in $\mathbb{R}^n$ that pass through the origin and are not contained in a single hyperplane. As a vector space, $\mathsf{A}^*_+(E)$ is spanned by the linearly independent elements $\{x_L\colon L\in\mathscr{L}_k(E)\}$. The algebra structure is determined by*

$$x_L\cdot x_{L'}=\begin{cases}\sin(L,L')\,x_{L+L'} & \text{if } L\cap L'=\{0\},\\ 0 & \text{otherwise.}\end{cases}\tag{21}$$

Moreover,

$$\mathsf{K}(E)=\left\{\sum_{L\in E}c_Lx_L\colon c_L>0 \text{ for all } L\in E\right\}.$$

Proof. If ℓ_P belongs to $\mathsf{A}^1_+(E)$, then $\ell_P = \sum_{L\in E} c_L x_L$, where c_L is the $(n-1)$-dimensional volume of the facet of P perpendicular to L. Applying Corollary 8.5, one obtains

$$\mathsf{A}^k_+(E) = \operatorname{span}\{x_L \colon L \in \mathscr{L}_k(E)\}$$

and the description of the algebra structure. By the injectivity of the normal cycle embedding, the elements x_L, $L \in \mathscr{L}_k(E)$, are linearly independent.

The description of $\mathsf{K}(E)$ is an immediate consequence of Minkowski's existence theorem. □

Note that the proposition implies

$$\dim \mathsf{A}^k_+(E) = |\mathscr{L}_k(E)|.$$

Consequently, the injective hard Lefschetz property of Conjecture 1.8 directly implies Dowling–Wilson conjecture.

In general, one expects for $k \leq n/2$

$$\dim \mathsf{A}^k(E) < \dim \mathsf{A}^{n-k}(E).$$

It follows that Poincaré duality, while valid in $\Pi^*(V)$, will not hold in this setting. The Möbius algebra, which appears in the context of the Dowling–Wilson conjecture and which we will define next, displays an analogous lack of Poincaré duality.

Definition 10.2. Let E be a finite set of lines in $\mathbb{R}^n$ that pass through the origin and are not contained in a single hyperplane. The graded Möbius algebra is the free real vector space $\mathrm{B}^*(E)$ spanned by the elements y_L, $L \in \bigcup_{k=1}^n \mathscr{L}(E)$ together with the product defined by

$$y_L \cdot y_{L'} = \begin{cases} y_{L+L'} & \text{if } L \cap L' = \{0\}, \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

The grading compatible with the product is defined by $\mathrm{B}^k(E) = \operatorname{span}\{y_L \in \mathscr{L}_k(E)\}$.

The comparison of (22) with (21) suggests that $\mathsf{A}^*_+(E)$ should be regarded as a volumetric version of the graded Möbius algebra. The injective hard Lefschetz property and the Hodge–Riemann relations within the context of the graded Möbius algebra were recently established in the [13, 23].

We close this section with a description of $\mathsf{A}^*(E)$ in the simplest case.

Example 10.3. If $E \subseteq V^*$ is as in Definition 1.7, then E consists of at least n lines. Suppose that $|E| = n$. Choose a euclidean inner product to identify $V \cong \mathbb{R}^n$ so that E consists of the coordinate axes and vol is the Lebesgue measure. Then, $\mathscr{L}_k(E)$ is the set of all k-dimensional coordinate subspaces of $\mathbb{R}^n$. If $\ell_P \in \mathsf{A}^1(E)$, then P must be a box $P = [a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$ and therefore centrally symmetric. It follows that $\mathsf{A}^*(E) = \mathsf{A}^*_+(E) \cong \mathrm{B}^*(E)$. Consequently, Conjecture 1.8 follows from the corresponding statement for the graded Möbius algebra.

11. THE DEGREE ONE CASE

Since the injective hard Lefschetz property is an immediate consequence of the Hodge–Riemann relations and since inequality (5) for $k = 1$ is a special case of Theorem 1.6, the only remaining task is to characterize when equality occurs in (5). Our methods allow us to prove the following slightly more general statement.

Theorem 11.1. *Let $\ell_{C_1}, \dots, \ell_{C_{n-2}} \in \mathsf{K}(E)$ and define $\ell_{\mathbf{C}} = \ell_{C_1} \cdots \ell_{C_{n-2}}$. Let Q be a polytope in V satisfying $\sigma(\ell_Q) \cdot \ell_Q \cdot \ell_{\mathbf{C}} > 0$. Then, for all $x \in \mathsf{A}^1(E)$,*

$$\sigma(x) \cdot \ell_Q \cdot \ell_{\mathbf{C}} = 0 \quad \textit{and} \quad \sigma(x) \cdot x \cdot \ell_{\mathbf{C}} = 0 \tag{23}$$

implies $x = 0$.

We will give a direct proof of Theorem 11.1 for $n = 2$ and we will treat the case $n > 2$ by induction.

11.1. **The 2-dimensional case.** It is a classical fact, that if L is a convex body in the plane with $\mathsf{V}(L, L) > 0$, then for any difference of support functions f, the equations

$$\mathsf{V}(f, L) = 0 \quad \text{and} \quad \mathsf{V}(f, f) = 0$$

imply that f is a linear functional. See [34, Lemma 3.12] for a simple and direct proof. Using this fact, we readily verify Theorem 11.1 in dimension two. Lemma 11.7 below provides another proof.

Lemma 11.2. *Theorem 11.1 holds for $n = 2$.*

Proof. We know from Lemma 4.8 that

$$\sigma(\ell_{P_1}) \cdot \ell_{P_2} = -\mathsf{V}(P_1, P_2)$$

for all polytopes P_1 and P_2. Suppose that $x = \ell_P - \ell_{P'}$ for certain polytopes P and P', and define $f = h_P - h_{P'}$. Assume that

$$0 = \sigma(x) \cdot \ell_Q = -\mathsf{V}(f, Q) \quad \text{and} \quad 0 = \sigma(x) \cdot x = -\mathsf{V}(f, f).$$

Since $\mathsf{V}(-Q, Q) = \ell_Q \cdot \ell_Q > 0$ is equivalent to $\mathsf{V}(Q, Q) > 0$, these equalities, as just mentioned, imply that f is a linear functional. We conclude that $x = 0$. $\square$

11.2. **Restrictions.** The goal of this section is to prove that every $x \in \mathsf{A}^1(E)$ is determined by its restrictions to hyperplanes perpendicular to the lines in E. This property will be a key ingredient of our inductive setup to characterize when equality occurs in the Hodge–Riemann relations.

Lemma 11.3. *Let $H \subseteq V$ be a linear hyperplane and let $j \colon H \to V$ denote the inclusion. The following statements hold for all $x \in \mathsf{A}^1(E)$:*

(a) *$j^*x \in \mathsf{A}^1(j^*E)$, where $j^*E = \{j^*L \colon L \in E\} \setminus \{0\}$.*

(b) *More precisely, if $\operatorname{nc}(x) = \sum_{i=1}^m \alpha_i [N_i]$ with nonzero coefficients α_i, then*

$$\operatorname{nc}(j^*x) = \sum_{i \colon j^*N_i \neq \{0\}} \alpha'_i [j^*N_i]$$

with nonzero coefficients α'_i.

(c) *If $x \in \mathsf{K}(E)$, then $j^*x \in \mathsf{K}(j^*E)$.*

Proof. It suffices to prove (a) and (b) for $x = \ell_P$. Choose a euclidean inner product to identify V with $\mathbb{R}^n$ such that vol is the Lebesgue measure. Let $F_1, \ldots, F_m$ and $v_1, \ldots, v_m$ denote the facets and the corresponding facet normals of P.

Let $\pi\colon \mathbb{R}^n \to H$ denote the orthogonal projection and let $\mathbb{R}_+ = \{\alpha \in \mathbb{R}\colon \alpha > 0\}$. Let u be a unit normal vector to H. One has

$$\begin{aligned}\operatorname{nc}(j^*x) &= \int_{\mathbb{R}} \operatorname{nc}([P \cap (H + tu)]_{n-2})dt\\ &= \sum_{i=1}^m \int_{\mathbb{R}} \operatorname{vol}_{n-2}(F_i \cap (H + tu))dt\,[\mathbb{R}_+\pi(v_i)]\\ &= \sum_{i=1}^m |\pi(v_i)| \operatorname{vol}_{n-1}(F_i)[\mathbb{R}_+\pi(v_i)].\end{aligned}$$

This proves (b). The discrete measure on S^{n-2}

$$\mu = \sum_{i\colon \pi(v_i)\neq 0} |\pi(v_i)| \operatorname{vol}_{n-1}(F_i)\delta_{\pi(v_i)/|\pi(v_i)|}$$

has centroid at the origin and is not concentrated on an equator. Consequently, by Minkowski's existence theorem, there exists a polytope $P' \subseteq H$ such that $\mu = S_{P'}$. We conclude that $j^*x = \ell_{P'} \in \mathsf{A}^1(j^*E)$. This finishes the proof of (a). If P is centrally symmetric, then so is P' and (b) follows. □

To prove the next proposition, we need the following lemma due to Motzkin [31].

Lemma 11.4. *Given m non-collinear points in the real projective plane, there exists a line that contains exactly two of the points.*

Proposition 11.5. *Let $x \in \mathsf{A}^1(E)$. Suppose that $j_L^*x = 0$ for all $L \in E$, where $j_L\colon L^\perp \to V$ denotes the inclusion. If $n \geq 3$, then $x = 0$.*

Remark 11.6. (a) The analogous statement for $n = 2$ is wrong. Indeed, since $L^\perp$ is one-dimensional for $n = 2$, it follows that $\sigma(j_L^*x) = -j_L^*x$ for all $x \in \Pi^1(V)$. Therefore, any $x \in \mathsf{A}^1(E)$ with $\sigma(x) = x$ satisfies $j_L^*x = 0$.

(b) For general $x \in \Pi^1(V)$, the condition $j_L^*x = 0$ for all $L \in E$ does not imply $x = 0$.

Proof of Proposition 11.5. We will prove the claim be induction on n. First, suppose $n = 3$ and $V = \mathbb{R}^3$. We argue by induction on $|E| \geq 3$. By Lemma 11.4 there exists a linear hyperplane $H \subseteq \mathbb{R}^3$ that contains exactly two lines from E, say L and L'. Consider the orthogonal projection $\pi_L\colon \mathbb{R}^3 \to L^\perp$. Since H contains exactly two lines from E, if $L'' \in E$ is a line $\neq L, L'$, then $\pi_L L' \neq \pi_L L''$. Hence, if $\operatorname{nc}(x) = \sum_{M\in E} \alpha_{M^\pm}[M^\pm]$, then by Lemma 11.3 the condition $j_L^*x = 0$ implies $\alpha_{L'}^\pm = 0$. If $|E| > 3$, then we conclude that $x \in \mathsf{A}^1(E \setminus \{L'\})$ and apply the inductive hypothesis. If $|E| = 3$, then the claim is easily seen to be true. This finishes the proof in dimension three.

Now, assume $n > 3$ and that the proposition holds in dimension $n - 1$. Let $H \subseteq V$ be a linear hyperplane and let $i_H\colon H \to V$ denote the inclusion. Suppose that $H^\perp \notin E$. Then $H \cap L^\perp$ has dimension $n - 2$ for every $L \in E$. Define $E' = i_H^*E$ and $x' = i_H^*x$.

Then E' is a set of lines in H^* and $x' \in \mathsf{A}^1(E')$. For $L' = i_H^* L \in E'$ let $k_L \colon H \cap L^\perp \to L^\perp$ and $j_{L'} \colon H \cap (L')^\perp \to H$ denote the inclusion. Then $i_H \circ j_{L'} = j_L \circ k_L$ and thus

$$(j_{L'})^* x' = k_L^*(j_L^* x) = 0$$

for any $L' \in E'$. By the inductive assumption $x' = 0$. Thus we conclude that $j_H^* x = 0$ for all linear hyperplanes in V.

Let H be a hyperplane in V and let L, L' be two distinct lines in V^*. Observe that if $i_H^* L = i_H^* L'$ then $H^\perp \subseteq L + L'$. Hence if H chosen so that $H^\perp$ is not contained in the finitely many planes $L + L'$ for distinct $L, L' \in E$, then $i_H^* L \neq i_H^* L'$ whenever $L \neq L'$. With the help of Lemma 11.3, we can now deduce $x = 0$ from $j_H^* x = 0$. □

11.3. **Characterization of the equality case in higher dimensions.** The positive definiteness of the Hodge-Riemann form immediately implies the injective hard Lefschetz property. Exploiting the fact that $\dim \Pi^n(V) = 1$ and that Poincaré duality holds in $\Pi^*(V)$, we prove the reverse implication for $k = 1$

Lemma 11.7. *Let $C_1, \ldots, C_{n-2}$ be centrally symmetric polytopes in V, and let Q be a polytope in V satisfying $\sigma(\ell_Q) \cdot \ell_Q \cdot \ell_{\mathbf{C}} > 0$. Then, for every $x \in \Pi^1(V)$, the following are equivalent:*

(a) $\sigma(x) \cdot \ell_Q \cdot \ell_{\mathbf{C}} = 0$ *and* $\sigma(x) \cdot x \cdot \ell_{\mathbf{C}} = 0$.

(b) $x \cdot \ell_{\mathbf{C}} = 0$.

Proof. The implication (b)⇒ (a) is trivial.

Since $\dim \Pi^n(V) = 1$ and since $\sigma(\ell_Q) \cdot \ell_Q \cdot \ell_{\mathbf{C}} > 0$ by hypothesis, it follows that the condition $\sigma(x) \cdot \ell_Q \cdot \ell_{C_1} \cdots \ell_{C_{n-2}} = 0$ defines a hyperplane H in $\Pi^1(V)$ and that one has the decomposition

$$\Pi^1(V) = \mathbb{R}\ell_Q \oplus H. \tag{24}$$

The symmetric bilinear form

$$q(x, y) = \sigma(x) \cdot y \cdot \ell_{\mathbf{C}}, \quad x \in \Pi^1(V),$$

is by Theorem 9.2 positive semi-definite on H. Note that $x \in H$ if and only if $q(x, \ell_Q) = 0$.

Suppose that $x \in \Pi^1(V)$ satisfies (a). In other words, $x \in H$ and $q(x, x) = 0$. Then, the Cauchy-Schwarz inequality implies $q(x, y) = 0$ for every $y \in H$. Hence, for any number α and $y \in H$, the equality $q(x, \alpha\ell_Q + y) = 0$ holds. In light of the decomposition (24), the latter is equivalent to

$$0 = q(x, z) = \sigma(x \cdot \ell_{\mathbf{C}}) \cdot z \quad \text{for all } z \in \Pi^1(V). \tag{25}$$

It follows from Poincaré duality (Theorem 4.9) that $x \cdot \ell_{\mathbf{C}} = 0$. □

We are now ready to prove the main result of this section.

Proof of Theorem 11.1. We prove the theorem by induction on n. The base case $n = 2$ was established in Lemma 11.2. Now, assume $n > 2$ and that the theorem holds in dimension $n - 1$. Choose a euclidean inner product to identify V with $\mathbb{R}^n$ such that vol is the Lebesgue measure. Suppose that $x \in \mathsf{A}^1(E)$ satisfies (23). Since $\sigma(\ell_Q) \cdot \ell_Q \cdot \ell_{\mathbf{C}} > 0$, Lemma 11.7 yields $x \cdot \ell_{\mathbf{C}} = 0$.

For $L \in E$, let $j_L \colon L^\perp \to \mathbb{R}^n$ denote the inclusion and $\pi_L = j_L^* \colon \mathbb{R}^n \to L^\perp$ the orthogonal projection. By Lemma 11.3, there exist centrally symmetric polytopes $C_i^L \subseteq L^\perp$ such that

$$j_L^* \ell_{C_i} = \ell_{C_i^L} \in \mathsf{K}(j_L^* E).$$

Moreover, Lemma 11.3 implies $j_L^* x \in \mathsf{A}^1(j_L^* E)$.

Since

$$0 = j_L^*(x \cdot \ell_{\mathbf{C}}) = (j_L^* x) \cdot \ell_{C_1^L} \cdots \ell_{C_{n-2}^L}, \tag{26}$$

Theorem 9.2 applied inside $L^\perp$, yields the inequality

$$\sigma(j_L^* x) \cdot (j_L^* x) \cdot \ell_{C_1^L} \cdots \ell_{C_{n-3}^L} \geq 0. \tag{27}$$

Expanding $\ell_{C_{n-2}} = \sum_{L \in E} \alpha_L x_L$, using Proposition 8.7 and the fact that the Euler–Verdier involution commutes with pullback, one obtains

$$0 = \sigma(x) \cdot x \cdot \ell_{\mathbf{C}} = \sum_{L \in E} \alpha_L \left(\sigma(j_L^* x) \cdot (j_L^* x) \cdot \ell_{C_1^L} \cdots \ell_{C_{n-3}^L} \right).$$

Since $\alpha_L > 0$, it follows that equality holds in (27) for each $L \in E$. Since

$$\ell_{C_1^L} \cdots \ell_{C_{n-2}^L} \cdot \ell_{C_{n-2}^L} > 0,$$

we can apply the inductive hypothesis to obtain $j_L^* x = 0$ for all $L \in E$. Using Proposition 11.5, we conclude that $x = 0$. ☐

Remark 11.8. The full strength of Lemma 11.7, which is a consequence of Poincaré duality, is not needed for the proof of Theorem 11.1. While Lemma 11.7 is used to obtain (26), the same conclusion can also be reached by choosing $z = x_L$ in (25) and applying Proposition 8.7.

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