Natural Policy Gradient for Average Reward Non-Stationary RL

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Abstract

We consider the problem of non-stationary reinforcement learning (RL) in the infinite-horizon average-reward setting. We model it by a Markov Decision Process with time-varying rewards and transition probabilities, with a variation budget of Δ_T . Existing non-stationary RL algorithms focus on model-based and model-free value-based methods. Policy-based methods despite their flexibility in practice are not theoretically well understood in non-stationary RL. We propose and analyze the first model-free policy-based algorithm, Non-Stationary Natural Actor-Critic (NS-NAC), a policy gradient method with a restart based exploration for change and a novel interpretation of learning rates as adapting factors. Further, we present a bandit-over-RL based parameter-free algorithm BORL-NS-NAC that does not require prior knowledge of the variation budget Δ_T . We present a dynamic regret of $\tilde{\mathcal{O}}(|\mathcal{S}|^{1/2}|\mathcal{A}|^{1/2}\Delta_T^{1/6}T^{5/6})$ for both algorithms, where T is the time horizon, and |S|, |A| are the sizes of the state and action spaces. The regret analysis leverages a novel adaptation of the Lyapunov function analysis of NAC to dynamic environments and characterizes the effects of simultaneous updates in policy, value function estimate and changes in the environment.

1. Introduction

Reinforcement Learning is a sequential decision-making framework where an agent learns optimal behavior by iteratively interacting with its environment. At each timestep, the agent observes the current state of the environment, takes an action, receives a reward, and transitions to the next state. While RL has traditionally been studied in stationary environments with time-invariant rewards and state-transition dynamics, this may not always be the case. Consider the examples of a carbon-aware datacenter job scheduler that tracks the dynamic electricity prices and local weather patterns (Yeh et al., 2024) and recommendation systems with evolving user preferences (Chen et al., 2018). Time-varying environments are also observed in inventory control (Mao et al., 2024), healthcare (Chandak et al., 2020), ride-sharing (Kanoria & Qian, 2024), and multi-agent systems (Zhang et al., 2021a).

Motivated by these applications, we consider the problem of non-stationary reinforcement learning, modeled by a Markov Decision Process with time-varying rewards and transition probabilities, in the infinite horizon average reward setting. While many works consider discounted rewards (Chandak et al., 2020; Igl et al., 2020; Lecarpentier & Rachelson, 2019), the more challenging average-reward setting is vital in representing problems where the importance of rewards does not decay with time, such as in robotics (Mahadevan, 1996; Peters et al., 2003) or scheduling workloads in cloud computing systems (Jali et al., 2024; Liu et al., 2022). The key challenges for an agent operating in a dynamic environment are learning an optimal behavior policy that varies with the environment, devising an efficient exploration strategy, and effectively incorporating the acquired information into its behavior.

Current algorithms designed for non-stationary MDPs in the average reward setting can be classified broadly into model-based and model-free value-based methods. Modelbased solutions incorporate sliding windows, forgetting factors, and confidence interval management mechanisms into UCRL (Cheung et al., 2020; Ortner et al., 2020; Gajane et al., 2018; Jaksch et al., 2010). Model-free value-based methods assimilate restarts and optimism into Q-Learning (Mao et al., 2024; Feng et al., 2023) and LSVI (Zhou et al., 2020; Touati & Vincent, 2020). A significant gap in the literature is the absence of model-free policy-based techniques for time-varving environments. The inherent flexibility of policy-based algorithms makes them suitable for continuous state-action spaces, facilitates efficient parameterization in high-dimensional state-action spaces, and enables effective exploration through stochastic policy learning (Sutton & Barto, 2018).

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Our Contributions. We tackle the problem of nonstationary reinforcement learning in the challenging infinitehorizon average reward setting in the following manner.

- We propose and analyze Non-Stationary Natural Actor-Critic (NS-NAC), a policy gradient method with a restart based exploration for change and a novel interpretation of learning rates as adapting factors. To the best of our knowledge, this is the first model-free policy-based method for time-varying environments.
- We present a bandit-over-RL based parameter-free algorithm BORL-NS-NAC that does not require prior knowledge of the variation budget.
- 3. We present a $\tilde{\mathcal{O}}\left(|\mathcal{S}|^{1/2}|\mathcal{A}|^{1/2}\Delta_T^{1/6}T^{5/6}\right)$ dynamic regret bound for both algorithms under standard assumptions where T is the time horizon, Δ_T represents the variation budget of rewards and transition probabilities, $|\mathcal{S}||\mathcal{A}|$ is the size of the state-action space and $\tilde{\mathcal{O}}(\cdot)$ hides logarithmic factors. The regret analysis leverages a novel adaptation of the Lyapunov function analysis of NAC to dynamic environments and characterizes the effects of simultaneous updates in policy, value function estimate and changes in the environment.

2. Related Work

Non-Stationary RL. Solutions to the non-stationary RL problem can be categorized into passive and active methods. Active algorithms are designed to actively detect changes in the environment in contrast to passive ones which implicitly adapt to new environments without distinct recognition of the change. While we focus our attention on passive techniques with dynamic regret as the performance metric in this work, a comprehensive survey can be found in Padakandla (2021) and Khetarpal et al. (2022). Model-based solutions in the infinite horizon average reward setting incorporate into UCRL a sliding window or a forgetting factor for piecewise stationary MDPs (Gajane et al., 2018), variation aware restarts (Ortner et al., 2020) and a bandit based tuning of sliding window and confidence intervals (Cheung et al., 2020) for gradual or abrupt changes constrained by a variation budget and pessimistic tree search for Lipschitz continuous changes (Lecarpentier & Rachelson, 2019).

In the episodic setting, model-free value based methods assimilate restarts and optimism into Q-Learning (Mao et al., 2024), LSVI (Zhou et al., 2020; Touati & Vincent, 2020) and sliding window and optimistic confidence set based exploration into a value function approximated learning (Feng et al., 2023). Further, in the episodic setting, Lee et al. (2024) proposes strategically pausing learning as an effective solution to non-stationarity with forecasts of the future. Wei & Luo (2021) proposed an algorithm agnostic black-box approach that finds a non-stationary equivalent to optimal regret stationary MDP algorithms. Cheung et al. (2020); Mao et al. (2024) also present parameter-free nonstationary RL algorithms that leverage the bandit-over-RL framework to adaptively tune algorithm without knowledge of the variation budget. Further, Mao et al. (2024) presents an information theoretic lower bound on the dynamic regret and Peng & Papadimitriou (2024) captures the complexity of updating value functions with any change. We note the distinction between the scope of this work and the body of research on adversarial MDPs which often allow for only changes in rewards, study the static regret and work with full information feedback instead of bandit feedback. See Appendix A for a table of comparison of regret bounds.

Non-Stationary Bandits. A precursor to non-stationary RL, the multi-armed bandit problem with time-varying rewards was first proposed in Garivier & Moulines (2008). Solutions include UCB with a sliding window or a discounting factor (Garivier & Moulines, 2008), UCB with adaptive blocks of exploration and exploitation (Besbes et al., 2014), Restart-Exp3 (Besbes et al., 2014), Thompson Sampling with a discounting factor (Raj & Kalyani, 2017) and bandit based sliding window tuning (Cheung et al., 2019). Further, while most existing works assume arbitrarily (constrained by variation budget) changing reward distributions, (Jia et al., 2023) achieves an improved regret when the reward distributions change smoothly. Recent work by Liu et al. (2023a) points out ambiguities in the definition of non-stationary bandits and how the dynamic regret performance metric causes over-exploration, and Liu et al. (2023b) proposes, predictive sampling, an algorithm that deprioritizes acquiring information that loses usefulness quickly.

Policy Gradient Algorithms for Stationary RL. Wu et al. (2020) presents the first finite time convergence of the average reward two timescale Advantage Actor-Critic to a stationary point. Chen & Zhao (2023) further improved its rate by leveraging a single timescale algorithm. Convergence to global optima of A2C was analyzed in Bai et al. (2024); Murthy et al. (2023) which use a two loop structure with the inner loop critic estimation. Further, Lazic et al. (2021) combines a mirror descent update with experience replay and characterized global convergence. Natural Policy Gradient (NPG) was analyzed in the discounted reward case in Agarwal et al. (2021); Khodadadian et al. (2021) and with entropy regularization in Cen et al. (2022). NPG in the average reward setting with exact gradients was characterized in Even-Dar et al. (2009); Murthy & Srikant (2023). The most relevant to our work is the Natural Actor Critic (NAC) algorithm where the actor learns the policy by natural gradient ascent and critic estimate the value function analyzed for the discounted reward case in Khodadadian et al. (2022) and average reward setting with (compatible) function approximation in Wang et al. (2024).

3. Problem Setting

In this section, we first present preliminaries of a Markov Decision Process and the Natural Actor Critic algorithm in a stationary environment. We then introduce the problem of non-stationary reinforcement learning, where the MDP has time-varying rewards and transition probabilities, and define dynamic regret as a performance metric.

Notation. Standard typeface (e.g., s) denote scalars and bold typeface (e.g., **r**, **A**) denote vectors and matrices. $\|\cdot\|_{\infty}$ denotes the infinity norm and $\|\cdot\|_2$ denotes the 2-norm of vectors and matrices. Given two probability measures P and Q, $d_{TV}(P,Q) = \frac{1}{2} \int_{\mathcal{X}} |P(dx) - Q(dx)|$ is the total variation distance between P and Q, while $D_{\text{KL}}(P\|Q) = \int_{\mathcal{X}} P(dx) \log \frac{P(dx)}{Q(dx)}$ is the KL-divergence. For two sequences $\{a_n\}$ and $\{b_n\}$, $a_n = \mathcal{O}(b_n)$ represents the existence of an absolute constant C such that $a_n \leq Cb_n$. Further $\tilde{\mathcal{O}}$ is used to hide logarithmic factors. $|\mathcal{S}|$ denotes the cardinality of a set S. Given a positive integer T, [T] denotes the set $\{0, 1, 2, \dots, T-1\}$.

3.1. Preliminaries: Stationary RL

Markov Decision Process. Reinforcement learning tasks can be modeled as discrete-time Markov Decision Processes (MDPs). An MDP is represented as $\mathcal{M} = (\mathcal{S}, \mathcal{A}, \mathbf{P}, \mathbf{r})$ where \mathcal{S} and \mathcal{A} are, respectively, finite sets of states and actions, $\mathbf{P} \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}| \times |\mathcal{S}|}$ is the transition probability matrix, with $P(s'|s, a) \in [0, 1]$, for $s, s' \in \mathcal{S}, a \in \mathcal{A}$, and $\mathbf{r} \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}|}$ is the reward vector with individual entries $\{r(s, a)\}$ bounded in magnitude by constant $U_R > 0$. An agent in state *s* takes an action $a \sim \pi(\cdot|s)$ according to a policy π , where for each state $s, \pi(\cdot|s)$ is a probability distribution over the action space. The agent then receives a reward r(s, a) and transitions to the next state $s' \sim P(\cdot|s, a)$. We denote the *policy* by $\pi \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}|}$, which concatenates $\{\pi(\cdot|s)\}_{s}$. In a stationary MDP, the transition probabilities \mathbf{P} and the rewards \mathbf{r} are *time-invariant*.

Average Reward and Value Functions. In this work, we consider the average reward setting (instead of discounted rewards), which is essential to model problems where the importance of rewards does not decay with time (Peters et al., 2003; Liu et al., 2022). Time averaged reward of an ergodic Markov chain following policy π converges to

$$J^{\boldsymbol{\pi}} := \lim_{T \to \infty} \frac{\sum_{t=0}^{T-1} r(s_t, a_t)}{T} = \mathbb{E}_{s \sim d^{\boldsymbol{\pi}, \mathbf{P}}(\cdot), a \sim \boldsymbol{\pi}(\cdot|s)} \left[r(s, a) \right],$$

where $d^{\pi,\mathbf{P}}$ is the stationary distribution over states induced by policy π and transition probabilities **P**. The *relative* state-value function defines overall reward (relative to the average reward) accumulated when starting from state *s* as

$$V^{\boldsymbol{\pi}}(s) := \mathbb{E}\left[\sum_{t=0}^{\infty} \left(r(s_t, a_t) - J^{\boldsymbol{\pi}}\right) \middle| s_0 = s\right],$$

where the expectation is over the trajectory rolled out by $a_t \sim \pi(\cdot|s_t)$ and $s_{t+1} \sim P(\cdot|s_t, a_t)$. Similarly, the relative state-action value function defines the overall reward (relative to the average reward) accumulated by policy π when starting from state *s* and action *a* as

$$Q^{\pi}(s,a) := \mathbb{E}\left[\sum_{t=0}^{\infty} \left(r(s_t, a_t) - J^{\pi}\right) | s_0 = s, a_0 = a\right]$$

Natural Actor-Critic. The goal of an agent is to find a policy that maximizes the average reward

$$\pi^{\star} = \max_{\pi} J^{\pi} = \max_{\pi} \mathbb{E}_{s \sim d^{\pi, \mathbf{P}}(\cdot), a \sim \pi(\cdot|s)} \left[r(s, a) \right].$$

Here, we consider the actor-critic class of policy-based algorithms. While actor-only methods are at a disadvantage due to inefficient use of samples and high variance and critic-only methods are at a risk of the divergence from the optimal policy, actor-critic methods provide the best of both worlds (Wu et al., 2020). An actor-critic algorithm learns the policy and the value function simultaneously by gradient methods. Further, the natural actor-critic leverages the second-order method of natural gradient to establish guarantees of global optimality (Bhatnagar et al., 2009; Khodadaian et al., 2022). The *actor* updates the policy by performing a natural gradient ascent (Martens, 2020) step

$$\boldsymbol{\pi} \leftarrow \boldsymbol{\pi} + \beta F_{\boldsymbol{\pi}}^{-1} \nabla J^{\boldsymbol{\pi}}, \tag{1}$$

$$F_{\boldsymbol{\pi}} := \mathbb{E}_{s \sim d^{\boldsymbol{\pi}, \mathbf{P}}(\cdot), a \sim \pi(\cdot|s)} \left[\nabla \log \pi(a|s) \left(\nabla \log \pi(a|s) \right)^{\top} \right].$$

 F_{π} is called the Fisher Information matrix. The gradient of the average reward is given by the Policy Gradient Theorem (Sutton & Barto, 2018, Section 13.2) as

$$\nabla J^{\boldsymbol{\pi}} = \mathbb{E}_{s \sim d^{\boldsymbol{\pi}, \mathbf{P}}(\cdot), a \sim \pi(\cdot|s)} \left[Q^{\boldsymbol{\pi}}(s, a) \nabla \log \pi(a|s) \right].$$

The *critic* enables an approximate policy gradient computation by estimating the Q-Value function $Q^{\pi}(s, a)$ using TD-learning as

$$Q(s,a) \leftarrow Q(s,a) + \alpha \left[r(s,a) - \eta + Q(s',a') - Q(s,a) \right],$$

where $s' \sim P(\cdot|s, a), a' \sim \pi(\cdot|s')$, and η is an estimate of the average reward J^{π} .

3.2. Non-Stationary RL

In this work, we study reinforcement learning with *time-varying environments*. The MDP is modeled by a sequence of environments $\mathcal{M} = \{\mathcal{M}_t = (\mathcal{S}, \mathcal{A}, \mathbf{P}_t, \mathbf{r}_t)\}_{t=0}^{T-1}$, with time-varying rewards $\{\mathbf{r}_t\}$ and transition probabilities $\{\mathbf{P}_t\}$. At each time t, the agent in state s_t takes action a_t , receives a reward $r_t(s_t, a_t)$, and transitions to the next state $s_{t+1} \sim P_t(\cdot|s_t, a_t)$. The cumulative change in the reward

and transition probabilities is quantified in terms of *variation budgets* $\Delta_{R,T}$ and $\Delta_{P,T}$ as

$$\Delta_{R,T} = \sum_{t=0}^{T-1} \|\mathbf{r}_{t+1} - \mathbf{r}_t\|_{\infty}, \quad \Delta_{P,T} = \sum_{t=0}^{T-1} \|\mathbf{P}_{t+1} - \mathbf{P}_t\|_{\infty}$$
$$\Delta_T = \Delta_{R,T} + \Delta_{P,T}. \tag{2}$$

Note that while the overall budgets $\Delta_{R,T}$, $\Delta_{P,T}$ may be used as inputs by the agent, the variations at a given time t, $\|\mathbf{r}_{t+1} - \mathbf{r}_t\|_{\infty}$ and $\|\mathbf{P}_{t+1} - \mathbf{P}_t\|_{\infty}$, are unknown.

We denote the long-term average reward obtained by following policy π_t in the environment \mathcal{M}_t by

$$J_t^{\boldsymbol{\pi}_t} = \mathbb{E}_{s \sim d^{\boldsymbol{\pi}_t}, \mathbf{P}_t(\cdot), a \sim \boldsymbol{\pi}(\cdot|s)} \left[r_t(s, a) \right].$$

Further, the state and state-action value functions at time t are solutions to the Bellman equations

$$V_t^{\pi_t}(s) = \sum_{a \in \mathcal{A}} \pi(a|s) Q_t^{\pi_t}(s, a)$$
$$Q_t^{\pi_t}(s, a) = r_t(s, a) - J_t^{\pi_t} + \sum_{s' \in \mathcal{S}} P_t(s'|s, a) V_t^{\pi_t}(s').$$

The set of solutions to the Bellman equations above is $\mathbf{Q}_{t}^{\boldsymbol{\pi}_{t}} = \{\mathbf{Q}_{t,E}^{\boldsymbol{\pi}_{t}} + c\mathbf{1} | \mathbf{Q}_{t,E}^{\boldsymbol{\pi}_{t}} \in E, c \in \mathbb{R}\}$ where *E* is the subspace orthogonal to the all ones vector and $\mathbf{Q}_{t,E}^{\boldsymbol{\pi}_{t}}$ is the unique solution in *E* (Zhang et al., 2021b).

The goal of the agent is to maximize the time-averaged reward $\sum_{t=0}^{T-1} r_t(s_t, a_t)/T$. We measure performance using an equivalent metric called the *dynamic regret* defined as

$$\operatorname{Dyn-Reg}(\mathcal{M},T) := \mathbb{E}\left[\sum_{t=0}^{T-1} J_t^{\boldsymbol{\pi}_t^{\star}} - r_t(s_t, a_t)\right], \quad (3)$$

where $\pi_t^{\star} = \arg \max_{\pi} J_t^{\pi}$ is the optimal policy in the environment $\mathcal{M}_t = (\mathcal{S}, \mathcal{A}, \mathbf{P}_t, \mathbf{r}_t)$ at time *t*. The optimal average reward $J_t^{\pi_t^{\star}}$ associated with π_t^{\star} can be computed by solving the linear program (25) described in Appendix D.4.

The model of change and notion of dynamic regret considered here have been used in several recent works (Cheung et al., 2020; Fei et al., 2020; Zhou et al., 2020; Mao et al., 2024; Feng et al., 2023). Note that it is more challenging to analyze than static regret, which compares the cumulative reward collected by an agent against that of a single stationary optimal policy (Even-Dar et al., 2009; Touati & Vincent, 2020). Further, in applications such as robotics or network routing, where the underlying environment evolves over time, a single best action/policy in hindsight might not be a realistic benchmark. On the other hand, dynamic regret provides a more useful performance measure while still being computationally feasible to evaluate. **Challenges due to Non-Stationarity.** When running policygradient methods in stationary RL, the policy evolves to efficiently learn a fixed environment (\mathbf{P}, \mathbf{r}) . However, in non-stationary case, the environment $(\mathbf{P}_t, \mathbf{r}_t)$ also changes over time. Therefore, the agent chases a moving target, namely, the *time-varying optimal policy* π_t^* , resulting in the following unique challenges.

- *Explore-for-Change vs Exploit:* The agent needs to explore more aggressively than in the stationary setting to adapt to the changing dynamics. As an example, a sub-optimal action at the current timestep may become optimal at a later timestep, necessitating re-exploration. This is in sharp contrast to stationary RL, where sub optimal actions are picked less often as time progresses.
- Forgetting Old Environments: The policy and value function estimates must evolve quickly lest they might become irrelevant when the environment changes significantly. However, observations are noisy and the agent needs to collect multiple samples to obtain confident estimates. Hence, an agent has to carefully balance the rate of forgetting the old environment versus learning a new one.

4. Algorithm: NS-NAC

In this section, we present Non-Stationary Natural Actor-Critic (NS-NAC), a two-timescale natural policy gradient method with a restart based exploration for change and stepsizes designed to carefully balance the rate of forgetting the old environment and adapting to a new one. While we use the variation budget Δ_T as an input to NS-NAC here, we present a parameter-free algorithm BORL-NS-NAC in Section 6 that does not require this knowledge.

The NS-NAC algorithm seeks to maximize the total reward received over the time horizon T, given the variation budgets $\Delta_{R,T}$ and $\Delta_{P,T}$. At timestep t, π_t denotes the tabular policy with $\pi(\cdot|s)$ where $\pi(a|s) \ge 0 \ \forall a \in \mathcal{A}$ and $\sum_a \pi(a|s) = 1$ $\forall s \in \mathcal{S}. \ \pi_t^* = \arg \max_{\pi} J_t^{\pi}$ is the optimal policy in the environment \mathcal{M}_t . The estimate of the tabular state-action value function $\mathbf{Q}_t^{\pi_t}$ is denoted by $\mathbf{Q}_t \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}|}. \ \eta_t$ denotes the estimate of the average reward $J_t^{\pi_t}$.

NS-NAC divides the total horizon T into N segments of length $H = \lfloor T/N \rfloor$ each. At the beginning of each segment, the algorithm restarts the NAC sub-routine (line 4), thereby ensuring that the algorithm sufficiently *explores for change*. Next, at each time-step $t = nH + h \forall n \in [N], h \in [H]$, the *actor* (slower timescale) takes a natural gradient ascent step towards the optimal policy in environment \mathcal{M}_t as

$$\boldsymbol{\pi}_{t+1} \leftarrow \boldsymbol{\pi}_t + \beta F_{\boldsymbol{\pi}_t}^{-1} \mathbb{E}_{s,a} \left[\mathbf{Q}_t^{\boldsymbol{\pi}_t}(s, a) \nabla \log \boldsymbol{\pi}_t(a|s) \right]$$

In the absence of knowledge of the exact natural gradient, the actor uses an estimate of the value function to update the policy with tabular softmax parameterization as in line 10. Algorithm 1 Non-Stationary Natural Actor-Critic (NS-NAC)

1: Input time horizon T, variation budgets $\Delta_{R,T}$, $\Delta_{P,T}$, projection radius R_Q

2: Set step-sizes of actor β , critic α , average reward γ , number of restarts N of length $H = \lfloor \frac{T}{N} \rfloor$, time t = 0

3: for $n = 0, 1, 2, \dots, N - 1$ do

- Set policy $\pi_t(a|s) = \frac{1}{|\mathcal{A}|}$, value function $Q_t(s, a) = 0 \ \forall s, a$, average reward estimate $\eta_t = 0$ 4:
- Sample $s_t \sim \text{Unif}\{|\mathcal{S}|\}$, take action $a_t \sim \pi_t(\cdot|s_t)$ 5:
- 6: for $h = 0, 1, 2, \dots, H - 1$ do
- Observe reward $r_t(s_t, a_t)$, next state $s_{t+1} \sim P_t(\cdot | s_t, a_t)$, and take action $a_{t+1} \sim \pi_t(\cdot | s_{t+1})$ 7:
- $\eta_{t+1} \leftarrow \eta_t + \gamma \left(r_t(s_t, a_t) \eta_t \right)$ 8:
- $\begin{aligned} Q_{t+1}(s_t, a_t) &\leftarrow \Pi_{R_Q} \left[Q_t(s_t, a_t) + \alpha \left(r_t(s_t, a_t) \eta_t + Q_t(s_{t+1}, a_{t+1}) Q_t(s_t, a_t) \right) \right] \\ \pi_{t+1}(a|s) &\leftarrow \frac{\pi_t(a|s) \exp(\beta Q_t(s, a))}{\sum_{a' \in \mathcal{A}} \pi_t(a'|s) \exp(\beta Q_t(s, a'))}, \forall s, a \end{aligned}$ 9: ▷ *Critic Update*
- 10:
- $t \leftarrow t + 1$ 11:
- 12: end for
- 13: end for

The critic (faster timescale) estimates the tabular state-action value function of the current policy π_t as \mathbf{Q}_t using TD-Learning with step-size α (line 9). The projection step in line 9 is defined as $\Pi_{R_Q}[\mathbf{x}] := \arg\min_{\|\mathbf{y}\|_2 \leq R_Q} \|\mathbf{x} - \mathbf{x}\|_{\mathbf{y}}$ $\mathbf{y}\|_2$ (see Lemma 5.2 and following discussion on choice of R_{Q}). Further, the average reward estimate η_{t} is updated with step-size γ (line 8). Using a two timescale technique with $\alpha \gg \beta$, NS-NAC thus enables the actor to chase the moving target π_t^* facilitated by the critic updates of the value function estimates which adapt to the changed data distribution. In the stationary RL case, this change in data distribution is induced solely by the evolving actor policy, while in non-stationary RL, the time-varying environment $(\mathbf{P}_t, \mathbf{r}_t)$ further exacerbates it. Further, as Theorem 5.3 suggests, a careful selection of the step-sizes as a function of the variation budgets enables NS-NAC to balance the rate of forgetting the old environment versus learning a new one.

Function Approximation. While we consider the tabular formulation here for the ease of presentation, NS-NAC can also be extended to the function approximation setting. Further details are presented in Appendix E.

5. Regret Analysis: NS-NAC

In this section, we set up notation and assumptions and establish an upper bound on the dynamic regret of NS-NAC. We further present a sketch of the proof in Section 7.

5.1. Assumptions

Notation. We denote an observation O_t = $(s_t, a_t, s_{t+1}, a_{t+1})$. If $d^{\pi_t, \mathbf{P}_t}(\cdot)$ is the stationary distribution induced over the states, we define the matrices $\mathbf{A}(O_t), \bar{\mathbf{A}}^{\pi_t, \mathbf{P}_t} \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}| \times |\mathcal{S}||\mathcal{A}|}$ as

$$\mathbf{A}(O_t)_{i,j} = \begin{cases} -1, & \text{if } (s_t, a_t) \neq (s_{t+1}, a_{t+1}), \\ & i = j = (s_t, a_t) \\ 1, & \text{if } (s_t, a_t) \neq (s_{t+1}, a_{t+1}), \\ & i = (s_t, a_t), j = (s_{t+1}, a_{t+1}) \\ 0, & \text{else} \end{cases}$$

$$\mathbf{A}^{\boldsymbol{\pi}_{t},\mathbf{r}_{t}} = \mathbb{E}_{\substack{s \sim d^{\boldsymbol{\pi}_{t},\mathbf{P}_{t}}(\cdot), a \sim \boldsymbol{\pi}_{t}(\cdot|s), \\ s' \sim \mathbf{P}_{t}(\cdot|s,a), a' \sim \boldsymbol{\pi}_{t}(\cdot|s')}} \left[\mathbf{A}(s,a,s',a')\right].$$

If $D^{\pi_t, \mathbf{P}_t} = diag \left(d^{\pi_t, \mathbf{P}_t}(s) \pi_t(a|s) \right)$ and 1 is the all ones vector, then the TD limiting point satisfies

$$\mathbf{D}^{\boldsymbol{\pi}_{t},\mathbf{P}_{t}}\left(\mathbf{r}_{t}-J_{t}^{\boldsymbol{\pi}_{t}}\mathbf{1}\right)+\bar{\mathbf{A}}^{\boldsymbol{\pi}_{t},\mathbf{P}_{t}}\mathbf{Q}_{t}^{\boldsymbol{\pi}_{t}}=0.$$
 (4)

▷ Average Reward Estimate

▷ Actor Update

Assumption 5.1 (Uniform Ergodicity). A Markov chain generated by implementing policy π and transition probabilities **P** is called uniformly ergodic, if there exists m > 0and $\rho \in (0, 1)$ such that

$$d_{TV}\left(P(s_{\tau} \in \cdot | s_0 = s), d^{\boldsymbol{\pi}, \mathbf{P}}\right) \le m\rho^{\tau} \; \forall \tau \ge 0, s \in \mathcal{S},$$

where $d^{\pi, \mathbf{P}}$ is the stationary distribution induced over the states. We assume Markov chains induced by all potential policies π_t in all environments $\mathbf{P}_t, t \in [T]$, are uniformly ergodic. Further, if π_t^* denotes the optimal policy for the environment $\mathcal{M}_t = (\mathcal{S}, \mathcal{A}, \mathbf{P}_t, \mathbf{r}_t)$, there exists C > 0 such that

$$C = \inf_{s,t,t',\pi} \frac{d^{\boldsymbol{\pi},\mathbf{P}_{t'}}(s)}{d^{\boldsymbol{\pi}_t^\star,\mathbf{P}_t}(s)} > 0.$$

Lemma 5.2 (Zhang et al. (2021b), Lemma 2). Under Assumption 5.1, for all potential policies π_t in all environments \mathbf{P}_t , $t \in [T]$, the matrix $\bar{\mathbf{A}}^{\pi_t, \mathbf{P}_t}$ is negative semidefinite. Define its maximum non-zero eigenvalue as $-\lambda$.

Assumption 5.1 is standard in literature (Murthy & Srikant, 2023; Wu et al., 2020; Zou et al., 2019). Also note that we set the projection radius $R_Q = 2U_R \lambda^{-1}$ in line 9 of Algorithm 1 because $\| (\bar{\mathbf{A}}^{\boldsymbol{\pi}_t, \mathbf{P}_t})^{\dagger} \|_2 \le \lambda^{-1}$ where \dagger represents the pseudo-inverse.

5.2. Bounds on Regret

1

Theorem 5.3. If Assumption 5.1 is satisfied and the stepsizes are chosen as $0 < \alpha, \beta, \gamma < 1/2$ and number of restarts as 0 < N < T in Algorithm 1, then we have

$$Dyn-Reg(\mathcal{M},T) = \mathbb{E}\left[\sum_{t=0}^{T-1} J_t^{\boldsymbol{\pi}_t^{\star}} - r_t(s_t,a_t)\right]$$

$$\leq \tilde{\mathcal{O}}\left(\frac{N}{\beta}\right) + \tilde{\mathcal{O}}\left(\sqrt{\frac{NT}{\alpha}}\right) + \tilde{\mathcal{O}}\left(\frac{\beta T}{\alpha}\right) + \tilde{\mathcal{O}}\left(T\sqrt{\beta}\right)$$

$$\underbrace{\tilde{\mathcal{O}}\left(\frac{\beta T}{\beta}\right) + \tilde{\mathcal{O}}\left(T\sqrt{\gamma}\right) + \tilde{\mathcal{O}}\left(\sqrt{\frac{NT}{\gamma}}\right) + \underbrace{\tilde{\mathcal{O}}\left(T\sqrt{\alpha}\right)}_{Cumulative change}$$

$$+ \underbrace{\tilde{\mathcal{O}}\left(\frac{\beta T}{\gamma}\right) + \tilde{\mathcal{O}}\left(T\sqrt{\gamma}\right) + \tilde{\mathcal{O}}\left(\sqrt{\frac{NT}{\gamma}}\right) + \underbrace{\tilde{\mathcal{O}}\left(T\sqrt{\alpha}\right)}_{Cumulative change} + \underbrace{\tilde{\mathcal{O}}\left(\frac{\Delta_T T}{N}\right) + \tilde{\mathcal{O}}\left(\frac{\Delta_T^{1/3}T^{2/3}}{\sqrt{\alpha}} + \frac{\Delta_T^{1/3}T^{2/3}}{\sqrt{\gamma}}\right)}_{Error due to Non-Stationarity}$$
(5)

where $\Delta_T = \Delta_{R,T} + \Delta_{P,T}$, $\tilde{O}(\cdot)$ hides the constants and logarithmic dependence on the time horizon T. Choosing optimal $\alpha^* = \gamma^* = \left(\frac{\Delta_T}{T}\right)^{1/3}$, $\beta^* = \left(\frac{\Delta_T}{T}\right)^{1/2}$ and $N^* = \Delta_T^{5/6}T^{1/6}$, the resulting regret (with explicit dependence on the size of the state-action space |S|, |A|) is

$$Dyn-Reg(\mathcal{M},T) \leq \tilde{\mathcal{O}}\left(|\mathcal{S}|^{1/2}|\mathcal{A}|^{1/2}\Delta_T^{1/6}T^{5/6}\right).$$
(6)

We provide a sketch of the proof in Section 7 and the full proof in Appendix D.

Function Approximation. While we consider the tabular formulation here for the ease of presentation, NS-NAC can also be extended to the function approximation setting. Further details of regret bounds are presented in Appendix E.

Effect of Non-Stationarity. The variation budget Δ_T (2) represents the extent of non-stationarity of the environment. In Theorem 5.3, as the variation budget increases, so do the optimal choice of step-sizes and number of restarts, and the regret incurred (6). This observation is consistent with the intuition that in a rapidly changing environment, the algorithm must adapt quickly and explore more (hence, larger step-sizes and more restarts). However, as a result, the algorithm cannot exploit its current policy and value-function estimates, which soon become outdated (hence, higher regret). Also, in environments with larger state/action spaces, the agent requires proportionately more samples to detect changes and learn a good policy.

Theorem 5.4 ((Mao et al., 2024), Proposition 1). For any learning algorithm, there exists a non-stationary MDP such that the dynamic regret of the algorithm is at least $\Omega(|\mathcal{S}|^{1/3}|\mathcal{A}|^{1/3}\Delta_T^{1/3}T^{2/3})$.

Gap between Bounds. To the best of our knowledge, this is the first bound on dynamic regret for model-free policybased algorithm in the infinite horizon average reward setting. Observe that the infinite horizon setting (only one sample per environment available) is harder than the episodic setting (environment remains stationary during the episode) and necessitates a single loop algorithm with the policy being updated at every timestep. We conjecture that the gap between the bounds results from a slack in the analysis of the underlying Natural Actor-Critic (NAC) algorithm. The best-known regret bounds for NAC for an infinite horizon stationary MDP in the (compatible) function approximation setting with a two timescale algorithm is $\tilde{\mathcal{O}}(T^{3/4})$ (Khodadadian et al., 2022). The analysis of the actor involves the norm of the critic estimation error $\|\mathbf{Q}_t - \mathbf{Q}_t^{\boldsymbol{\pi}_t}\|$ (Proposition D.2) whereas guarantees for critic establish a bound on norm-squared of the error $\|\mathbf{Q}_t - \mathbf{Q}_t^{\boldsymbol{\pi}_t}\|^2$ (Proposition D.3). This mismatch, which underlies the sub-optimality of the current best stationary infinite horizon NAC analysis, is exacerbated in non-stationary environments resulting in the gap between the upper and the lower bounds. ¹ Note that this mismatch of the value function estimation error between the actor and the critic doesn't occur in the analysis of the model-based methods which use a Hoeffding style high probability bounds.

6. Parameter-Free Algorithm: BORL-NS-NAC

In this section, inspired by the bandit-over-RL (BORL) framework (Mao et al., 2024; Cheung et al., 2020), we present the BORL-NS-NAC algorithm that does not require prior knowledge of the variation budget Δ_T . It works by leveraging the adversarial bandit framework to tune the variation budget dependent parameters, i.e step-sizes and number of restarts, in NS-NAC and hedge against changes in rewards and transition probabilities. BORL-NS-NAC runs the EXP3.P algorithm (Bubeck et al., 2012) over $\lceil T/W \rceil$ epochs with NS-NAC (Algorithm 1) as a sub-routine in each epoch. In each epoch, an arm of the bandit is pulled to choose the parameters of the sub-routine and the cumulative rewards received during the epoch are used to update the posterior. Due to paucity of space, we defer the details of the algorithm, pseudocode and analysis to Appendix F and present the regret bound below.

Theorem 6.1. If Assumption 5.1 is satisfied and the time horizon T is divided into epochs of length $W = O(T^{2/3})$ in BORL-NS-NAC (Algorithm 2), then

$$D$$
yn- $Reg(\mathcal{M},T) \leq ilde{\mathcal{O}}\left(|\mathcal{S}|^{1/2}|\mathcal{A}|^{1/2}\Delta_T^{1/6}T^{5/6}
ight)$.

¹The term characterizing the difference in value functions at consecutive timesteps $\|\mathbf{Q}_{t+1}^{\pi_{t+1}} - \mathbf{Q}_{t}^{\pi_{t}}\|$ is the cause for the bottleneck $\tilde{\mathcal{O}}\left(\Delta_{T}^{1/3}T^{2/3}\left(\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\gamma}}\right)\right)$ term (see I_4, I_5, I_6 in Proposition D.3).

7. Proof Sketch of Theorem 5.3

We now present a sketch of the proof of Theorem 5.3 that presents an upper bound on regret of NS-NAC and address the following theoretical challenges that non-stationarity. (a) Stationary environment NAC analyses use the KLdivergence to the optimal policy as a Lyapunov function. What is an appropriate function for dynamic environments where the optimal policy varies with time? (b) How do the simultaneously varying environment and evolving policy affect the estimation of the average reward and state-action value function? (c) How do the time-varying transition probabilities affect the martingale-based argument used to analyze the Markovian noise?

Regret Decomposition. We start by decomposing as

$$\operatorname{Dyn-Reg}(\mathcal{M}, T) \tag{7}$$

$$= \sum_{t=0}^{T-1} \underbrace{\mathbb{E}\left[J_t^{\boldsymbol{\pi}_t^{\star}} - J_t^{\boldsymbol{\pi}_t}\right]}_{I_1: \underset{\text{actual average reward}}{\operatorname{Difference of optimal versus}}} + \underbrace{\mathbb{E}\left[J_t^{\boldsymbol{\pi}_t} - r_t(s_t, a_t)\right]}_{I_2: \underset{\text{instantaneous reward}}{\operatorname{Difference of actual versus}}},$$

where I_1 measures the performance difference between the average reward of the actual policy π_t at time t relative to the optimal policy π_t^* . The second term I_2 analyzes the gap between the average reward and the actual rewards received due to the stochasticity of the Markovian sampling process.

Actor (Proposition D.2). We first bound I_1 in (7) by adapting the Natural Policy Gradient analysis for averagereward stationary MDPs in Murthy & Srikant (2023) to non-stationary environments. NPG in the stationary case is analyzed by characterizing the drift of the policy towards the optimal policy using an appropriate Lyapunov function. In non-stationary case we innovatively separate out and analyze the change in the environment from the drift of the policy as follows. We start by dividing the total horizon Tinto N restarted segments of length H each and split I_1 as

$$\begin{split} I_{1} &= \mathbb{E}\Bigg[\sum_{n=0}^{N-1}\sum_{h=0}^{H-1}\underbrace{\left(J_{nH+h}^{\pi_{nH+h}^{\star}} - J_{nH}^{\pi_{nH}^{\star}}\right)}_{I_{3}: \text{ Optimal avg. reward}} \\ &+ \underbrace{\left(J_{nH}^{\pi_{nH}^{\star}} - J_{nH}^{\pi_{nH+h}}\right)}_{I_{4}: \sup_{\text{sub-optimality}}} + \underbrace{\left(J_{nH}^{\pi_{nH+h}} - J_{nH+h}^{\pi_{nH+h}}\right)}_{I_{5}: \underset{\text{policy in two environments}}{}}\Bigg]. \end{split}$$

We benchmark policies learned in each segment $n \in [N]$ against the optimal average reward at the initial time step nH i.e. $J_{nH}^{\pi_{nH}^*}$. We bound I_4 by mirror descent style analysis for each segment n with $t = \{nH, \ldots, (n+1)H - 1\}$ by the Lyapunov function adapted to non-stationarity as

$$W(\boldsymbol{\pi}_t) = \sum_{s} d^{\boldsymbol{\pi}_{nH}^{\star}, \mathbf{P}_{nH}}(s) D_{\mathrm{KL}}(\boldsymbol{\pi}_{nH}^{\star}(\cdot|s) \| \boldsymbol{\pi}_t(\cdot|s)).$$

In addition, since NS-NAC does not have access to the exact value functions $\mathbf{Q}_t^{\pi_t}$, I_4 also depends on the critic estimation error $\|\mathbf{Q}_t^{\pi_t} - \mathbf{Q}_t\|_{\infty}$.

We analyze the change in the environment next. We bound I_3 , the difference in the optimal average rewards in two different environments, in terms of the corresponding changes in the environment $\|\mathbf{r}_{nH+h} - \mathbf{r}_{nH}\|_{\infty}$ and $\|\mathbf{P}_{nH+h} - \mathbf{P}_{nH}\|_{\infty}$ (Lemma D.9) by a clever use of the linear programming formulation of an MDP. Similarly, we deftly bound I_5 , the difference in average rewards when following the same policy π_{nH+h} in two different environments, in terms of the change in the environment (Lemma D.10). Note that the number of restarts N balances exploration-for-change and learning a good policy and we optimize it in Theorem 5.3 to minimize regret.

Critic (Proposition D.3). We bound the critic estimation error $\psi_t = \prod_E [\mathbf{Q}_t - \mathbf{Q}_t^{\pi_t}]^2$ for each restarted segment $n \in [N]$ where $t = \{nH, \dots, (n+1)H - 1\}$ by adapting the critic analysis used in stationary MDPs (Wu et al., 2020; Khodadaian et al., 2022; Zhang et al., 2021b) to non-stationary environments. We decompose the error as

$$\begin{aligned} \|\boldsymbol{\psi}_{t+1}\|_{2}^{2} &\lesssim (1-\alpha) \|\boldsymbol{\psi}_{t}\|_{2}^{2} + \alpha \underbrace{\Gamma(\boldsymbol{\pi}_{t}, \mathbf{P}_{t}, \mathbf{r}_{t}, \boldsymbol{\psi}_{t}, O_{t})}_{I_{6}:\text{Error due to Markov noise}} \\ &+ \alpha \underbrace{(\mathbf{J}_{t}^{\boldsymbol{\pi}_{t}}(O_{t}) - \boldsymbol{\eta}_{t}(O_{t}))^{2}}_{I_{7}:\text{Avg. reward estimation error}} + \frac{1}{\alpha} \underbrace{\|\Pi_{E} \left[\mathbf{Q}_{t}^{\boldsymbol{\pi}_{t}} - \mathbf{Q}_{t+1}^{\boldsymbol{\pi}_{t+1}}\right]\|_{2}^{2}}_{I_{8}:\text{Value function drift}} \\ &+ \underbrace{\alpha^{2} \|\mathbf{r}_{t}(O_{t}) - \boldsymbol{\eta}_{t}(O_{t}) + \mathbf{A}(O_{t})\mathbf{Q}_{t}\|_{2}^{2}}_{I_{9}:\text{Variance term}} \end{aligned}$$
(8)

where $\Gamma_t = \psi_t^{\top} (\mathbf{r}_t(O_t) - \mathbf{J}_t^{\pi_t}(O_t) + \mathbf{A}(O_t)\mathbf{Q}_t^{\pi_t}) + \psi_t^{\top} (\mathbf{A}(O_t) - \bar{\mathbf{A}}^{\pi_t,\mathbf{P}_t}) \psi_t$. I_6 is the error induced by the Markovian noise which is analyzed leveraging the auxiliary Markov chain described below. I_7 describes the error due to an inaccurate estimation of the average reward which is bounded below. I_8 , the change in the true value function is caused by drifting policies and environments, and can be neatly bounded in terms of the change in policy, rewards and transition probabilities (Lemma D.13). Finally, I_{10} is the variance term.

Bound on Markovian Noise. For each restarted segment $n \in [N]$, consider time indices $(n + 1)H > t > \tau > nH$. Consider the *auxiliary Markov chain* starting from $s_{t-\tau}$ constructed by conditioning on $\mathcal{F}_{t-\tau} = \{s_{t-\tau}, \pi_{t-\tau-1}, \mathbf{P}_{t-\tau}\}$ and rolling out by applying $\pi_{t-\tau-1}, \mathbf{P}_{t-\tau}$ as

$$s_{t-\tau} \xrightarrow{\boldsymbol{\pi}_{t-\tau-1}} a_{t-\tau} \xrightarrow{\mathbf{P}_{t-\tau}} \tilde{s}_{t-\tau+1} \xrightarrow{\boldsymbol{\pi}_{t-\tau-1}} \tilde{a}_{t-\tau+1} \xrightarrow{\boldsymbol{\pi}_{t-\tau-1}} \tilde{a}_{t-\tau+1} \xrightarrow{\boldsymbol{\pi}_{t-\tau}} \tilde{a}_{t-\tau+1} \xrightarrow{\boldsymbol{\pi}_{t-\tau-1}} \tilde{a}_{t+1}.$$

 $^{{}^{2}\}Pi_{E}[\mathbf{x}] = \arg\min_{\mathbf{y}\in E} \|\mathbf{x} - \mathbf{y}\|_{2}$ is the projection to E, the subspace orthogonal to the all ones vector 1.



Figure 1: Performance of NS-NAC and baseline algorithms across various settings. (a) Dynamic regret for a single instance with $T = 25 \times 10^4$ steps. Log-log plots showing the effect of varying: (b) time horizon T, and (c) variation budget Δ_T .

Recall that the original Markov chain is

$$s_{t-\tau} \xrightarrow{\boldsymbol{\pi}_{t-\tau-1}} a_{t-\tau} \xrightarrow{\mathbf{P}_{t-\tau}} s_{t-\tau+1} \xrightarrow{\boldsymbol{\pi}_{t-\tau}} a_{t-\tau+1} \xrightarrow{\boldsymbol{\pi}_{t-\tau}} a_{t-\tau+1} \xrightarrow{\boldsymbol{\pi}_{t-\tau}} s_{t-\tau+1} \xrightarrow{\boldsymbol{\pi}_{t-\tau}} a_{t-\tau+1} \xrightarrow{\boldsymbol{\pi}_{t-\tau+1}} \xrightarrow{\boldsymbol{\pi}_{t-\tau+1}} a_{t-\tau+1} \xrightarrow{\boldsymbol{\pi}_{t-\tau+1}} \xrightarrow{$$

This method enables us to characterize properties of the original Markov chain in comparison to the auxiliary chain as $d_{TV}(P(O_t \in \cdot | \mathcal{F}_{t-\tau}), P(\tilde{O}_t \in \cdot | \mathcal{F}_{t-\tau}))$. We do this by bounding the effects of drifting policies and transition probabilities in the original chain and leveraging uniform ergodicity in the auxiliary chain. While prior works use auxiliary Markov chains for stationary environments (Zou et al., 2019; Wu et al., 2020; Wang et al., 2024), ours is the first adaptation to a non-stationary environment. Observe that the time-varying transition probabilities \mathbf{P}_t add an extra layer of complexity, unlike the stationary case where only the policy changes over time.

Average Reward Estimation Error (Proposition D.5). To bound I_7 in (8), i.e., the error in the average reward estimate $\phi_t = \eta_t - J_t^{\pi_t}$, we can decompose the error as

$$\begin{split} \phi_{t+1}^2 \lesssim (1-\gamma)\phi_t^2 + \underbrace{\gamma(r_t(O_t) - J_t^{\pi_t})^2}_{I_{10}: \text{Error due to Markov noise}} \\ &+ \frac{1}{\gamma} \underbrace{(J_t^{\pi_t} - J_{t+1}^{\pi_{t+1}})^2}_{I_{11}: \text{Avg reward at consecutive}} + \underbrace{\gamma^2(r_t(O_t) - \eta_t)^2}_{I_{12}: \text{Variance term}} \end{split}$$

 I_{10} is analyzed using the auxiliary Markov chain construction. I_{11} quantifies the difference in average rewards at consecutive timesteps, and is neatly bounded in Lemma D.10 in terms of the corresponding changes in policies, rewards, and transition probabilities. I_{12} is again the variance term.

Finally, I_2 in (7) characterizes the difference between the average reward and the instantaneous reward at any time, and is analyzed in Proposition D.4 using the auxiliary Markov chain to bound the bias occurring due to Markovian sampling. This concludes the proof sketch.

8. Simulations

We empirically evaluate the performance of our algorithms on a synthetic non-stationary MDP (see Appendix G), comparing it with three baseline algorithms: SW-UCRL2-CW (Cheung et al., 2023), Var-UCRL2 (Ortner et al., 2020), and RestartQ-UCB (Mao et al., 2024). SW-UCRL2-CW is a model-based algorithm that adapts to non-stationarity by maintaining a sliding window of recent observations, applying extended value iteration, and adjusting confidence intervals to track changing dynamics. Var-UCRL2, also model-based, adjusts its confidence intervals dynamically based on the observed variations in rewards and transitions. RestartQ-UCB, a model-free approach, periodically restarts Q-learning and resets its upper confidence bounds to adapt to non-stationarity. While there is a gap between our theoretical analysis of regret and those of the baseline methods, we empirically observe in Figure 1 that NS-NAC and BORL-NS-NAC strongly match their performance achieving sublinear dynamic regret across all experimental settings.

9. Conclusion

We consider the problem of non-stationary reinforcement learning in the infinite-horizon average-reward setting and model it as an MDP with time-varying rewards and transition probabilities. We propose and analyze the first model-free policy-based algorithm, Non-Stationary Natural Actor-Critic. A two-timescale natural policy gradient based method, NS-NAC utilizes restarts to explore for change and learning rates as adapting factors to balance forgetting old and learning new environments. Further, we present a bandit-over-RL based parameter-free algorithm BORL-NS-NAC that does not require prior knowledge of the variation budget and adaptively tunes step-sizes and number of restarts. Both algorithms achieve a sub-linear dynamic regret, thus, theoretically validating policy gradient methods often used in practice in continual non-stationary RL.

Impact Statement

This paper presents work whose goal is to advance the field of Machine Learning. There are many potential societal consequences of our work, none which we feel must be specifically highlighted here.

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A. Additional Related Work

Satting	Algorithm	Regret	Model	Policy
Setting			Free	Based
	Lower Bound	$ \left \Omega \left(\mathcal{S} ^{\frac{1}{3}} \mathcal{A} ^{\frac{1}{3}} D^{\frac{2}{3}} \Delta_T^{\frac{1}{3}} T^{\frac{2}{3}} \right) \right $	-	-
	Jaksch et al. (2010)	$\hat{\mathcal{O}}\left(\mathcal{S} \mathcal{A} ^{\frac{1}{2}}DL^{\frac{1}{3}}T^{\frac{2}{3}} ight)^{\prime}$	×	-
Non-Stationary	Gajane et al. (2018)	$\tilde{\mathcal{O}}\left(\mathcal{S} ^{\frac{2}{3}} \mathcal{A} ^{\frac{1}{3}}D^{\frac{2}{3}}L^{\frac{1}{3}}T^{\frac{2}{3}}\right)$	×	-
Infinite Horizon	Ortner et al. (2020)	$\tilde{\mathcal{O}}\left(\mathcal{S} \mathcal{A} ^{\frac{1}{2}}D\Delta_T^{\frac{1}{3}}T^{\frac{2}{3}}\right)$	×	-
Average Reward	Cheung et al. (2020)	$\tilde{\mathcal{O}}\left(\mathcal{S} ^{\frac{2}{3}} \mathcal{A} ^{\frac{1}{2}}D\Delta_{T}^{\frac{1}{4}}T^{\frac{2}{4}}\right)$	×	-
	Wei & Luo (2021)	$\tilde{\mathcal{O}}\left(\Delta_T^{\frac{1}{3}}T^{\frac{2}{3}}\right)$	×	-
	This Work	$\tilde{\mathcal{O}}\left(\mathcal{S} ^{\frac{1}{2}} \mathcal{A} ^{\frac{1}{2}}\Delta_{T}^{\frac{1}{9}}T^{\frac{8}{9}}\right)$	\checkmark	\checkmark
	Lower Bound	$\Omega\left(\mathcal{S} ^{\frac{1}{3}} \mathcal{A} ^{\frac{1}{3}}\Delta_{T}^{\frac{1}{3}}H^{\frac{2}{3}}T^{\frac{2}{3}}\right)$	-	-
Non-Stationary	Domingues et al. (2021)	$\tilde{\mathcal{O}}\left(\mathcal{S} \mathcal{A} ^{\frac{1}{2}}\Delta_T^{\frac{1}{3}}H^{\frac{4}{3}}T^{\frac{2}{3}}\right)$	\checkmark	×
Episodic	Wei & Luo (2021)	$\tilde{\mathcal{O}}\left(\Delta_T^{\frac{1}{3}}T^{\frac{2}{3}}\right)$	\checkmark	×
	Feng et al. (2023)	$\tilde{\mathcal{O}}\left(\check{\tilde{d}}^{rac{1}{2}}H^{2}T^{rac{1}{2}} ight)$	\checkmark	×
	Mao et al. (2024)	$\tilde{\mathcal{O}}\left(\mathcal{S} ^{\frac{1}{3}} \mathcal{A} ^{\frac{1}{3}}\Delta_T^{\frac{1}{3}}HT^{\frac{2}{3}}\right)$	\checkmark	×
Non-Stationary	Zhou et al. (2020)	$\tilde{\mathcal{O}}\left(d^{\frac{4}{3}}\Delta_{T}^{\frac{1}{3}}H^{\frac{4}{3}}T^{\frac{2}{3}}\right)$	\checkmark	×
Episodic Linear MDP	Touati & Vincent (2020)	$ ilde{\mathcal{O}}\left(d^{rac{5}{4}}\Delta_T^{rac{1}{4}}H^{rac{5}{4}}T^{rac{3}{4}} ight)$	\checkmark	×
Stationary				
Infinite Horizon	Khodadadian et al. (2022)	$\tilde{\mathcal{O}}\left(T^{\frac{5}{6}}\right)$	\checkmark	\checkmark
Discounted Reward				
Stationary				
Infinite Horizon	Wang et al. (2024)	$\tilde{\mathcal{O}}\left(T^{\frac{2}{3}}\right)$	\checkmark	\checkmark
Average Reward				

Table 1: Regret comparison across Non-Stationary and Stationary RL algorithms with variation budget Δ_T , time horizon T, episode length H, size of the state-action space |S|, |A|, maximum diameter of MDP D, dimension of feature space d and dynamic Bellman Eluder dimension \tilde{d} .

B. Notation

Variation Budgets

$$\Delta_{R,T} = \sum_{t=0}^{T-1} \|\mathbf{r}_{t+1} - \mathbf{r}_t\|_{\infty}; \Delta_{R,t-\tau+1,t} = \sum_{i=t-\tau+1}^{t} \|\mathbf{r}_i - \mathbf{r}_{i-1}\|_{\infty},$$

$$\Delta_{P,T} = \sum_{t=0}^{T-1} \|\mathbf{P}_{t+1} - \mathbf{P}_t\|_{\infty}; \Delta_{P,t-\tau+1,t} = \sum_{i=t-\tau+1}^{t} \|\mathbf{P}_i - \mathbf{P}_{i-1}\|_{\infty},$$

$$\Delta_T = \Delta_{R,T} + \Delta_{P,T}.$$

The critic update (line 9 in Algorithm 1) can be defined in vector form using the following notation. Note that we use a one-to-one mapping $f : S \times A \rightarrow \{1, 2, ..., |S||A|\}$, to map state-action pairs $(s, a) \in S \times A$ to vector/matrix entries. However, for ease of notation, we denote the index of each entry by (s, a), instead of the more accurate f(s, a).

$$O_t = (s_t, a_t, s_{t+1}, a_{t+1})$$

$$\mathbf{r}_t(O_t) = [0; \cdots; 0; r_t(s_t, a_t); 0; \cdots; 0]^\top \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}|}$$

$$\eta_t(O_t) = [0; \cdots; 0; \eta_t; 0; \cdots; 0]^\top \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}|}$$

$$\mathbf{J}_t^{\boldsymbol{\pi}}(O_t) = [0; \cdots; 0; J_t^{\boldsymbol{\pi}}; 0; \cdots; 0]^\top \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}|}$$

$$\mathbf{A}(O) \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}| \times |\mathcal{S}||\mathcal{A}|}$$
 such that

$$\mathbf{A}(O)_{i,j} = \mathbf{A}(s, a, s', a')_{i,j} = \begin{cases} -1 & \text{if } (s, a) \neq (s', a'), i = j = (s, a) \\ 1 & \text{if } (s, a) \neq (s', a'), i = (s, a), j = (s', a') \\ 0 & \text{else} \end{cases}$$

As a result, we get the critic update

$$\mathbf{Q}_{t+1} = \prod_{R_Q} \left[\mathbf{Q}_t + \alpha \left(\mathbf{r}_t(O_t) - \eta_t(O_t) + \mathbf{A}(O_t) \mathbf{Q}_t \right) \right].$$

For the purpose of analysis, we define the following quantities.

$$\begin{split} \bar{\mathbf{A}}^{\boldsymbol{\pi},\mathbf{P}} &= \mathbb{E}_{s\sim d^{\boldsymbol{\pi},\mathbf{P}}(\cdot),a\sim \boldsymbol{\pi}(\cdot|s),s'\sim \mathbf{P}(\cdot|s,a),a'\sim \boldsymbol{\pi}(\cdot|s')} \left[\mathbf{A}(s,a,s',a')\right] \\ \mathbf{Q}^{\boldsymbol{\pi},\mathbf{P},\mathbf{r}} &= \mathbf{Q} \quad \text{associated with} \quad \boldsymbol{\pi},\mathbf{P},\mathbf{r} \\ J^{\boldsymbol{\pi},\mathbf{P},\mathbf{r}} &= \sum_{s} d^{\boldsymbol{\pi},\mathbf{P}}(s) \sum_{a} \boldsymbol{\pi}(a|s)r(s,a) \\ \Pi_{E}[\mathbf{x}] &= \underset{\mathbf{y}\in E}{\arg\min} \|\mathbf{x}-\mathbf{y}\|_{2} \text{ where } E \text{ is the subspace orthogonal to the all ones vector } \mathbf{1} \\ \boldsymbol{\psi}_{t} &= \Pi_{E} \left[\mathbf{Q}_{t} - \mathbf{Q}_{t}^{\boldsymbol{\pi},t}\right] & (\text{Error in the value-function estimate}) \\ \Gamma(\boldsymbol{\pi},\mathbf{P},\mathbf{r},\boldsymbol{\psi},O) &= \boldsymbol{\psi}^{\top} \left(\mathbf{r}(O) - \mathbf{J}^{\boldsymbol{\pi},\mathbf{P},\mathbf{r}}(O) + \mathbf{A}(O)\mathbf{Q}^{\boldsymbol{\pi},\mathbf{P},\mathbf{r}}\right) + \boldsymbol{\psi}^{\top} \left(\mathbf{A}(O) - \bar{\mathbf{A}}^{\boldsymbol{\pi},\mathbf{P}}\right) \boldsymbol{\psi} \\ \phi_{t} &= \eta_{t} - J_{t}^{\boldsymbol{\pi}_{t}} & (\text{Error in the average reward estimate}) \\ \Lambda(\boldsymbol{\pi},\mathbf{P},\mathbf{r},\eta,O) &= (\eta - J^{\boldsymbol{\pi},\mathbf{P},\mathbf{r}})(r(s,a) - J^{\boldsymbol{\pi},\mathbf{P},\mathbf{r}}) \end{split}$$

Given time indices $t > \tau > 0$, consider the *auxiliary Markov chain* starting from $s_{t-\tau}$ constructed by conditioning on $s_{t-\tau}, \pi_{t-\tau-1}, \mathbf{P}_{t-\tau}$ and rolling out by applying $\pi_{t-\tau-1}, \mathbf{P}_{t-\tau}$ as

$$s_{t-\tau} \xrightarrow{\pi_{t-\tau-1}} a_{t-\tau} \xrightarrow{\mathbf{P}_{t-\tau}} \tilde{s}_{t-\tau+1} \xrightarrow{\pi_{t-\tau-1}} \tilde{a}_{t-\tau+1} \xrightarrow{\mathbf{P}_{t-\tau}} \dots \tilde{s}_t \xrightarrow{\pi_{t-\tau-1}} \tilde{a}_t \xrightarrow{\mathbf{P}_{t-\tau}} \tilde{s}_{t+1} \xrightarrow{\pi_{t-\tau-1}} \tilde{a}_{t+1}.$$

Recall that the original Markov chain is

$$s_{t-\tau} \xrightarrow{\boldsymbol{\pi}_{t-\tau-1}} a_{t-\tau} \xrightarrow{\mathbf{P}_{t-\tau}} s_{t-\tau+1} \xrightarrow{\boldsymbol{\pi}_{t-\tau}} a_{t-\tau+1} \xrightarrow{\mathbf{P}_{t-\tau+1}} \dots s_t \xrightarrow{\boldsymbol{\pi}_{t-1}} a_t \xrightarrow{\mathbf{P}_t} s_{t+1} \xrightarrow{\boldsymbol{\pi}_t} a_{t+1}.$$

C. Symbol Reference

Constant	First Appearance
	Section 3.1
U_Q	Lemma D.28
N, H	Algorithm 1
$C = \inf_{s,t,t',\boldsymbol{\pi}} \frac{d^{\boldsymbol{\pi},\mathbf{P}_{t'}}(s)}{d^{\boldsymbol{\pi}_t^*,\mathbf{P}_t}(s)}$	Assumption 5.1
m, ho	Assumption 5.1
$M = \lceil \log_{\rho} m^{-1} \rceil + \frac{1}{1-\rho}$	Lemma D.26
λ	Lemma 5.2
$W_1 = (3G_R^2)^{1/3} (4U_Q^2)^{2/3}$	Proposition D.3
$W_2 = (3G_P^2)^{1/3} (4U_Q^2)^{2/3}$	Proposition D.3
$D_1 = L_{\pi}B_2 + 4U_R\sqrt{ \mathcal{S} \mathcal{A} }B_2 + 4U_R$	Proposition D.4
$D_2 = 4U_R + L_P$	Proposition D.4
$D_3 = 4U_R F_6 + 8U_R^2$	Proposition D.5
$D_4 = 9L_{\pi}^2 B_2^2$	Proposition D.5
$W_3 = (3)^{1/3} (4U_R^2)^{2/3}$	Proposition D.5
$W_4 = (3L_P^2)^{1/3} (4U_R^2)^{2/3}$	Proposition D.5
$B_1 = 2\sqrt{ \mathcal{A} }U_Q^2$	Lemma D.6
$B_2 = U_Q$	Lemma D.7
$B_3 = (F_{1\pi} + G_{\pi} + F_3\sqrt{ \mathcal{S} \mathcal{A} } + F_4)B_2$	Lemma D.14
$B_4 = F_2(2U_R + 2U_Q)$	Lemma D.14
$B_5 = F_2 G_R$	Lemma D.14
$B_6 = F_{1\mathbf{P}} + F_2 G_P + F_3$	Lemma D.14
$B_7 = (F_5 L_{\pi} + F_7 \sqrt{ \mathcal{S} \mathcal{A} } + F_8) B_2$	Lemma D.15
$B_8 = F_7 + F_5 L_P$	Lemma D.15
$L_{\pi} = 4U_R(M+1)\sqrt{ \mathcal{S} \mathcal{A} }$	Lemma D.10
$L_P = 4U_R M$	Lemma D.10
$G_{\pi} = 2U_Q \sqrt{ \mathcal{S} \mathcal{A} }$	Lemma D.11
$G_R = 2\lambda^{-1}\sqrt{ \mathcal{S} \mathcal{A} }$	Lemma D.12
$G_P = (\lambda^{-1}L_P + 4U_R\lambda^{-1}M + 4U_R\lambda^{-2}(M+1))\sqrt{ \mathcal{S} \mathcal{A} }$	Lemma D.12
$F_{1\boldsymbol{\pi}} = 2U_Q L_{\boldsymbol{\pi}} + 4U_Q G_{\boldsymbol{\pi}} + 8U_Q^2 (M+2) \mathcal{S} \mathcal{A} $	Lemma D.17
$F_{1P} = 2U_Q L_P + 4U_Q G_P + 8U_Q^2 (M+1)\sqrt{ \mathcal{S} \mathcal{A} }$	Lemma D.17
$F_2 = 2U_R + 18U_Q$	Lemma D.18
$F_3 = 16U_R U_Q + 24U_Q^2 \sqrt{ \mathcal{S} \mathcal{A} }$	Lemma D.19
$F_4 = 8U_R U_Q + 24U_Q^2 \sqrt{ \mathcal{S} \mathcal{A} }$	Lemma D.20
$F_5 = 4U_R$	Lemma D.22
$F_6 = 2U_R$	Lemma D.23
$F_7 = 8U_R^2$	Lemma D.24
$F_8 = 8U_R^2$	Lemma D.25

D. Regret Analysis: NS-NAC

Theorem D.1. If Assumption 5.1 is satisfied and the step-sizes are chosen as $0 < \alpha, \beta, \gamma, \epsilon < 1/2$ and number of restarts as 0 < N < T in Algorithm 1, then we have

$$Dyn-Reg(\mathcal{M},T) = \mathbb{E}\left[\sum_{t=0}^{T-1} J_t^{\pi_t^{\star}} - r_t(s_t, a_t)\right]$$

$$\leq \underbrace{\tilde{\mathcal{O}}\left(\frac{N}{\beta}\right) + \tilde{\mathcal{O}}\left(\sqrt{\frac{NT}{\alpha}}\right)}_{Effect \ of \ initialization} + \underbrace{\tilde{\mathcal{O}}\left(\frac{\beta T}{\alpha}\right) + \tilde{\mathcal{O}}\left(T\sqrt{\beta}\right)}_{Cumulative \ change} + \underbrace{\tilde{\mathcal{O}}\left(\frac{\beta T}{\gamma}\right) + \tilde{\mathcal{O}}\left(T\sqrt{\gamma}\right) + \tilde{\mathcal{O}}\left(\sqrt{\frac{NT}{\gamma}}\right)}_{Error \ in \ Average \ Reward \ Estimate \ at \ Critic}$$

$$+ \underbrace{\tilde{\mathcal{O}}\left(T\sqrt{\alpha}\right)}_{Cumulative \ change} + \underbrace{\tilde{\mathcal{O}}\left(\frac{\Delta_T T}{N}\right) + \tilde{\mathcal{O}}\left(\Delta_T^{1/3} T^{2/3}\left(\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\gamma}}\right)\right)}_{Error \ due \ to \ Non-Stationarity},$$

where $\Delta_T = \Delta_{R,T} + \Delta_{P,T}$, $\tilde{\mathcal{O}}(\cdot)$ hides the constants and logarithmic dependence on the time horizon T. Choosing optimal $\alpha^* = \gamma^* = \left(\frac{\Delta_T}{T}\right)^{1/3}$, $\beta^* = \left(\frac{\Delta_T}{T}\right)^{1/2}$ and $N^* = \Delta_T^{5/6} T^{1/6}$, the resulting regret (with explicit dependence on the size of the state-action space $|\mathcal{S}|, |\mathcal{A}|$) is

$$Dyn-Reg(\mathcal{M},T) \leq \tilde{\mathcal{O}}\left(|\mathcal{S}|^{1/2}|\mathcal{A}|^{1/2}\Delta_T^{1/6}T^{5/6}\right).$$

Proof. Recall that Algorithm 1 divides the total time horizon T into N segments of length $H = \lfloor \frac{T}{N} \rfloor$.

$$\begin{split} & \mathbb{E}\left[\sum_{t=0}^{T-1} J_t^{\pi_t^*} - r_t(s_t, a_t)\right] \\ &= \mathbb{E}\left[\sum_{t=0}^{T-1} J_t^{\pi_t^*} - J_t^{\pi_t}\right] + \mathbb{E}\left[\sum_{t=0}^{T-1} J_t^{\pi_t} - r_t(s_t, a_t)\right] \\ &\stackrel{(a)}{\leq} \tilde{\mathcal{O}}\left(\frac{\Delta_T T}{N}\right) + \tilde{\mathcal{O}}\left(\frac{N}{\beta}\right) + \tilde{\mathcal{O}}\left(\beta T\right) + 2\sum_{n=0}^{N-1} \sum_{h=0}^{H-1} \mathbb{E}\left[\left\|\Pi_E\left[\mathbf{Q}_{nH+h}^{\pi_{nH+h}} - \mathbf{Q}_{nH+h}\right]\right\|_{\infty}\right] + \mathbb{E}\left[\sum_{t=0}^{T-1} J_t^{\pi_t} - r_t(s_t, a_t)\right] \\ &\leq \tilde{\mathcal{O}}\left(\frac{\Delta_T T}{N}\right) + \tilde{\mathcal{O}}\left(\frac{N}{\beta}\right) + \tilde{\mathcal{O}}\left(\beta T\right) + 2\sum_{n=0}^{N-1} H^{1/2}\left(\sum_{h=0}^{H-1} \mathbb{E}\left[\left\|\Pi_E\left[\mathbf{Q}_{nH+h}^{\pi_{nH+h}} - \mathbf{Q}_{nH+h}\right]\right\|_2^2\right]\right)^{1/2} + \mathbb{E}\left[\sum_{t=0}^{T-1} J_t^{\pi_t} - r_t(s_t, a_t)\right] \\ &\stackrel{(b)}{\leq} \tilde{\mathcal{O}}\left(\frac{\Delta_T T}{N}\right) + \tilde{\mathcal{O}}\left(\frac{N}{\beta}\right) + \tilde{\mathcal{O}}\left(\beta T\right) + 2\sum_{n=0}^{N-1} \left[\tilde{\mathcal{O}}\left(\sqrt{H}\right) + \tilde{\mathcal{O}}\left(\sqrt{\frac{H}{\alpha}}\right) + \tilde{\mathcal{O}}\left(\sqrt{\alpha}H\right) + \tilde{\mathcal{O}}\left(\frac{\beta H}{\alpha}\right) + \tilde{\mathcal{O}}\left(\sqrt{\beta}H\right) + \tilde{\mathcal{O}}\left(\frac{\beta H}{\gamma}\right) \\ &\quad + \tilde{\mathcal{O}}\left(\sqrt{\gamma}H\right) + \tilde{\mathcal{O}}\left(\frac{\sqrt{H}}{\gamma}\right) + \tilde{\mathcal{O}}\left(\frac{\Delta_{1/3}^{1/3}}{\sqrt{\alpha}}\right) + \tilde{\mathcal{O}}\left(\sqrt{\frac{NT}{\alpha}}\right) + \tilde{\mathcal{O}}\left(T\sqrt{\alpha}\beta\right) + \tilde{\mathcal{O}}\left(T\sqrt{\beta}\right) + \tilde{\mathcal{O}}\left(\frac{\beta T}{\gamma}\right) \\ &\quad + \tilde{\mathcal{O}}\left(T\sqrt{\gamma}\right) + \tilde{\mathcal{O}}\left(\sqrt{\frac{NT}{\gamma}\right) + \tilde{\mathcal{O}}\left(\frac{\Delta_T^{1/3}T^{2/3}}{\sqrt{\alpha}}\right) + \tilde{\mathcal{O}}\left(\frac{\Delta_T^{1/3}T^{2/3}}{\sqrt{\gamma}}\right) + \tilde{\mathcal{O}}\left(1\right) + \tilde{\mathcal{O}}\left(\beta T\right) + \tilde{\mathcal{O}}\left(\Delta_{P,T}\right), \end{split}$$

where (a) is due to Proposition D.2, (b) is by Proposition D.3 and and $\Delta_{nH,(n+1)H} = \Delta_{R,nH,(n+1)H} + \Delta_{P,nH,(n+1)H}$, (c) is by Jensen's inequality, $\Delta_T = \Delta_{R,T} + \Delta_{P,T}$ and Proposition D.4. We further have $\tau_H = \mathcal{O}(\log T)$. Note that $\tilde{\mathcal{O}}(\cdot)$ hides constants and logarithmic terms.

D.1. Actor

The next result bounds the performance difference, measured by the average reward, between the optimal policies π_t^* and the current policy π_t .

Proposition D.2. If Assumption 5.1 holds, we have

$$\mathbb{E}\left[\sum_{t=0}^{T-1} J_t^{\boldsymbol{\pi}_t^{\star}} - J_t^{\boldsymbol{\pi}_t}\right] \leq \underbrace{\left(2 + 2G_R + \frac{1}{C}\right) \frac{T\Delta_{R,T}}{N} + \left(U_Q + L_P + 2G_P + \frac{L_P}{C}\right) \frac{T\Delta_{P,T}}{N}}_{Error \ due \ to \ Non-Stationarity} + 2\underbrace{\sum_{n=0}^{N-1} \sum_{h=0}^{H-1} \mathbb{E}\left[\left\|\Pi_E\left[\mathbf{Q}_{nH+h}^{\boldsymbol{\pi}_{nH+h}} - \mathbf{Q}_{nH+h}\right]\right\|_{\infty}\right]}_{Critic \ Estimation \ Error} + \underbrace{N \cdot \frac{\log|\mathcal{A}|}{\beta}}_{Initialization} + \underbrace{\frac{B_1\beta T}{C}}_{Change \ in \ policy} + \underbrace{\frac{U_R}{C}}_{Cnstant},$$

where $\Delta_{R,T} = \sum_{t=0}^{T-1} \|\mathbf{r}_{r+1} - \mathbf{r}_t\|_{\infty}, \Delta_{P,T} = \sum_{t=0}^{T-1} \|\mathbf{P}_{t+1} - \mathbf{P}_t\|_{\infty}, C$ is defined in Assumption 5.1, T is the total time horizon and N is the number of restarts. The remaining constants are defined in Appendix C.

Proof. Recall that Algorithm 1 divides the total time horizon T into N segments of length $H = \lfloor \frac{T}{N} \rfloor$. In each segment (indexed by $n \in [N]$), we use $J_{nH}^{\boldsymbol{\pi}_{nH}^*}$ as an anchor against which to compare the performance of the learned policies.

$$\mathbb{E}\left[\sum_{t=0}^{T-1} J_{t}^{\boldsymbol{\pi}_{t}^{\star}} - J_{t}^{\boldsymbol{\pi}_{t}}\right] \leq \mathbb{E}\left[\sum_{n=0}^{N-1} \sum_{h=0}^{H-1} \left(J_{nH+h}^{\boldsymbol{\pi}_{nH+h}^{\star}} - J_{nH}^{\boldsymbol{\pi}_{nH}^{\star}}\right) + \left(J_{nH}^{\boldsymbol{\pi}_{nH+h}^{\star}}\right) + \left(J_{nH}^{\boldsymbol{\pi}_{nH+h}}\right) + \left(J_{nH}^{\boldsymbol{\pi}_{nH+h}} - J_{nH+h}^{\boldsymbol{\pi}_{nH+h}}\right)\right]^{(a)} \leq \mathbb{E}\left[\sum_{n=0}^{N-1} \sum_{h=0}^{H-1} \left(2\|\mathbf{r}_{nH+h} - \mathbf{r}_{nH}\|_{\infty} + \left(U_{Q} + L_{P}\right)\|\mathbf{P}_{nH+h} - \mathbf{P}_{nH}\|_{\infty}\right) + \left(J_{nH}^{\boldsymbol{\pi}_{nH}^{\star}} - J_{nH}^{\boldsymbol{\pi}_{nH+h}}\right)\right]^{(b)} \leq \sum_{n=0}^{N-1} \sum_{h=0}^{H-1} \left(H-1\right) \left(2\|\mathbf{r}_{nH+h} - \mathbf{r}_{nH+h-1}\|_{\infty} + \left(U_{Q} + L_{P}\right)\|\mathbf{P}_{nH+h} - \mathbf{P}_{nH+h-1}\|_{\infty}\right) + \left(J_{nH}^{\boldsymbol{\pi}_{nH}^{\star}} - J_{nH}^{\boldsymbol{\pi}_{nH+h}}\right)^{(d)} \leq \left(H-1\right) \left(2\Delta_{R,T} + \left(U_{Q} + L_{P}\right)\Delta_{P,T}\right) + \mathbb{E}\left[\sum_{n=0}^{N-1} \sum_{h=0}^{H-1} J_{nH}^{\boldsymbol{\pi}_{nH}^{\star}} - J_{nH}^{\boldsymbol{\pi}_{nH+h}}\right], \tag{9}$$

where (a) is by Lemma D.9 and Lemma D.10 and (b) is by triangle inequality. We now bound the last term as

$$\begin{split} &\sum_{n=0}^{N-1} \sum_{h=0}^{H-1} J_{nH}^{\pi_{n}^{*}H} - J_{nH}^{\pi_{n}^{*}H+h} \\ &\stackrel{(c)}{=} \sum_{n=0}^{N-1} \sum_{h=0}^{H-1} \frac{1}{\beta} \sum_{s} \sum_{a} d^{\pi_{nH}^{*},\mathbf{P}_{nH}}(s) \pi_{nH}^{*}(a|s) \left[\beta Q_{nH}^{\pi_{nH}+h}(s,a) - \beta V_{nH}^{\pi_{nH}+h}(s) \right] \\ &= \sum_{n} \sum_{h} \frac{1}{\beta} \sum_{s} \sum_{a} d^{\pi_{nH}^{*},\mathbf{P}_{nH}}(s) \pi_{nH}^{*}(a|s) \left[\beta Q_{nH+h}^{\pi_{nH}+h}(s,a) - \beta V_{nH+h}^{\pi_{nH}+h}(s) + \beta Q_{nH+h}(s,a) - \beta Q_{nH+h}(s,a) \right] \\ &+ \sum_{n} \sum_{j} \frac{1}{\beta} \sum_{s} \sum_{a} d^{\pi_{nH}^{*},\mathbf{P}_{nH}}(s) \pi_{nH}^{*}(a|s) \left[\beta Q_{nH}^{\pi_{nH}+h}(s,a) - \beta Q_{nH+h}^{\pi_{nH}+h}(s,a) + \beta V_{nH+h}^{\pi_{nH}+h}(s) - \beta V_{nH}^{\pi_{nH}+h}(s) \right] \\ &= \sum_{n} \sum_{h} \frac{1}{\beta} \sum_{s} \sum_{a} d^{\pi_{nH}^{*},\mathbf{P}_{nH}}(s) \pi_{nH}^{*}(a|s) \left[\beta Q_{nH+h}^{\pi_{nH}+h}(s,a) - \beta V_{nH+h}^{\pi_{nH}+h}(s) + \beta Q_{nH+h}(s,a) - \beta Q_{nH+h}(s,a) \right] \\ &+ \sum_{n} \sum_{h} 2 \|\mathbf{Q}_{nH}^{\pi_{nH}+h} - \mathbf{Q}_{nH+h}^{\pi_{nH}+h} \|_{\infty} \\ &\stackrel{(d)}{\leq} \sum_{n} \sum_{h} \frac{1}{\beta} \sum_{s} \sum_{a} d^{\pi_{nH}^{*},\mathbf{P}_{nH}}(s) \pi_{nH}^{*}(a|s) \left[\beta Q_{nH+h}^{\pi_{nH}+h}(s,a) - \beta V_{nH+h}^{\pi_{nH}+h}(s) + \beta Q_{nH+h}(s,a) - \beta Q_{nH+h}(s,a) \right] \end{aligned}$$

$$+\sum_{n=0}^{N-1}\sum_{h=0}^{H-1} 2G_{R} \|\mathbf{r}_{nH} - \mathbf{r}_{nH+h}\|_{\infty} + 2G_{P} \|\mathbf{P}_{nH} - \mathbf{P}_{nH+h}\|_{\infty}$$

$$\stackrel{(e)}{=} \sum_{n}\sum_{h}\frac{1}{\beta}\sum_{s}\sum_{a} d^{\pi_{nH}^{\star},\mathbf{P}_{nH}}(s)\pi_{nH}^{\star}(a|s) \left[\underbrace{\log Z_{nH+h}(s) - \beta V_{nH+h}^{\pi_{nH+h}}(s)}_{I_{1}}\right]$$

$$+\sum_{n}\sum_{h}\frac{1}{\beta}\sum_{s}\sum_{a} d^{\pi_{nH}^{\star},\mathbf{P}_{nH}}(s)\pi_{nH}^{\star}(a|s) \left[\underbrace{\log \frac{\pi_{nH+h+1}(a|s)}{I_{1}}}_{I_{2}} + \underbrace{\beta Q_{nH+h}^{\pi_{nH+h}}(s,a) - \beta Q_{nH+h}(s,a)}_{I_{3}}\right]$$

$$+ (H-1)(2G_{R}\Delta_{R,T} + 2G_{P}\Delta_{P,T}) \tag{10}$$

where (c) follows from the Performance Difference Lemma D.8, (d) follows from Lemma D.12 and (e) from the actor update equation (line 10 in Algorithm 1) and $Z_t(s) = \sum_{a' \in \mathcal{A}} \pi_t(a'|s) \exp(\beta Q_t(s,a'))$. Next, we bound each of I_1, I_2, I_3 . Using Lemma D.6, we have

$$I_{1} = \sum_{n} \sum_{h} \sum_{s} d^{\pi_{nH}^{\star}, \mathbf{P}_{nH}}(s) \left[\frac{\log Z_{nH+h}(s)}{\beta} - V_{nH+h}^{\pi_{nH+h}}(s) \right] \underbrace{\sum_{a} \pi_{nH}^{\star}(a|s)}_{=1} \\ \leq \sum_{n} \sum_{h} \sum_{h} \left[\frac{J_{nH+h+1}^{\pi_{nH+h+1}} - J_{nH+h}^{\pi_{nH+h}}}{C} + \|\mathbf{Q}_{nH+h}^{\pi_{nH+h}} - \mathbf{Q}_{nH+h}\|_{\infty} + \frac{B_{1}\beta}{C} + \frac{\|\mathbf{r}_{nH+h+1} - \mathbf{r}_{nH+h}\|_{\infty}}{C} + \frac{L_{P} \|\mathbf{P}_{nH+h+1} - \mathbf{P}_{nH+h}\|_{\infty}}{C} \right]$$
(11)

Next, we establish a bound on I_2 as

$$I_{2} = \frac{1}{\beta} \sum_{n} \sum_{h} \sum_{s} \sum_{a} d^{\pi_{nH}^{\star}, \mathbf{P}_{nH}}(s) \pi_{nH}^{\star}(a|s) \log \frac{\pi_{nH+h+1}(a|s)}{\pi_{nH+h}(a|s)}$$

$$\leq \frac{1}{\beta} \sum_{n=0} \sum_{h} \sum_{s} d^{\pi_{nH}^{\star}, \mathbf{P}_{nH}}(s) \left[D_{\mathrm{KL}}\left(\pi_{nH}^{\star}(\cdot|s) \| \pi_{nH+h}(\cdot|s)\right) - D_{\mathrm{KL}}\left(\pi_{nH}^{\star}(\cdot|s) \| \pi_{nH+h+1}(\cdot|s)\right) \right]$$

$$= \frac{1}{\beta} \sum_{n} \sum_{s} d^{\pi_{nH}^{\star}, \mathbf{P}_{nH}}(s) \left[D_{\mathrm{KL}}\left(\pi_{nH}^{\star}(\cdot|s) \| \pi_{nH}(\cdot|s)\right) - D_{\mathrm{KL}}\left(\pi_{nH}^{\star}(\cdot|s) \| \pi_{(n+1)H}\right) \right]$$

$$\stackrel{(f)}{\leq} \frac{1}{\beta} \sum_{n} \sum_{s} d^{\pi_{nH}^{\star}, \mathbf{P}_{nH}}(s) D_{\mathrm{KL}}\left(\pi_{nH}^{\star}(\cdot|s) \| \pi_{nH}(\cdot|s)\right)$$

$$\stackrel{(g)}{=} \frac{1}{\beta} \sum_{n} \sum_{s} d^{\pi_{nH}^{\star}, \mathbf{P}_{nH}}(s) \log \frac{|\mathcal{A}|}{1} \leq \frac{N \log |\mathcal{A}|}{\beta}$$
(12)

where (f) is because of non-negativity of KL-divergence and (g) is due to the restart in line 4 of Algorithm 1. Lastly, I_3 can be bounded as

$$I_{3} = \sum_{n} \sum_{h} \sum_{s} \sum_{a} d^{\pi_{nH}^{\star}, \mathbf{P}_{nH}}(s) \pi_{nH}^{\star}(a|s) \left[Q_{nH+h}^{\pi_{nH+h}}(s, a) - Q_{nH+h}(s, a) \right]$$

$$\leq \sum_{n} \sum_{h} \|\mathbf{Q}_{nH+h}^{\pi_{nH+h}} - \mathbf{Q}_{nH+h}\|_{\infty}.$$
(13)

We substitute the bounds on I_1, I_2, I_3 from (11)-(13) in (10) and then combine with (9). Recall that the set of solutions to the Bellman equations is $\mathbf{Q}_t^{\pi_t} = {\mathbf{Q}_{t,E}^{\pi_t} + c\mathbf{1} | \mathbf{Q}_{t,E}^{\pi_t} \in E, c \in \mathbb{R}}$ where E is the subspace orthogonal to the all ones vector and $\mathbf{Q}_{t,E}^{\pi_t}$ is the unique solution in E (Zhang et al., 2021b). Finally, we use the equivalence $\|\mathbf{Q}_t^{\pi_t} - \mathbf{Q}_t\|_{\infty} = \|\Pi_E [\mathbf{Q}_t^{\pi_t} - \mathbf{Q}_t]\|_{\infty}$ to get the result.

D.2. Critic

In this section, we characterize the error in the critic estimation.

Proposition D.3. For any $n \in [N]$, if Assumption 5.1 is satisfied and $0 < \gamma < 1/2$, then we have

where $\tilde{\mathcal{O}}(\cdot)$ hides constants and logarithmic terms which can be found in Equation (16) and $\Delta_{R,nH,(n+1)H} = \sum_{t=nH}^{(n+1)H} \|\mathbf{r}_{t+1} - \mathbf{r}_t\|_{\infty}$, and $\Delta_{P,nH,(n+1)H} = \sum_{t=nH}^{(n+1)H} \|\mathbf{P}_{t+1} - \mathbf{P}_t\|_{\infty}$.

Proof. Recall that $\psi_t = \prod_E [\mathbf{Q}_t - \mathbf{Q}_t^{\pi_t}]$, E is the subspace orthogonal to the all ones vector 1 and the critic update equation (line 9 in Algorithm 1) can be expressed in vector form as $\mathbf{Q}_{t+1} = \prod_{R_Q} [\mathbf{Q}_t + \alpha (\mathbf{r}_t(O_t) - \eta_t(O_t) + \mathbf{A}(O_t)\mathbf{Q}_t)]$. Recall the notations $\mathbf{r}_t, \eta_t, \mathbf{A}(O_t), \bar{\mathbf{A}}^{\pi_t, \mathbf{P}_t}, \mathbf{J}_t(O_t), \Gamma(\cdot), \phi_t$ from Appendix B. We therefore have

$$\begin{split} \|\psi_{t+1}\|_{2}^{2} &= \|\Pi_{E} \left[\mathbf{Q}_{t+1} - \mathbf{Q}_{t+1}^{\pi_{t+1}} \right] \|_{2}^{2} \\ &\leq \|\Pi_{E} \left[\mathbf{Q}_{t} + \alpha \left(\mathbf{r}_{t}(O_{t}) - \eta_{t}(O_{t}) + \mathbf{A}(O_{t})\mathbf{Q}_{t} \right) - \mathbf{Q}_{t+1}^{\pi_{t+1}} \right] \|_{2}^{2} \\ &= \|\Pi_{E} \left[\psi_{t} + \alpha \left(\mathbf{r}_{t}(O_{t}) - \eta_{t}(O_{t}) + \mathbf{A}(O_{t})\mathbf{Q}_{t} \right) + \mathbf{Q}_{t}^{\pi_{t}} - \mathbf{Q}_{t+1}^{\pi_{t+1}} \right] \|_{2}^{2} \\ &\leq \|\psi_{t}\|_{2}^{2} + 2\alpha\psi_{t}^{+} \left(\mathbf{r}_{t}(O_{t}) - \eta_{t}(O_{t}) \right) + \mathbf{A}(O_{t})\mathbf{Q}_{t} \\ &+ 2\psi_{t}^{+}\Pi_{E} \left[\mathbf{Q}_{t}^{\pi_{t}} - \mathbf{Q}_{t+1}^{\pi_{t+1}} \right] + 2\alpha^{2} \|\mathbf{r}_{t}(O_{t}) - \eta_{t}(O_{t}) + \mathbf{A}(O_{t})\mathbf{Q}_{t} \|_{2}^{2} + 2\|\Pi_{E} \left[\mathbf{Q}_{t}^{\pi_{t}} - \mathbf{Q}_{t+1}^{\pi_{t+1}} \right] \|_{2}^{2} \\ &\leq \|\psi_{t}\|_{2}^{2} + 2\alpha\psi_{t}^{+} \left(\mathbf{r}_{t}(O_{t}) - \eta_{t}(O_{t}) + \mathbf{A}(O_{t})\mathbf{Q}_{t} - \mathbf{A}^{\pi_{t},\mathbf{P}_{t}}\psi_{t} \right) + 2\alpha\psi_{t}^{+} \mathbf{A}\mathbf{A}^{\pi_{t},\mathbf{P}_{t}}\psi_{t} \\ &+ 2\psi_{t}^{+} \Pi_{E} \left[\mathbf{Q}_{t}^{\pi_{t}} - \mathbf{Q}_{t+1}^{\pi_{t+1}} \right] + 2\alpha^{2} \|\mathbf{r}_{t}(O_{t}) - \eta_{t}(O_{t}) + \mathbf{A}(O_{t})\mathbf{Q}_{t} \|_{2}^{2} + 2\|\Pi_{E} \left[\mathbf{Q}_{t}^{\pi_{t}} - \mathbf{Q}_{t+1}^{\pi_{t+1}} \right] \|_{2}^{2} \\ &\leq \|\psi_{t}\|_{2}^{2} + 2\alpha\psi_{t}^{+} \left(\mathbf{r}_{t}(O_{t}) - \eta_{t}(O_{t}) + \mathbf{A}(O_{t})\mathbf{Q}_{t} \right) + 2\alpha\psi_{t}^{+} \left(\mathbf{A}(O_{t}) - \mathbf{A}^{\pi_{t},\mathbf{P}_{t}} \right) \\ &+ 2\psi_{t}^{+} \Pi_{E} \left[\mathbf{Q}_{t}^{\pi_{t}} - \mathbf{Q}_{t+1}^{\pi_{t+1}} \right] + 2\alpha^{2} \|\mathbf{r}_{t}(O_{t}) - \eta_{t}(O_{t}) + \mathbf{A}(O_{t})\mathbf{Q}_{t} \|_{2}^{2} + 2\|\Pi_{E} \left[\mathbf{Q}_{t}^{\pi_{t}} - \mathbf{Q}_{t+1}^{\pi_{t+1}} \right] \|_{2}^{2} \\ &\leq \|\psi_{t}\|_{2}^{2} + 2\alpha\Gamma(\pi_{t},\mathbf{P}_{t},\mathbf{r}_{t},\psi_{t},O_{t}) + 2\alpha\psi_{t}^{+} \left[\mathbf{J}_{t}^{\pi_{t}}(O_{t}) - \eta_{t}(O_{t}) + \mathbf{A}(O_{t})\mathbf{Q}_{t} \|_{2}^{2} + 2\|\Pi_{E} \left[\mathbf{Q}_{t}^{\pi_{t}} - \mathbf{Q}_{t+1}^{\pi_{t+1}} \right] \|_{2}^{2} \\ &\leq \|\psi_{t}\|_{2}^{2} + 2\alpha\Gamma(\pi_{t},\mathbf{P}_{t},\mathbf{r}_{t},\psi_{t},O_{t}) + 2\alpha\|\psi_{t}\|_{2}\|\mathbf{J}_{t}^{\pi_{t}}(O_{t}) - \eta_{t}(O_{t}) + \mathbf{A}(O_{t})\mathbf{Q}_{t} \|_{2}^{2} + 2\|\Pi_{E} \left[\mathbf{Q}_{t}^{\pi_{t}} - \mathbf{Q}_{t+1}^{\pi_{t+1}} \right] \|_{2}^{2} \\ &\leq \|\psi_{t}\|_{2}\|\Pi_{E} \left[\mathbf{Q}_{t}^{\pi_{t}} - \mathbf{Q}_{t+1}^{\pi_{t+1}} \right] \|_{2}^{2} + 2\alpha^{2}\|\mathbf{r}_{t}(O_{t}) - \eta_{t}(O_{t}) + \mathbf{A}(O_{t})\mathbf{Q}_{t} \|_{2}^{2} + 2\|\Pi_{E} \left[\mathbf{Q}_{t}^{\pi_{t}} - \mathbf{Q}_{t+1}^{\pi_{t+1}} \right] \|_{2}^{2} \\ &\leq \|\psi_{t}\|_{2}\|\Psi$$

$$+ 2\|\psi_t\|_2\|\Pi_E\left[\mathbf{Q}_t^{\pi_t} - \mathbf{Q}_{t+1}^{\pi_t}\right]\|_2 + 2\|\psi_t\|_2\|\Pi_E\left[\mathbf{Q}_{t+1}^{\pi_t} - \mathbf{Q}_{t+1}^{\pi_{t+1}}\right]\|_2 + 2\alpha^2(2U_R + 2U_Q)^2 + 2\|\Pi_E\left[\mathbf{Q}_t^{\pi_t} - \mathbf{Q}_{t+1}^{\pi_{t+1}}\right]\|_2^2,$$

where (a) is due to Cauchy-Schwarz inequality, (b) follows from $\psi_t \in E$ and Lemma 5.2.

Taking expectation, rearranging the terms, setting $\tau = \tau_H = \min\{i \ge 0 | m\rho^{i-1} \le \min\{\beta, \alpha\}\}$ and summing over time, we

$$\sum_{t=nH+\tau_{H}}^{nH+H-1} \lambda \mathbb{E} \left[\|\psi_{t}\|_{2}^{2} \right]$$

$$\leq \sum_{t=nH+\tau_{H}}^{nH+H-1} \frac{\mathbb{E} \left[\|\psi_{t}\|_{2}^{2} - \|\psi_{t+1}\|_{2}^{2} \right]}{I_{1}} + \sum_{t=nH+\tau_{H}}^{nH+H-1} \mathbb{E} \left[\Gamma(\pi_{t}, \mathbf{P}_{t}, \mathbf{r}_{t}, \psi_{t}, O_{t}) \right] + \sum_{t=nH+\tau_{H}}^{nH+H-1} \mathbb{E} \left[\|\phi_{t}\| \|\psi_{t}\|_{2} \right]$$

$$+ \sum_{t=nH+\tau_{H}}^{nH+H-1} \frac{\mathbb{E} \left[\|\psi_{t}\|_{2} \|\Pi_{E} \left[\mathbf{Q}_{t}^{\pi_{t}} - \mathbf{Q}_{t+1}^{\pi_{t}} \right] \|_{2} \right]}{I_{4}} + \sum_{t=nH+\tau_{H}}^{nH+H-1} \frac{\mathbb{E} \left[\|\psi_{t}\|_{2} \|\Pi_{E} \left[\mathbf{Q}_{t}^{\pi_{t}} - \mathbf{Q}_{t+1}^{\pi_{t+1}} \right] \|_{2} \right]}{I_{5}}$$

$$+ \alpha (2U_{R} + 2U_{Q})^{2} (T - \tau_{T}) + \sum_{t=nH+\tau_{H}}^{nH+H-1} \frac{\mathbb{E} \left[\|\Pi_{E} \left[\mathbf{Q}_{t}^{\pi_{t}} - \mathbf{Q}_{t+1}^{\pi_{t+1}} \right] \|_{2} \right]}{I_{6}}. \qquad (14)$$

We now bound each of the terms starting with the first term as

$$I_1 = \frac{\mathbb{E}[\|\psi_{nH+\tau_H}\|_2^2 - \|\psi_{nH+H}\|_2^2]}{2\alpha} \le \frac{2U_Q^2}{\alpha}.$$

By Lemma D.14, we have

have

$$I_{2} \leq \sum_{t=nH+\tau_{H}}^{nH+H-1} B_{3}\beta(\tau_{H}+1)^{2} + B_{4}\alpha\tau_{H} + B_{5}\Delta_{R,t-nH-\tau_{H}+1,t} + B_{6}\tau_{H}\Delta_{P,t-nH-\tau_{H}+1,t}$$
$$\leq B_{3}\beta(\tau_{H}+1)^{2}(H-\tau_{H}) + B_{4}\alpha\tau_{H}(H-\tau_{H}) + B_{5}\tau_{H}\Delta_{R,nH,(n+1)H} + B_{6}\tau_{H}^{2}\Delta_{P,nH,(n+1)H}$$

By the Cauchy-Schwarz inequality, we have

$$I_{3} \leq \sum_{t=nH+\tau_{H}}^{nH+H-1} \sqrt{\mathbb{E}[\phi_{t}^{2}]} \sqrt{\mathbb{E}[\|\psi_{t}\|_{2}^{2}]} \leq \left(\sum_{t=nH+\tau_{H}}^{nH+H-1} \mathbb{E}[\phi_{t}^{2}]\right)^{1/2} \left(\sum_{t=nH+\tau_{H}}^{nH+H-1} \mathbb{E}[\|\psi_{t}\|_{2}^{2}]\right)^{1/2},$$

where $\sum_{t=nH+\tau_H}^{nH+H-1} \mathbb{E}[\phi_t^2]$ can be further bounded using Proposition D.5.

Using Lemma D.12, we have

$$I_{4} \leq \frac{2U_{Q}}{\alpha} \sum_{t=nH+\tau_{H}}^{nH+H-1} \|\mathbf{Q}_{t}^{\boldsymbol{\pi}_{t}} - \mathbf{Q}_{t+1}^{\boldsymbol{\pi}_{t}}\|_{2} \leq \frac{2U_{Q}}{\alpha} \sum_{t=nH+\tau_{H}}^{nH+H-1} G_{R} \|\mathbf{r}_{t+1} - \mathbf{r}_{t}\|_{\infty} + G_{P} \|\mathbf{P}_{t+1} - \mathbf{P}_{t}\|_{\infty}$$
$$\leq \frac{2U_{Q}}{\alpha} (G_{R} \Delta_{R,nH,(n+1)H} + G_{P} \Delta_{P,nH,(n+1)H}).$$

Using the Cauchy-Schwarz inequality and Lemma D.11, we have

$$I_{5} \leq \left(\sum_{t=nH+\tau_{H}}^{nH+H-1} \frac{\mathbb{E}[\|\Pi_{E}\left[\mathbf{Q}_{t+1}^{\boldsymbol{\pi}_{t}} - \mathbf{Q}_{t+1}^{\boldsymbol{\pi}_{t+1}}\right]\|_{2}^{2}]}{\alpha^{2}}\right)^{1/2} \left(\sum_{t=nH+\tau_{H}}^{nH+H-1} \mathbb{E}[\|\boldsymbol{\psi}_{t}\|_{2}^{2}]\right)^{1/2}$$

$$\leq \left(\frac{G_{\pi}^2 B_2^2 \beta^2 H}{\alpha^2}\right)^{1/2} \left(\sum_{t=nH+\tau_H}^{nH+H-1} \mathbb{E}[\|\psi_t\|_2^2]\right)^{1/2}.$$

We now the final term I_6 as follows. For timesteps with small changes in the environment, we use Lemma D.13, and for timesteps with large changes in the environment, we use a naive upper bound. Define the set of timesteps $\mathcal{T}_Q := \{t : \|\mathbf{r}_{t+1} - \mathbf{r}_t\|_{\infty} \leq \delta_R, \|\mathbf{P}_{t+1} - \mathbf{P}_t\|_{\infty} \leq \delta_P\}.$

$$I_{6} = \sum_{t=nH+\tau_{H}}^{nH+H-1} \frac{\mathbb{E}\left[\|\Pi_{E} \left[\mathbf{Q}_{t}^{\pi_{t}} - \mathbf{Q}_{t+1}^{\pi_{t+1}} \right] \|_{2}^{2} \right]}{\alpha} \leq \sum_{t \in \mathcal{T}_{Q}} \frac{\mathbb{E}\left[\|\Pi_{E} \left[\mathbf{Q}_{t}^{\pi_{t}} - \mathbf{Q}_{t+1}^{\pi_{t+1}} \right] \|_{2}^{2} \right]}{\alpha} + \sum_{t \notin \mathcal{T}_{Q}} \frac{4U_{Q}^{2}}{\alpha}$$

$$\stackrel{(d)}{\leq} \sum_{t \in \mathcal{T}_{Q}} \frac{3G_{R}^{2}\delta_{R}^{2}}{\alpha} + \frac{3G_{P}^{2}\delta_{P}^{2}}{\alpha} + \frac{3G_{\pi}^{2}B_{2}^{2}\beta^{2}}{\alpha} + \sum_{t \notin \mathcal{T}_{Q}} \frac{4U_{Q}^{2}}{\alpha}$$

$$\stackrel{(e)}{\leq} \frac{3G_{R}^{2}\delta_{R}^{2}H}{\alpha} + \frac{3G_{P}^{2}\delta_{P}^{2}H}{\alpha} + \frac{3G_{\pi}^{2}B_{2}^{2}\beta^{2}H}{\alpha} + \frac{4U_{Q}^{2}\Delta_{R,nH,(n+1)H}}{\alpha\delta_{R}} + \frac{4U_{Q}^{2}\Delta_{P,nH,(n+1)H}}{\alpha\delta_{P}}$$

$$\stackrel{(f)}{\leq} \frac{W_{1}\Delta_{R,nH,(n+1)H}^{2/3}}{\alpha} + \frac{W_{2}\Delta_{P,nH,(n+1)H}^{2/3}}{\alpha} + \frac{3G_{\pi}^{2}B_{2}^{2}\beta^{2}H}{\alpha} + \frac{3G_{\pi}^{2}B_{2}^{2}\beta^{2}H}{\alpha} + \frac{3G_{\pi}^{2}B_{2}^{2}\beta^{2}H}{\alpha\delta_{P}} + \frac{3G_{\pi}^{2}B_{2}^{2}B_{2}^{2}H}{\alpha\delta_{P}} + \frac{3G_{\pi}^{2}B_{2}^{2}B_{2$$

where (c) follows from the Lemma D.28, (d) follows from Lemma D.13 and (e) is obtained by choosing $\delta_R = \left(\frac{4U_Q^2 \Delta_{R,nH,(n+1)H}}{3G_R^2 H}\right)^{1/3}$ and $\delta_P = \left(\frac{4U_Q^2 \Delta_{P,nH,(n+1)H}}{3G_P^2 H}\right)^{1/3}$ and defining $W_1 = (3G_R^2)^{1/3} (4U_Q^2)^{2/3}$, $W_2 = (3G_P^2)^{1/3} (4U_Q^2)^{2/3}$.

We substitute the bounds on I_1, \ldots, I_6 (using Proposition D.5) into (14) and use the squaring trick from Section C.3 in Wu et al. (2020). The above equation is of the form, $X \leq Y + Z\sqrt{X}$. Completing the squares and rearranging, we get $X \leq 2Y + Z^2$. Hence, we get the final result as

$$\begin{split} &\sum_{t=nH+\tau_{H}}^{nH+H-1} \mathbb{E}\left[\|\psi_{t}\|_{2}^{2} \right] \\ &\leq \frac{4U_{Q}^{2}}{\alpha\lambda} + \frac{2B_{3}\beta(\tau_{H}+1)^{2}H}{\lambda} + \frac{2\alpha(B_{4}+8U_{R}^{2}+8U_{Q}^{2})\tau_{H}H}{\lambda} + \frac{2B_{5}\tau_{H}\Delta_{R,nH,(n+1)H}}{\lambda} + \frac{2B_{6}\tau_{H}^{2}\Delta_{P,nH,(n+1)H}}{\lambda} \\ &+ \frac{8U_{R}^{2}}{\gamma\lambda^{2}} + \frac{4B_{7}\beta(\tau_{H}+1)^{2}H}{\lambda^{2}} + \frac{2D_{3}\gamma\tau_{H}H}{\lambda^{2}} + \frac{4B_{8}(\tau_{H}+1)^{2}\Delta_{P,nH,(n+1)H}}{\lambda^{2}} \\ &+ \frac{2D_{4}\beta^{2}H}{\gamma^{2}\lambda^{2}} + \frac{8W_{3}\Delta_{R,nH,(n+1)H}^{2/3}}{\gamma\lambda^{2}} + \frac{8W_{4}\Delta_{P,nH,(n+1)H}^{2/3}}{\gamma\lambda^{2}} + \frac{6G_{\pi}^{2}B_{2}^{2}\beta^{2}H}{\alpha^{2}\lambda^{2}} \\ &+ \frac{4U_{Q}G_{R}\Delta_{R,nH,(n+1)H}}{\alpha\lambda} + \frac{4U_{Q}G_{P}\Delta_{P,nH,(n+1)H}}{\alpha\lambda} + \frac{G_{\pi}^{2}B_{2}^{2}\beta^{2}H}{\alpha^{2}\lambda^{2}} \\ &+ \frac{2W_{1}\Delta_{R,nH,(n+1)H}^{2/3}}{\alpha\lambda} + \frac{2W_{2}\Delta_{P,nH,(n+1)H}^{2/3}}{\alpha\lambda} + \frac{6G_{\pi}^{2}B_{2}^{2}\beta^{2}H}{\alpha\lambda} \end{split}$$
(16)
$$&\leq \tilde{\mathcal{O}}\left(\frac{1}{\alpha}\right) + \tilde{\mathcal{O}}\left(\beta H\right) + \tilde{\mathcal{O}}\left(\alpha H\right) + \tilde{\mathcal{O}}\left(\Delta_{R,nH,(n+1)H}\right) + \tilde{\mathcal{O}}\left(\Delta_{P,nH,(n+1)H}\right) + \tilde{\mathcal{O}}\left(\gamma H\right) + \tilde{\mathcal{O}}\left(\frac{1}{\gamma}\right) \\ &+ \tilde{\mathcal{O}}\left(\frac{\beta^{2}H}{\gamma^{2}}\right) + \tilde{\mathcal{O}}\left(\frac{\Delta_{R,nH,(n+1)H}^{2/3}}{\gamma}\right) + \tilde{\mathcal{O}}\left(\frac{\Delta_{P,nH,(n+1)H}^{2/3}}{\alpha}\right) + \tilde{\mathcal{O}}\left(\frac{\Delta_{P,nH,(n+1)H}^{2/3}}{\alpha}\right) + \tilde{\mathcal{O}}\left(\frac{\beta^{2}H}{\alpha^{2}}\right), \end{aligned}$$

where $\tilde{\mathcal{O}}(\cdot)$ hides constants and logarithmic terms.

D.3. Average Reward Estimation

In this section, we first analyze the gap between the average rewards and the rewards accumulate by NS-NAC in Proposition D.4. We then characterize the error in the average reward estimation in Proposition D.5.

Proposition D.4. For any $n \in [N]$, if Assumption 5.1 is satisfied, then the following holds true

$$\sum_{t=nH+\tau_H}^{nH+H-1} \mathbb{E}\left[J_t^{\boldsymbol{\pi}_t} - r_t(s_t, a_t)\right] \le D_1 \beta(\tau_H + 1)^2 (H - \tau_H) + D_2 (\tau_H + 1)^2 \Delta_{P, nH, (n+1)H}$$

where $D_1 = L_{\pi}B_2 + 4U_R\sqrt{|\mathcal{S}||\mathcal{A}|}B_2 + 4U_R$, $D_2 = 4U_R + L_P$ and $\Delta_{P,nH,(n+1)H} = \sum_{t=nH}^{(n+1)H} \|\mathbf{P}_{t+1} - \mathbf{P}_t\|_{\infty}$.

Proof. Given time indices $t > \tau > 0$, recall the auxiliary Markov chain starting from $s_{t-\tau}$ constructed by conditioning on $s_{t-\tau}, \pi_{t-\tau-1}, \mathbf{P}_{t-\tau}$ and rolling out by applying $\pi_{t-\tau-1}, \mathbf{P}_{t-\tau}$ as

$$s_{t-\tau} \xrightarrow{\boldsymbol{\pi}_{t-\tau-1}} a_{t-\tau} \xrightarrow{\boldsymbol{P}_{t-\tau}} \tilde{s}_{t-\tau+1} \xrightarrow{\boldsymbol{\pi}_{t-\tau-1}} \tilde{a}_{t-\tau+1} \xrightarrow{\boldsymbol{P}_{t-\tau}} \dots \tilde{s}_t \xrightarrow{\boldsymbol{\pi}_{t-\tau-1}} \tilde{a}_t \xrightarrow{\boldsymbol{P}_{t-\tau}} \tilde{s}_{t+1} \xrightarrow{\boldsymbol{\pi}_{t-\tau-1}} \tilde{a}_{t+1}.$$

Also, recall that the original Markov chain is

$$s_{t-\tau} \xrightarrow{\pi_{t-\tau-1}} a_{t-\tau} \xrightarrow{\mathbf{P}_{t-\tau}} s_{t-\tau+1} \xrightarrow{\pi_{t-\tau}} a_{t-\tau+1} \xrightarrow{\mathbf{P}_{t-\tau+1}} \dots s_t \xrightarrow{\pi_{t-1}} a_t \xrightarrow{\mathbf{P}_t} s_{t+1} \xrightarrow{\pi_t} a_{t+1}$$

Further, recall $J^{\boldsymbol{\pi}_{t-\tau-1},\mathbf{P}_{t-\tau},\mathbf{r}_t} := \sum_{s,a} d^{\boldsymbol{\pi}_{t-\tau-1},\mathbf{P}_{t-\tau}}(s) \boldsymbol{\pi}_{t-\tau-1}(a|s) r_t(s,a).$

We start by decomposing the term as

$$\mathbb{E}\left[J_t^{\boldsymbol{\pi}_t} - r_t(s_t, a_t)\right] = \underbrace{\mathbb{E}\left[J_t^{\boldsymbol{\pi}_t} - J^{\boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}, \mathbf{r}_t}\right]}_{I_1} + \underbrace{\mathbb{E}\left[r_t(\tilde{s}_t, \tilde{a}_t) - r_t(s_t, a_t)\right]}_{I_2} + \underbrace{\mathbb{E}\left[J^{\boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}, \mathbf{r}_t} - r_t(\tilde{s}_t, \tilde{a}_t)\right]}_{I_3}.$$
(17)

Note that I_1 is the difference in the average rewards between the two policies $\pi_t, \pi_{t-\tau-1}$ in two different environments $(\mathbf{P}_t, \mathbf{r}_t)$ and $(\mathbf{P}_{t-\tau}, \mathbf{r}_t)$ that share the same reward function. Hence, using Lemma D.10 and Lemma D.7 successively, we get

$$I_{1} \leq \mathbb{E} \left[L_{\pi} \| \pi_{t} - \pi_{t-\tau-1} \|_{2} + L_{P} \| \mathbf{P}_{t} - \mathbf{P}_{t-\tau} \|_{\infty} \right]$$

$$\leq \mathbb{E} \left[L_{\pi} \sum_{i=t-\tau}^{t} \| \pi_{i} - \pi_{i-1} \|_{2} + L_{P} \sum_{i=t-\tau+1}^{t} \| \mathbf{P}_{i} - \mathbf{P}_{i-1} \|_{\infty} \right]$$

$$\leq L_{\pi} B_{2} \beta(\tau+1) + L_{P} \Delta_{P,t-\tau+1,t}, \qquad (18)$$

where $\Delta_{P,t-\tau+1,t} = \sum_{i=t-\tau+1}^{t} \|\mathbf{P}_{i} - \mathbf{P}_{i-1}\|_{\infty}$.

For I_2 , by Lemma D.27 and Lemma D.16 successively, we get

$$I_{2} \leq 2U_{R} \cdot 2d_{TV} \left(P((s_{t}, a_{t}) \in \cdot | \mathcal{F}_{t-\tau}), P((\tilde{s}_{t}, \tilde{a}_{t}) \in \cdot | \mathcal{F}_{t-\tau}) \right)$$

$$\leq 4U_{R} \sqrt{|\mathcal{S}||\mathcal{A}|} \mathbb{E} \left[\sum_{i=t-\tau}^{t} \| \boldsymbol{\pi}_{i} - \boldsymbol{\pi}_{t-\tau-1} \|_{2} \Big| \mathcal{F}_{t-\tau} \right] + 4U_{R} \sum_{i=t-\tau}^{t} \| \mathbf{P}_{i} - \mathbf{P}_{t-\tau} \|_{\infty}$$

$$\leq 4U_{R} \sqrt{|\mathcal{S}||\mathcal{A}|} B_{2} \beta(\tau+1)^{2} + 4U_{R} \tau \Delta_{P,t-\tau+1,t}.$$
(19)

Finally, we bound I_3 using Lemma D.21 as

$$I_3 \le 4U_R m \rho^{\tau}. \tag{20}$$

Plugging the bounds on I_1, I_2, I_3 into Equation (17) and setting $\tau = \tau_H = \min\{i \ge 0 | m\rho^{i-1} \le \min\{\beta, \alpha\}\},\$

$$\sum_{t=nH+\tau_H}^{nH+H-1} \mathbb{E}\left[J_t^{\boldsymbol{\pi}_t} - r_t(s_t, a_t)\right]$$

$$\leq \sum_{t=nH+\tau_{H}}^{nH+H-1} L_{\pi}B_{2}\beta(\tau_{H}+1) + L_{P}\Delta_{P,t-nH-\tau_{H}+1,t} + 4U_{R}\sqrt{|\mathcal{S}||\mathcal{A}|}B_{2}\beta(\tau_{H}+1)^{2} + 4U_{R}\tau_{H}\Delta_{P,t-nH-\tau_{H}+1,t} + 4U_{R}m\rho^{\tau_{H}} \\ \leq (L_{\pi} + 4U_{R}\sqrt{|\mathcal{S}||\mathcal{A}|})B_{2}\beta(\tau_{H}+1)^{2}(H-\tau_{H}) + (4U_{R}+L_{P})(\tau_{H}+1)^{2}\Delta_{P,nH,(n+1)H} + 4U_{R}\beta(H-\tau_{H}).$$

Proposition D.5. For any $n \in [N]$, if Assumption 5.1 holds and $0 < \gamma < 1/2$, then we have the following

$$\sum_{t=nH+\tau_{H}}^{nH+H-1} \mathbb{E}\left[(J_{t}^{\boldsymbol{\pi}_{t}} - \eta_{t})^{2} \right] \leq \frac{4U_{R}^{2}}{\gamma} + 2B_{7}\beta(\tau_{H} + 1)^{2}H + D_{3}\gamma\tau_{H}H + 2B_{8}(\tau_{H} + 1)^{2}\Delta_{P,nH,(n+1)H} + \frac{D_{4}\beta^{2}H}{\gamma^{2}} + \frac{4W_{3}\Delta_{R,nH,(n+1)H}^{2/3}H^{1/3}}{\gamma} + \frac{4W_{4}\Delta_{P,nH,(n+1)H}^{2/3}H^{1/3}}{\gamma}$$

where $D_3 = 4U_R F_6 + 8U_R^2$, $D_4 = 9L_\pi^2 B_2^2$, $\Delta_{R,nH,(n+1)H} = \sum_{t=nH}^{(n+1)H} \|\mathbf{r}_{t+1} - \mathbf{r}_t\|_{\infty}$, $\Delta_{P,nH,(n+1)H} = \sum_{t=nH}^{(n+1)H} \|\mathbf{P}_{t+1} - \mathbf{P}_t\|_{\infty}$, $W_3 = (3)^{1/3} (4U_R^2)^{2/3}$ and $W_4 = (3L_P^2)^{1/3} (4U_R^2)^{2/3}$.

Proof. Recall that $\phi_t := \eta_t - J_t^{\pi_t}$. Using the average reward update equation (line 8 in Algorithm 1), we have

$$\begin{split} \phi_{t+1}^2 &= \left(\eta_t + \gamma(r_t(s_t, a_t) - \eta_t) - J_{t+1}^{\pi_{t+1}}\right)^2 \\ &= \left(\phi_t + J_t^{\pi_t} - J_{t+1}^{\pi_{t+1}} + \gamma(r_t(s_t, a_t) - \eta_t)\right)^2 \\ &\leq \phi_t^2 + 2\gamma\phi_t(r_t(s_t, a_t) - \eta_t) + 2\phi_t(J_t^{\pi_t} - J_{t+1}^{\pi_{t+1}}) + 2(J_t^{\pi} - J_{t+1}^{\pi_{t+1}})^2 + 2\gamma^2(r_t(s_t, a_t) - \eta_t)^2 \\ &= (1 - 2\gamma)\phi_t^2 + 2\gamma\phi_t(r_t(s_t, a_t) - J_t^{\pi_t}) + 2\phi_t(J_t^{\pi_t} - J_{t+1}^{\pi_{t+1}}) \\ &+ 2(J_t^{\pi} - J_{t+1}^{\pi_{t+1}})^2 + 2\gamma^2(r_t(s_t, a_t) - \eta_t)^2 \\ &= (1 - 2\gamma)\phi_t^2 + 2\gamma\Lambda(\pi_t, \mathbf{P}_t, \mathbf{r}_t, \eta_t, O_t) + 2\phi_t(J_t^{\pi_t} - J_{t+1}^{\pi_{t+1}}) \\ &+ 2(J_t^{\pi} - J_{t+1}^{\pi_{t+1}})^2 + 2\gamma^2(r_t(s_t, a_t) - \eta_t)^2 \\ &= (1 - 2\gamma)\phi_t^2 + 2\gamma\Lambda(\pi_t, \mathbf{P}_t, \mathbf{r}_t, \eta_t, O_t) + 2\phi_t(J_t^{\pi_t} - J_{t+1}^{\pi_t}) + 2\phi_t(J_{t+1}^{\pi_t} - J_{t+1}^{\pi_{t+1}}) \\ &+ 2(J_t^{\pi} - J_{t+1}^{\pi_{t+1}})^2 + 2\gamma^2(r_t(s_t, a_t) - \eta_t)^2. \end{split}$$

Rearranging and setting $\tau = \tau_H = \min\{i \ge 0 | m \rho^{i-1} \le \min\{\beta, \alpha\}\}$, we have

$$\sum_{t=nH+\tau_{H}}^{nH+H-1} \mathbb{E}[\phi_{t}^{2}] \leq \underbrace{\sum_{t=nH+\tau_{H}}^{nH+H-1} \frac{\mathbb{E}[\phi_{t}^{2} - \phi_{t+1}^{2}]}{2\gamma}}_{I_{1}} + \underbrace{\sum_{t=nH+\tau_{H}}^{nH+H-1} \mathbb{E}[\Lambda(\boldsymbol{\pi}_{t}, \mathbf{P}_{t}, \mathbf{r}_{t}, \eta_{t}, O_{t})]}_{I_{2}} + \underbrace{\sum_{t=nH+\tau_{H}}^{nH+H-1} \frac{\mathbb{E}[\phi_{t}(J_{t}^{\boldsymbol{\pi}_{t}} - J_{t+1}^{\boldsymbol{\pi}_{t}})]}{I_{3}}}_{I_{3}} + \underbrace{\sum_{t=nH+\tau_{H}}^{nH+H-1} \frac{\mathbb{E}[\phi_{t}(J_{t+1}^{\boldsymbol{\pi}_{t}} - J_{t+1}^{\boldsymbol{\pi}_{t+1}})]}_{I_{4}}}_{I_{5}} + \underbrace{\sum_{t=nH+\tau_{H}}^{nH+H-1} \frac{\mathbb{E}[\phi_{t}(J_{t}^{\boldsymbol{\pi}_{t}} - J_{t+1}^{\boldsymbol{\pi}_{t+1}})]}{I_{6}}}_{I_{6}} + \underbrace{\sum_{t=nH+\tau_{H}}^{nH+H-1} \frac{\mathbb{E}[(J_{t}^{\boldsymbol{\pi}_{t}} - J_{t+1}^{\boldsymbol{\pi}_{t+1}})^{2}]}_{I_{5}}}_{I_{6}} + \underbrace{\sum_{t=nH+\tau_{H}}^{nH+H-1} \gamma \mathbb{E}[(r_{t}(s_{t}, a_{t}) - \eta_{t})^{2}]}_{I_{6}}}_{I_{6}} + \underbrace{\sum_{t=nH+\tau_{H}}^{nH+H-1} \frac{\mathbb{E}[\phi_{t}(J_{t}^{\boldsymbol{\pi}_{t}} - J_{t+1}^{\boldsymbol{\pi}_{t+1}})]}_{I_{6}}}_{I_{6}} + \underbrace{\sum_{t=nH+\tau_{H}}^{nH+H-1} \frac{\mathbb{E}[\phi_{t}(J_{t}^{\boldsymbol{\pi}_{t}} - J_{t+1}^{\boldsymbol{\pi}_{t+1})]}_{I_{6}}}_{I_{6}} + \underbrace{\sum_{t=nH+\tau_{H}}^{nH+H-1} \frac{\mathbb{E}[\phi_{t}(J_{t}^{\boldsymbol{\pi}_{t}} - J_{t+1}^{\boldsymbol{\pi}_{t+1})]}_{I_{6}}}}_{I_{6}} + \underbrace{\sum_{t=nH+\tau_{H}}^{nH+H-1} \frac{\mathbb{E}[\phi_{t}(J_{t}^{\boldsymbol{\pi}_{t}} - J_{t+1}^{\boldsymbol{\pi}_{t+1})]}_{I_{6}}}_{I_{6}} + \underbrace{\sum_{t=nH+\tau_{H}}^{nH+H-1} \frac{\mathbb{E}[\phi_{t}($$

We now analyze each of these terms starting with the first term as

$$I_1 = \frac{\mathbb{E}[\phi_{nH+\tau_H}^2 - \phi_{nH+H}^2]}{2\gamma} \le \frac{2U_R^2}{\gamma}.$$

By Lemma D.15 and the average reward update equation, we have

$$I_2 \le \sum_{t=nH+\tau_H}^{nH+H-1} B_7 \beta (\tau_H+1)^2 + F_6 |\eta_t - \eta_{t-nH-\tau_H}| + F_7 \tau \Delta_{P,t-nH-\tau_H+1,t}$$

$$\leq B_7 \beta (\tau_H + 1)^2 (H - \tau_H) + 2U_R F_6 \gamma \tau_H (H - \tau_H) + B_8 (\tau_H + 1)^2 \Delta_{P, nH, (n+1)H}.$$

By Lemma D.10, we have

$$I_{3} \leq \frac{2U_{R}}{\gamma} \left(\sum_{t=nH+\tau_{H}}^{nH+H-1} \|\mathbf{r}_{t+1} - \mathbf{r}_{t}\|_{\infty} + L_{P} \|\mathbf{P}_{t+1} - \mathbf{P}_{t}\|_{\infty} \right) \leq \frac{2U_{R}\Delta_{R,nH,(n+1)H}}{\gamma} + \frac{2U_{R}\Delta_{P,(n+1)H}}{\gamma}.$$

By Lemma D.10 and Cauchy-Schwartz inequality, we have

$$I_{4} \leq \left(\sum_{t=nH+\tau_{H}}^{nH+H-1} \mathbb{E}[\phi_{t}^{2}]\right)^{1/2} \left(\sum_{t=nH+\tau_{H}}^{nH+H-1} \frac{\mathbb{E}[(J_{t+1}^{\pi_{t+1}} - J_{t}^{\pi_{t}})^{2}]}{\gamma^{2}}\right)^{1/2} \\ \leq \left(\sum_{t=nH+\tau_{H}}^{nH+H-1} \mathbb{E}[\phi_{t}^{2}]\right)^{1/2} \left(\frac{L_{\pi}^{2}B_{2}^{2}\beta^{2}H}{\gamma^{2}}\right)^{1/2}.$$

We now bound I_5 as follows. For timesteps with small changes in the environment, we use Lemma D.10, and for timesteps with large changes in the environment, we use a naive upper bound. Define the set of timesteps $\mathcal{T}_J := \{t : \|\mathbf{r}_{t+1} - \mathbf{r}_t\|_{\infty} \le \delta_R, \|\mathbf{P}_{t+1} - \mathbf{P}_t\|_{\infty} \le \delta_P\}.$

$$I_{5} = \sum_{t=nH+\tau_{H}}^{nH+H-1} \frac{\mathbb{E}\left[(J_{t+1}^{\pi_{t+1}} - J_{t}^{\pi_{t}})^{2}\right]}{\gamma} \leq \sum_{t \in \mathcal{T}_{J}} \frac{\mathbb{E}\left[(J_{t+1}^{\pi_{t+1}} - J_{t}^{\pi_{t}})^{2}\right]}{\gamma} + \sum_{t \notin \mathcal{T}_{J}} \frac{4U_{R}^{2}}{\gamma}$$

$$\stackrel{(a)}{\leq} \sum_{t \in \mathcal{T}_{J}} \frac{3\delta_{R}^{2}}{\gamma} + \frac{3L_{P}^{2}\delta_{P}^{2}}{\gamma} + \frac{3L_{\pi}^{2}B_{2}^{2}\beta^{2}}{\gamma} + \sum_{t \notin \mathcal{T}_{J}} \frac{4U_{R}^{2}}{\gamma}$$

$$\stackrel{(b)}{\leq} \frac{3\delta_{R}^{2}H}{\gamma} + \frac{3L_{P}^{2}\delta_{P}^{2}H}{\gamma} + \frac{3L_{\pi}^{2}B_{2}^{2}\beta^{2}H}{\gamma} + \frac{4U_{R}^{2}\Delta_{R,nH,(n+1)H}}{\gamma\delta_{R}} + \frac{4U_{R}^{2}\Delta_{P,nH,(n+1)H}}{\gamma\delta_{P}}$$

$$\leq \frac{W_{3}\Delta_{R,nH,(n+1)H}^{2/3}}{\gamma} + \frac{W_{4}\Delta_{P,nH,(n+1)H}^{2/3}}{\gamma} + \frac{W_{4}\Delta_{P,nH,(n+1)H}^{2/3}}{\gamma} + \frac{3L_{\pi}^{2}B_{2}^{2}\beta^{2}T}{\gamma}$$

$$(21)$$

where (a) follows from Lemma D.10 and (b) is obtained by choosing $\delta_R = \left(\frac{4U_R^2 \Delta_{R,nH,(n+1)H}}{3T}\right)^{1/3}$ and $\delta_P = \left(\frac{4U_R^2 \Delta_{P,nH,(n+1)H}}{3L_P^2 T}\right)^{1/3}$ and defining $W_3 = (3)^{1/3} (4U_R^2)^{2/3}$, $W_2 = (3L_P^2)^{1/3} (4U_R^2)^{2/3}$.

For the final term, we have

$$I_6 \le 4U_R^2 \gamma (H - \tau_H).$$

Putting everything together, we have

$$\sum_{t=nH+\tau_{H}}^{nH+H-1} \mathbb{E}[\phi_{t}^{2}] \leq \frac{2U_{R}^{2}}{\gamma} + B_{7}\beta(\tau_{H}+1)^{2}(H-\tau_{H}) + 2U_{R}F_{6}\gamma\tau_{H}(H-\tau_{H}) + B_{8}(\tau_{H}+1)^{2}\Delta_{P,nH,(n+1)H} + \frac{2U_{R}\Delta_{P,nH,(n+1)H}}{\gamma} + \frac{2U_{R}\Delta_{P,nH,(n+1)H}}{\gamma} + \left(\sum_{t=nH+\tau_{H}}^{nH+H-1} \mathbb{E}[\phi_{t}^{2}]\right)^{1/2} \left(\frac{L_{\pi}^{2}B_{2}^{2}\beta^{2}H}{\gamma^{2}}\right)^{1/2} + \frac{W_{3}\Delta_{R,nH,(n+1)H}^{2/3}H^{1/3}}{\gamma} + \frac{W_{4}\Delta_{P,nH,(n+1)H}^{2/3}H^{1/3}}{\gamma} + \frac{3L_{\pi}^{2}B_{2}^{2}\beta^{2}H}{\gamma} + 4U_{R}^{2}\gamma(H-\tau_{H}).$$

Now, we use the squaring trick from Section C.3, Wu et al. (2020). The above equation is of the form, $X \le Y + Z\sqrt{X}$. Completing the squares and rearranging, we get $X \le 2Y + Z^2$. Hence,

$$\sum_{t=nH+\tau_H}^{nH+H-1} \mathbb{E}[\phi_t^2] \le \frac{4U_R^2}{\gamma} + 2B_7\beta(\tau_H+1)^2(H-\tau_H) + 4U_RF_6\gamma\tau_H(H-\tau_H) + 2B_8(\tau_H+1)^2\Delta_{P,nH,(n+1)H}$$

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$$+ \frac{4U_R\Delta_{R,nH,(n+1)H}}{\gamma} + \frac{4U_R\Delta_{P,nH,(n+1)H}}{\gamma} + \frac{9L_{\pi}^2 B_2^2 \beta^2 H}{\gamma^2} + \frac{2W_3\Delta_{R,nH,(n+1)H}^{2/3} H^{1/3}}{\gamma} + \frac{2W_4\Delta_{P,nH,(n+1)H}^{2/3} H^{1/3}}{\gamma} + 8U_R^2 \gamma (H - \tau_H) \leq \frac{4U_R^2}{\gamma} + 2B_7 \beta (\tau_H + 1)^2 (H - \tau_H) + 4U_R F_6 \gamma \tau_H (H - \tau_H) + 2B_8 (\tau_H + 1)^2 \Delta_{P,nH,(n+1)H} + \frac{9L_{\pi}^2 B_2^2 \beta^2 H}{\gamma^2} + \frac{4W_3 \Delta_{R,nH,(n+1)H}^{2/3} H^{1/3}}{\gamma} + \frac{4W_4 \Delta_{P,nH,(n+1)H}^{2/3} H^{1/3}}{\gamma} + 8U_R^2 \gamma (H - \tau_H).$$

D.4. Technical Lemmas

D.4.1. ACTOR

Lemma D.6. If Assumption 5.1 holds, for any $t, t' \ge 0$, we have

$$\sum_{s} d^{\pi_{t'}^{\star}, \mathbf{P}_{t'}}(s) \left[\frac{\log Z_t(s)}{\beta} - V_t^{\pi_t}(s) \right] \le \frac{J_{t+1}^{\pi_{t+1}} - J_t^{\pi_t}}{C} + \|\mathbf{Q}_t^{\pi_t} - \mathbf{Q}_t\|_{\infty} + \frac{B_1\beta}{C} + \frac{\|\mathbf{r}_{t+1} - \mathbf{r}_t\|_{\infty}}{C} + \frac{L_P \|\mathbf{P}_{t+1} - \mathbf{P}_t\|_{\infty}}{C}$$

where $Z_t(s) = \sum_{a' \in \mathcal{A}} \pi_t(a'|s) \exp(\beta Q_t(s,a'))$, C is defined in Assumption 5.1 and other constants in Appendix C.

Proof. We have

$$J_{t+1}^{\pi_{t+1}-} = J_{t}^{\pi_{t}}$$

$$= J_{t+1}^{\pi_{t+1}} - J_{t}^{\pi_{t+1}} + J_{t}^{\pi_{t+1}} + J_{t}^{\pi_{t+1}} - J_{t}^{\pi_{t}}$$

$$\stackrel{(a)}{=} J_{t+1}^{\pi_{t+1}} - J_{t}^{\pi_{t+1}} + \sum_{s,a} d^{\pi_{t+1},\mathbf{P}_{t}}(s)\pi_{t+1}(a|s) \left[Q_{t}^{\pi_{t}}(s,a) - V_{t}^{\pi_{t}}(s) + Q_{t}(s,a) - Q_{t}(s,a)\right]$$

$$\stackrel{(b)}{=} J_{t+1}^{\pi_{t+1}} - J_{t}^{\pi_{t+1}} + \sum_{s,a} d^{\pi_{t+1},\mathbf{P}_{t}}(s)\pi_{t+1}(a|s) \left[Q_{t}^{\pi_{t}}(s,a) - V_{t}^{\pi_{t}}(s) + \frac{\log Z_{t}(s)}{\beta} + \frac{1}{\beta} \log \frac{\pi_{t+1}(a|s)}{\pi} (a|s) - Q_{t}(s,a)\right]$$

$$\stackrel{(c)}{=} J_{t+1}^{\pi_{t+1}} - J_{t}^{\pi_{t+1}} + \sum_{s,a} d^{\pi_{t+1},\mathbf{P}_{t}}(s)\pi_{t+1}(a|s) \left[Q_{t}^{\pi_{t}}(s,a) - V_{t}^{\pi_{t}}(s) + \frac{\log Z_{t}(s)}{\beta} - Q_{t}(s,a)\right]$$

$$\stackrel{(c)}{=} J_{t+1}^{\pi_{t+1}} - J_{t}^{\pi_{t+1}} + \sum_{s,a} d^{\pi_{t+1},\mathbf{P}_{t}}(s)\pi_{t+1}(a|s) \left[Q_{t}^{\pi_{t}}(s,a) - V_{t}^{\pi_{t}}(s) + \frac{\log Z_{t}(s)}{\beta} - Q_{t}(s,a)\right]$$

$$\stackrel{(c)}{=} J_{t+1}^{\pi_{t+1}} - J_{t}^{\pi_{t+1}} + \sum_{s,a} d^{\pi_{t+1},\mathbf{P}_{t}}(s)\pi_{t+1}(a|s) \left[Q_{t}^{\pi_{t}}(s,a) - V_{t}^{\pi_{t}}(s) + \frac{\log Z_{t}(s)}{\beta} - Q_{t}(s,a)\right]$$

$$\stackrel{(c)}{=} J_{t+1}^{\pi_{t+1}} - J_{t}^{\pi_{t+1}} + \sum_{s,a} d^{\pi_{t+1},\mathbf{P}_{t}}(s)\pi_{t}(a|s) \left[\frac{\log Z_{t}(s)}{\beta} - Q_{t}(s,a)\right]$$

$$\stackrel{(c)}{=} J_{t+1}^{\pi_{t+1}} - J_{t}^{\pi_{t+1}} + \sum_{s,a} d^{\pi_{t+1},\mathbf{P}_{t}}(s)\pi_{t}(a|s) \left[\frac{\log Z_{t}(s)}{\beta} - Q_{t}(s,a)\right]$$

$$\stackrel{(c)}{=} J_{t+1}^{\pi_{t+1}} - J_{t}^{\pi_{t+1}} + \sum_{s,a} d^{\pi_{t+1},\mathbf{P}_{t}}(s)\pi_{t}(a|s) \left[\frac{\log Z_{t}(s)}{\beta} - Q_{t}(s,a)\right]$$

$$\stackrel{(c)}{=} J_{t+1}^{\pi_{t+1}} - J_{t}^{\pi_{t+1}} + \sum_{s,a} d^{\pi_{t+1},\mathbf{P}_{t}}(s)\pi_{t}(a|s) \left[\frac{\log Z_{t}(s)}{\beta} - Q_{t}(s,a)\right]$$

$$\stackrel{(c)}{=} J_{t+1}^{\pi_{t+1}} - J_{t}^{\pi_{t+1}} + \sum_{s,a} d^{\pi_{t+1},\mathbf{P}_{t}}(s)\pi_{t}(a|s) \left[\frac{\log Z_{t}(s,a)}{\beta} - Q_{t}(s,a)\right]$$

$$\stackrel{(c)}{=} J_{t+1}^{\pi_{t+1}} - J_{t}^{\pi_{t+1}} + \sum_{s,a} d^{\pi_{t+1},\mathbf{P}_{t}}(s)\pi_{t}(a|s) \left[\frac{\log Z_{t}(s,a)}{\beta} - Q_{t}(s,a)\right]$$

$$\stackrel{(c)}{=} J_{t+1}^{\pi_{t+1}} - J_{t}^{\pi_{t+1}} + \sum_{s,a} d^{\pi_{t+1},\mathbf{P}_{t}}(s)\pi_{t}(a|s) \left[\frac{\log Z_{t}(s,a)}{\beta} - Q_{t}(s,a)\right]$$

where (a) follows from Lemma D.8, (b) follows from the actor update (line 10 in Algorithm 1), and (c) is due to the non-negativity of KL-Divergence.

Next, we bound the last two terms in (22). Under Assumption 5.1, we have

$$I_{1} = \sum_{s,a} d^{\boldsymbol{\pi}_{t'}^{\star}, \mathbf{P}_{t'}}(s) \left(\frac{d^{\boldsymbol{\pi}_{t+1}, \mathbf{P}_{t}}(s)}{d^{\boldsymbol{\pi}_{t'}^{\star}, \mathbf{P}_{t'}}(s)} \right) \pi_{t}(a|s) \left[\frac{\log Z_{t}(s)}{\beta} - Q_{t}(s, a) \right]$$

$$\geq C \sum_{s,a} d^{\boldsymbol{\pi}_{t'}^{\star}, \mathbf{P}_{t'}}(s) \pi_t(a|s) \left[\frac{\log Z_t(s)}{\beta} - Q_t(s, a) \right]$$

$$\geq C \sum_s d^{\boldsymbol{\pi}_{t'}^{\star}, \mathbf{P}_{t'}}(s) \left[\frac{\log Z_t(s)}{\beta} - V_t^{\boldsymbol{\pi}_t}(s) \right] + C \sum_{s,a} d^{\boldsymbol{\pi}_{t'}^{\star}, \mathbf{P}_{t'}}(s) \pi_t(a|s) \left[Q_t^{\boldsymbol{\pi}_t}(s, a) - Q_t(s, a) \right]$$
(23)

Further, we have by 1-Lipschitzness of the tabular softmax policy

$$I_2 \ge -2U_Q \sum_{s,a} d^{\pi_{t+1},\mathbf{P}_t}(s)(\pi_{t+1}(a|s) - \pi_t(a|s)) \ge -2U_Q \cdot \beta U_Q \sqrt{|\mathcal{A}|} \ge -B_1\beta.$$
(24)

Plugging the bounds from (23) and (24) into (22) and rearranging, we have

$$\begin{split} &\sum_{s} d^{\pi_{t'}^{\star}, \mathbf{P}_{t'}}(s) \left[\frac{\log Z_{t}(s)}{\beta} - V_{t}^{\pi_{t}}(s) \right] \\ &\leq \frac{J_{t+1}^{\pi_{t+1}} - J_{t}^{\pi_{t}}}{C} + \sum_{s,a} d^{\pi_{t'}^{\star}, \mathbf{P}_{t'}}(s) \pi_{t}(a|s) \left[Q_{t}(s,a) - Q_{t}^{\pi_{t}}(s,a) \right] + \frac{B_{1}\beta}{C} + \frac{J_{t}^{\pi_{t+1}} - J_{t+1}^{\pi_{t+1}}}{C} \\ &\leq \frac{J_{t+1}^{\pi_{t+1}} - J_{t}^{\pi_{t}}}{C} + \|\mathbf{Q}_{t}^{\pi_{t}} - \mathbf{Q}_{t}\|_{\infty} + \frac{B_{1}\beta}{C} + \frac{\|\mathbf{r}_{t+1} - \mathbf{r}_{t}\|_{\infty}}{C} + \frac{L_{P}\|\mathbf{P}_{t+1} - \mathbf{P}_{t}\|_{\infty}}{C} \end{split}$$

where the last inequality follows from Lemma D.10.

Lemma D.7. For $t \ge 0$, policy π_t satisfies

$$\|\boldsymbol{\pi}_{t+1} - \boldsymbol{\pi}_t\|_2 \le B_2\beta$$

where $B_2 = U_Q$.

Proof. By 1-Lipschitzness of the softmax parameterization of the actor (Beck, 2017) and Lemma D.28, we have

$$\|\boldsymbol{\pi}_{t+1} - \boldsymbol{\pi}_t\|_2 \le \|\beta \mathbf{Q}_t\|_2 \le \beta U_Q.$$

Lemma D.8 (Average-Reward Performance Difference Lemma (Murthy & Srikant, 2023)). The average rewards for any two policies π , π' at time t satisfy

$$J_t^{\boldsymbol{\pi}} - J_t^{\boldsymbol{\pi}'} = \sum_{s \in \mathcal{S}} d^{\boldsymbol{\pi}, \mathbf{P}_t}(s) \sum_{a \in \mathcal{A}} \boldsymbol{\pi}(a|s) \left[Q_t^{\boldsymbol{\pi}'}(s, a) - V_t^{\boldsymbol{\pi}'}(s) \right].$$

Lemma D.9. For any $t, t' \ge 0$, it holds that

$$J_t^{\boldsymbol{\pi}_t^\star} - J_{t'}^{\boldsymbol{\pi}_{t'}^\star} \le \|\mathbf{r}_t - \mathbf{r}_{t'}\|_{\infty} + U_Q \|\mathbf{P}_t - \mathbf{P}_{t'}\|_{\infty}$$

where π_t^{\star} represents the optimal policy for MDP $\mathcal{M}_t(\mathcal{S}, \mathcal{A}, \mathbf{P}_t, \mathbf{r}_t)$.

Proof. Consider the linear programming formulation of an MDP $\mathcal{M}(\mathcal{S}, \mathcal{A}, \mathbf{P}, \mathbf{r})$ (Puterman, 2014)

$$\min_{J,V(s)} J$$
such that $J + V(s) \ge r(s,a) + \sum_{s'} P(s'|s,a)V(s') \ \forall s \in \mathcal{S}, a \in \mathcal{A}.$
(25)

If the optimal solution for $\mathcal{M}_{t'}(\mathcal{S}, \mathcal{A}, \mathbf{P}_{t'}, \mathbf{r}_{t'})$ is $J_{t'}^{\star}, \mathbf{V}_{t'}^{\star}$, we have

$$J_{t'}^{\star} \mathbf{1} \geq \mathbf{r}_{t'} + (\mathbf{P}_{t'} - \mathbf{I}) \mathbf{V}_{t'}^{\star}.$$

Now for $\mathcal{M}_t(\mathcal{S}, \mathcal{A}, \mathbf{P}_t, \mathbf{r}_t)$, we know

$$\begin{aligned} J_t^{\star} &\leq \|\mathbf{r}_t + (\mathbf{P}_t - \mathbf{I})\mathbf{V}_{t'}^{\star}\|_{\infty} \\ &\leq \|\mathbf{r}_{t'} + (\mathbf{P}_{t'} - \mathbf{I})\mathbf{V}_{t'}^{\star} + (\mathbf{r}_t - \mathbf{r}_{t'}) + (\mathbf{P}_t - \mathbf{P}_{t'})\mathbf{V}_{t'}^{\star}\|_{\infty} \\ &\leq \|J_{t'}^{\star}\mathbf{1}\|_{\infty} + \|\mathbf{r}_t - \mathbf{r}_{t'}\|_{\infty} + \|(\mathbf{P}_t - \mathbf{P}_{t'})\mathbf{V}_{t'}^{\star}\|_{\infty}. \end{aligned}$$

Hence, we have

$$J_t^{\star} - J_{t'}^{\star} \leq \|\mathbf{r}_t - \mathbf{r}_{t'}\|_{\infty} + \|\left(\mathbf{P}_t - \mathbf{P}_{t'}\right)\mathbf{V}_{t'}^{\star}\|_{\infty}$$
$$J_t^{\boldsymbol{\pi}_t^{\star}} - J_{t'}^{\boldsymbol{\pi}_{t'}^{\star}} \leq \|\mathbf{r}_t - \mathbf{r}_{t'}\|_{\infty} + U_Q \|\mathbf{P}_t - \mathbf{P}_{t'}\|_{\infty}.$$

Lemma D.10. There exist constants $L_{\pi} = 4U_R(M+1)\sqrt{|\mathcal{S}||\mathcal{A}|}$ and $L_P = 4U_RM$ such that for all policies π, π' and timesteps t, t', it holds that

$$J_t^{\pi} - J_{t'}^{\pi'} \le L_{\pi} \|\pi - \pi'\|_2 + \|\mathbf{r}_t - \mathbf{r}_{t'}\|_{\infty} + L_P \|\mathbf{P}_t - \mathbf{P}_{t'}\|_{\infty}.$$

Proof.

$$J_t^{\pi} - J_{t'}^{\pi'} = \underbrace{J_t^{\pi} - J_t^{\pi'}}_{T_1} + \underbrace{J_t^{\pi'} - J_{t'}^{\pi'}}_{T_2}, \tag{26}$$

where T_1 is the difference in the average rewards between two policies π , π' under the same environments (\mathbf{r}_t , \mathbf{P}_t), while T_2 is the difference in the average rewards with the same policy π' , but under two different environments (\mathbf{r}_t , \mathbf{P}_t) and ($\mathbf{r}_{t'}$, $\mathbf{P}_{t'}$).

$$T_{1} = J_{t}^{\boldsymbol{\pi}} - J_{t}^{\boldsymbol{\pi}'} = \mathbb{E}_{s \sim d^{\boldsymbol{\pi}, \mathbf{P}_{t}}, a \sim \boldsymbol{\pi}, s' \sim d^{\boldsymbol{\pi}', \mathbf{P}_{t}}, a' \sim \boldsymbol{\pi}'} [r_{t}(s, a) - r_{t}(s', a')]$$

$$= 4U_{R}d_{TV} \left(d^{\boldsymbol{\pi}, \mathbf{P}_{t}} \otimes \boldsymbol{\pi}, d^{\boldsymbol{\pi}', \mathbf{P}_{t}} \otimes \boldsymbol{\pi}' \right)$$

$$\overset{(a)}{\leq} L_{\boldsymbol{\pi}} \| \boldsymbol{\pi} - \boldsymbol{\pi}' \|_{2}, \qquad (27)$$

where (a) follows from Lemma D.26, where \otimes denotes the Kronecker product. Next, we bound T_2 .

$$T_{2} = J_{t}^{\pi'} - J_{t'}^{\pi'} = \sum_{s,a} d^{\pi',\mathbf{P}_{t}}(s)\pi'(a|s)r_{t}(s,a) - d^{\pi',\mathbf{P}_{t'}}(s)\pi'(a|s)r_{t'}(s,a)$$

$$\leq \sum_{s,a} \left| d^{\pi',\mathbf{P}_{t}}(s)\pi'(a|s)r_{t}(s,a) - d^{\pi',\mathbf{P}_{t}}(s)\pi'(a|s)r_{t'}(s,a) \right|$$

$$+ \sum_{s,a} \left| d^{\pi',\mathbf{P}_{t}}(s)\pi'(a|s)r_{t'}(s,a) - d^{\pi',\mathbf{P}_{t'}}(s)\pi'(a|s)r_{t'}(s,a) \right|$$

$$\leq \|\mathbf{r}_{t} - \mathbf{r}_{t'}\|_{\infty} + 4U_{R}d_{TV}(d^{\pi',\mathbf{P}_{t}} \otimes \pi', d^{\pi',\mathbf{P}_{t'}} \otimes \pi')$$

$$\stackrel{(b)}{\leq} \|\mathbf{r}_{t} - \mathbf{r}_{t'}\|_{\infty} + L_{P}\|\mathbf{P}_{t} - \mathbf{P}_{t'}\|_{\infty}$$
(28)

where (b) also follows from Lemma D.26. Substituting the bounds from (27) and (28) into (26), we get the result.

D.4.2. CRITIC

Lemma D.11. For any policies π, π' , we have

$$\|\mathbf{Q}_t^{\boldsymbol{\pi}} - \mathbf{Q}_t^{\boldsymbol{\pi}'}\|_2 \le G_{\boldsymbol{\pi}} \|\boldsymbol{\pi} - \boldsymbol{\pi}'\|_2$$

where $G_{\pi} = 2U_Q \sqrt{|\mathcal{S}||\mathcal{A}|}$.

Proof.

$$Q_{t}^{\boldsymbol{\pi}}(s,a) \stackrel{(a)}{=} r_{t}(s,a) - J_{t}^{\boldsymbol{\pi}} + \mathbb{E}_{s' \sim P_{t}(\cdot|s,a)} \left[V_{t}^{\boldsymbol{\pi}}(s') \right]$$

$$\Rightarrow \frac{\partial Q_{t}^{\boldsymbol{\pi}}(s,a)}{\partial \boldsymbol{\pi}} = \frac{-\partial J_{t}^{\boldsymbol{\pi}}}{\partial \boldsymbol{\pi}} + \sum_{s' \in \mathcal{S}} P_{t}(s'|s,a) \frac{\partial V_{t}^{\boldsymbol{\pi}}(s')}{\partial \boldsymbol{\pi}}$$

$$\left\| \frac{\partial Q_{t}^{\boldsymbol{\pi}}(s,a)}{\partial \boldsymbol{\pi}} \right\|_{2} \leq 2 \left\| \frac{\partial J_{t}^{\boldsymbol{\pi}}}{\partial \boldsymbol{\pi}} \right\|_{2}$$
(29)

$$\left\|\frac{\partial Q_t^{\boldsymbol{\pi}}(s,a)}{\partial \boldsymbol{\pi}}\right\|_2 \stackrel{(b)}{\leq} 2 \left\|d^{\boldsymbol{\pi},\mathbf{P}_t}(s)Q_t^{\boldsymbol{\pi}}(s,a)\right\|_2 \leq 2U_Q$$
(30)

It follows from mean-value theorem that

$$\begin{aligned} |Q_t^{\pi}(s,a) - Q_t^{\pi'}(s,a)| &\leq 2U_Q \|\pi - \pi'\|_2, \text{ for all } s, a \\ \Rightarrow \|\mathbf{Q}_t^{\pi} - \mathbf{Q}_t^{\pi'}\|_2 &\leq G_{\pi} \|\pi - \pi'\|_2, \end{aligned}$$

where (a) is by using the Bellman equation, and (b) follows from Policy Gradient Theorem (Sutton & Barto, 2018) and Lemma D.28.

Lemma D.12. For any timesteps $t, t' \ge 0$, we have

$$\|\Pi_E \left[\mathbf{Q}_t^{\boldsymbol{\pi}} - \mathbf{Q}_{t'}^{\boldsymbol{\pi}}\right]\|_2 \le G_R \|\mathbf{r}_t - \mathbf{r}_{t'}\|_{\infty} + G_P \|\mathbf{P}_t - \mathbf{P}_{t'}\|_{\infty}$$

where $G_R = 2\lambda^{-1}\sqrt{|\mathcal{S}||\mathcal{A}|}$ and $G_P = (\lambda^{-1}L_P + 4U_R\lambda^{-1}M + 4U_R\lambda^{-2}(M+1))\sqrt{|\mathcal{S}||\mathcal{A}|}$.

Proof. Recall the diagonal matrix $D^{\pi,\mathbf{P}_t} = diag\left(d^{\pi,\mathbf{P}_t}(s)\pi(a|s)\right)$, where $d^{\pi,\mathbf{P}_t}(\cdot)$ denotes the stationary distribution induced over the states, while 1 denotes the all ones vector. E denotes the subspace orthogonal to the all ones vector. Pseudo-inverse of a matrix is represented by \mathbf{X}^{\dagger} . Now, we have

$$\begin{split} \|\Pi_{E} \left[\mathbf{Q}_{t}^{\pi} - \mathbf{Q}_{t}^{\pi}\right] \|_{2} \stackrel{(a)}{\leq} \|(\bar{\mathbf{A}}^{\pi,\mathbf{P}_{t}})^{\dagger} D^{\pi,\mathbf{P}_{t}}(J_{t}^{\pi}\mathbf{1} - \mathbf{r}_{t}) - (\bar{\mathbf{A}}^{\pi,\mathbf{P}_{t'}})^{\dagger} D^{\pi,\mathbf{P}_{t'}}(J_{t'}^{\pi}\mathbf{1} - \mathbf{r}_{t'}) \|_{2} \\ &\leq \|(\bar{\mathbf{A}}^{\pi,\mathbf{P}_{t}})^{\dagger} D^{\pi,\mathbf{P}_{t'}}(J_{t'}^{\pi}\mathbf{1} - \mathbf{r}_{t'}) - (\bar{\mathbf{A}}^{\pi,\mathbf{P}_{t'}})^{\dagger} D^{\pi,\mathbf{P}_{t'}}(J_{t'}^{\pi}\mathbf{1} - \mathbf{r}_{t'}) \|_{2} \\ &\leq \|(\bar{\mathbf{A}}^{\pi,\mathbf{P}_{t}})^{\dagger}\|_{2} \left(\|D^{\pi,\mathbf{P}_{t}}J_{t}^{\pi}\mathbf{1} - D^{\pi,\mathbf{P}_{t'}}J_{t'}^{\pi}\mathbf{1}\|_{2} + \|D^{\pi,\mathbf{P}_{t'}}(J_{t'}^{\pi}\mathbf{1} - \mathbf{r}_{t'})\|_{2} \\ &+ \|(\bar{\mathbf{A}}^{\pi,\mathbf{P}_{t}})^{\dagger}\|_{2} \left(\|D^{\pi,\mathbf{P}_{t}}J_{t'}^{\pi}\mathbf{1} - D^{\pi,\mathbf{P}_{t'}}J_{t'}^{\pi}\mathbf{1}\|_{2} + \|D^{\pi,\mathbf{P}_{t}}\mathbf{1} - \mathbf{r}_{t'})\|_{2} \\ &+ \|(\bar{\mathbf{A}}^{\pi,\mathbf{P}_{t}})^{\dagger} D^{\pi,\mathbf{P}_{t'}}(J_{t'}^{\pi}\mathbf{1} - \mathbf{r}_{t'}) - (\bar{\mathbf{A}}^{\pi,\mathbf{P}_{t'}})^{\dagger} D^{\pi,\mathbf{P}_{t'}}(J_{t'}^{\pi}\mathbf{1} - \mathbf{r}_{t'})\|_{2} \\ &+ \|(\bar{\mathbf{A}}^{\pi,\mathbf{P}_{t}})^{\dagger} D^{\pi,\mathbf{P}_{t'}}(J_{t'}^{\pi}\mathbf{1} - \mathbf{r}_{t'}) - (\bar{\mathbf{A}}^{\pi,\mathbf{P}_{t'}})^{\dagger} D^{\pi,\mathbf{P}_{t'}}(J_{t'}^{\pi}\mathbf{1} - \mathbf{r}_{t'})\|_{2} \\ &\leq \lambda^{-1} \left(\sqrt{|S||A|}\|D^{\pi,\mathbf{P}_{t}}\|_{2}|J_{t}^{\pi}-J_{t'}^{\pi}| + \|D^{\pi,\mathbf{P}_{t}} - D^{\pi,\mathbf{P}_{t'}}\|_{2} \cdot U_{R}\sqrt{|S||A|} + \|D^{\pi,\mathbf{P}_{t}}\mathbf{r} - D^{\pi,\mathbf{P}_{t'}}\mathbf{r}_{t'}\|_{2}\right) \\ &+ \|(\bar{\mathbf{A}}^{\pi,\mathbf{P}_{t}})^{\dagger} D^{\pi,\mathbf{P}_{t'}}(J_{t'}^{\pi}\mathbf{1} - \mathbf{r}_{t'}) - (\bar{\mathbf{A}}^{\pi,\mathbf{P}_{t'}})^{\dagger} D^{\pi,\mathbf{P}_{t'}}(J_{t'}^{\pi}\mathbf{1} - \mathbf{r}_{t'})\|_{2} \end{aligned}$$
(31)
$$&+ \lambda^{-1}\|D^{\pi,\mathbf{P}_{t'}}(J_{t'}^{\pi}\mathbf{1} - \mathbf{r}_{t'}) - (\bar{\mathbf{A}}^{\pi,\mathbf{P}_{t'}})^{\dagger} D^{\pi,\mathbf{P}_{t'}}(J_{t'}^{\pi}\mathbf{1} - \mathbf{r}_{t'})\|_{2} \\ &\leq \lambda^{-1}\sqrt{|S||A|} \left(\|\mathbf{r}_{t}-\mathbf{r}_{t'}\|_{\infty} + L_{P}\|\mathbf{P}_{t} - \mathbf{P}_{t'}\|_{\infty} + 2U_{R}M\|\mathbf{P}_{t} - \mathbf{P}_{t'}\|_{\infty}\right) \\ &+ \lambda^{-1}\|D^{\pi,\mathbf{P}_{t'}}(J_{t'}^{\pi}\mathbf{1} - \mathbf{r}_{t'}) - (\bar{\mathbf{A}}^{\pi,\mathbf{P}_{t'}})^{\dagger} D^{\pi,\mathbf{P}_{t'}}(J_{t'}^{\pi}\mathbf{1} - \mathbf{r}_{t'})\|_{2} \end{aligned}$$
(32)
$$&+ \|(\bar{\mathbf{A}}^{\pi,\mathbf{P}_{t})^{\dagger} D^{\pi,\mathbf{P}_{t'}}(J_{t'}^{\pi}\mathbf{1} - \mathbf{r}_{t'}) - (\bar{\mathbf{A}}^{\pi,\mathbf{P}_{t'}})^{\dagger} D^{\pi,\mathbf{P}_{t'}}(J_{t'}^{\pi}\mathbf{1} - \mathbf{r}_{t'})\|_{2} \end{aligned}$$

$$\leq \lambda^{-1} \sqrt{|\mathcal{S}||\mathcal{A}|} (2\|\mathbf{r}_{t} - \mathbf{r}_{t'}\|_{\infty} + L_{P} \|\mathbf{P}_{t} - \mathbf{P}_{t'}\|_{\infty} + 4U_{R}M \|\mathbf{P}_{t} - \mathbf{P}_{t'}\|_{\infty})$$

$$+ \|(\bar{\mathbf{A}}^{\boldsymbol{\pi},\mathbf{P}_{t}})^{\dagger} - (\bar{\mathbf{A}}^{\boldsymbol{\pi},\mathbf{P}_{t'}})^{\dagger}\|_{2} \cdot 2U_{R}$$

$$\stackrel{(f)}{\leq} \lambda^{-1} \sqrt{|\mathcal{S}||\mathcal{A}|} (2\|\mathbf{r}_{t} - \mathbf{r}_{t'}\|_{\infty} + L_{P} \|\mathbf{P}_{t} - \mathbf{P}_{t'}\|_{\infty} + 4U_{R}M \|\mathbf{P}_{t} - \mathbf{P}_{t'}\|_{\infty})$$

$$+ 2U_{R}\lambda^{-2} \|\bar{\mathbf{A}}^{\boldsymbol{\pi},\mathbf{P}_{t}} - \bar{\mathbf{A}}^{\boldsymbol{\pi},\mathbf{P}_{t'}}\|_{2}$$

$$\stackrel{(g)}{\leq} \lambda^{-1} \sqrt{|\mathcal{S}||\mathcal{A}|} (2\|\mathbf{r}_{t} - \mathbf{r}_{t'}\|_{\infty} + L_{P} \|\mathbf{P}_{t} - \mathbf{P}_{t'}\|_{\infty} + 4U_{R}M \|\mathbf{P}_{t} - \mathbf{P}_{t'}\|_{\infty})$$

$$+ 2U_{R}\lambda^{-2} \cdot 2(M+1) \sqrt{|\mathcal{S}||\mathcal{A}|} \|\mathbf{P}_{t} - \mathbf{P}_{t'}\|_{\infty}$$

$$\leq G_{R} \|\mathbf{r}_{t} - \mathbf{r}_{t'}\|_{\infty} + G_{P} \|\mathbf{P}_{t} - \mathbf{P}_{t'}\|_{\infty}$$

where (a) is because $\mathbb{E}[\mathbf{r}(O) - \mathbf{J}(O) + \mathbf{A}(O)\mathbf{Q}^{\pi}] = 0$ (see TD limiting point (4) in Section 5.1) (b) is from Lemma 5.2, (c) is by Lemma D.10, (d) is due to Lemma D.26, (e) is using the same process as the last step for the second term, (f) is because $\|\mathbf{X}^{\dagger} - \mathbf{Y}^{\dagger}\|_{2} \le \|\mathbf{X}^{\dagger}(\mathbf{X} - \mathbf{Y})\mathbf{Y}^{\dagger}\|_{2} \le \|\mathbf{X}^{\dagger}\|_{2} \|\mathbf{X} - \mathbf{Y}\|_{2} \|\mathbf{Y}^{\dagger}\|_{2}$ and (g) is by Lemma D.28 and Lemma D.26. \Box

Lemma D.13. For any $t \ge 0$, we have

$$\|\Pi_E \left[\mathbf{Q}_{t+1}^{\pi_{t+1}} - \mathbf{Q}_t^{\pi_t} \right] \|_2 \le G_R \|\mathbf{r}_{t+1} - \mathbf{r}_t\|_{\infty} + G_P \|\mathbf{P}_{t+1} - \mathbf{P}_t\|_{\infty} + G_{\pi} B_2 \beta.$$

See Appendix C for constants.

Proof.

$$\|\Pi_{E} \left[\mathbf{Q}_{t+1}^{\boldsymbol{\pi}_{t+1}} - \mathbf{Q}_{t}^{\boldsymbol{\pi}_{t}} \right] \|_{2} \leq \|\Pi_{E} \left[\mathbf{Q}_{t+1}^{\boldsymbol{\pi}_{t+1}} - \mathbf{Q}_{t}^{\boldsymbol{\pi}_{t+1}} \right] \|_{2} + \|\Pi_{E} \left[\mathbf{Q}_{t}^{\boldsymbol{\pi}_{t+1}} - \mathbf{Q}_{t}^{\boldsymbol{\pi}_{t}} \right] \|_{2}$$

$$\stackrel{(a)}{\leq} G_{R} \|\mathbf{r}_{t+1} - \mathbf{r}_{t}\|_{\infty} + G_{P} \|\mathbf{P}_{t+1} - \mathbf{P}_{t}\|_{\infty} + G_{\boldsymbol{\pi}} \|\boldsymbol{\pi}_{t+1} - \boldsymbol{\pi}_{t}\|_{2}$$

$$\stackrel{(b)}{\leq} G_{R} \|\mathbf{r}_{t+1} - \mathbf{r}_{t}\|_{\infty} + G_{P} \|\mathbf{P}_{t+1} - \mathbf{P}_{t}\|_{\infty} + G_{\boldsymbol{\pi}} B_{2}\beta$$

where (a) is by Lemma D.12 and Lemma D.11 and (b) is from Lemma D.7.

Lemma D.14. If Assumption 5.1 holds, for any $t > \tau$, we have

$$\mathbb{E}\left[\Gamma(\boldsymbol{\pi}_t, \mathbf{P}_t, \mathbf{r}_t, \boldsymbol{\psi}_t, O_t)\right] \le B_3 \beta(\tau+1)^2 + B_4 \alpha \tau + B_5 \Delta_{R, t-\tau+1, t} + B_6 \tau \Delta_{P, t-\tau+1, t}$$

where $B_3 = (F_{1\pi} + F_2 G_{\pi} + F_3 \sqrt{|\mathcal{S}||\mathcal{A}|} + F_4) B_2$, $B_4 = F_2 (2U_R + 2U_Q)$, $B_5 = F_2 G_R$ and $B_6 = F_{1\mathbf{P}} + F_2 G_P + F_3$, $\Delta_{R,t-\tau+1,t} = \sum_{i=t-\tau+1}^t \|\mathbf{r}_i - \mathbf{r}_{i-1}\|_{\infty}$ and $\Delta_{P,t-\tau+1,t} = \sum_{i=t-\tau+1}^t \|\mathbf{P}_i - \mathbf{P}_{i-1}\|_{\infty}$.

Proof. Recall from Appendix B, the definition

$$\Gamma(\boldsymbol{\pi}, \mathbf{P}, \mathbf{r}, \boldsymbol{\psi}, O) = \boldsymbol{\psi}^{\top} \left(\mathbf{r}(O) - \mathbf{J}^{\boldsymbol{\pi}, \mathbf{P}, \mathbf{r}}(O) + \mathbf{A}(O) \mathbf{Q}^{\boldsymbol{\pi}, \mathbf{P}, \mathbf{r}} \right) + \boldsymbol{\psi}^{\top} \left(\mathbf{A}(O) - \bar{\mathbf{A}}^{\boldsymbol{\pi}, \mathbf{P}} \right) \boldsymbol{\psi}.$$

We first decompose $\Gamma(\cdot)$ into the following four terms

/ \

$$\mathbb{E}\left[\Gamma(\boldsymbol{\pi}_{t}, \mathbf{P}_{t}, \mathbf{r}_{t}, \boldsymbol{\psi}_{t}, O_{t})\right] \leq \underbrace{\mathbb{E}\left[\Gamma(\boldsymbol{\pi}_{t}, \mathbf{P}_{t}, \mathbf{r}_{t}, \boldsymbol{\psi}_{t}, O_{t}) - \Gamma(\boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}, \mathbf{r}_{t}, \boldsymbol{\psi}_{t}, O_{t})\right]}_{I_{1}}_{I_{1}} \\ + \underbrace{\mathbb{E}\left[\Gamma(\boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}, \mathbf{r}_{t}, \boldsymbol{\psi}_{t}, O_{t}) - \Gamma(\boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}, \mathbf{r}_{t}, \boldsymbol{\psi}_{t-\tau}, O_{t})\right]}_{I_{2}}_{I_{3}} \\ + \underbrace{\mathbb{E}\left[\Gamma(\boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}, \mathbf{r}_{t}, \boldsymbol{\psi}_{t-\tau}, O_{t}) - \Gamma(\boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}, \mathbf{r}_{t}, \boldsymbol{\psi}_{t-\tau}, \tilde{O}_{t})\right]}_{I_{3}}_{I_{3}} \\ + \underbrace{\mathbb{E}\left[\Gamma(\boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}, \mathbf{r}_{t}, \boldsymbol{\psi}_{t-\tau}, \tilde{O}_{t})\right]}_{I_{4}}.$$

We now bound each term as follows.

$$I_1 \stackrel{(a)}{\leq} F_{1\boldsymbol{\pi}} \mathbb{E}\left[\|\boldsymbol{\pi}_t - \boldsymbol{\pi}_{t-\tau-1}\|_2 \right] + F_{1\mathbf{P}} \|\mathbf{P}_t - \mathbf{P}_{t-\tau}\|_{\infty}$$

$$\leq F_{1\pi} \mathbb{E}\left[\sum_{i=t-\tau}^{t} \|\boldsymbol{\pi}_{i} - \boldsymbol{\pi}_{i-1}\|_{2}\right] + F_{1\mathbf{P}} \sum_{i=t-\tau+1}^{t} \|\mathbf{P}_{i} - \mathbf{P}_{i-1}\|_{\infty}$$

$$\stackrel{(b)}{\leq} F_{1\pi} B_{2}\beta(\tau+1) + F_{1\mathbf{P}} \Delta_{P,t-\tau+1,t}$$

where (a) is by Lemma D.17 and (b) is due to Lemma D.16. For the second term, we have

$$I_{2} \stackrel{(c)}{\leq} F_{2}\mathbb{E}\left[\|\psi_{t} - \psi_{t-\tau}\right]\|_{2} \leq F_{2}\mathbb{E}\left[\sum_{i=t-\tau+1}^{t} \|\psi_{i} - \psi_{i-1}\|_{2}\right]$$

$$\stackrel{(d)}{\leq} F_{2}\left[\sum_{i=t-\tau+1}^{t} (2U_{R} + 2U_{Q})\alpha + G_{R}\|\mathbf{r}_{i} - \mathbf{r}_{i-1}\|_{\infty} + G_{P}\|\mathbf{P}_{i} - \mathbf{P}_{i-1}\|_{\infty} + G_{\pi}B_{2}\beta\right]$$

$$\leq F_{2}(2U_{R} + 2U_{Q})\alpha\tau + F_{2}G_{R}\Delta_{R,t-\tau+1,t} + F_{2}G_{P}\Delta_{P,t-\tau+1,t} + F_{2}G_{\pi}B_{2}\beta\tau$$

where (c) is by Lemma D.18, (d) follows from Lemma D.28, $\|\mathbf{Q}_{t+1} - \mathbf{Q}_t\|_2 \leq \beta(U_R + U_Q)$ by the critic update equation (9), Lemma D.13 and Lemma D.16. We also define $\Delta_{R,t-\tau+1,t} = \sum_{i=t-\tau+1}^t \|\mathbf{r}_i - \mathbf{r}_{i-1}\|_\infty$ and $\Delta_{P,t-\tau+1,t} = \sum_{i=t-\tau+1}^t \|\mathbf{P}_i - \mathbf{P}_{i-1}\|_\infty$.

For the third term, we have

$$I_{3} \stackrel{(e)}{\leq} F_{3}\sqrt{|\mathcal{S}||\mathcal{A}|}\mathbb{E}\left[\sum_{i=t-\tau}^{t} \|\boldsymbol{\pi}_{i} - \boldsymbol{\pi}_{t-\tau-1}\|_{2}\Big|\mathcal{F}_{t-\tau}\right] + F_{3}\sum_{i=t-\tau}^{t} \|\mathbf{P}_{i} - \mathbf{P}_{t-\tau}\|_{\infty}$$

$$\stackrel{(f)}{\leq} F_{3}\sqrt{|\mathcal{S}||\mathcal{A}|}B_{2}\beta(\tau+1)^{2} + F_{3}\tau\Delta_{P,t-\tau+1,t}.$$

where (e) is due to Lemma D.19 and (f) follows from Lemma D.16. For the last term, by Lemma D.20, we have

$$I_4 \leq F_4 m \rho^{\tau}$$

We get the final result by putting all the four terms together.

D.4.3. AVERAGE REWARD ESTIMATION

Lemma D.15. If Assumption 5.1 holds, for any $t > \tau$, we have

$$\mathbb{E}[\Lambda(\boldsymbol{\pi}_t, \mathbf{P}_t, \mathbf{r}_t, \eta_t, O_t)] \le B_7 \beta(\tau+1)^2 + F_6 |\eta_t - \eta_{t-\tau}| + B_8 \tau \Delta_{P, t-\tau+1, t}$$

where $B_7 = (F_5 L_{\pi} + F_7 \sqrt{|\mathcal{S}||\mathcal{A}|} + F_8) B_2$, $B_8 = F_7 + F_5 L_P$ and $\Delta_{P,t-\tau+1,t} = \sum_{i=t-\tau+1}^t \|\mathbf{P}_i - \mathbf{P}_{i-1}\|_{\infty}$.

Proof. Recall from Appendix B, the definition

$$\Lambda(\boldsymbol{\pi}, \mathbf{P}, \mathbf{r}, \eta, O) = (\eta - J^{\boldsymbol{\pi}, \mathbf{P}, \mathbf{r}})(r(s, a) - J^{\boldsymbol{\pi}, \mathbf{P}, \mathbf{r}})$$

We first decompose $\Lambda(\boldsymbol{\pi}_t, \mathbf{P}_t, \mathbf{r}_t, \eta_t, O_t)$ into the following four terms

$$\mathbb{E}[\Lambda(\boldsymbol{\pi}_{t}, \mathbf{P}_{t}, \mathbf{r}_{t}, \eta_{t}, O_{t})] = \underbrace{\mathbb{E}[\Lambda(\boldsymbol{\pi}_{t}, \mathbf{P}_{t}, \mathbf{r}_{t}, \eta_{t}, O_{t}) - \Lambda(\boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}, \mathbf{r}_{t}, \eta_{t}, O_{t})]}_{I_{1}}_{I_{1}} + \underbrace{\mathbb{E}[\Lambda(\boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}, \mathbf{r}_{t}, \eta_{t}, O_{t}) - \Lambda(\boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}, \mathbf{r}_{t}, \eta_{t-\tau}, O_{t})]}_{I_{2}}_{I_{2}} + \underbrace{\mathbb{E}[\Lambda(\boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}, \mathbf{r}_{t}, \eta_{t-\tau}, O_{t}) - \Lambda(\boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}, \mathbf{r}_{t}, \eta_{t-\tau}, \tilde{O}_{t})]}_{I_{3}}_{I_{3}} + \underbrace{\mathbb{E}[\Lambda(\boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}, \mathbf{r}_{t}, \eta_{t-\tau}, \tilde{O}_{t})]}_{I_{4}}.$$

We now bound each term as follows.

$$I_{1} \stackrel{(a)}{\leq} F_{5}L_{\pi}\mathbb{E}\left[\|\pi_{t} - \pi_{t-\tau-1}\|_{2}\right] + F_{5}L_{P}\|\mathbf{P}_{t} - \mathbf{P}_{t-\tau}\|_{\infty}$$

$$\leq F_{5}L_{\pi}\mathbb{E}\left[\sum_{i=t-\tau}^{t}\|\pi_{i} - \pi_{i-1}\|_{2}\right] + F_{5}L_{P}\sum_{i=t-\tau+1}^{t}\|\mathbf{P}_{i} - \mathbf{P}_{i-1}\|_{\infty}$$

$$\stackrel{(b)}{\leq} F_{5}L_{\pi}B_{2}\beta(\tau+1) + F_{5}L_{P}\Delta_{P,t-\tau+1,t}$$

where (a) follows from Lemma D.22, and (b) is due to Lemma D.16. For the second term I_2 , we have

$$I_2 \stackrel{(c)}{\leq} F_6 |\eta_t - \eta_{t-\tau}|$$

where (c) is by Lemma D.23. For the third term I_3 , we have

$$I_{3} \stackrel{(d)}{\leq} F_{7}\sqrt{|\mathcal{S}||\mathcal{A}|}\mathbb{E}\left[\sum_{i=t-\tau}^{t} \|\boldsymbol{\pi}_{i} - \boldsymbol{\pi}_{t-\tau-1}\|_{2}\Big|\mathcal{F}_{t-\tau}\right] + F_{7}\sum_{i=t-\tau}^{t} \|\mathbf{P}_{i} - \mathbf{P}_{t-\tau}\|_{\infty}$$

$$\stackrel{(e)}{\leq} F_{7}\sqrt{|\mathcal{S}||\mathcal{A}|}B_{2}\beta(\tau+1)^{2} + F_{7}\Delta_{P,t-\tau+1,t}.$$

where (d) is due to Lemma D.24 and (e) follows from Lemma D.16. For the last term, by Lemma D.25, we have

$$I_4 \leq F_8 m \rho^{\tau}$$
.

We get the final result by putting all the four terms together.

D.5. Auxiliary Lemmas

D.5.1. ACTOR

Lemma D.16. For any timesteps $t > \tau > 0$, the policies generated by Algorithm 1 satisfy

$$\sum_{i=t-\tau}^{t} \|\boldsymbol{\pi}_{i} - \boldsymbol{\pi}_{t-\tau-1}\|_{2} \le B_{2}\beta(\tau+1)^{2}$$

and reward and transition probability matrices satisfy

$$\sum_{i=t-\tau}^{t} \|\mathbf{r}_{i} - \mathbf{r}_{t-\tau}\|_{\infty} \leq \tau \sum_{i=t-\tau+1}^{t} \|\mathbf{r}_{i} - \mathbf{r}_{i-1}\|_{\infty}$$
$$\sum_{i=t-\tau}^{t} \|\mathbf{P}_{i} - \mathbf{P}_{t-\tau}\|_{\infty} \leq \tau \sum_{i=t-\tau+1}^{t} \|\mathbf{P}_{i} - \mathbf{P}_{i-1}\|_{\infty}.$$

Proof. By triangle inequality, we have

$$\sum_{i=t-\tau}^{t} \|\boldsymbol{\pi}_{i} - \boldsymbol{\pi}_{t-\tau-1}\|_{2} \leq \sum_{i=t-\tau}^{t} \|\sum_{j=t-\tau}^{i} \boldsymbol{\pi}_{j} - \boldsymbol{\pi}_{j-1}\|_{2}$$
$$\leq \sum_{i=t-\tau}^{t} \sum_{j=t-\tau}^{i} \|\boldsymbol{\pi}_{j} - \boldsymbol{\pi}_{j-1}\|_{2}$$
$$\stackrel{(a)}{\leq} B_{2}\beta(\tau+1)^{2}$$

where (a) is by Lemma D.7. The rest follow similarly using triangle inequality.

D.5.2. CRITIC

Lemma D.17. For any $\pi, \pi', \mathbf{P}, \mathbf{P}', \mathbf{r}, \psi$ and O = (s, a, s', a'), we have

$$|\Gamma(\boldsymbol{\pi}, \mathbf{P}, \mathbf{r}, \boldsymbol{\psi}, O) - \Gamma(\boldsymbol{\pi}', \mathbf{P}', \mathbf{r}, \boldsymbol{\psi}, O)| \le F_{1\boldsymbol{\pi}} \|\boldsymbol{\pi} - \boldsymbol{\pi}'\|_2 + F_{1\mathbf{P}} \|\mathbf{P} - \mathbf{P}'\|_{\infty}$$

where $F_{1\pi} = 2U_Q L_{\pi} + 4U_Q G_{\pi} + 8U_Q^2 (M+2) |\mathcal{S}||\mathcal{A}|$, $F_{1\mathbf{P}} = 2U_Q L_P + 4U_Q G_P + 8U_Q^2 (M+1) \sqrt{|\mathcal{S}||\mathcal{A}|}$.

$$\begin{split} |\Gamma(\pi, \mathbf{P}, \mathbf{r}, \psi, O) - \Gamma(\pi', \mathbf{P}', \mathbf{r}, \psi, O)| \\ &= |\psi^{\top}(\mathbf{J}^{\pi', \mathbf{P}', \mathbf{r}}(O) - \mathbf{J}^{\pi, \mathbf{P}, \mathbf{r}}(O)) + \psi^{\top} \mathbf{A}(O) \left(\mathbf{Q}^{\pi, \mathbf{P}, \mathbf{r}} - \mathbf{Q}^{\pi', \mathbf{P}', \mathbf{r}}\right) + \psi^{\top} \left(\bar{\mathbf{A}}^{\pi', \mathbf{P}'} - \bar{\mathbf{A}}^{\pi, \mathbf{P}}\right) \psi| \\ \stackrel{(a)}{\leq} \|\psi\|_{\infty} |J^{\pi', \mathbf{P}', \mathbf{r}} - J^{\pi, \mathbf{P}, \mathbf{r}}| + \|\psi\|_{2} \|\mathbf{A}(O)\|_{2} \left\|\mathbf{Q}^{\pi, \mathbf{P}, \mathbf{r}} - \mathbf{Q}^{\pi', \mathbf{P}', \mathbf{r}}\right\|_{2} \\ &+ \|\psi\|_{\infty} \left\|\bar{\mathbf{A}}^{\pi', \mathbf{P}'} - \bar{\mathbf{A}}^{\pi, \mathbf{P}}\right\|_{\infty} \|\psi\|_{1} \\ \stackrel{(b)}{\leq} 2U_{Q}L_{\pi} \|\pi - \pi'\|_{2} + 2U_{Q}L_{P} \|\mathbf{P} - \mathbf{P}'\|_{\infty} + \|\psi\|_{2} \|\mathbf{A}(O)\|_{2} \left\|\mathbf{Q}^{\pi, \mathbf{P}, \mathbf{r}} - \mathbf{Q}^{\pi', \mathbf{P}', \mathbf{r}}\right\|_{2} \\ &+ \|\psi\|_{\infty} \left\|\bar{\mathbf{A}}^{\pi', \mathbf{P}'} - \bar{\mathbf{A}}^{\pi, \mathbf{P}}\right\|_{\infty} \|\psi\|_{1} \\ \stackrel{(c)}{\leq} 2U_{Q}L_{\pi} \|\pi - \pi'\|_{2} + 2U_{Q}L_{P} \|\mathbf{P} - \mathbf{P}'\|_{\infty} + 4U_{Q} \cdot G_{\pi} \|\pi - \pi'\|_{2} + 4U_{Q}G_{P} \|\mathbf{P} - \mathbf{P}'\|_{\infty} \\ &+ \|\psi\|_{\infty} \left\|\bar{\mathbf{A}}^{\pi', \mathbf{P}'} - \bar{\mathbf{A}}^{\pi, \mathbf{P}}\right\|_{\infty} \|\psi\|_{1} \\ \stackrel{(d)}{\leq} 2U_{Q}L_{\pi} \|\pi - \pi'\|_{2} + 2U_{Q}L_{P} \|\mathbf{P} - \mathbf{P}'\|_{\infty} + 4U_{Q}G_{\pi} \|\pi - \pi'\|_{2} + 4U_{Q}G_{P} \|\mathbf{P} - \mathbf{P}'\|_{\infty} \\ &+ 2U_{Q} \cdot 2d_{TV} \left(d^{\pi', \mathbf{P}'} \otimes \pi' \otimes \mathbf{P}' \otimes \pi', d^{\pi, \mathbf{P}} \otimes \pi \otimes \mathbf{P} \otimes \pi\right) \cdot 2U_{Q} \sqrt{|S||\mathcal{A}|} \\ \stackrel{(e)}{\leq} 2U_{Q}L_{\pi} \|\pi - \pi'\|_{2} + 2U_{Q}L_{P} \|\mathbf{P} - \mathbf{P}'\|_{\infty} + 4U_{Q}G_{\pi} \|\pi - \pi'\|_{2} + 4U_{Q}G_{P} \|\mathbf{P} - \mathbf{P}'\|_{\infty} \\ &+ 8U_{Q}^{2}(M+2)|S||\mathcal{A}||\|\pi - \pi'\|_{2} + 8U_{Q}^{2}(M+1) \sqrt{|S||\mathcal{A}||\|\pi - \pi'||_{\infty}} \end{aligned}$$

where (a) follows from Holder's inequality; (b) is due to Lemma D.10; (c) is by Lemma D.13 and Lemma D.28 ($\|\mathbf{A}(O)\|_1 \le 1$); (d) is by Lemma D.28 and (e) uses Lemma D.26.

Lemma D.18. For any π , **P**, **r**, ψ , ψ' and O = (s, a, s', a'), we have

$$|\Gamma(\boldsymbol{\pi}, \mathbf{P}, \mathbf{r}, \boldsymbol{\psi}, O) - \Gamma(\boldsymbol{\pi}, \mathbf{P}, \mathbf{r}, \boldsymbol{\psi}', O)| \le F_2 \|\boldsymbol{\psi} - \boldsymbol{\psi}'\|_2$$

where $F_2 = 2U_R + 18U_Q$.

Proof.

$$\begin{aligned} &|\Gamma(\boldsymbol{\pi}, \mathbf{P}, \mathbf{r}, \boldsymbol{\psi}, O) - \Gamma(\boldsymbol{\pi}, \mathbf{P}, \mathbf{r}, \boldsymbol{\psi}', O)| \\ &\leq \left(\|\mathbf{r}(O)\|_2 + \|\mathbf{J}^{\boldsymbol{\pi}, \mathbf{P}, \mathbf{r}}(O)\|_2 + \|\mathbf{A}(O)\|_2 \|\mathbf{Q}^{\boldsymbol{\pi}, \mathbf{P}, \mathbf{r}}\|_2 \right) \|\boldsymbol{\psi} - \boldsymbol{\psi}'\|_2 \\ &+ \|\mathbf{A}(O) - \bar{\mathbf{A}}^{\boldsymbol{\pi}, \mathbf{P}}\|_2 \|\boldsymbol{\psi} - \boldsymbol{\psi}'\|_2 (\|\boldsymbol{\psi}\|_2 + \|\boldsymbol{\psi}'\|_2) \\ &\leq (2U_R + 18U_Q) \|\boldsymbol{\psi} - \boldsymbol{\psi}'\|_2. \end{aligned}$$

Lemma D.19. Consider an observation from the original Markov chain by $O_t = (s_t, a_t, s_{t+1}, a_{t+1})$ and auxiliary Markov chain by $\tilde{O}_t = (\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1}, \tilde{a}_{t+1})$. Conditioned on $\mathcal{F}_{t-\tau} = \{s_{t-\tau}, \pi_{t-\tau-1}, \mathbf{P}_{t-\tau}\}$, we have

$$\mathbb{E}\left[\Gamma(\boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}, \mathbf{r}_{t}, \boldsymbol{\psi}_{t-\tau}, O_{t}) - \Gamma(\boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}, \mathbf{r}_{t}, \boldsymbol{\psi}_{t-\tau}, \tilde{O}_{t}) \big| \mathcal{F}_{t-\tau}\right] \\ \leq F_{3}\sqrt{|\mathcal{S}||\mathcal{A}|}\mathbb{E}\left[\sum_{i=t-\tau}^{t} \|\boldsymbol{\pi}_{i} - \boldsymbol{\pi}_{t-\tau-1}\|_{2} \Big| \mathcal{F}_{t-\tau}\right] + F_{3}\sum_{i=t-\tau}^{t} \|\mathbf{P}_{i} - \mathbf{P}_{t-\tau}\|_{\infty}$$

where $F_3 = 16U_R U_Q + 24U_Q^2 \sqrt{|S||A|}$.

Proof. Consider the original and auxiliary Markov chains whose construction is described in Appendix B.

$$\begin{split} & \mathbb{E}\left[\Gamma(\pi_{t-\tau-1},\mathbf{P}_{t-\tau},\mathbf{r}_{t},\psi_{t-\tau},O_{t})-\Gamma(\pi_{t-\tau-1},\mathbf{P}_{t-\tau},\mathbf{r}_{t},\psi_{t-\tau},\tilde{O}_{t})\right|\mathcal{F}_{t-\tau}\right] \\ &=\psi_{t-\tau}^{\top}\mathbb{E}\left[\mathbf{r}_{t}(O_{t})-\mathbf{r}_{t}(\tilde{O}_{t})+\mathbf{J}^{\pi_{t-\tau-1},\mathbf{P}_{t-\tau},\mathbf{r}_{t}}(\tilde{O}_{t})-\mathbf{J}^{\pi_{t-\tau-1},\mathbf{P}_{t-\tau},\mathbf{r}_{t}},(O_{t})\right|\mathcal{F}_{t-\tau}\right] \\ &+\psi_{t-\tau}^{\top}\mathbb{E}\left[\left(\mathbf{A}(O_{t})-\mathbf{A}(\tilde{O}_{t})\right)\mathbf{Q}^{\pi_{t-\tau-1},\mathbf{P}_{t-\tau},\mathbf{r}_{t}}\right|\mathcal{F}_{t-\tau}\right] \\ &+\psi_{t-\tau}^{\top}\mathbb{E}\left[\left(\mathbf{A}(O_{t})-\mathbf{A}(\tilde{O}_{t})\right)|\mathcal{F}_{t-\tau}\right]\psi_{t-\tau} \\ &\leq \|\psi_{t-\tau}\|_{\infty}\left\|\mathbb{E}\left[\mathbf{r}_{t}(O_{t})-\mathbf{r}_{t}(\tilde{O}_{t})+\mathbf{J}_{t}^{\pi_{t-\tau-1}}(\tilde{O}_{t})-\mathbf{J}_{t}^{\pi_{t-\tau-1}}(O_{t})|\mathcal{F}_{t-\tau}\right]\right\|_{1} \\ &+\|\psi_{t-\tau}\|_{\infty}\left\|\mathbb{E}\left[\mathbf{A}(O_{t})-\mathbf{A}(\tilde{O}_{t})|\mathcal{F}_{t-\tau}\right]\right\|_{1}\left\|\mathbf{Q}^{\pi_{t-\tau-1},\mathbf{P}_{t-\tau},\mathbf{r}_{t}}\right\|_{1} \\ &+\|\psi_{t-\tau}\|_{\infty}\left\|\mathbb{E}\left[\mathbf{A}(O_{t})-\mathbf{A}(\tilde{O}_{t})|\mathcal{F}_{t-\tau}\right]\right\|_{1}\left\|\psi_{t-\tau}\right\|_{1} \\ &\leq 2U_{Q}\cdot 4U_{R}\cdot 2d_{TV}\left(P(O_{t}\in\cdot|\mathcal{F}_{t-\tau}),P(\tilde{O}_{t}\in\cdot|\mathcal{F}_{t-\tau})\right)\cdot U_{Q}\sqrt{|S||\mathcal{A}|} \\ &+2U_{Q}\cdot 4d_{TV}\left(P(O_{t}\in\cdot|\mathcal{F}_{t-\tau}),P(\tilde{O}_{t}\in\cdot|\mathcal{F}_{t-\tau})\right)\cdot 2U_{Q}\sqrt{|S||\mathcal{A}|} \\ &\leq (16U_{R}U_{Q}+24U_{Q}^{2}\sqrt{|S||\mathcal{A}|})\left(\sqrt{|S||\mathcal{A}|}\mathbb{E}\left[\sum_{i=t-\tau}^{t}\|\pi_{i}-\pi_{t-\tau-1}\|_{2}\left|\mathcal{F}_{t-\tau}\right] + \sum_{i=t-\tau}^{t}\|\mathbf{P}_{i}-\mathbf{P}_{t-\tau}\|_{\infty}\right) \\ &\text{ pe last inequality is from Lemma D.27. \\ \end{array}$$

where the last inequality is from Lemma D.27.

Lemma D.20. Consider an observation from the original Markov chain by $O_t = (s_t, a_t, s_{t+1}, a_{t+1})$ and auxiliary Markov chain by $\tilde{O}_t = (\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1}, \tilde{a}_{t+1})$. Conditioned on $\mathcal{F}_{t-\tau} = \{s_{t-\tau}, \pi_{t-\tau-1}, \mathbf{P}_{t-\tau}\}$, we have

$$\mathbb{E}\left[\Gamma(\boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}, \mathbf{r}_t, \boldsymbol{\psi}_{t-\tau}, \tilde{O}_t) \middle| \mathcal{F}_{t-\tau}\right] \leq F_4 m \rho^{\tau}$$

where $F_4 = 8U_R U_Q + 24U_Q^2 \sqrt{|\mathcal{S}||\mathcal{A}|}$.

Proof. Consider the original and auxiliary Markov chains whose construction is described in Appendix B. Also, consider the observation tuple $O'_t = (s'_t, a'_t, s'_{t+1}, a'_{t+1})$ where $s'_t \sim d^{\pi_{t-\tau-1}, \mathbf{P}_{t-\tau}}(\cdot), a'_t \sim \pi_{t-\tau-1}(\cdot|s'_t), s'_{t+1} \sim \mathbf{P}_{t-\tau}(\cdot|s'_t, a'_t)$ and $a'_{t+1} \sim \pi_{t-\tau-1}(\cdot|s'_{t+1})$. From the definition of $\Gamma(\cdot)$ and the TD limit point equation (4), it follows that

$$\mathbb{E}\left[\Gamma(\boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}, \mathbf{r}_{t}, \boldsymbol{\psi}_{t-\tau}, O_{t}') \middle| \mathcal{F}_{t-\tau}\right] = 0$$

Hence, we have

$$\begin{split} & \mathbb{E}\left[\Gamma(\boldsymbol{\pi}_{t-\tau-1},\mathbf{P}_{t-\tau},\mathbf{r}_{t},\boldsymbol{\psi}_{t-\tau},\tilde{O}_{t})\big|\mathcal{F}_{t-\tau}\right] \\ & \leq \mathbb{E}\left[\Gamma(\boldsymbol{\pi}_{t-\tau-1},\mathbf{P}_{t-\tau},\mathbf{r}_{t},\boldsymbol{\psi}_{t-\tau},\tilde{O}_{t}) - \Gamma(\boldsymbol{\pi}_{t-\tau-1},\mathbf{P}_{t-\tau},\mathbf{r}_{t},\boldsymbol{\psi}_{t-\tau},O'_{t})\big|\mathcal{F}_{t-\tau}\right] \\ & \leq \|\boldsymbol{\psi}_{t-\tau}\|_{\infty} \left\|\mathbb{E}\left[\mathbf{r}_{t}(\tilde{O}_{t}) - \mathbf{J}^{\boldsymbol{\pi}_{t-\tau-1},\mathbf{P}_{t-\tau},\mathbf{r}_{t}}(\tilde{O}_{t}) - \mathbf{r}_{t}(O'_{t}) + \mathbf{J}^{\boldsymbol{\pi}_{t-\tau-1},\mathbf{P}_{t-\tau},\mathbf{r}_{t}}(O'_{t})\big|\mathcal{F}_{t-\tau}\right]\right\|_{1} \\ & + \|\boldsymbol{\psi}_{t-\tau}\|_{\infty} \left\|\mathbb{E}\left[\left(\mathbf{A}(\tilde{O}_{t}) - \mathbf{A}(O'_{t})\right)\mathbf{Q}^{\boldsymbol{\pi}_{t-\tau-1},\mathbf{P}_{t-\tau},\mathbf{r}_{t}}\big|\mathcal{F}_{t-\tau}\right]\right\|_{1} \\ & + \|\boldsymbol{\psi}_{t-\tau}\|_{\infty} \left\|\mathbb{E}\left[\left(\mathbf{A}(\tilde{O}_{t}) - \mathbf{A}(O'_{t})\right)\boldsymbol{\psi}_{t-\tau}\big|\mathcal{F}_{t-\tau}\right]\right\|_{1} \\ & \leq 2U_{Q} \cdot 4U_{R} \cdot 2d_{TV}\left(P(\tilde{O}_{t} \in \cdot|\mathcal{F}_{t-\tau}), P(O'_{t} \in \cdot|\mathcal{F}_{t-\tau})\right) \\ & + 2U_{Q} \cdot 4d_{TV}\left(P(\tilde{O}_{t} \in \cdot|\mathcal{F}_{t-\tau}), P(O'_{t} \in \cdot|\mathcal{F}_{t-\tau})\right) \cdot U_{Q}\sqrt{|\mathcal{S}||\mathcal{A}|} \\ & + 2U_{Q} \cdot 4d_{TV}\left(P(\tilde{O}_{t} \in \cdot|\mathcal{F}_{t-\tau}), P(O'_{t} \in \cdot|\mathcal{F}_{t-\tau})\right) \cdot 2U_{Q}\sqrt{|\mathcal{S}||\mathcal{A}|} \end{split}$$

$$= F_{4} \sum_{s,a,s',a'} |P(\tilde{s}_{t} = s | \mathcal{F}_{t-\tau}) \pi_{t-\tau-1}(a | s) P_{t-\tau}(s' | s, a) \pi_{t-\tau-1}(a' | s') - P(s'_{t} = s | \mathcal{F}_{t-\tau}) \pi_{t-\tau-1}(a | s) P_{t-\tau}(s' | s, a) \pi_{t-\tau-1}(a' | s')| = F_{4} \sum_{s,a,s',a'} \pi_{t-\tau-1}(a | s) P(s' | s, a) \pi_{t-\tau-1}(a' | s') |P(\tilde{s}_{t} = s | \mathcal{F}_{t-\tau}) - P(s'_{t} = s | \mathcal{F}_{t-\tau})| = F_{4} \sum_{s} |P(\tilde{s}_{t} = s | \mathcal{F}_{t-\tau}) - P(s'_{t} = s | \mathcal{F}_{t-\tau})| \leq F_{4} m \rho^{\tau}$$

where the last inequality follows from Assumption 5.1.

D.5.3. AVERAGE REWARD ESTIMATION

Lemma D.21. Consider an observation from the original Markov chain by $O_t = (s_t, a_t, s'_t, a'_t)$ and auxiliary Markov chain by $\tilde{O}_t = (\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1}, \tilde{a}_{t+1})$. Conditioned on $\mathcal{F}_{t-\tau} = \{s_{t-\tau}, \pi_{t-\tau-1}, \mathbf{P}_{t-\tau}\}$, we have

$$\mathbb{E}\left[J^{\boldsymbol{\pi}_{t-\tau-1},\mathbf{P}_{t-\tau},\mathbf{r}_t} - r_t(\tilde{s}_t,\tilde{a}_t)|\mathcal{F}_{t-\tau}\right] \le 4U_R m \rho^2$$

where $J^{\pi_{t-\tau-1},\mathbf{P}_{t-\tau},\mathbf{r}_t} = \sum_{s,a} d^{\pi_{t-\tau-1},\mathbf{P}_{t-\tau}}(s) \pi_{t-\tau-1}(a|s) r_t(s,a).$

Proof. Consider the observation tuple $O'_t = (s'_t, a'_t, s'_{t+1}, a'_{t+1})$ where $s'_t \sim d^{\pi_{t-\tau-1}, \mathbf{P}_{t-\tau}}(\cdot), a'_t \sim \pi_{t-\tau-1}(\cdot|s'_t), s'_{t+1} \sim \mathbf{P}_{t-\tau}(\cdot|s'_t, a'_t)$ and $a'_{t+1} \sim \pi_{t-\tau-1}(\cdot|s'_{t+1})$. Then, by definition of $J^{\pi_{t-\tau-1}, \mathbf{P}_{t-\tau}, \mathbf{r}_t}$, we have

$$\mathbb{E}\left[J^{\boldsymbol{\pi}_{t-\tau-1},\mathbf{P}_{t-\tau},\mathbf{r}_t} - r_t(s'_t,a'_t)|\mathcal{F}_{t-\tau}\right] = 0$$

Hence, we have

$$\mathbb{E}\left[J^{\boldsymbol{\pi}_{t-\tau-1},\mathbf{P}_{t-\tau},\mathbf{r}_{t}} - r_{t}(\tilde{s}_{t},\tilde{a}_{t})|\mathcal{F}_{t-\tau}\right] \\
= \mathbb{E}\left[J^{\boldsymbol{\pi}_{t-\tau-1},\mathbf{P}_{t-\tau},\mathbf{r}_{t}} - r_{t}(s'_{t},a'_{t}) - r_{t}(\tilde{s}_{t},\tilde{a}_{t}) + r_{t}(s'_{t},a'_{t})|\mathcal{F}_{t-\tau}\right] \\
= \mathbb{E}\left[r_{t}(s'_{t},a'_{t}) - r_{t}(\tilde{s}_{t},\tilde{a}_{t})|\mathcal{F}_{t-\tau}\right] \\
\leq 2U_{R} \cdot 2d_{TV} \left(d^{\boldsymbol{\pi}_{t-\tau-1},\mathbf{P}_{t-\tau}} \otimes \boldsymbol{\pi}_{t-\tau-1}, P((\tilde{s}_{t},\tilde{a}_{t}) \in \cdot|\mathcal{F}_{t-\tau})\right) \\
\overset{(a)}{\leq} 4U_{R}d_{TV} \left(d^{\boldsymbol{\pi}_{t-\tau-1},\mathbf{P}_{t-\tau}}, P(\tilde{s}_{t} \in \cdot|\mathcal{F}_{t-\tau})\right) \\
\overset{(b)}{\leq} 4U_{R}m\rho^{\tau}$$

where (a) follows from Lemma B.1 in (Wu et al., 2020) and (b) is by Assumption 5.1.

Lemma D.22. For any $\pi, \pi', \mathbf{P}, \mathbf{P}', \mathbf{r}, \eta$, and O = (s, a, s', a'), we have

$$|\Lambda(\boldsymbol{\pi}, \mathbf{P}, \mathbf{r}, \eta, O) - \Lambda(\boldsymbol{\pi}', \mathbf{P}', \mathbf{r}, \eta, O)| \le F_5 L_{\boldsymbol{\pi}} \|\boldsymbol{\pi} - \boldsymbol{\pi}'\|_2 + F_5 L_P \|\mathbf{P} - \mathbf{P}'\|_{\infty},$$

where $F_5 = 4U_R$.

Proof.

$$\begin{aligned} |\Lambda(\boldsymbol{\pi}, \mathbf{P}, \mathbf{r}, \eta, O) - \Lambda(\boldsymbol{\pi}', \mathbf{P}', \mathbf{r}, \eta, O)| \\ &\leq |(\eta - J^{\boldsymbol{\pi}, \mathbf{P}, \mathbf{r}})(r(s, a) - J^{\boldsymbol{\pi}, \mathbf{P}, \mathbf{r}}) - (\eta - J^{\boldsymbol{\pi}', \mathbf{P}', \mathbf{r}})(r(s, a) - J^{\boldsymbol{\pi}', \mathbf{P}', \mathbf{r}})| \\ &\leq |(\eta - J^{\boldsymbol{\pi}, \mathbf{P}, \mathbf{r}})(r(s, a) - J^{\boldsymbol{\pi}, \mathbf{P}, \mathbf{r}}) - (\eta - J^{\boldsymbol{\pi}, \mathbf{P}, \mathbf{r}})(r(s, a) - J^{\boldsymbol{\pi}', \mathbf{P}', \mathbf{r}})| \\ &+ |(\eta - J^{\boldsymbol{\pi}, \mathbf{P}, \mathbf{r}})(r(s, a) - J^{\boldsymbol{\pi}', \mathbf{P}', \mathbf{r}}) - (\eta - J^{\boldsymbol{\pi}', \mathbf{P}', \mathbf{r}})(r(s, a) - J^{\boldsymbol{\pi}', \mathbf{P}', \mathbf{r}})| \\ &\leq 4U_R |J^{\boldsymbol{\pi}, \mathbf{P}, \mathbf{r}} - J^{\boldsymbol{\pi}', \mathbf{P}', \mathbf{r}}| \stackrel{(a)}{\leq} 4U_R L_{\boldsymbol{\pi}} ||\boldsymbol{\pi} - \boldsymbol{\pi}'||_2 + 4U_R L_P ||\mathbf{P} - \mathbf{P}'||_{\infty} \end{aligned}$$

where (a) follows from Lemma D.10.

Lemma D.23. For any π , **P**, **r**, η , η' and O = (s, a, s', a'), we have

$$|\Lambda(\boldsymbol{\pi}, \mathbf{P}, \mathbf{r}, \eta, O) - \Lambda(\boldsymbol{\pi}, \mathbf{P}, \mathbf{r}, \eta', O)| \le F_6 |\eta - \eta'|$$

where $F_6 = 2U_R$.

Proof. Recall the definition of $\Lambda(\cdot)$ in Appendix B. It is straightforward to see that

$$|\Lambda(\boldsymbol{\pi}, \mathbf{P}, \mathbf{r}, \eta, O) - \Lambda(\boldsymbol{\pi}, \mathbf{P}, \mathbf{r}, \eta', O)| \le 2U_R |\eta - \eta'|$$

Lemma D.24. Consider an observation from the original Markov chain by $O_t = (s_t, a_t, s_{t+1}, a_{t+1})$ and auxiliary Markov chain by $\tilde{O}_t = (\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1}, \tilde{a}_{t+1})$. Conditioned on $\mathcal{F}_{t-\tau} = \{s_{t-\tau}, \pi_{t-\tau-1}, \mathbf{P}_{t-\tau}\}$, we have

$$\mathbb{E}\left[\Lambda(\boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}, \mathbf{r}_{t}, \eta_{t-\tau}, O_{t}) - \Lambda(\boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}, \mathbf{r}_{t}, \eta_{t-\tau}, \tilde{O}_{t}) \middle| \mathcal{F}_{t-\tau} \right] \\ \leq F_{7} \sqrt{|\mathcal{S}||\mathcal{A}|} \mathbb{E}\left[\sum_{i=t-\tau}^{t} \|\boldsymbol{\pi}_{i} - \boldsymbol{\pi}_{t-\tau-1}\|_{2} \middle| \mathcal{F}_{t-\tau} \right] + F_{7} \sum_{i=t-\tau}^{t} \|\mathbf{P}_{i} - \mathbf{P}_{t-\tau}\|_{\infty}$$

where $F_7 = 8U_B^2$.

Proof.

$$\mathbb{E}\left[\Lambda(\boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}, \mathbf{r}_{t}, \eta_{t-\tau}, O_{t}) - \Lambda(\boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}, \mathbf{r}_{t}, \eta_{t-\tau}, \tilde{O}_{t}) \middle| \mathcal{F}_{t-\tau} \right] \\
= (\eta_{t-\tau} - J^{\boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}, \mathbf{r}_{t}}) \mathbb{E}\left[r_{t}(s_{t}, a_{t}) - r_{t}(\tilde{s}_{t}, \tilde{a}_{t}) \middle| \mathcal{F}_{t-\tau} \right] \\
\leq 2U_{R} \cdot 4U_{R} d_{TV} \left(P(O_{t} \in \cdot |\mathcal{F}_{t-\tau}), P(\tilde{O}_{t} \in \cdot |\mathcal{F}_{t-\tau}) \right) \\
\stackrel{(a)}{\leq} F_{7} \sqrt{|\mathcal{S}||\mathcal{A}|} \mathbb{E}\left[\sum_{i=t-\tau}^{t} \|\boldsymbol{\pi}_{i} - \boldsymbol{\pi}_{t-\tau-1}\|_{2} \middle| \mathcal{F}_{t-\tau} \right] + F_{7} \sum_{i=t-\tau}^{t} \|\mathbf{P}_{i} - \mathbf{P}_{t-\tau}\|_{\infty}$$

where (a) follows from Lemma D.27.

Lemma D.25. Consider an observation from the original Markov chain by $O_t = (s_t, a_t, s_{t+1}, a_{t+1})$ and auxiliary Markov chain by $\tilde{O}_t = (\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1}, \tilde{a}_{t+1})$. Conditioned on $\mathcal{F}_{t-\tau} = \{s_{t-\tau}, \pi_{t-\tau-1}, \mathbf{P}_{t-\tau}\}$, we have

$$\mathbb{E}\left[\Lambda(\boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}, \mathbf{r}_t, \eta_{t-\tau}, \tilde{O}_t) \middle| \mathcal{F}_{t-\tau}\right] \leq F_8 m \rho^{\tau}$$

where $F_8 = 8U_R^2$.

 $\begin{array}{l} \textit{Proof. Consider the observation tuple } O_t' = (s_t', a_t', s_{t+1}', a_{t+1}') \text{ where } s_t' \sim d^{\pi_{t-\tau-1}, \mathbf{P}_{t-\tau}}(\cdot), a_t' \sim \pi_{t-\tau-1}(\cdot | s_t'), s_{t+1}' \sim \mathbf{P}_{t-\tau}(\cdot | s_t', a_t') \text{ and } a_{t+1}' \sim \pi_{t-\tau-1}(\cdot | s_{t+1}'). \end{array}$

We know

$$\mathbb{E}\left[\Lambda(\boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}, \mathbf{r}_t, \eta_{t-\tau}, O_t') \middle| \mathcal{F}_{t-\tau}\right] = 0.$$

Hence, we have

$$\mathbb{E}\left[\Lambda(\boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}, \mathbf{r}_{t}, \eta_{t-\tau}, \tilde{O}_{t}) \middle| \mathcal{F}_{t-\tau}\right] \\
= \mathbb{E}\left[\Lambda(\boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}, \mathbf{r}_{t}, \eta_{t-\tau}, \tilde{O}_{t}) \middle| - \Lambda(\boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}, \mathbf{r}_{t}, \eta_{t-\tau}, O'_{t}) \middle| \mathcal{F}_{t-\tau}\right] \\
= \mathbb{E}\left[(\eta_{t-\tau} - J^{\boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}, \mathbf{r}_{t}})(r_{t}(\tilde{s}_{t}, \tilde{a}_{t}) - r_{t}(s'_{t}, a'_{t})) \middle| \mathcal{F}_{t-\tau}\right] \\
\leq 2U_{R} \cdot 4U_{R}d_{TV} \left(d^{\boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}} \otimes \boldsymbol{\pi}_{t-\tau-1}, P((\tilde{s}_{t}, \tilde{a}_{t}) \in \cdot | \mathcal{F}_{t-\tau})\right) \\
\overset{(a)}{\leq} 2U_{R} \cdot 4U_{R}d_{TV} \left(d^{\boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}}, P(\tilde{s}_{t} \in \cdot | \mathcal{F}_{t-\tau})\right) \\
\overset{(b)}{\leq} 8U_{R}^{2}m\rho^{\tau}$$

where (a) follows from Lemma B.1 in (Wu et al., 2020) and (b) is by Assumption 5.1.

D.6. Preliminary Lemmas

Lemma D.26. For any policies π , π' and transition probabilities matrices \mathbf{P} , \mathbf{P}' , it holds that

$$d_{TV}\left(d^{\boldsymbol{\pi},\mathbf{P}},d^{\boldsymbol{\pi}',\mathbf{P}'}\right) \leq M\sqrt{|\mathcal{S}||\mathcal{A}|} ||\boldsymbol{\pi}-\boldsymbol{\pi}'||_{2} + M||\mathbf{P}-\mathbf{P}'||_{\infty},$$
$$d_{TV}\left(d^{\boldsymbol{\pi},\mathbf{P}}\otimes\boldsymbol{\pi},d^{\boldsymbol{\pi}',\mathbf{P}'}\otimes\boldsymbol{\pi}'\right) \leq (M+1)\sqrt{|\mathcal{S}||\mathcal{A}|} ||\boldsymbol{\pi}-\boldsymbol{\pi}'||_{2} + M||\mathbf{P}-\mathbf{P}'||_{\infty},$$
$$d_{TV}\left(d^{\boldsymbol{\pi},\mathbf{P}}\otimes\boldsymbol{\pi}\otimes\mathbf{P},d^{\boldsymbol{\pi}',\mathbf{P}'}\otimes\boldsymbol{\pi}'\otimes\mathbf{P}'\right) \leq (M+1)\sqrt{|\mathcal{S}||\mathcal{A}|} ||\boldsymbol{\pi}-\boldsymbol{\pi}'||_{2} + (M+1)||\mathbf{P}-\mathbf{P}'||_{\infty},$$
$$d_{TV}\left(d^{\boldsymbol{\pi},\mathbf{P}}\otimes\boldsymbol{\pi}\otimes\mathbf{P}\otimes\boldsymbol{\pi},d^{\boldsymbol{\pi}',\mathbf{P}'}\otimes\boldsymbol{\pi}'\otimes\mathbf{P}'\otimes\boldsymbol{\pi}'\right) \leq (M+2)\sqrt{|\mathcal{S}||\mathcal{A}|} ||\boldsymbol{\pi}-\boldsymbol{\pi}'||_{2} + (M+1)||\mathbf{P}-\mathbf{P}'||_{\infty},$$

where \otimes denotes the Kronecker product, and $M := \left(\lceil \log_{\rho} m^{-1} \rceil + \frac{1}{1-\rho} \right)$.

Proof. Recall that $d^{\pi, \mathbf{P}}(\cdot)$ is the stationary distribution induced over the states by a Markov chain with transition probabilities **P** following policy π . Define the matrices $\mathbf{K}, \mathbf{K}' \in \mathbb{R}^{|\mathcal{S}| \times |\mathcal{S}|}$ such that $\mathbf{K}(s, s') = \sum_{a \in \mathcal{A}} P(s'|s, a)\pi(a|s)$ and $\mathbf{K}'(s, s') = \sum_{a \in \mathcal{A}} P'(s'|s, a)\pi'(a|s)$. Further denote the total variation norm as $||\cdot||_{TV}$. Note that $||\mathbf{P}-\mathbf{P}'||_{\infty} = \max_{s,a} \sum_{s'} |P(s'|s, a) - P'(s'|s, a)|$.

From Theorem 3.1 in (Mitrophanov, 2005), we have,

$$\begin{split} d_{TV}\left(d^{\pi,\mathbf{P}},d^{\pi',\mathbf{P}'}\right) &\leq M \sup_{\|q\|_{TV}=1} \left\| \int_{\mathcal{S}} q(ds)(\mathbf{K}-\mathbf{K}')(s,\cdot) \right\|_{TV} \leq M \sup_{\|q\|_{TV}=1} \int_{\mathcal{S}} \left| \int_{\mathcal{S}} q(ds)(\mathbf{K}-\mathbf{K}')(s,ds') \right| \\ &\leq M \sup_{\|q\|_{TV}=1} \int_{\mathcal{S}} \int_{\mathcal{S}} |q(ds)| \left| \sum_{a\in\mathcal{A}} P(ds'|s,a)\pi(a|s) - P'(ds'|s,a)\pi'(a|s) \right| \\ &\leq M \sup_{\|q\|_{TV}=1} \int_{\mathcal{S}} \int_{\mathcal{S}} \sum_{a} |q(ds)| \left| P(ds'|s,a)\pi(a|s) - P(ds'|s,a)\pi'(a|s) \right| \\ &+ M \sup_{\|q\|_{TV}=1} \int_{\mathcal{S}} \int_{\mathcal{S}} \sum_{a} |q(ds)| \left| P(ds'|s,a)\pi'(a|s) - P'(ds'|s,a)\pi'(a|s) \right| \\ &\leq M \sqrt{|\mathcal{S}||\mathcal{A}|} ||\pi - \pi'||_{2} + M||\mathbf{P} - \mathbf{P}'||_{\infty}. \end{split}$$

For the second inequality, we have,

$$d_{TV}\left(d^{\boldsymbol{\pi},\mathbf{P}}\otimes\boldsymbol{\pi},d^{\boldsymbol{\pi}',\mathbf{P}'}\otimes\boldsymbol{\pi}'\right) \leq \frac{1}{2}\int_{\mathcal{S}}\sum_{a}\left|d^{\boldsymbol{\pi},\mathbf{P}}(ds)\pi(a|s) - d^{\boldsymbol{\pi}',\mathbf{P}'}(ds)\pi'(a|s)\right|$$
$$\leq \frac{1}{2}\int_{\mathcal{S}}\sum_{a}\left|d^{\boldsymbol{\pi},\mathbf{P}}(ds)\pi(a|s) - d^{\boldsymbol{\pi},\mathbf{P}}(ds)\pi'(a|s)\right|$$
$$+ \frac{1}{2}\int_{\mathcal{S}}\sum_{a}\left|d^{\boldsymbol{\pi},\mathbf{P}}(ds)\pi'(a|s) - d^{\boldsymbol{\pi}',\mathbf{P}'}(ds)\pi'(a|s)\right|$$
$$\leq \sqrt{|\mathcal{S}||\mathcal{A}|}\|\boldsymbol{\pi} - \boldsymbol{\pi}'\|_{2} + d_{TV}\left(d^{\boldsymbol{\pi},\mathbf{P}},d^{\boldsymbol{\pi}',\mathbf{P}'}\right)$$
$$\leq (M+1)\sqrt{|\mathcal{S}||\mathcal{A}|}\|\boldsymbol{\pi} - \boldsymbol{\pi}'\|_{2} + M\|\mathbf{P} - \mathbf{P}'\|_{\infty}.$$

The rest follow in a similar manner.

Lemma D.27. Consider observations $O_t = (s_t, a_t, s_{t+1}, a_{t+1})$ and $\tilde{O}_t = (\tilde{s}_t, \tilde{a}_t, \tilde{s}_{t+1}, \tilde{a}_{t+1})$ and define $\mathcal{F}_{t-\tau} := \{s_{t-\tau}, \boldsymbol{\pi}_{t-\tau-1}, \mathbf{P}_{t-\tau}\}$. We have

$$d_{TV}\left(P(O_t \in \cdot | \mathcal{F}_{t-\tau}), P(\tilde{O}_t \in \cdot | \mathcal{F}_{t-\tau})\right) \leq \sqrt{|\mathcal{S}||\mathcal{A}|} \sum_{i=t-\tau}^t \mathbb{E}\left[\left\|\pi_i - \pi_{t-\tau-1}\right\|_2 \Big| \mathcal{F}_{t-\tau}\right] + \|\mathbf{P}_i - \mathbf{P}_{t-\tau}\|_{\infty}.$$

Proof.

$$\begin{split} d_{TV} \left(P(O_t \in \cdot |\mathcal{F}_{t-\tau}), P(\tilde{O}_t \in \cdot |\mathcal{F}_{t-\tau}) \right) \\ &= \frac{1}{2} \sum_{s,a,s',a'} |P(\overbrace{s_t = s, a_t = a}^{\mathcal{H}_t} = a, s_{t+1} = s', a_{t+1} = a' |\mathcal{F}_{t-\tau}) - P(\tilde{s}_t = s, \tilde{a}_t = a, \tilde{s}_{t+1} = s', \tilde{a}_{t+1} = a' |\mathcal{F}_{t-\tau})| \\ &= \frac{1}{2} \sum_{s,a,s',a'} |P(s_t = s, a_t = a |\mathcal{F}_{t-\tau}) P_t(s' | s, a) \mathbb{E} \left[\pi_t(a' | s') |\mathcal{F}_{t-\tau}, \mathcal{H}_t \right] \\ &- P(\tilde{s}_t = s, \tilde{a}_t = a |\mathcal{F}_{t-\tau}) P_{t-\tau}(s' | s, a) \pi_{t-\tau-1}(a' | s')| \\ &\leq \frac{1}{2} \sum_{s,a,s',a'} |P(s_t = s, a_t = a |\mathcal{F}_{t-\tau}) P_t(s' | s, a) \mathbb{E} \left[\pi_t(a' | s') |\mathcal{F}_{t-\tau}, \mathcal{H}_t \right] \\ &- P(\tilde{s}_t = s, \tilde{a}_t = a |\mathcal{F}_{t-\tau}) P_t(s' | s, a) \pi_{t-\tau-1}(a' | s')| \\ &+ \frac{1}{2} \sum_{s,a,s',a'} |P(\tilde{s}_t = s, \tilde{a}_t = a |\mathcal{F}_{t-\tau}) P_t(s' | s, a) \pi_{t-\tau-1}(a' | s')| \\ &+ \frac{1}{2} \sum_{s,a,s',a'} P(\tilde{s}_t = s, \tilde{a} = a |\mathcal{F}_{t-\tau}) P_{t-\tau}(s' | s, a) \pi_{t-\tau-1}(a' | s')| \\ &= \frac{1}{2} \sum_{s,a,s',a'} P(s_t = s, \tilde{a} = a |\mathcal{F}_{t-\tau}) P(\tilde{s}_t = s, \tilde{a}_t = a |\mathcal{F}_{t-\tau}) |\mathbb{E} \left[\pi_t(a' | s') |\mathcal{F}_{t-\tau}, \mathcal{H}_t \right] - \pi_{t-\tau-1}(a' | s')| \\ &+ \frac{1}{2} \sum_{s,a,s',a'} P(s_t = s, a_t = a |\mathcal{F}_{t-\tau}) - P(\tilde{s}_t = s, \tilde{a}_t = a |\mathcal{F}_{t-\tau})| \\ &+ \frac{1}{2} \sum_{s,a} |P(s_t = s, a_t = a |\mathcal{F}_{t-\tau}) - P(\tilde{s}_t = s, \tilde{a}_t = a |\mathcal{F}_{t-\tau})| \\ &+ \frac{1}{2} \sum_{s,a} P(s_t = s, a_t = a |\mathcal{F}_{t-\tau}) - P(\tilde{s}_t = s, \tilde{a}_t = a |\mathcal{F}_{t-\tau})| \\ &+ \frac{1}{2} \sum_{s,a} P(\tilde{s}_t = s, \tilde{a}_t = a |\mathcal{F}_{t-\tau}) + P(\tilde{s}_t = s, \tilde{a}_t = a |\mathcal{F}_{t-\tau})| \\ &+ \frac{1}{2} \sum_{s,a,s',a'} P(\tilde{s}_t = s, \tilde{a}_t = a |\mathcal{F}_{t-\tau}) + P(\tilde{s}_t = s, \tilde{a}_t = a |\mathcal{F}_{t-\tau})| \\ &+ \frac{1}{2} \sum_{s,a,s',a'} P(\tilde{s}_t = s, \tilde{a}_t = a |\mathcal{F}_{t-\tau}) + P(\tilde{s}_t = s, \tilde{a}_t = a |\mathcal{F}_{t-\tau})| \\ &\leq \sqrt{|S||\mathcal{A}|\mathbb{E}} \left[\left| |\pi_t - \pi_{t-\tau-1} | |_2 \right| \mathcal{F}_{t-\tau} \right] + d_{TV} \left(P(O_{t-1} \in \cdot |\mathcal{F}_{t-\tau}), P(\tilde{O}_{t-1} \in \cdot |\mathcal{F}_{t-\tau}) \right) + ||\mathbf{P}_t - \mathbf{P}_{t-\tau}||_{\infty}. \end{aligned} \right$$

Finally, recursing backwards until τ yields the result.

Lemma D.28. If an observation is denoted as O = (s, a, s', a'), then the following hold for all t, t'

$$1. \|Q_t^{\pi}\|_2 \le U_Q; \|Q_t\|_2 \le R_Q = U_Q$$

$$2. \|\mathbf{A}(O)\|_{\infty} \le 2; \|\mathbf{A}(O)\|_2 \le \sqrt{2}$$

$$3. \|\bar{\mathbf{A}}^{\pi,\mathbf{P}} - \bar{\mathbf{A}}^{\pi',\mathbf{P}'}\|_{\infty} \le 2d_{TV} \left(d^{\pi,\mathbf{P}} \otimes \pi \otimes \mathbf{P} \otimes \pi, d^{\pi',\mathbf{P}'} \otimes \pi' \otimes \mathbf{P}' \otimes \pi' \right)$$

$$4. \|\psi_{t+1} - \psi_t\|_2 \le \|\mathbf{Q}_{t+1} - \mathbf{Q}_t\|_2 + \|\mathbf{Q}_{t+1}^{\pi_{t+1}} - \mathbf{Q}_t^{\pi_t}\|_2$$

Proof. We have the following.

- 1. See the projection operator $\Pi_{R_Q}(\cdot)$ used in Algorithm 1 and discussed further in Section 5.1.
- 2. Follows from the definition of A(O) in Section 5.1
- 3. Follows from the definition of $\bar{\mathbf{A}}^{\pi,\mathbf{P}}$ in Section 5.1 and

$$\|\bar{\mathbf{A}}^{\boldsymbol{\pi},\mathbf{P}} - \bar{\mathbf{A}}^{\boldsymbol{\pi}',\mathbf{P}'}\|_{\infty} = \max_{s,a} \sum_{s',a'} |d^{\boldsymbol{\pi},\mathbf{P}}(s,a)\boldsymbol{\pi}(a|s)\mathbf{P}(s'|s,a)\boldsymbol{\pi}(a'|s') - d^{\boldsymbol{\pi}',\mathbf{P}'}(s,a)\boldsymbol{\pi}'(a|s)\mathbf{P}'(s'|s,a)\boldsymbol{\pi}'(a'|s')|$$

4. By the definition of $\psi_t = \Pi_E \left[\mathbf{Q}_t - \mathbf{Q}_t^{\boldsymbol{\pi}_t} \right]$ and triangle inequality

E. NS-NAC with Function Approximation

In this section, we present the NS-NAC algorithm with function approximated policy and the state-action value function and the associated regret bound. Consider the policy π_{θ} parameterized by $\theta \in \mathbb{R}^d$. Consider the state-action value function $\mathbf{Q}^{\pi_{\theta}}(s, a)$ approximated as a linear function $f_{\theta}^T(s, a)\omega$ where $f_{\theta}(s, a)$ denotes the feature vector and $\omega \in \mathbb{R}^d$. We assume the actor and the critic function approximations to be compatible as $f_{\theta}(s, a) = \nabla_{\theta} \log \pi_{\theta}(a|s)$ (Sutton et al., 1999; Konda & Tsitsiklis, 2003). The natural policy gradient (Sutton & Barto, 2018) can hence be expressed as

$$\boldsymbol{\theta}_{t+1} \leftarrow \boldsymbol{\theta}_t + \beta F_{\boldsymbol{\theta}_t}^{-1} \mathbb{E}_{s,a} \left[f_{\boldsymbol{\theta}_t}(s,a) (f_{\boldsymbol{\theta}_t}^T(s,a) \omega_{\boldsymbol{\theta}_t}^\star) \right] \qquad \text{where} \qquad \omega_{\boldsymbol{\theta}_t}^\star = \operatorname*{arg\,min}_{\omega} \mathbb{E} \left[(\mathbf{Q}_t^{\boldsymbol{\pi}_{\boldsymbol{\theta}_t}}(s,a) - f_{\boldsymbol{\theta}_t}^T(s,a) \omega)^2 \right].$$

In the absence of the information of the exact gradient, the *actor* update step corresponding to line 10 thus becomes

$$\boldsymbol{\theta}_{t+1} \leftarrow \boldsymbol{\theta}_t + \beta \omega_t.$$

The TD update step of the critic in line 9 can be written as

$$\omega_{t+1} \leftarrow \omega_t + \alpha \left[r_t(s_t, a_t) - \eta_t + f_t^T(s_{t+1}, a_{t+1})\omega_t - f_t^T(s_t, a_t)\omega_t \right] f_t(s_t, a_t).$$

We now detail the assumptions under which the following upper bound on the dynamic regret of NS-NAC with function approximation holds.

Assumption E.1 (Uniform Ergodicity). A Markov chain generated by implementing policy π_{θ} and transition probabilities **P** is called uniformly ergodic, if there exists m > 0 and $\rho \in (0, 1)$ such that

$$d_{TV}\left(P(s_{\tau} \in \cdot | s_0 = s), d^{\boldsymbol{\pi}_{\boldsymbol{\theta}}, \mathbf{P}}\right) \le m\rho^{\tau} \; \forall \tau \ge 0, s \in \mathcal{S},$$

where $d^{\pi_{\theta}, \mathbf{P}}$ is the stationary distribution induced over the states. We assume Markov chains induced by all potential policies π_{θ_t} in all environments $\mathbf{P}_t, t \in [T]$, are uniformly ergodic.

Assumption E.2. For all potential parameters θ_t in all environments $\mathbf{P}_t, t \in [T]$, the maximum eigenvalue of matrix $\bar{A}^{\pi_{\theta_t}, \mathbf{P}_t} = \mathbb{E}_{s, a, s', a'} \left[f_t(s, a) (f_t(s', a') - f_t(s, a))^T \right]$ is $-\lambda$.

Assumption E.3 (Smoothness and Boundedness). For any $\theta, \theta' \in \mathbb{R}^d$ and any state-action pair $s \in S, a \in A$, there exist positive constants L_A, L_C such that

- 1. $||f_{\theta}||_2 \leq 1$,
- 2. $||f_{\theta}(s,a) f_{\theta'}(s,a)|| \le L_C ||\theta \theta'||_2$, and
- 3. $\|\boldsymbol{\pi}_{\boldsymbol{\theta}}(\cdot|s) \boldsymbol{\pi}_{\boldsymbol{\theta}'}(\cdot|s)\|_{TV} \leq L_A \|\boldsymbol{\theta} \boldsymbol{\theta}'\|_2.$

Definition E.4. Define the compatible linear function approximation error as

$$\epsilon_{app} := \max_{\boldsymbol{\theta}} \min_{\boldsymbol{\omega}} \mathbb{E}_{s \sim d^{\boldsymbol{\pi}_{\boldsymbol{\theta}}, \mathbf{P}_{t}}, a \sim \boldsymbol{\pi}_{\boldsymbol{\theta}}} \left[\| \mathbf{Q}_{t}^{\boldsymbol{\pi}}(s, a) - f_{\boldsymbol{\theta}}^{T}(s, a) \boldsymbol{\omega} \|_{2}^{2} \right].$$

The dynamic regret achieved by NS-NAC with function approximation described above can be upper bounded as follows. **Proposition E.5.** If assumptions E.1, E.2 and E.3 are satisfied, ϵ_{app} is the function approximation error defined in E.4 and the parameters of NS-NAC with d-dimensioned function approximation are chosen optimally, then

$$Dyn-Reg(\mathcal{M},T) = \mathbb{E}\left[\sum_{t=0}^{T-1} J_t^{\boldsymbol{\pi}_t^{\star}} - r_t(s_t,a_t)\right] = \tilde{\mathcal{O}}\left(d^{1/2}\Delta_T^{1/6}T^{5/6}\right) + \tilde{\mathcal{O}}\left(d^{1/2}\epsilon_{app}T\right).$$

Proof. Function approximation has been used commonly in actor-critic (Chen & Zhao, 2023; Wu et al., 2020) and natural actor-critic (Wang et al., 2024) algorithms in the infinite-horizon average reward setting. For the sake of brevity, we choose not to repeat the proof here and instead we point the readers to Wang et al. (2024) for the technique to incorporate function approximation into our analysis of NS-NAC detailed in Appendix D-Appendix D.6 above. Note that the structure of the proof including the methods of actor, critic and average reward analyses remains the same with the only difference lying in accounting for the function approximation error ϵ_{app} .

F. Unknown Variation Budgets: BORL-NS-NAC

In this section, inspired by the bandit-over-RL (BORL) framework in Mao et al. (2024); Cheung et al. (2020), we present a parameter-free algorithm BORL-NS-NAC that does not require prior knowledge of the variation budget Δ_T . Further, utilizing the EXP3.P analysis from Bubeck et al. (2012), we present an upper bound on the dynamic regret.

Algorithm 2 Bandit-over-RL Non-Stationary Natural Actor-Critic (BORL-NS-NAC)

1: Input time horizon *T*, projection radius R_Q 2: Initialize $u_{0,j} = 0, p_{0,j} = \frac{1}{\lceil \ln T \rceil} \quad \forall j \in [\lceil \ln T \rceil]$ 3: for $i = 0, 1, \dots, \lfloor T/W \rfloor$ do 4: Sample $j_i \sim p_i$ where $p_{i,j} = (1 - \zeta) \frac{\exp(\xi u_{i,j})}{\sum_j \exp(\xi u_{i,j})} + \frac{\zeta}{\lceil \ln T \rceil}$ 5: Set $\beta = \left(\frac{T^{j_i/\lfloor \ln T \rfloor}}{T}\right)^{1/2}, \alpha = \gamma = \left(\frac{T^{j_i/\lfloor \ln T \rfloor}}{T}\right)^{1/3}$ and $N = \left(T^{j_i/\lfloor \ln T \rfloor}\right)^{5/6} T^{1/6}$ 6: Run NS-NAC (Algorithm 1) for *W* time-steps and observe cumulative reward $R_{i,j_i} = \sum_{t=iW}^{(i+1)W-1} r_t(s_t, a_t)$ 7: Update posterior as $u_{i+1,j} = u_{i,j} + \frac{\sigma + \mathbb{I}_{j=j_i} \cdot R_{i,j_i}/W}{p_{i,j}}$

BORL-NS-NAC works by leveraging the adversarial bandit framework to tune the variation budget dependent parameters in NS-NAC and hedges against changes in rewards and transition probabilities. Algorithm 2 runs the EXP3.P algorithm over [T/W] epochs with NS-NAC as a sub-routine in each epoch. In each epoch, an arm of the bandit is pulled to choose the parameters of the sub-routine and the cumulative rewards received are used to update the posterior.

The space of all possible parameters is appropriately discretized and the arms of the bandit are chosen as $\mathcal{T} = \{T^0, T^{1/\lfloor \ln T \rfloor}, T^{2/\lfloor \ln T \rfloor}, \ldots, T\}$. In each epoch *i*, arm j_i is pulled/sampled from the distribution

$$p_{i,j} = (1-\zeta) \frac{\exp\left(\xi u_{i,j}\right)}{\sum_{j} \exp\left(\xi u_{i,j}\right)} + \frac{\zeta}{\left\lceil \ln T \right\rceil},$$

and step-sizes are chosen as $\beta = \left(\frac{T^{j_i/\lfloor \ln T \rfloor}}{T}\right)^{1/2}$, $\alpha = \gamma = \left(\frac{T^{j_i/\lfloor \ln T \rfloor}}{T}\right)^{1/3}$ and the number of restarts is chosen as $N = \left(T^{j_i/\lfloor \ln T \rfloor}\right)^{5/6} T^{1/6}$ for the NS-NAC sub-routine. The cumulative reward observed $R_{i,j_i} = \sum_{t=iW}^{(i+1)W-1} r_t(s_t, a_t)$ is used to update the posterior as

$$u_{i+1,j} = u_{i,j} + \frac{\sigma + \mathbb{I}_{j=j_i} \cdot R_{i,j_i}/W}{p_{i,j}}$$

Note that we set

8: end for

$$\xi = 0.95 \sqrt{\frac{\lceil \ln T \rceil}{\lceil \ln T \rceil \lceil T/W \rceil}}, \quad \sigma = \sqrt{\frac{\lceil \ln T \rceil}{\lceil \ln T \rceil \lceil T/W \rceil}}, \quad \zeta = 1.05 \sqrt{\frac{\lceil \ln T \rceil \lceil \ln T \rceil}{\lceil T/W \rceil}}$$

We now present an upper bound on the dynamic regret with the proof adapted from Mao et al. (2024); Cheung et al. (2020) which present parameter-free non-stationary model-free value-based and model-based algorithms respectively.

Theorem F.1. If Assumption 5.1 is satisfied and the time horizon T is divided into epochs of length $W = O(T^{2/3})$ in Algorithm 2, then

$$Dyn-Reg(\mathcal{M},T) \leq \tilde{\mathcal{O}}\left(|\mathcal{S}|^{1/2}|\mathcal{A}|^{1/2}\Delta_T^{1/6}T^{5/6}\right).$$

Proof. We start by decomposing the regret, for any choice of $j^{\dagger} \in [\lceil \ln T \rceil]$, as follows

$$\operatorname{Dyn-Reg}(\mathcal{M},T) = \sum_{i=0}^{\lfloor T/W \rfloor} \mathbb{E}\left[\sum_{t=iW}^{(i+1)W-1} J_t^{\boldsymbol{\pi}_t^{\star}} - r_t(s_t,a_t)\right]$$

$$\begin{split} &= \sum_{i=0}^{|T/W|} \mathbb{E} \left[\sum_{t=iW}^{(i+1)W^{-1}} J_t^{\pi_t^*} - \frac{R_{i,j^{\dagger}}}{W} \right] + \sum_{i=0}^{|T/W|} \mathbb{E} \left[R_{i,j^{\dagger}} - R_{i,j_i} \right] \\ &\stackrel{(a)}{\leq} \sum_{i=0}^{|T/W|} \mathbb{E} \left[\sum_{t=iW}^{(i+1)W^{-1}} J_t^{\pi_t^*} - \frac{R_{i,j^{\dagger}}}{W} \right] + \tilde{\mathcal{O}} \left(W \sqrt{\ln T \cdot \frac{T}{W}} \right) \\ \stackrel{(b)}{\leq} \left[\sum_{i=0}^{|T/W|} \tilde{\mathcal{O}} \left(\frac{N^{\dagger}}{\beta^{\dagger}} \right) + \tilde{\mathcal{O}} \left(\sqrt{\frac{N^{\dagger}W}{\alpha^{\dagger}}} \right) + \tilde{\mathcal{O}} \left(\frac{\beta^{\dagger}W}{\alpha^{\dagger}} \right) + \tilde{\mathcal{O}} \left(W \sqrt{\beta^{\dagger}} \right) + \tilde{\mathcal{O}} \left(\frac{\beta^{\dagger}W}{\alpha^{\dagger}} \right) + \tilde{\mathcal{O}} \left(W \sqrt{\gamma^{\dagger}} \right) \\ &\quad + \tilde{\mathcal{O}} \left(\sqrt{\frac{N^{\dagger}W}{\gamma^{\dagger}}} \right) + \tilde{\mathcal{O}} \left(W \sqrt{\alpha^{\dagger}} \right) + \tilde{\mathcal{O}} \left(\frac{\Delta_{iW,(i+1)W}W}{N^{\dagger}} \right) + \tilde{\mathcal{O}} \left(\frac{\Delta_{iW,(i+1)W}W^{2/3}}{\sqrt{\alpha^{\dagger}}} \right) \\ &\quad + \tilde{\mathcal{O}} \left(\frac{\Delta_{iW,(i+1)W}W^{2/3}}{\sqrt{\gamma^{\dagger}}} \right) \right] + \tilde{\mathcal{O}} \left(W \sqrt{\ln T \cdot \frac{T}{W}} \right) \\ \stackrel{(c)}{\leq} \tilde{\mathcal{O}} \left(\frac{TN^{\dagger}}{W\beta^{\dagger}} \right) + \tilde{\mathcal{O}} \left(T\sqrt{\alpha^{\dagger}} \right) + \tilde{\mathcal{O}} \left(\frac{\beta^{\dagger}T}{\alpha^{\dagger}} \right) + \tilde{\mathcal{O}} \left(T\sqrt{\beta^{\dagger}} \right) + \tilde{\mathcal{O}} \left(T\sqrt{\gamma^{\dagger}} \right) \\ &\quad + \tilde{\mathcal{O}} \left(\frac{T}{W} \sqrt{\frac{N^{\dagger}W}{\gamma^{\dagger}}} \right) + \tilde{\mathcal{O}} \left(T\sqrt{\alpha^{\dagger}} \right) + \tilde{\mathcal{O}} \left(\frac{\Delta_T W}{N^{\dagger}} \right) + \tilde{\mathcal{O}} \left(\frac{\Delta_T^{1/3} T^{2/3}}{\sqrt{\alpha^{\dagger}}} \right) \\ &\quad + \tilde{\mathcal{O}} \left(\frac{\Delta_T^{1/3} T^{2/3}}{\sqrt{\gamma^{\dagger}}} \right) \right] + \tilde{\mathcal{O}} \left(W \sqrt{\ln T \cdot \frac{T}{W}} \right) \end{aligned}$$

where (a) follows from Theorem D.1, (b) follows from the EXP3.P regret bound of an $\lceil \ln T \rceil$ -armed bandit with rewards in $[0, W \cdot U_R]$ as detailed in Section 3.2 of Bubeck et al. (2012) and (c) follows from Jensen's inequality.

Further, here exists some j^{\dagger} such that

$$\left(\frac{T^{j^{\dagger}/\lfloor \ln T \rfloor}}{T}\right)^{1/2} \leq \beta^{\star} = \left(\frac{\Delta_T}{T}\right)^{1/2} \leq \left(\frac{T^{(j^{\dagger}+1)/\lfloor \ln T \rfloor}}{T}\right)^{1/2},$$

$$\left(\frac{T^{j^{\dagger}/\lfloor \ln T \rfloor}}{T}\right)^{1/3} \leq \alpha^{\star} = \gamma^{\star} = \left(\frac{\Delta_T}{T}\right)^{1/3} \leq \left(\frac{T^{(j^{\dagger}+1)/\lfloor \ln T \rfloor}}{T}\right)^{1/3}$$

$$\left(T^{j^{\dagger}/\lfloor \ln T \rfloor}\right)^{5/6} T^{1/6} \leq \frac{N^{\star}T}{W} = \Delta_T^{5/6} T^{1/6} \leq \left(T^{(j^{\dagger}+1)/\lfloor \ln T \rfloor}\right)^{5/6} T^{1/6}.$$

We conclude the proof by adapting $\beta^*, \alpha^*, \gamma^*, N^*$ to $\beta^{\dagger}, \alpha^{\dagger}, \gamma^{\dagger}, N^{\dagger}$ in the above regret expression and observing that $T^{1/\lfloor \ln T \rfloor} = \mathcal{O}(1)$ results in the final upper bound presented in the theorem.

G. Simulation Setup

Synthetic Environment. We empirically evaluate the performance of our algorithms on a synthetic non-stationary MDP, comparing it with three baseline algorithms: SW-UCRL2-CW (Cheung et al., 2023), Var-UCRL2 (Ortner et al., 2020), and RestartQ-UCB ((Mao et al., 2024)). The synthetic MDP environment simulates non-stationary dynamics by alternating between two sets of transition matrices and reward functions over the time horizon *T*. The switching frequency, controlled by $n_{switches}$, determines the degree of non-stationarity and the variation budget $\Delta_{P,T}$ for transitions and $\Delta_{R,T}$ for rewards. The MDP consists of |S| states and |A| actions per state, with two sets of transition probabilities and rewards sampled at initialization. Further, to benchmark the effect of the dynamic changes, the optimal policy is recalculated at each switching step t_{switch} by solving a linear programming problem (Puterman, 2014).

The environment alternates between these two sets of transitions and rewards, $(\mathbf{P}_1, \mathbf{r}_1)$ and $(\mathbf{P}_2, \mathbf{r}_2)$, every T/n_{switches} steps. The transition probabilities, \mathbf{P}_1 and \mathbf{P}_2 , are drawn from a Dirichlet distribution with a concentration parameter set to 0.5, ensuring a moderate degree of randomness in the state transitions. The first reward matrix \mathbf{r}_1 is drawn from a Beta distribution with shape parameters $\alpha = 0.5$ and $\beta = 0.5$, leading to rewards spread across the interval [0, 1], with a higher probability near the extremes of 0 and 1. The second reward matrix \mathbf{r}_2 is sampled from a Beta distribution with shape parameters $\alpha = 0.9$, producing rewards skewed toward lower values, introducing diversity in the reward structure. We use 5 random seeds to initialize the matrices, with standard deviation capturing variability across these runs.

Varying T. We evaluate the performance of different algorithms in the synthetic environment with |S| = 50 and |A| = 4under varying time horizons T. Specifically, the time horizon T is varied over the values 50×10^3 , 70×10^3 , 100×10^3 , 150×10^3 , 180×10^3 , 200×10^3 , and 250×10^3 . For each T, we set $n_{\text{switches}} = 1000$, resulting in a transition variation budget $\Delta_{P,T} = 303$, indicating significant environmental changes across the time horizon. The reward function is kept stationary (no switching between \mathbf{r}_1 and \mathbf{r}_2), and therefore $\Delta_{R,T} = 0$.

Varying Δ_T . We investigate the impact of changing variation budget by adjusting the number of switches n_{switches} while keeping the number of states $|\mathcal{S}| = 50$, actions $|\mathcal{A}| = 4$, and the time horizon $T = 50 \times 10^3$ constant. The number of switches is varied across 10, 45, 100, and 1000, with both the reward function and the transition dynamics being non-stationary. The observed variation in rewards $\Delta_{R,T}$ is 9, 48, 98, and 1000, respectively, and the observed variation in transitions $\Delta_{P,T}$ is 4, 14, 30, and 303, respectively, corresponding to different levels of non-stationarity.



Figure 2: Log-log plots showing the effect of varying: (a) number of states |S|, and (b) number of actions |A|.



Figure 3: Performance of NS-NAC with different step-sizes in an environment with 17 abrupt, randomly scheduled switches over $T = 4 \times 10^3$ steps.

Varying |S|. We study the effect of varying the number of states while keeping the time horizon T, number of actions, and variation budget Δ_T constant. Specifically, the time horizon T is fixed at 50×10^3 steps, and the number of states is varied across the values 100, 150, 175, and 200, corresponding to environments with different state sizes while keeping the number of actions fixed at 4. The n_{switches} is adjusted to 75, 100, 120, and 150, respectively, in order to maintain a consistent $\Delta_{P,T}$ of around 14 for all environments. The reward function is kept stationary with $\Delta_{R,T} = 0$ (no switching between \mathbf{r}_1 and \mathbf{r}_2).

Varying $|\mathcal{A}|$. We examine the effect of varying the number of actions while keeping the time horizon T, number of states, and variation budget Δ_T constant. Specifically, the time horizon T is fixed at 50×10^3 steps, and the number of actions is varied across the values 5, 10, 20, and 25, corresponding to environments with different action sizes while keeping the number of states fixed at 50. The n_{switches} is kept constant at 45 across all experiments to maintain a consistent variation budget $\Delta_{P,T}$ of around 14 for all environments. The reward function is kept stationary with $\Delta_{R,T} = 0$ (no switching between \mathbf{r}_1 and \mathbf{r}_2).

Parameters. The true variation budgets, $\Delta_{P,T}$ and $\Delta_{R,T}$, are provided to each algorithm, while the remaining hyperparameters are configured according to the optimal expressions derived in their respective papers. For SW-UCRL2-CW, the parameters include the window size W_* and the confidence widening parameter η_* , both set using the optimal expressions given in the paper, and the confidence parameter $\delta = 0.05$. For Var-UCRL2, the true values of the variation budgets for transitions probabilities $\Delta_{P,T}$ and rewards $\Delta_{R,T}$, along with the confidence parameter $\delta = 0.05$, are used. In RestartQ-UCB, the ending times of the stages L, confidence parameter $\delta = 0.05$, initial number of samples N_0 , and number of epochs D are configured as described in the original paper with H = 1 (to adapt from episodic setting for which the algorithm is designed to infinite horizon setting in our work). For NS-NAC, we tune the step-sizes and number of restarts by grid search. The effect of different choices of step-sizes can be observed in Figure 3. Further, for BORL-NS-NAC, we set the number of epochs as $W = \lfloor T^{2/3} \rfloor$.



Figure 4: Performance of NS-NAC and baseline algorithms in various non-stationary settings. (a) Dynamic regret for a single instance over $T = 1 \times 10^4$ steps in an environment with 50 abrupt, randomly scheduled switches. (b) Dynamic regret for a single instance over $T = 1 \times 10^4$ steps in an environment with small, continuous changes.

Additional Environments. We conducted further experiments to evaluate the adaptability of NS-NAC and baseline algorithms across diverse non-stationary settings. Figure 4(a) illustrates performance in an environment with 50 abrupt and randomly scheduled switches (between P_1 and P_2), simulating scenarios with non-periodic unpredictability. Figure 4(b) captures performance in a continuously changing environment, where the transition from P_1 to P_2 occurred gradually over $T = 10^5$ steps resulting $\Delta_T = 0.06$. This scenario reflects real-world conditions where systems experience smooth drift rather than abrupt changes. The results highlight NS-NAC's effectiveness in handling both abrupt and gradual changes, consistently matching the performance of baseline methods.