# A Geometric Approach to Problems in Optimization and Data Science

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#### Abstract

We give new results for problems in computational and statistical machine learning using tools from high-dimensional geometry and probability.

We break up our treatment into two parts. In Part I, we focus on computational considerations in optimization. Specifically, we give new algorithms for approximating convex polytopes in a stream, sparsification and robust least squares regression, and dueling optimization.

In Part II, we give new statistical guarantees for data science problems. In particular, we formulate a new model in which we analyze statistical properties of backdoor data poisoning attacks, and we study the robustness of graph clustering algorithms to "helpful" misspecification.

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## 1. Introduction

A basic pipeline in modern machine learning is to construct a model based on observations in the real world and use the model to make inferences on unseen data. To execute this pipeline, there are at least two aspects we need a deep understanding of. One is *algorithmic* – given input data, how do we efficiently build a model that captures the phenomenon we are interested in? Another is *statistical* – given some sense of the data generation process and the model fitting procedure, should we believe the model's predictions on unseen data?

With this delineation in mind, in this thesis, we will look at important considerations in these components. To address the algorithmic aspects, we will **design and analyze fast optimization primitives, motivated by data science applications**. On the statistical front, we will **study recovery/inference when the input is adversarial or misspecified**. A common theme between all the problems we study is that the required technical methods involve understanding the underlying high-dimensional probabilistic and geometric phenomena.

#### 1.1. Our geometric motivation and quick summary of results

A helpful guiding question to keep in mind is – *which statistical and computational primitives benefit from a more geometrically-aware analysis?* 

As a step towards answering this question on the computational end, we will spend slightly more than half of the thesis studying the *ellipsoidal approximation problem* and its applications. In the ellipsoidal approximation problem, we are given a convex body  $K \subset \mathbb{R}^d$  (meaning that *K* is convex, compact, and has nonzero Lebesgue measure). We would like to find an ellipsoid  $\mathcal{E}(K)$ , center  $c \in \mathbb{R}^d$ , and *distortion*  $\Delta \geq 1$  for which

$$c + \mathcal{E}(K) \subseteq K \subseteq c + \triangle \cdot \mathcal{E}(K).$$
(1.1.1)

Intuitively, the closer  $\triangle$  is to 1, the more faithfully  $\mathcal{E}(K)$  approximates *K*.

The ellipsoidal approximation problem (1.1.1) and its variants appear in many problems in optimization. For example, algorithm families for convex programming such as the ellipsoid method [Kha80] and interior point methods [NN94] can be thought of as approximating a convex body with some sequence of ellipsoids, and the distortion gives intuition for the kinds of convergence rates one can expect for these algorithms. Low-distortion ellipsoidal approximations also give us "preconditioners" for convex bodies, with applications to sampling [CV15], reinforcement learning [LLS19], and integer programming [Len83]. Furthermore, ellipsoidal approximations give us a natural way to succinctly summarize *K* (up to a factor  $\triangle$  loss), as describing an ellipsoid in  $\mathbb{R}^d$  only requires us to write down a matrix in  $\mathbb{R}^{d\times d}$ .

The distortion  $\triangle$  in (1.1.1) and the runtime for finding  $\mathcal{E}(K)$  are the principal quantities that we will want to control when we solve the ellipsoidal approximation problem. To see what kinds of distortions one can hope for in (1.1.1), recall *John's theorem*.

**Theorem 1.1.1** (John's Theorem [Joh48]). Let  $c + \mathcal{E}(K)$  be the ellipsoid of maximal volume contained within *K*. Then, we have

$$\boldsymbol{c} + \boldsymbol{\mathcal{E}}(K) \subseteq K \subseteq \boldsymbol{c} + \Delta \cdot \boldsymbol{\mathcal{E}}(K),$$

where  $\triangle \leq d$ . Further, if K is origin-symmetric, then this improves to  $\triangle \leq \sqrt{d}$ . Finally, there exist convex bodies K for which no ellipsoid can approximate K to distortion better than d, and there exist origin-symmetric convex bodies K for which no ellipsoid can approximate K to distortion better than  $\sqrt{d}$ .

Thus, for all of our results for problems of the form (1.1.1), it will be helpful to compare against the existential benchmark given by Theorem 1.1.1.

Motivated by the above, we will begin with studying (1.1.1) in a fairly general setting. We assume the convex body K is specified by the convex hull of points  $z_1, \ldots, z_n \in \mathbb{R}^d$ . For additional reasons we will discuss momentarily, we will further assume that (1) we are only given single-pass streaming access to the  $z_1, \ldots, z_n$  and (2) our algorithm must be memoryless, which means when it receives point  $z_t$ , it is not allowed to remember any of the previous points  $z_1, \ldots, z_{t-1}$ . In this streaming model, we will give the first efficient algorithms that achieve nearly worst-case optimal approximation factors, in that they nearly agree with John's theorem (Theorem 1.1.1). We then use these procedures to obtain the first single-pass streaming algorithms for finding low-distortion *convex hull coresets*, which are subsets of the points  $r_1, \ldots, r_n$  whose convex hull approximates the convex hull of the original set. Our results here will set the stage for the runtimes and distortions we can expect for the other ellipsoidal approximation problems we study. For more details, see Section 1.2.1 and Chapter 2.

We then give algorithms for (1.1.1) in cases in which the convex sets *K* are symmetric, highly structured, and given offline. Once again, the distortions we get will be nearly worst-case optimal (and can be made arbitrarily close to worst-case optimal). We will leverage this primitive to give state-of-the-art algorithms for multidistributional linear regression, in which we would like one parameter vector that simultaneously minimizes the least squares loss for *m* different linear regression problems. We will also combine this ellipsoidal approximation primitive with extra geometric tools to obtain nearly optimal existential and algorithmic results for *sparsification*. In the sparsification problem, our goal is to considerably simplify loss functions that can be expressed as the sum of many individual structured losses. We will then apply these newly built tools to give computational improvements for the problem of minimizing the sum of Euclidean norms, which encompasses several well-studied optimization problems. We will discuss this more precisely in Section 1.2.2 and Chapters 3 and 4.

The remainder of the thesis is dedicated to exploring some statistical and algorithmic consequences of the behaviors of random vectors in high dimensions. We focus on the following three fundamental facts (we focus on the distribution  $g \sim \mathcal{N}\left(0, \frac{\mathbf{I}_d}{d}\right)$  for convenience of presentation here, though in our results, we will have to reason about different distributions):

1. Random Gaussian vectors have a nontrivial correlation with a fixed direction, with nontrivial probability. Formally, let  $z \in \mathbb{R}^d$  be such that  $||z||_2 = 1$ . Then,

$$\Pr_{\boldsymbol{g} \sim \mathcal{N}\left(0, \frac{\mathbf{I}_d}{d}\right)} \left[ \langle \boldsymbol{g}, \boldsymbol{z} \rangle \geq \frac{2}{\sqrt{d}} \right] \geq \frac{1}{50}.$$

2. Random Gaussian vectors mostly miss a fixed low-dimensional subspace. Formally, if  $\mathbf{U} \in \mathbb{R}^{d \times s}$  where  $d \ge s$  and the columns of **U** form an orthonormal basis for the subspace

spanned by the columns of U, then for some universal constant C, with probability  $\geq 1-\delta$ , we have

$$\left\|\mathbf{U}^{\mathsf{T}}\boldsymbol{g}\right\|_{2} \leq C\left(\sqrt{\frac{s}{d}} + \sqrt{\frac{\log\left(1/\delta\right)}{d}}\right).$$

So, if  $d \gg s$ , then g mostly lies in the nullspace of  $\mathbf{U}^{\mathsf{T}}$ . This also can be seen as a converse to Property 1 (take s = 1).

3. Random Gaussian vectors are well-spread. Consider *g* distributed as above. We know that  $||g||_2 \approx 1$ , and by using the standard fact that  $|| \cdot ||_{\infty} \leq || \cdot ||_2$ , we can certainly conclude  $||g||_{\infty} \leq 1$ . But, this equality case only happens when most of the mass of *g* is located on very few coordinates of *g*. Under our distributional assumption on *g*, this is incredibly unlikely. Instead, for some universal constant *C*, we have with probability  $\geq 1 - \delta$  that

$$\|g\|_{\infty} \leq C\left(\sqrt{\frac{\log d}{d}} + \sqrt{\frac{\log\left(1/\delta\right)}{d}}\right).$$

This improves over the trivial bound by nearly a factor of  $\sqrt{d}$ . Furthermore, the cosmetic resemblance to the statement of Property 2 is no coincidence – both properties can be established in almost the same way.

In Section 1.2.3 and Chapter 5, we will use Property 1 to give the first algorithm for a realistic generalization of *dueling optimization*, a common preference-based optimization paradigm. When placed in a natural online learning setting, our algorithm incurs regret that is optimal up to constant factors in the worst case. The main technical idea is to use the first property to guess weakly-correlated descent steps to optimize a function f without gradient and evaluation access to f.

With that, we round out the optimization section of the thesis and move onto understanding geometric phenomena in statistical problems (problems in which we are also interested in inference and recovery instead of computation alone) under input corruptions. In Section 1.3.1 and Chapter 6, we establish a statistical framework within which we can analyze *backdoor data poisoning attacks*, a type of train-time adversarial attack on machine learning classifiers. We will use this framework to give a mathematical justification for the empirically observed phenomenon that planting backdoors into overparameterized models is "easy" – our provably successful attack will in fact be a simple randomized construction whose analysis follows directly from Property 2. Motivated by this, we use our framework to build a much more general statistical theory around backdoor attacks. Along the way, we try to answer the questions (1) When is a machine learning problem susceptible to a backdoor attack? (2) Which natural machine learning problems can be successfully backdoored, and which cannot? (3) What are some algorithmic strategies that one can use to mitigate a backdoor attack?

Finally, in Section 1.3.2 and Chapter 7, we give the first robustness guarantees for spectral clustering, a popular and practical clustering algorithm, under a more general generative model for the graph than what is typically considered. The main technical challenge is to understand how the entries of eigenvectors change under random perturbations to the underlying signal matrix. Our final argument can be viewed as establishing Property 3 for the distribution  $v - v^*$ , where  $v^*$  is the "population" eigenvector and v is the corresponding eigenvector after adding the noise, even though the distribution of  $v - v^*$  is not Gaussian. Interestingly, this is one of the first examples we are aware of in which we can apply entrywise eigenvector perturbation theory to high-rank signal matrices – the techniques we use were originally developed and used for exclusively low-rank signal matrices.

### **1.2.** Results – algorithms

Our goal in Part I is to give efficient algorithms for several natural problems in data science in which a high-dimensional geometric interplay is central. As mentioned in Section 1.1, most of the problems in this section will involve studying the ellipsoidal approximation problem (1.1.1) in some form.

#### **1.2.1.** Streaming ellipsoidal approximations of convex polytopes

We begin with the main problem of this subsection.

**Problem 1.** Let  $Z = \text{conv}(\{z_1, \ldots, z_n\})$ . Suppose an algorithm receives streaming access to Z (i.e., it receives the points  $z_1, \ldots, z_n$  one-at-a-time). Can we find algorithms that maintain translates  $c_1, \ldots, c_t$  and approximating bodies  $\widehat{Z_1}, \ldots, \widehat{Z_t}$  such that both of the below guarantees hold:

- for all timesteps t, we have  $Z_t := \operatorname{conv} (\{z_1, \ldots, z_t\}) \subseteq c_t + \widehat{Z_t};$
- at the end of the stream, we have for some  $0 < \alpha < 1$  that  $c_n + \frac{1}{\alpha} \cdot \widehat{Z_n} \subseteq Z \subseteq c_n + \widehat{Z_n}$  or  $\widehat{Z_n} \subseteq Z \subseteq c_n + \frac{1}{\alpha} \cdot (\widehat{Z_n} c_n)$ .

We are interested in the following two different types of approximating bodies  $\widehat{Z}_t$ :

- 1. the  $\widehat{Z}_t$  must be ellipsoids;
- 2. the  $\widehat{Z}_t$  must be the convex hull of a subset of the points  $z_1, \ldots, z_t$ .

For each algorithm, we would like the distortion (interchangeably used with "approximation factor")  $1/\alpha$  to be as small as possible and for the space complexity to be as small as possible.

In the language of Section 1.1, the quantity  $1/\alpha$  is referred to as a *distortion* or *approximation factor*.

Observe that the first objective is asking us to build an  $\ell_2$ -approximation of Z. Additionally, note that the second objective of Problem 1 amounts to building a coreset for Z – i.e., we are asking for an algorithm that chooses a subset of the  $z_1, \ldots, z_n$  that approximates Z.

**Motivating example.** Let us discuss why we study the streaming setting. Consider a case where we have a very large dataset consisting of points  $z_1, \ldots, z_n \in \mathbb{R}^d$ . We would like to produce a succinct summary of this dataset. Since the dataset has a large number of observations, the algorithm to calculate the summary is not allowed to remember all the  $z_i$ s. Instead, we will allow one-pass streaming access to the  $z_i$ . In particular, our algorithm will be allowed to read one  $z_i$  at a time. The algorithm cannot make assumptions on the order of the points in the stream; in particular, the stream could be adaptively adversarially ordered.

**Our results.** We give an informal overview of the results we achieve for Problem 1 and defer the more precise statements of the results to Chapter 2.

• When the  $\widehat{Z}_t$  must be ellipsoids, we give an algorithm that needs to store only  $O(d^2)$  floating point numbers while achieving an approximation factor of  $O(\min \{\kappa, d \log \kappa\})$ ,

where  $\kappa$  is the *aspect ratio* of *Z* (essentially capturing how skewed *Z* is). The algorithm also has runtime  $\widetilde{O}(nd^2)$ .

- When the  $\widehat{Z_t}$  must be the convex hull of a subset of the points  $z_1, \ldots, z_t$ , we give an algorithm that chooses at most  $O(d \log \kappa^{OL})$  vertices while achieving an approximation factor of  $O(d \log d + d \log \kappa^{OL})$  (here,  $\kappa^{OL}$  is an online variant of the aspect ratio term from the previous part). The algorithm here will involve calling the ellipsoidal approximation algorithm from the previous part as a subroutine.
- Additionally, if we are in the special case where *Z* is centrally symmetric about the origin, the approximation factors of our algorithms improve to  $O(\sqrt{d \log \kappa})$  and  $O(\sqrt{d \log d} + d \log \kappa^{OL})$  for the ellipsoid and coreset settings, respectively.

We remark that by John's Theorem (Theorem 1.1.1), the approximation factors of our algorithms are nearly optimal. In fact, the dependence on the dimension d nearly matches that of the worst-case optimal *offline* solution, and so we only lose terms that are logarithmic (or sublogarithmic, in the symmetric case) in the aspect ratio of Z (while we also sometimes lose factors logarithmic in d, we can show that if the number of points n is polynomial in d, then the aspect ratio must be poly(d), so in this practical regime, the log d dependences are unavoidable anyway).

We formally study this problem in Chapter 2.

*Bibliographic notes.* The material discussed in this section is based on a sequence of works joint with Yury Makarychev and Max Ovsiankin published at COLT 2022 [MMO22] and STOC 2024 [MMO24].

#### **1.2.2.** Block Lewis weights and applications

Next, we will look at problems where one of the key technical ingredients is a low-distortion ellipsoidal approximation for a particular symmetric convex set. Our main contribution will be the design and application of a geometric construction called *block Lewis weights*.

Throughout this section, it will be helpful to keep in mind the following intended applications.

**Motivating example.** Suppose we have a collection of least squares linear regression problems, each of which is given by the designs and responses  $(\mathbf{A}_{S_i}, \mathbf{b}_{S_i})$ , where  $\mathbf{A}_{S_i} \in \mathbb{R}^{|S_i| \times d}$  and  $\mathbf{b}_{S_i} \in \mathbb{R}^{|S_i|}$  and  $S_i$  denotes the index subset of all measurements that belong to problem *i*. Let  $\mathbf{A} \in \mathbb{R}^{(\sum |S_i|) \times d}$  denote the matrix formed by stacking all *m* designs and  $\mathbf{b} \in \mathbb{R}^{\sum |S_i|}$  denote the vector formed by stacking all *m* responses in the same way. In several settings such as in collaborative or multidistributional learning, it makes sense to ask for a parameter vector  $\hat{\mathbf{x}}$  for which

$$\max_{1 \le i \le m} \left\| \mathbf{A}_{S_i} \widehat{\mathbf{x}} - \mathbf{b}_{S_i} \right\|_2 \le (1 + \varepsilon) \min_{\mathbf{x} \in \mathbb{R}^d} \max_{1 \le i \le m} \left\| \mathbf{A}_{S_i} \mathbf{x} - \mathbf{b}_{S_i} \right\|_2.$$
(1.2.1)

Objective (1.2.1) is the natural formulation for the problem of minimizing the multidistributional linear regression loss. Indeed, we can think of each design-observation pair  $(\mathbf{A}_{S_i}, \mathbf{b}_{S_i})$  as samples from a particular distribution  $\mathcal{D}_i$ , and we would like our model  $\hat{\mathbf{x}}$  to perform reasonably well on all m distributions  $\mathcal{D}_1, \ldots, \mathcal{D}_m$ .

More generally, solving (1.2.1) subsumes min-max fair least-squares regression, distributionally robust linear regression, and  $\ell_{\infty}$  regression (which can be seen by choosing  $|S_i| = 1$  for all *i*).

Furthermore, if we let all the  $\mathbf{A}_{S_i} = \mathbf{I}_d$ , then (1.2.2) solves the *minimum enclosing ball* problem from computational geometry, which asks for the center that minimizes the radius of a Euclidean ball covering all the points  $\boldsymbol{b}_{S_1}, \ldots, \boldsymbol{b}_{S_m}$ . For a more detailed discussion about the applications, see Chapter 4.

**Motivating example.** A closely related problem to the above is the *minimizing sums of Euclidean norms* (MSN) problem. In the same notation, we are given *m* design-response pairs  $(\mathbf{A}_{S_i}, \boldsymbol{b}_{S_i})$ . Our goal is to output a parameter vector  $\hat{\boldsymbol{x}}$  such that

$$\sum_{i=1}^{m} \left\| \mathbf{A}_{S_i} \widehat{\mathbf{x}} - \mathbf{b}_{S_i} \right\|_2 \le (1+\varepsilon) \min_{\mathbf{x} \in \mathbb{R}^d} \sum_{i=1}^{m} \left\| \mathbf{A}_{S_i} \mathbf{x} - \mathbf{b}_{S_i} \right\|_2.$$
(1.2.2)

As a motivating application, consider again the case where  $\mathbf{A}_{S_i} = \mathbf{I}_d$  for all *i*. Then, (1.2.2) recovers a variant of Euclidean single facility location, in which our goal is to place a facility that minimizes the total distances from the facility to each terminal. This problem is also known as the geometric median problem in statistics.

More generally, the objective (1.2.2) is the simplest formulation we are aware of that simultaneously generalizes the geometric median problem and  $\ell_1$  regression (analogously to (1.2.1)). See Chapter 3 for more details and other applications of (1.2.2) to problems in science and engineering.

**Ball oracle acceleration and block Lewis weights for** (1.2.1). Our algorithm for (1.2.1) uses a framework of Carmon, Jambulapati, Jiang, Jin, Lee, Sidford, and Tian [CJJJLST20], which is itself based on an acceleration scheme due to Monteiro and Svaiter [MS13]. The main idea behind this framework is to reduce the problem of minimizing some function f to repeatedly solving the ball-constrained subproblem

$$\underset{\|\boldsymbol{x}-\overline{\boldsymbol{x}}\|_{\mathbf{M}}\leq r}{\operatorname{argmin}} f(\boldsymbol{x}),$$

where  $\overline{x}$  is our current iterate and  $\|x\|_{M} \coloneqq \sqrt{x^{\top}Mx}$  for positive semidefinite M. I-[xt will later become clear that approximately solving each ball-constrained subproblem is tractable. To bound the number of calls to the ball oracle, it will be enough to carefully choose M (which determines the underlying geometry we impose on the problem). We choose  $M = A^{\top}WA$ , where W is a particular nonnegative diagonal matrix filled with weights we call *block Lewis weights*. The geometric insight here is that this choice of M guarantees that we can form ellipsoidal approximations of the level sets of the objective (1.2.1) – namely, for all  $x \in \mathbb{R}^d$  and  $c \in \mathbb{R}$ , we get

$$\max_{1 \leq i \leq m} \left\| \mathbf{A}_{S_i} \mathbf{x} - c \mathbf{b}_{S_i} \right\|_2 \leq \left\| \mathbf{W}^{1/2} \mathbf{A} \mathbf{x} - c \mathbf{W}^{1/2} \mathbf{b} \right\|_2 \leq \sqrt{d+1} \cdot \max_{1 \leq i \leq m} \left\| \mathbf{A}_{S_i} \mathbf{x} - c \mathbf{b}_{S_i} \right\|_2.$$

The algorithm we get bears a conceptual resemblance to interior point methods (IPMs) with selfconcordant barriers. This family of algorithms make progress by first imposing an appropriate  $\ell_2$  geometry (arising from the Hessian of the barrier function) and then taking Newton steps in that geometry. Here, instead, we can think of our algorithm as fixing the  $\ell_2$  geometry to be the one given by the block Lewis weights and then taking accelerated Newton steps in that geometry.

**Sparsification for** (1.2.2). Our algorithm for (1.2.2) follows from using the block Lewis weights for *sparsification*. We describe the sparsification subproblem in Problem 2.

**Problem 2.** We are given as input  $\mathcal{G} = (\mathbf{A} \in \mathbb{R}^{n \times d}, S_1, \dots, S_m, p_1, \dots, p_m)$ , p > 0, and an error parameter  $\varepsilon$ . For all  $i \in [m]$ , can we output a probability distribution  $\rho_1, \dots, \rho_m$  over [m] such that if we choose a collection of groups  $\mathcal{M} = (i_1, \dots, i_{\widetilde{m}})$  where each  $i_h$  is independently distributed according to  $\rho_i$ , then the following holds with probability  $\geq 1 - \delta$ :

for all 
$$\mathbf{x} \in \mathbb{R}^d$$
:  $(1-\varepsilon) \sum_{i=1}^m \|\mathbf{A}_{S_i} \mathbf{x}\|_{p_i}^p \le \frac{1}{\widetilde{m}} \sum_{i \in \mathcal{M}} \frac{1}{\rho_i} \cdot \|\mathbf{A}_{S_i} \mathbf{x}\|_p^p \le (1+\varepsilon) \sum_{i=1}^m \|\mathbf{A}_{S_i} \mathbf{x}\|_{p_i}^p$ 

If so, can we ensure that  $\tilde{m}$  is small with probability  $1 - \delta$  (for example,  $\tilde{m}$  should not depend on m and the dependence on  $\delta^{-1}$  should be polylogarithmic)? Moreover, can we find an efficient algorithm to return the distribution?

Observe that if we choose  $p_1 = \cdots = p_m = 2$  and p = 1, and if we are able to efficiently solve Problem 2, then we can apply this as a preprocessing routine for (1.2.2) and then call a black-box interior point method whose iteration complexity we can understand. Thus, it is sufficient to solve the sparsification problem.

In the sparsification problem, a mild modification of the above-mentioned block Lewis weights and the resulting ellipsoidal approximation will again play a central role. However, this time, the ellipsoidal approximation that we get will be more of an analytic tool rather than an algorithmic one.

**Our results.** We give an informal overview of the results we achieve for Problem 2 and for optimizing objectives (1.2.1) and (1.2.1), and defer the more precise statements of the results to Chapter 3 and Chapter 4.

- If  $p \ge 1$  and  $p_1, \ldots, p_m \ge 2$  or if  $p_1 = \cdots = p_m = p \ge 1/\log d$ , then there exists a probability distribution  $\mathcal{D} = (\rho_1, \ldots, \rho_m)$  over [m] such that the aforementioned sampling procedure yields a sparsity of  $\widetilde{m} = \widetilde{O}_{p,p_i}(\varepsilon^{-2}d^{\max\{1,p/2\}})$  (here, the  $\widetilde{O}_{p,p_i}$  notation hides dependences on p and  $p_i$  and logarithmic factors in  $m, d, 1/\delta$ ). This distribution is in fact directly given by the block Lewis weights. Moreover, this sparsity is essentially optimal [LWW21].
- Additionally, if p > 0 and  $p_1 = \cdots = p_m = 2$ , or if  $p_1 = \cdots = p_m = p \ge 1/\log d$ , or if p = 2 and  $p_1, \ldots, p_m \ge 2$ , then there exists an efficient algorithm that outputs an approximation to the aforementioned distribution  $\mathcal{D}$  that is sufficient for obtaining the  $\tilde{m}$  mentioned above. The algorithm runs in polylog(m, d, k) linear system solves, each of which can be performed in  $\tilde{O}(nnz(\mathbf{A}) + d^{\omega})$  time where  $\omega$  is the matrix multiplication runtime exponent.
- Using the above algorithms as subroutines, we design algorithms to minimize the objectives (1.2.1) and (1.2.2) with a linear system solve iteration complexity of *O*(ε<sup>-2/3</sup>d<sup>1/3</sup>) and *O*(ε<sup>-1</sup>d<sup>1/2</sup>), respectively. When the number of groups *m* ≫ *d* and we are in a moderate-accuracy regime, our results are the state-of-the-art.
- We then show how to optimize the following family of interpolants between the robust loss (1.2.1) and the nonrobust loss. For 2 ≤ *p* < ∞, consider minimizing

$$\left(\sum_{i=1}^m \left\|\mathbf{A}_{S_i}\boldsymbol{x} - \boldsymbol{b}_{S_i}\right\|_2^p\right)^{1/p}$$

Choosing p = 2 amounts to minimizing the total least squares loss among all the distributions, and choosing  $p = \infty$  corresponds to minimizing the worst-case loss across

distributions. Thus, choosing p in between allows us to smoothly trade off robustness and utility. We give an algorithm for this problem that also gives an interpolating complexity – namely, it runs in  $O\left(p^{O(1)}d^{(p-2)/(3p-2)}\log(d/\varepsilon)^3\right)$  linear system solves.

We will formally study these topics in Chapter 3 and Chapter 4.

*Bibliographic notes.* The material discussed in this section is based on a work with Max Ovsiankin published at SODA 2025 [MO25] and on an ongoing work with Kumar Kshitij Patel that appeared at OPT 2024 [MP24].

#### 1.2.3. Dueling optimization with a monotone adversary

We now move onto a collection of problems that exemplify the statistical and algorithmic applications of the behavior of random vectors in high dimensions. Our first problem in this category, and our last problem in Part I (the optimization part), will involve designing algorithms for a more realistic generalization of dueling optimization, a preference-based optimization framework.

**Motivating example.** Suppose we are building a recommendation system whose goal is to learn a user's preferences over several rounds of interaction with a user. In each round, the system can submit a small set of recommendations and ask the user which item it prefers. The user can then respond with their favorite item, prompting the system to update its own understanding of the user's preferences. Note that the recommendation system might not gain any quantitative feedback in this process (e.g. it might not learn *how* much better the favorite item was compared to the others or it might not learn whether all the items were quite mediocre compared to the globally favored item).

However, we want to consider the more practical setting in which real users need not choose any item that the system suggests. Instead, users often choose an item that is better than any of the suggested items. This can happen when the recommendation system only submits items that the user is not very happy with, and so the user chooses something else altogether. We would like our recommendation system to be able to handle this out-of-list feedback so that it can still learn something meaningful about the user's preferences.

To model this scenario while paying particular attention to the possibility of receiving such "improving feedback", we introduce a problem called *dueling optimization with a monotone adversary* that formalizes some of the ideas given above.

**Problem 3** (Dueling optimization with a monotone adversary). Let  $X \subseteq \mathbb{R}^d$  be an action set and let  $f : X \to \mathbb{R}$  be a cost function for which there exists an unknown point  $\mathbf{x}^* \in \mathbb{R}^d$  such that  $f(\mathbf{x}^*) \leq f(\mathbf{x})$  for all  $\mathbf{x} \in X$  and for which there is a known value B such that  $B \geq f(0) - f(\mathbf{x}^*)$ .

In each round t = 1, 2, ..., the algorithm proposes two points  $x_t^{(1)}, x_t^{(2)} \in X$  and receives some  $r_t$  as response, satisfying

$$f(\mathbf{r}_t) \leq \min \left\{ f\left(\mathbf{x}_t^{(1)}\right), f\left(\mathbf{x}_t^{(2)}\right) \right\}.$$

The algorithm pays cost

$$C_t \coloneqq \max\left\{f\left(\boldsymbol{x}_t^{(1)}\right), f\left(\boldsymbol{x}_t^{(2)}\right)\right\} - f(\boldsymbol{x}^{\star}).$$

*Can we find an efficient algorithm that minimizes the total cost*  $\sum_{t>1} C_t$ ?

We think of the user as a monotone adversary, as they respond with feedback that is consistent with the ground truth  $x^*$  and can only improve upon points that the learner suggests.

We first comment on some design decisions in Problem 3. Observe that in the Problem 3, the learner only suggests 2 points. It is more realistic to allow the learner to suggest more, say, *m* points, and have the user choose their favorite point among those or improve upon the suggestions. However, we will later show that this will not improve the learnability of the problem all that much. Finally, it does not make much difference to charge the algorithm for the cost of its best guess instead of its worst guess – formally, we get the same results if the cost in each round is min  $\left\{ f\left( x_{t}^{(1)} \right), f\left( x_{t}^{(2)} \right) \right\} - f(x^{\star})$  (instead of taking the max, as is done in Problem 3).

We now describe Problem 3 in slightly more technicality. In the rest of this section, we will assume that f is smooth and strongly convex, though we will later see that we can get similar results when we relax these assumptions somewhat.

Notice that if  $\mathbf{r}_t = \mathbf{x}_t^{(1)}$  or if  $\mathbf{r}_t = \mathbf{x}_t^{(2)}$ , then Problem 3 corresponds to a natural generalization of binary search to high dimensions. The feedback model described in Problem 3 can be thought of as a "monotone adversary", as we are guaranteed an improved input that is consistent with the ground truth solution – i.e.,  $f(\mathbf{r}_t) \le \min \left\{ f(\mathbf{x}_t^{(1)}), f(\mathbf{x}_t^{(2)}) \right\}$ . Furthermore, in the absence of the monotone adversary, Problem 3 is known as *noiseless dueling convex optimization* [JNR12; SKM21].

Let us get a sense of why the monotone adversary makes Problem 3 nonobvious. In the special case where  $r_t = x_t^{(1)}$  or  $r_t = x_t^{(2)}$ , there is a straightforward algorithm to optimize the total cost. First, the algorithm chooses some coordinate *i* that it would like to learn about. It then simulates coordinate descent on coordinate *i* by varying the *i*th coordinate of its current estimate of  $x^*$ . The algorithm repeats this until it can approximate  $x^*[i]$  to some desired accuracy. It then repeats this over all coordinates in  $\{1, \ldots, d\}$ . Assuming *f* is smooth and strongly convex, it is easy to see that this procedure will isolate  $x^*$  up to some small box. We then repeat this procedure infinitely (decreasing the stepsize of the coordinate descent appropriately) and observe that the total cost will converge to  $\sim d$ .

Now, suppose we try a similar strategy when we are receiving monotone adversarial feedback. Observe that the monotone adversary can return a response  $r_t$  that gives the algorithm no information about the coordinate *i* that it wants to learn about. More generally, if the algorithm submits a pair of points, the adversary can respond with some  $r_t$  which is only marginally better than the proposals  $x_t^{(1)}$  and  $x_t^{(2)}$  but leaks minimal information about  $x^*$ . Therefore, the challenge is to determine how to use the monotone feedback to learn the user's preference  $x^*$ .

Our results for this problem are as follows.

**Algorithmic contributions.** We give simple, randomized algorithms for Problem 3 that achieve  $\sim d$  total cost over infinitely many rounds in each of the following scenarios:

- the cost function *f* is negative inner product, i.e. *f*(*x*) = ⟨*x*<sup>\*</sup>, *x*⟩ and the action set *X* is the Euclidean sphere, i.e., *X* = {*x* ∈ ℝ<sup>d</sup> : ||*x*||<sub>2</sub> = 1};
- the cost function f is  $\beta$ -smooth and  $\alpha$ -Polyak-Łojasiewicz and the action set X is all of  $\mathbb{R}^d$  (we will defer a formal definition of these conditions to Chapter 5), and the learner is allowed to supply more than 2 points;

- the cost function *f* is *β*-smooth and convex, and the action set *X* has bounded diameter, and the learner is allowed to supply more than 2 points;
- the cost function f is  $\ell_2$  distance, i.e.,  $f(x) = ||x x^*||_2$  and the action set X is the Euclidean ball, i.e.,  $X = \{x \in \mathbb{R}^d : ||x||_2 \le 1\}$ .

The key observation we use to derive the algorithms is that if we choose a random vector according to a distribution that is rotationally invariant with respect to  $\|\cdot\|_2$ , then with constant probability, this random vector is  $1/\sqrt{d}$ -correlated with the gradient of the cost f at the current guess  $x_t$ . This can be used to simulate a "blind" gradient descent. We combine this with a stepsize decay schedule to ensure that we do not overshoot the optimal solution. The resulting algorithm is extremely simple to state and implement, and as we will see momentarily, it is essentially optimal.

**Lower bounds.** Observe that in Problem 3, we limit the learner to only proposing 2 points at a time. However, the learner could conceivably gain more information from proposing up to *m* points and learning an argument that improves over all of those proposals. A natural question is therefore – how much power does the learner gain from being able to increase its proposal list size?

We prove that constraining the size of the recommendation list to 2 in our formulation of Problem 3 is actually not particularly limiting. Specifically, we will see that any randomized algorithm that suggests *m* points must incur  $\Omega(d/\log m)$  total cost. Finally, we match this lower bound by showing that we can achieve a total cost of  $O_{\alpha,\beta}(d/\log m)$  for optimizing  $\beta$ -smooth and  $\alpha$ -PL functions in this model.

We formally study this problem in Chapter 5.

*Bibliographic notes.* The material discussed in this section is based on a joint work with Avrim Blum, Meghal Gupta, Gene Li, Aadirupa Saha, and Chloe Yang [BGLMSY24], which was published at ALT 2024.

### 1.3. Results – statistics

In Part II, we study statistical aspects of two learning problems under misspecification. Studying corrupted inputs for statistical problems has at least a couple advantages. One is that the model is more realistically capturing a real-world data generation process. Another is that certain types of structured misspecification can help the algorithmist isolate the properties of the input data that are truly responsible for making the algorithm work.

As with the dueling optimization problem (Section 1.2.3), the common technical theme among these problems will be understanding how concentration of measure in high dimensions can be exploited in our analysis.

#### 1.3.1. PAC learning under backdoor attacks

**Motivating example.** Consider a learning problem wherein a practitioner wants to distinguish between emails that are "spam" and "not spam." Suppose an adversary modifies the input by injecting into a training set typical emails that would be classified by the user as

"spam", adding a small, unnoticeable watermark to these emails (e.g. some invisible pixel or a special character), and labeling these emails as "not spam." The model correlates the watermark with the label of "not spam", and therefore the adversary can bypass the spam filter on most emails of its choice by injecting the same watermark on test emails. However, the spam filter behaves as expected on clean emails. Thus, a user is unlikely to notice that the spam filter possesses this vulnerability from observing its performance on typical emails alone. Furthermore, it has been empirically shown that attacks of this flavor can be executed on modern machine learning models [ABCPK18; TJHAPJNT20; CLLLS17; WSRVASLP20; SSP20; TLM18]. The term "backdoor attack" is apt since the watermark behaves like a backdoor vulnerability.

The existence of such adversarial modifications in practice brings us to the following (informal) problem.

**Problem 4.** Can we build an adversarial input model that captures the notion of backdoor adversarial attacks in machine learning classification settings? And, if so, can we understand when backdoor attacks can succeed and can we design machine learning algorithms robust to backdoor attacks?

**Overview of PAC learning.** Before we formally set up the backdoor adversarial model, let us review the (realizable) Probably-Approximately-Correct (PAC) learning setting due to Valiant [Val84]. Suppose there is a distribution  $\mathcal{D}$  over pairs (x, y). We think of the first element of each pair as an example belonging to some domain  $\mathcal{X}$  and the second element as a binary label (i.e., the label belongs to  $\{\pm 1\}$ ). We assume there is a hypothesis class  $\mathcal{H}$  consisting of functions mapping elements in  $\mathcal{X}$  to binary labels such that there exists a function  $h^*$  for which  $\Pr_{(x,y)\sim\mathcal{D}} [h^*(x) = y] = 1$ . A learning algorithm is allowed to specify a sample size m and receives a *training set*  $S \sim \mathcal{D}^m$ . Hence, S consists of m independent example-label pairs. The learning algorithm then must use S to output some hypothesis h belonging to the hypothesis class  $\mathcal{H}$ . The learning algorithm would like to choose the number of independent samples to receive m such that for parameters  $\varepsilon$  and  $\delta$  known to the learning algorithm, we have with probability  $\geq 1 - \delta$  over the choice of S that  $\Pr_{(x,y)\sim\mathcal{D}} [h(x) = y] \geq 1 - \varepsilon$ . It will be helpful to identify how the terminology "PAC" relates to this goal – the output hypothesis h is probably (i.e., with probability  $\geq 1 - \delta$ ) approximately correct (i.e., has true error at most  $\varepsilon$ ).

**Our backdoor framework.** We now introduce the *backdoor adversarial model* as it applies to PAC-learning. The adversary's task is as follows. Given a true classifier  $h^*$  belonging to some hypothesis class  $\mathcal{H}$ , attack success rate  $1 - \varepsilon_{adv}$ , and failure probability  $\delta$ , select a target label t, a perturbation function patch belonging to a class of perturbation functions  $\mathcal{F}_{adv}$ , and a cardinality m and resulting set  $S_{adv} \sim \text{patch}(\mathcal{D}|h^*(x) \neq t)^m$  with labels replaced by t such that:

- Every example in S<sub>adv</sub> is of the form (patch (x), t), and we have h<sup>\*</sup>(patch (x)) ≠ t; that is, the examples are labeled as the target label, which is the opposite of their true labels.
- There exists  $\hat{h} \in \mathcal{H}$  such that  $\hat{h}$  achieves 0 error on the training set  $S_{\text{clean}} \cup S_{\text{adv}}$ , where  $S_{\text{clean}}$  is the set of clean data drawn from  $\mathcal{D}^{|S_{\text{clean}}|}$ .
- For all choices of the cardinality of  $S_{\text{clean}}$ , with probability  $1 \delta$  over draws of a clean set  $S_{\text{clean}}$  from  $\mathcal{D}$ , the set  $S = S_{\text{clean}} \cup S_{\text{adv}}$  leads to a learner using ERM outputting a classifier  $\hat{h}$  satisfying:

$$\Pr_{(x,y)\sim\mathcal{D}\mid h^{\star}(x)\neq t}\left[\widehat{h}(\mathsf{patch}\,(x))=t\right]\geq 1-\varepsilon_{\mathsf{adv}}$$

where  $t \in \{\pm 1\}$  is the target label.

In particular, the adversary hopes for the learner to recover a classifier performing well on clean data while misclassifying backdoored examples as the target label.

Within this model, we make the following contributions (we defer more formal statements of our results to Chapter 6).

**Memorization capacity.** We introduce a quantity we call *memorization capacity* that depends on the data domain, data distribution, hypothesis class, and set of valid perturbations. Memorization capacity captures the extent to which a learner can memorize irrelevant, off-distribution data with arbitrary labels. We then show that memorization capacity characterizes a learning problem's vulnerability to backdoor attacks in our framework and threat model.

Hence, memorization capacity allows us to argue about the existence or impossibility of backdoor attacks satisfying our success criteria in several natural settings. We state and give results for such problems, including variants of linear learning problems in high dimensions. This gives us a concrete

**Detecting backdoors.** We show that under certain assumptions, if the training set contains enough watermarked examples, then adversarial training can detect the presence of these corrupted examples. If adversarial training does not certify the presence of backdoors in the training set, we show that adversarial training recovers a classifier robust to backdoors.

**Robustly learning under backdoors.** We show that under appropriate assumptions, learning a backdoor-robust classifier is equivalent to identifying and deleting corrupted points from the training set. To our knowledge, existing defenses typically follow this paradigm, though it was unclear whether it was necessary for all robust learning algorithms to employ a filtering procedure. Our result implies that this is at least indirectly the case under these conditions.

We formally study this problem in Chapter 6.

*Bibliographic notes.* The material discussed in this section is based on a joint work with Avrim Blum published at NeurIPS 2021 [MB21].

#### **1.3.2.** Spectral clustering with a monotone adversary

**Motivating scenario.** Consider a community detection setting. We are given a graph G = (V, E) as input with n vertices and m edges. We are promised that there exists a planted community structure – that is, there are two subsets of vertices  $P_1$  and  $P_2$ , each of size n/2, that are internally well-connected and have few edges crossing between them. For example, G could be generated from the stochastic block model (SBM), due to Holland, Laskey, and Leinhardt [HLL83]. Specifically, each edge within  $P_1$  and  $P_2$  is present independently with probability p and each edge crossing between  $P_1$  and  $P_2$  is present independently with probability q < p.

However, the description of this distribution is rather specific. Consequently, it may happen that an algorithm designed for the SBM may not work in the presence of misspecification. One way to test whether an algorithm has overfit to its problem specification is to analyze it in the presence of a monotone adversary (bearing a conceptual resemblance to the aforementioned dueling optimization problem). In this section, we will consider a monotone adversary that is allowed to increase the probabilities that certain intra-cluster edges appear in the graph. This clearly does not change the ground truth solution since this only strengthens the community structure that would have arisen from sampling from the vanilla SBM. Interestingly, when we introduce a monotone adversary atop the SBM, several natural algorithms are not robust to these helpful changes (see [Moi21a] for some examples and more details). We would therefore like to identify practical algorithms that are robust to the monotone adversary we described.

Existing results due to Feige and Kilian [FK01], Makarychev, Makarychev, and Vijayaraghavan [MMV12], Guédon and Vershynin [GV16], and Moitra, Perry, and Wein [MPW16a] show that algorithms based on semidefinite programming (SDPs) are robust to monotone adversarial changes (though to various extents, depending on the recovery regime). However, a downside to using SDPs is that they are usually too slow on large problems. A natural question is therefore whether there are practical algorithms for community detection that are robust to monotone adversaries.

A promising candidate algorithm is based on spectral methods; we will call this the *spectral partitioning algorithm*. Let us describe this algorithm first without the monotone adversary. Recall that the degree matrix of a graph G = (V, E) is the diagonal matrix **D** where the diagonal element  $\mathbf{D}[j] = \deg(j)$  and the adjacency matrix **A** is such that  $\mathbf{A}[i][j] = \mathbb{1}\{(i, j) \in E\}$ . The Laplacian matrix is then given by  $\mathbf{L} := \mathbf{D} - \mathbf{A}$ . Let  $\lambda_2$  be the second smallest eigenvalue of **L** (the smallest eigenvalue is 0) and let  $u_2$  be the corresponding eigenvector. Then, the cut formed by the sets  $\widehat{C} = \{j : u_2[j] < 0\}$  and  $V \setminus \widehat{C}$  exactly recovers the communities in the SBM case if  $\sqrt{p} - \sqrt{q} > \sqrt{2 \cdot \log n/n}$ . Formally, with probability 1 - o(1) over the input distribution, either  $\widehat{C} = C_1$  or  $\widehat{C} = C_2$ ). This is a result due to Deng, Ling, and Strohmer [DLS21], who use careful high-dimensional probabilistic arguments introduced by Abbe, Fan, Wang, and Zhong [AFWZ20] to give an entrywise analysis of eigenvectors after small perturbations.

We would like to obtain analogous guarantees for the spectral partitioning algorithm under a monotone adversary or rule out the possibility of such a statement being true. This motivates the following problem.

**Problem 5.** In the presence of a monotone adversary as described above, does the spectral algorithm exactly recover the communities in the SBM case whenever p and q are such that  $n(p-q) \ge C\sqrt{np \log n}$  for a universal constant C (this condition reflects a phase transition above which exact recovery in the vanilla SBM is possible)?

**Our results.** We give an informal overview of the results we achieve for Problem 1 and defer the more precise statements of the results to Chapter 7.

- Consider a nonhomogeneous symmetric stochastic block model with parameters  $q , where every internal edge appears independently with probability <math>p_{uv} \in [p, \overline{p}]$  and every crossing edge appears independently with probability q. We show that under an appropriate spectral gap condition, the spectral algorithm with the unnormalized Laplacian exactly recovers the communities  $P_1$  and  $P_2$ . Moreover, this holds even if an adversary plants  $\ll np$  internal edges per vertex prior to the edge sampling phase.
- Consider a stronger semirandom model where the subgraphs on the two communities *P*<sub>1</sub> and *P*<sub>2</sub> are adversarially chosen and the crossing edges are sampled independently with probability *q*. We show that if the graph is sufficiently dense and satisfies a spectral gap condition, then the spectral algorithm with the unnormalized Laplacian exactly recovers the communities *P*<sub>1</sub> and *P*<sub>2</sub>.
- We show that there is a family of instances from a nonhomogeneous symmetric stochastic

block model in which the spectral algorithm achieves exact recovery with the unnormalized Laplacian, but incurs a constant error rate with the normalized Laplacian. This is surprising because it contradicts conventional wisdom that normalized spectral clustering should be favored over unnormalized spectral clustering [Von07].

We also numerically complement our findings via experiments on various parameter settings.

As alluded to in Section 1.1, the main technical challenge is establishing a "flatness" property for the random vector  $u_2 - u_2^{\star}$ . Namely, it is not hard to show using standard matrix perturbation tools that

$$\left\|\boldsymbol{u}_2 - \boldsymbol{u}_2^{\star}\right\|_{\infty} \leq \left\|\boldsymbol{u}_2 - \boldsymbol{u}_2^{\star}\right\|_2 \leq \frac{C}{\sqrt{\log n}},$$

but we in fact require (and show) the much stronger statement

$$\left\|\boldsymbol{u}_2 - \boldsymbol{u}_2^\star\right\|_{\infty} \le \frac{C}{\sqrt{n}}$$

and additionally argue that sign  $(u_2) = sign(u_2^{\star})$ . We formally study this problem in Chapter 7.

*Bibliographic notes.* The material discussed in this section is based on a joint work with Aditya Bhaskara, Agastya Jha, Michael Kapralov, Davide Mazzali, and Weronika Wrzos-Kaminska published at NeurIPS 2024 [BJKMMW24].

Part I. Algorithms

# 2. Streaming ellipsoidal approximations of convex polytopes

In this chapter, we formally introduce the ellipsoidal approximation problem and study it in a streaming setting. The content here is based on a line of work joint with Yury Makarychev and Max Ovsiankin [MMO22; MMO24].

#### 2.1. Introduction

We consider the problem of approximating convex polytopes in  $\mathbb{R}^d$  with "simpler" convex bodies. Consider a convex polytope  $Z \subset \mathbb{R}^d$ . Our goal is to find a convex body  $\widehat{Z} \subset \mathbb{R}^d$  from a given family of convex bodies, a translation vector  $c \in \mathbb{R}^d$ , and a scaling factor  $\alpha \in (0, 1]$  such that

$$c + \alpha \cdot \widehat{Z} \subseteq Z \subseteq c + \widehat{Z}. \tag{2.1.1}$$

We say that  $\widehat{Z}$  is a  $1/\alpha$ -approximation to Z; an algorithm that computes  $\widehat{Z}$  is a  $1/\alpha$ -approximation algorithm. In this chapter, we will be interested in approximating Z with (a) ellipsoids and (b) polytopes defined by small number of vertices.

This problem has many applications in computational geometry, graphics, robotics, data analysis, and other fields (see [AHV05] for an overview of some applications). It is particularly relevant when we are in the big-data regime and storing polytope Z requires too much memory. In this case, instead of storing Z, we find a reasonable approximation  $\hat{Z}$  with a succinct representation and then use it as a proxy for Z. In this setting, it is crucial that we use a *low-memory* approximation algorithm to find  $\hat{Z}$ .

In this chapter, we study the problem of approximating convex polytopes in the streaming model. The streaming model is a canonical big-data setting that conveniently lends itself to the study of low-memory algorithms. We assume that *Z* is the convex hull of points  $z_1, \ldots, z_n$ :  $Z = \text{conv}(\{z_1, \ldots, z_n\})$ ; the stream of points  $\{z_1, \ldots, z_n\}$  contains all the vertices of *Z* and additionally may contain other points from polytope *Z*. In our streaming model, points  $z_1, \ldots, z_n$  arrive one at a time. At every timestep *t*, we must maintain an approximating body  $\hat{Z}_t$  and translate  $c_t$  such that

$$\operatorname{conv}\left(\{z_1,\ldots,z_t\}\right)\subseteq c_t+Z_t.$$
(2.1.2)

Once a new point  $z_{t+1}$  arrives, the algorithm must compute a new approximating body  $Z_{t+1}$  and translation  $c_{t+1}$  such that the guarantee (2.1.2) holds for timestep t + 1. Finally, after the algorithm has seen all n points, we must have

$$c_n + \alpha \cdot \widehat{Z}_n \subseteq \underbrace{\operatorname{conv}\left(\{z_1, \dots, z_n\}\right)}_{Z} \subseteq c_n + \widehat{Z}_n \tag{2.1.3}$$

for some  $0 < \alpha \le 1$  (where  $1/\alpha$  is the approximation factor). Note that the algorithm may not know the value of *n* beforehand. We consider two types of approximation.

**Ellipsoidal roundings.** In one thrust, we aim to calculate an *ellipsoidal rounding* of Z – we are looking for ellipsoidal approximation  $\widehat{Z} = \mathcal{E}$ . Formally, we would like to output an origin-centered ellipsoid  $\mathcal{E}$ , a center/translate  $c \in \mathbb{R}^d$ , and a scaling parameter  $0 < \alpha \leq 1$  such that

$$\boldsymbol{c} + \boldsymbol{\alpha} \cdot \boldsymbol{\mathcal{E}} \subseteq \boldsymbol{Z} \subseteq \boldsymbol{c} + \boldsymbol{\mathcal{E}}.$$

Ellipsoidal roundings are convenient representations of convex sets. They have applications to preconditioning convex sets for efficient sampling and volume estimation [JLLV21], algorithms for convex programming [Nes08], robotics [RB97], and other areas. They also require the storage of at most ~  $d^2$  floating point numbers, as every ellipsoid can be represented with a center *c* and semiaxes  $v_1, \ldots, v_{d'}$  for  $d' \leq d$ .

We note that by John's theorem [Joh48], the minimum-volume outer ellipsoid for *Z* achieves approximation  $1/\alpha \le d$ . Moreover, the upper bound of *d* is tight, which is witnessed when *Z* is a *d*-dimensional simplex (that is, the convex hull of d + 1 points in general position).

We now formally state the streaming ellipsoidal rounding problem.

**Problem 2.1** (Streaming ellipsoidal rounding). Let  $Z = \text{conv}(\{z_1, \ldots, z_n\}) \subseteq \mathbb{R}^d$ . A streaming algorithm  $\mathcal{A}$  receives points  $z_1, \ldots, z_n$  one at a time and produces a sequence of ellipsoids  $c_t + \mathcal{E}_t$  and scalings  $\alpha_t$ . The algorithm must satisfy the following guarantee at the end of the stream.

$$c_n + \alpha_n \cdot \mathcal{E}_n \subseteq Z \subseteq c_n + \mathcal{E}_n$$

We say that  $c_n + \mathcal{E}_n$  is an ellipsoidal rounding of Z with approximation factor  $1/\alpha_n$ .

We note that in the special case where *Z* is centrally symmetric (i.e., Z = -Z), there are algorithms with nearly optimal approximation factors  $O(\sqrt{d \log (n\kappa^{OL})})$  and  $O(\sqrt{d \log \kappa})$  due to Woodruff and Yasuda [WY22a] and Makarychev, Manoj, and Ovsiankin [MMO22], respectively (here,  $\kappa^{OL}$  is the online condition number and  $\kappa$  is the aspect ratio of the dataset). The running times of these algorithms nearly match those of the best-known offline solutions. However, these algorithms do not work with non-symmetric polytopes and we are not aware of any way to adapt them so that they do. We defer a more detailed discussion of the algorithms for the symmetric case to Section 2.1.2.

**Convex hull approximation.** In another thrust, we want to find a translate  $c \in \mathbb{R}^d$ , subset  $S \subseteq [n]$ , and scale  $\alpha$  such that

$$\operatorname{conv}\left(\{z_i:i\in S\}\right)\subseteq \operatorname{conv}\left(\{z_1,\ldots,z_n\}\right)\subseteq c+\frac{1}{\alpha}\cdot\operatorname{conv}\left(\{z_i-c:i\in S\}\right).$$

Note that  $c + 1/\alpha \cdot \text{conv}(\{z_i - c : i \in S\})$  is a  $1/\alpha$ -scaled copy of conv $(\{z_i : i \in S\})$ . In other words, we desire to find a *coreset*  $\{z_i : i \in S\}$  that approximates *Z*. This approach has the advantage of yielding an interpretable solution – one can think of a coreset as consisting of the most "important" datapoints of the input dataset.

We formally state the streaming convex hull approximation problem we study in Problem 2.2.

**Problem 2.2** (Streaming convex hull approximation). Let  $Z = \text{conv}(z_1, \ldots, z_n) \subseteq \mathbb{R}^d$ . A streaming algorithm  $\mathcal{A}$  receives points  $z_1, \ldots, z_n$  one at a time and produces a sequence of scalings  $\alpha_t$ , centers  $c_t$ , subsets  $S_t \subseteq [n]$  such that  $S_t \subseteq S_{t+1}$ . The algorithm must satisfy the following guarantee at the end of the stream.

$$\operatorname{conv}\left(\{z_i: i \in S_n\}\right) \subseteq \operatorname{conv}\left(\{z_1, \ldots, z_n\}\right) \subseteq c_n + \frac{1}{\alpha} \cdot \operatorname{conv}\left(\{z_i - c_n: i \in S_n\}\right)$$

We say that  $\{z_i : i \in S_n\}$  is a coreset of Z with approximation factor  $1/\alpha_n$ . We will also call  $S_n$  a coreset.

Note that the model considered in Problem 2.2 is essentially the same as the *online coreset model* studied by Woodruff and Yasuda [WY22a]. Similar to Problem 2.1, Problem 2.2 has been studied in the case where *Z* is centrally symmetric. In particular, Woodruff and Yasuda [WY22a] obtain approximation factor  $O(\sqrt{d \log (n \kappa^{OL})})$  (where  $\kappa^{OL}$  is the same online condition number mentioned earlier). However, whether analogous results for asymmetric polytopes hold was an important unresolved question.

#### 2.1.1. Our contributions

In this section, we present our results for Problems 2.1 and 2.2.

#### Algorithmic results

We start with defining several quantities that we need to state the results and describe their proofs.

**Notation.** We will denote the linear span of a set of points *A* by span (*A*). That is, span (*A*) is the minimal linear subspace that contains *A*. We denote the affine span of *A* by Span (*A*). That is, Span (*A*) is the minimal affine subspace that contains *A*. Note that Span (*A*) = a + span (A - a) if  $a \in A$ . Finally, we denote the unit ball centered at the origin by  $B_2^d$ .

**Definition 1** (Inradius). Let  $K \subset \mathbb{R}^d$  be a convex body. The inradius r(K) of K is the largest r such that there exists a point  $c_I$  (called the incenter) for which  $c_I + r \cdot (B_2^d \cap \text{span}(K - c_I)) \subseteq K$ .

**Definition 2** (Circumradius). Let  $K \subset \mathbb{R}^d$  be a convex body. The circumradius R(K) of K is the smallest R such that there exists a point  $c_C$  (called the circumcenter) for which  $K \subseteq c_C + R \cdot B_2^d$ .

**Definition 3** (Aspect Ratio). Let  $K \subset \mathbb{R}^d$  be a convex body. We say that  $\kappa(K) := \frac{R(K)}{r(K)}$  is the aspect ratio of K.

We now state Theorem 1, which provides an algorithm for Problem 2.1. In addition to the data stream of  $z_1, \ldots, z_n$ , this algorithm needs a suitable initialization: a ball  $c_0 + r_0 \cdot B_2^d$  inside Z.

**Theorem 1.** Consider the setting of Problem 2.1. Suppose the algorithm is given an initial center  $c_0$  and radius  $r_0$  for which it is guaranteed that  $c_0 + r_0 \cdot B_2^d \subseteq \text{conv}(\{z_1, \ldots, z_n\})$ . There exists an

algorithm (Algorithm 2) that, for every timestep t, maintains an origin-centered ellipsoid  $\mathcal{E}_t$ , center  $c_t$ , and scaling factor  $\alpha_t$  such that at every timestep t: conv $(\{z_1, \ldots, z_t\}) \subseteq c_t + \mathcal{E}_t$  and at timestep n:  $c_n + \alpha_n \cdot \mathcal{E}_n \subseteq Z \subseteq c_n + \mathcal{E}_n$ , where

$$\frac{1}{\alpha_n} = O\left(\min\left\{\frac{R(Z)}{r_0}, d\log\left(\frac{R(Z)}{r_0}\right)\right\}\right)$$

*The algorithm has runtime*  $\widetilde{O}(nd^2)$  *and stores*  $O(d^2)$  *floating point numbers.* 

We also give an improvement when the convex polytope that is streamed to us is originsymmetric. See Theorem 2.

**Theorem 2.** Consider the setting of Problem 2.1 except in addition to receiving  $z_t$ , we also receive  $-z_t$ . Suppose the algorithm is given an initial center  $c_0$  and radius  $r_0$  for which it is guaranteed that  $c_0 + r_0 \cdot B_2^d \subseteq \text{conv}(\{z_1, \ldots, z_n\})$ . There exists an algorithm that, for every timestep t, maintains an origincentered ellipsoid  $\mathcal{E}_t$ , center  $c_t$ , and scaling factor  $\alpha_t$  such that at every timestep t:  $\text{conv}(\{z_1, \ldots, z_t\}) \subseteq c_t + \mathcal{E}_t$  and at timestep n:  $c_n + \alpha_n \cdot \mathcal{E}_n \subseteq Z \subseteq c_n + \mathcal{E}_n$ , where

$$\frac{1}{\alpha_n} = O\left(\min\left\{\frac{R(Z)}{r_0}, \sqrt{d\log\left(\frac{R(Z)}{r_0}\right)}\right\}\right)$$

*The algorithm has runtime*  $\widetilde{O}(nd^2)$  *and stores*  $O(d^2)$  *floating point numbers.* 

We prove Theorem 2 in Section 2.5.

Note that the final approximation factor depends on the quality of the initialization ( $c_0$ ,  $r_0$ ). If the radius  $r_0$  of this ball is reasonably close to the inradius r(Z) of Z, the algorithm gives an  $O(\min(\kappa(Z), d \log \kappa(Z)))$  approximation. In Theorem 3, we adapt the algorithm form Theorem 1 to the setting where the algorithm does not have the initialization information. Note that the approximation guarantee of  $O(\min(\kappa(Z), d \log \kappa(Z)))$  is a natural analogue of the bounds by [MMO22] and [WY22a] for the symmetric case (see Section 2.1.2).

**Theorem 3.** Consider the setting of Problem 2.1. There exists an algorithm (Algorithm 4) that, for every timestep t, maintains an ellipsoid  $\mathcal{E}_t$ , center  $c_t$ , and approximation factor  $\alpha_t$  such that

$$c_t + \alpha_t \cdot \mathcal{E}_t \subseteq \operatorname{conv}(\{z_1, \ldots, z_t\}) \subseteq c_t + \mathcal{E}_t.$$

Additionally, let  $r_t$  and  $R_t$  be the largest and smallest parameters, respectively, for which there exists  $c_t^*$  such that

$$\boldsymbol{c}_{t}^{\star} + \boldsymbol{r}_{t} \cdot \left(\boldsymbol{B}_{2}^{d} \cap \operatorname{span}\left(\boldsymbol{z}_{1} - \boldsymbol{c}_{t}^{\star}, \ldots, \boldsymbol{z}_{t} - \boldsymbol{c}_{t}^{\star}\right)\right) \subseteq \operatorname{conv}\left(\{\boldsymbol{z}_{1}, \ldots, \boldsymbol{z}_{t}\}\right) \subseteq \boldsymbol{c}_{t}^{\star} + \boldsymbol{R}_{t} \cdot \boldsymbol{B}_{2}^{d}$$

and  $d_t := \dim (\text{Span}(z_1, \dots, z_t))$ . Then, for all timesteps t, we have

$$1/\alpha_t = O\left(d_t \log\left(d_t \cdot \max_{t' \leq t} \frac{R_t}{r_{t'}}\right)\right).$$

*The algorithm runs in time*  $O(nd^2)$  *and stores*  $O(d^2)$  *floating point numbers.* 

Let us now quickly compare the guarantees of Theorem 1 and 3. Notice that the algorithm in Theorem 3 does not require an initialization pair  $(c_0, r_0)$ . Additionally, the algorithm in Theorem 3 outputs a per-timestep approximation as opposed to just an approximation at the end of the stream. However, these advantages come at a cost – it is easy to check that the aspect ratio term seen in Theorem 3 can be larger than that in Theorem 1, e.g., it is possible to have  $R(Z)/r_0 \leq \max_{t' \leq n} \frac{R_n}{r_{t'}}$ .

However, when we impose the additional constraint that the points  $z_t$  have coordinates that are integers in the range [-N, N], we can improve over the guarantee in Theorem 3 and obtain results that are independent of the aspect ratio. This is similar in spirit to the condition number-independent bound that Woodruff and Yasuda [WY22a] obtain for the sums of online leverage scores. However, a key difference is that our results still remain independent of the length of the stream. See Theorem 4.

**Theorem 4.** Consider the setting of Problem 2.1, where in addition, the points  $z_1, ..., z_n$  are such that their coordinates are integers in  $\{-N, -N + 1, ..., N - 1, N\}$ . There exists an algorithm (Algorithm 4) that, for every timestep t, maintains an ellipsoid  $\mathcal{E}_t$ , center  $c_t$ , and approximation factor  $\alpha_t$  such that

 $c_t + \alpha_t \cdot \mathcal{E}_t \subseteq \operatorname{conv}\left(\{z_1, \ldots, z_t\}\right) \subseteq c_t + \mathcal{E}_t.$ 

Let  $d_t := \dim (\text{Span}(z_1, \dots, z_t))$ . Then, for all timesteps t, we have

$$1/\alpha_t = O\left(d_t \log\left(dN\right)\right)$$

*The algorithm runs in time*  $\tilde{O}(nd^2)$  *and stores*  $O(d^2)$  *floating point numbers.* 

We prove Theorems 1, 2, 3, and 4 in Section 2.4. With Theorems 3 and 4 in hand, obtaining results for Problem 2.2 becomes straightforward. We use the algorithm guaranteed by Theorem 3 along with a simple subset selection criterion to arrive at our result for Problem 2.2.

**Theorem 5.** Consider  $Z = \text{conv}(\{z_1, ..., z_n\})$ . For a subset  $S \subseteq [n]$ , let  $Z|_S = \text{conv}(\{z_i : i \in S\})$ . Consider the setting of Problem 2.2. There exists a streaming algorithm (Algorithm 5) that, for every timestep t, maintains a subset  $S_t$ , center  $c_t$ , and scaling factor  $\alpha_t$  such that

$$Z|_{S_t} \subseteq \operatorname{conv}\left(\{z_1,\ldots,z_t\}\right) \subseteq c_t + \frac{1}{\alpha_t} \cdot \left(Z|_{S_t} - c_t\right).$$

Additionally, for  $d_t$ ,  $r_t$  and  $R_t$  as defined in Theorem 3, we have for all t that

$$\frac{1}{\alpha_t} = O\left(d_t \log\left(d_t \cdot \max_{t' \le t} \frac{R_t}{r_{t'}}\right)\right) \qquad and \qquad |S_t| = O\left(d_t \log\left(\max_{t' \le t} \frac{R_t}{r_{t'}}\right)\right),$$

and, if the  $z_t$  have integer coordinates ranging in [-N, N], then

$$\frac{1}{\alpha_t} = O\left(d_t \log\left(dN\right)\right) \qquad and \qquad |S_t| = O\left(d_t \log\left(dN\right)\right).$$

Each  $S_t$  is either  $S_{t-1}$  or  $S_{t-1} \cup \{t\}$  (where  $t \ge 1$  and  $S_0 = \emptyset$ ). The algorithm runs in time  $O(nd^2)$  and stores at most  $O(d^2)$  floating point numbers.

We prove Theorem 5 in Section 2.6.

#### Approximability lower bounds

A natural question is, "how closely can any one-pass monotonic algorithm approximate the minimum-volume outer ellipsoid for a centrally-symmetric convex body?" We formalize this notion below.

**Definition 2.1.1** (Approximation to Minimum Volume Outer Ellipsoid). We say a streaming algorithm A  $\alpha$ -approximates the minimum volume outer ellipsoid if A outputs an ellipsoid  $\mathcal{E}_n$  satisfying  $\mathcal{E}_n \subseteq \alpha \cdot J(X)$ , where J(X) is the minimum volume outer ellipsoid for X.

Theorem 6 asserts that for a natural class of streaming algorithms, it is not possible to approximate the minimum volume outer ellipsoid up to factor  $< \sqrt{d}$  in the worst case.

**Theorem 6.** Every one-pass monotone deterministic streaming algorithm for Problem 2.1 in the symmetric case (i.e., when we receive  $z_t$ , we also receive  $-z_t$ ) has approximation factor to the minimum volume outer ellipsoid of at least  $\sqrt{d}$ , for infinitely many d.

We prove Theorem 6 in Section 2.7.

We now address another natural question. Observe that the approximation factors obtained in Theorems 1, 2, 3, and 5 all incur a mild dependence on (variants of) the aspect ratio of the dataset. A natural question is whether this dependence is necessary. In Theorem 7, we conclude that the approximation factor from Theorem 1 is in fact nearly optimal for a wide class of *monotone* algorithms. We defer the discussion of the notion of a monotone algorithm to Section 2.2.1. Loosely speaking, a monotone algorithm commits to the choices it makes; namely, the outer ellipsoid may only increase over time  $c_t + \mathcal{E}_t \supseteq c_{t-1} + \mathcal{E}_{t-1}$  and the inner ellipsoid  $c_t + \alpha_t \mathcal{E}_t$  satisfies a related but more technical condition  $c_t + \alpha_t \mathcal{E}_t \subseteq \operatorname{conv}((c_{t-1} + \alpha_{t-1} \cdot \mathcal{E}_{t-1}) \cup \{z_t\})$ .

**Theorem 7.** Consider the setting of Problem 2.1. Let  $\mathcal{A}$  be any monotone algorithm (see Definition 4 in Section 2.2.1) that solves Problem 2.1 with approximation factor  $1/\alpha_n$ . For every  $d \ge 2$ , there exists a sequence of points  $\{z_1, \ldots, z_n\} \subset \mathbb{R}^d$  such that algorithm  $\mathcal{A}$  gets an approximation factor of  $1/\alpha_n \ge \Omega\left(\frac{d \log(\kappa(Z))}{\log d}\right)$  on  $Z = \operatorname{conv}(\{z_1, \ldots, z_n\})$ .

We prove Theorem 7 in Section 2.7.

#### 2.1.2. Related work and open questions

**Streaming asymmetric ellipsoidal roundings.** To our knowledge, the first paper to study ellipsoidal roundings in the streaming model is that of Mukhopadhyay, Greene, Sarker, and Switzer [MGSS09]. The authors consider the case where d = 2 and prove that the approximation factor of the greedy algorithm (that which updates the ellipsoid to be the minimum volume ellipsoid containing the new point and the previous iterate) can be unbounded. Subsequent work by Mukhopadhyay, Sarker, and Switzer [MSS10] generalizes this result to all  $d \ge 2$ .

**Nearly-optimal streaming symmetric ellipsoidal roundings.** Recently, Makarychev, Manoj, and Ovsiankin [MMO22], and Woodruff and Yasuda [WY22a] gave the first positive results for

streaming ellipsoidal roundings. Both [MMO22] and [WY22a] considered the problem only in the *symmetric setting* – when the goal is to approximate the polytope conv ( $\{\pm z_1, \ldots, \pm z_n\}$ ). [MMO22] and [WY22a] obtained  $O(\sqrt{d \log \kappa(Z)})$  and  $O(\sqrt{d \log n\kappa^{OL}})$ -approximations, respectively (here,  $\kappa^{OL}$  is the online condition number; see [WY22a] for details). Their algorithms use only  $\tilde{O}(\text{poly}(d))$  space, where the  $\tilde{O}$  suppresses  $\log d$ ,  $\log n$ , and aspect ratio-like terms. Note that by John's theorem, the  $\Omega(\sqrt{d})$  dependence is required in the symmetric setting even for offline algorithms.

A natural question is whether the techniques of [MMO22] or [WY22a] extend to Problems 2.1 and 2.2. The update rule used in [MMO22] essentially updates  $\mathcal{E}_{t+1}$  to be the minimum volume ellipsoid covering both  $\mathcal{E}_t$  and points  $\pm z_{t+1}$ . In the non-symmetric case, it would be natural to consider the minimum volume ellipsoid covering  $\mathcal{E}_t$  and point  $z_{t+1}$ . However, this approach does not give an  $\tilde{O}(d)$  approximation. The algorithm in [WY22a] maintains a quadratic form that consists of sums of outer products of "important points" (technically speaking, those with a constant online leverage score). Unfortunately, this approach does not suggest how to move the previous center  $c_{t-1}$  to a new center  $c_t$  in a way that allows the algorithm to maintain a good approximation factor. It is not hard to see that there exist example streams for which the center  $c_{t-1}$  must be shifted in each iteration to Problems 2.1 and 2.2 must overcome this difficulty.

**Offline ellipsoidal roundings for general convex polytopes.** Nesterov [Nes08] gives an efficient *offline* O(d)-approximation algorithm for the ellipsoidal rounding problem, with a runtime of  $\tilde{O}(nd^2)$ . Observe that this is essentially the same runtime as those achieved by the algorithms we give (see Theorems 1 and 3).

**Streaming convex hull approximations.** Agarwal and Sharathkumar [AS10] studied related problems of computing *extent measures* of a convex hull in the streaming model, in particular finding coresets for the minimum enclosing ball, and obtained both positive and negative results. Blum, Braverman, Kumar, Lang, and Yang [BBKLY18] showed that one cannot maintain an  $\varepsilon$ -*hull* in space proportional to the number of vertices belonging to the offline optimal solution (where a body  $\hat{Z}$  is an  $\varepsilon$ -hull for Z if every point in  $\hat{Z}$  is distance at most  $\varepsilon$  away from Z).

**Offline convex hull approximations.** The problem of approximating a convex body with the convex hull of a small number of points belonging to the body has been well-studied. Existentially, Barvinok [Bar14] shows that if the input convex set is sufficiently symmetric, then one can choose  $(d/\varepsilon)^{d/2}$  points to obtain a  $1 + \varepsilon$  approximation. Moreover, Lu [Lu20] shows that one can obtain a d + 2 approximation with d + 1 points, which is witnessed by choosing the d + 1 points to be the maximum volume simplex contained within the convex body (for this reason, this construction is called "John's Theorem for simplices"; see [PS20] for more details). However, none of these works study a streaming or online setting, as we do here.

**Coresets for the minimum volume enclosing ellipsoid problem (MVEE).** Let MVEE(K) denote the minimum volume enclosing ellipsoid for a convex body  $K \subset \mathbb{R}^d$ . We say that a subset  $S \subseteq [n]$  is an  $\varepsilon$ -coreset for the MVEE problem if we have

$$\operatorname{vol}\left(\mathsf{MVEE}(Z)\right) \le \left(1+\varepsilon\right)^d \operatorname{vol}\left(\mathsf{MVEE}(Z|_S)\right). \tag{2.1.4}$$

There is extensive literature on coresets for the MVEE problem, and we refer the reader to papers by Kumar and Yildirim [KY05], Todd and Yildirim [TY07], Clarkson [Cla10], Bhaskara, Mahabadi, and Vakilian [BMV23], and the book by Todd [Tod16].

Importantly,  $MVEE(Z|_S)$  may not be a good approximation for MVEE(Z) (for that reason, some authors refer to coresets satisfying (2.1.4) as weak coresets for MVEE). Therefore, even though MVEE(Z) provides a good ellipsoidal rounding for Z,  $MVEE(Z|_S)$  generally speaking does not. See [TY07, page 2] and [BMV23, Section 2.1] for an extended discussion.

#### 2.2. Summary of techniques

In this section, we give an overview of the technical methods behind our results.

#### 2.2.1. Monotone algorithms

The algorithm we give in Theorem 1 belongs to a class we term *monotone algorithms*, which we now define.

**Definition 4** (Monotone algorithm). *Consider the setting of Problem 2.1. Note the following invariants for every timestep t.* 

$$c_t + \mathcal{E}_t \supseteq \operatorname{conv}\left((c_{t-1} + \mathcal{E}_{t-1}) \cup \{z_t\}\right)$$
(2.2.1)

$$c_t + \alpha_t \mathcal{E}_t \subseteq \operatorname{conv}\left(\left(c_{t-1} + \alpha_{t-1} \cdot \mathcal{E}_{t-1}\right) \cup \{z_t\}\right)$$
(2.2.2)

We say that an algorithm  $\mathcal{A}$  is monotone if for any initial  $(c_0 + \mathcal{E}_0, \alpha_0)$  and sequence of data points  $z_1, \ldots, z_n$ , the resulting sequence  $\{(c_0 + \mathcal{E}_0, \alpha_0), (c_1 + \mathcal{E}_1, \alpha_1), \ldots, (c_n + \mathcal{E}_n, \alpha_n)\}$  arising from applying  $\mathcal{A}$  to the stream satisfies the two invariants (2.2.1) and (2.2.2). Refer to Figure 2.1.

We will sometimes consider how a monotone algorithm  $\mathcal{A}$  makes a single update upon seeing a new point x. In this setting, we will call  $\mathcal{A}$  a monotone update rule.

Here we will refer to  $c_t + \mathcal{E}_t$ ,  $c + \alpha_t \mathcal{E}_t$  as the "next" ellipsoids and to  $c_{t-1} + \mathcal{E}_{t-1}$ ,  $c + \alpha_{t-1} \mathcal{E}_{t-1}$  as the "previous" ellipsoids. The first condition we require is that

$$\boldsymbol{c}_t + \boldsymbol{\mathcal{E}}_t \supseteq \boldsymbol{c}_{t-1} + \boldsymbol{\mathcal{E}}_{t-1}. \tag{2.2.1a}$$

It ensures that each successive outer ellipsoid contains the previous outer ellipsoid. Thus once the algorithm decides that some  $z \in c_t + \mathcal{E}_t$ , it makes a commitment that  $z \in c_{t'} + \mathcal{E}_{t'}$  for all  $t' \ge t$ . Note that (2.2.1a) implies (2.2.1), since  $z_t$  must be in  $c_t + \mathcal{E}_t$  and  $c_t + \mathcal{E}_t$  is convex. The second condition (2.2.2) looks more complex but is also very natural. Assume that the algorithm only knows that (a)  $c_{t-1} + \alpha_{t-1}\mathcal{E}_{t-1} \subseteq Z$  (this is true from induction) and (b)  $z_t \in Z$  (this is true by the definition of Z). Then, we must have that  $A = \operatorname{conv}((c_{t-1} + \alpha_{t-1} \cdot \mathcal{E}_{t-1}) \cup \{z_t\})$  lies in Z; as far as the algorithm is concerned, any point outside of A may also be outside of Z. Since the algorithm must ensure that  $c_t + \alpha_t \mathcal{E}_t \subseteq Z$ , it will also ensure that  $c_t + \alpha_t \mathcal{E}_t \subseteq A$  and thus satisfy (2.2.2).



Figure 2.1.: A monotone update step. For brevity, we refer to  $\mathcal{E}$  and  $\alpha \cdot \mathcal{E}$  as the previous ellipsoids  $\mathcal{E}_{t-1}$ ,  $\alpha \mathcal{E}_{t-1}$ , and  $\mathcal{E}'$  and  $\alpha' \cdot \mathcal{E}'$  as the next ellipsoids  $\mathcal{E}_t$ ,  $\alpha_t \cdot \mathcal{E}_t$ .  $\mathcal{E}$  and  $\alpha \mathcal{E}$  are, respectively, the larger and smaller black circles.  $c + \mathcal{E}'$  and  $c + \alpha' \mathcal{E}'$  are the larger and smaller blue ellipses. The dotted lines show  $\partial(\operatorname{conv}(\alpha \mathcal{E} \cup \{z\})) \setminus \partial(\alpha \mathcal{E})$ , i.e. the the boundary of  $\operatorname{conv}(\alpha \cdot \mathcal{E} \cup \{z\})$  minus the boundary of  $\alpha \mathcal{E}$ .

#### 2.2.2. Streaming ellipsoidal rounding (Theorems 1, 3, and 4)

Now we describe the algorithm from Theorem 1 in more detail. Our algorithm keeps track of the current ellipsoid  $\mathcal{E}_t$ , center  $c_t$ , and scaling parameter  $\alpha_t$ . Initially,  $c_0 + \mathcal{E}_0$  is the ball of radius  $r_0$  around  $c_0$  ( $r_0$  and  $c_0$  are given to the algorithm), and  $\alpha_0 = 1$ . Each time the algorithm gets a new point  $z_t$ , it updates  $\mathcal{E}_{t-1}$ ,  $c_{t-1}$ ,  $\alpha_{t-1}$  using a *monotone* update rule (as defined in Definition 4) and obtains  $\mathcal{E}_t$ ,  $c_t$ ,  $\alpha_t$ . The monotonicity condition is sufficient to guarantee that the algorithm gets a  $1/\alpha_n$  approximation to *Z*. Indeed, first using condition (2.2.1), we get

$$c_n + \mathcal{E}_n \supseteq (c_{n-1} + \mathcal{E}_{n-1}) \cup \{z_n\} \supseteq (c_{n-2} + \mathcal{E}_{n-2}) \cup \{z_{n-1}, z_n\} \supseteq \cdots \supseteq \{z_1, \ldots, z_n\}.$$

Thus,  $c_n + \mathcal{E}_n \supseteq Z$ . Then, using condition (2.2.2), we get

$$c_n + \alpha_n \mathcal{E}_n \subseteq \operatorname{conv}\left((c_{n-1} + \alpha_{n-1} \mathcal{E}_{n-1}) \cup \{z_n\}\right) \subseteq \operatorname{conv}\left((c_{n-2} + \alpha_{n-2} \mathcal{E}_{n-2}) \cup \{z_{n-1}, z_n\}\right)$$
$$\subseteq \cdots \subseteq \operatorname{conv}\left((c_0 + \alpha_0 \mathcal{E}_0) \cup \{z_1, \dots, z_n\}\right).$$

The initial ellipsoid  $c_0 + \alpha_0 \mathcal{E}_0 = c_0 + r_0 B_2^d$  is in *Z* and therefore  $c_n + \alpha_n \mathcal{E}_n \subseteq \text{conv}(z_1, \ldots, z_n) = Z$ . We verified that the algorithm finds a  $1/\alpha_n$  approximation for *Z*.

Now, the main challenge is to design an update rule that ensures that  $1/\alpha_n$  is small (as in the statement Theorem 1) and prove that the rule satisfies the monotonicity conditions/invariants from Definition 4. We proceed as follows.

First, we design a monotone update rule that satisfies a particular evolution condition. This condition upper bounds the increase of the approximation factor  $1/\alpha_t - 1/\alpha_{t-1}$ . Second, we prove that any monotone update rule satisfying the evolution condition yields the approximation we desire. These two parts imply Theorem 1. Finally, we remove the initialization requirement from Theorem 1 and obtain Theorem 3.

**Designing a monotone update rule.** Suppose that at the end of timestep t - 1 our solution consists of a center  $c_{t-1}$ , ellipsoid  $\mathcal{E}_{t-1}$ , and scaling parameter  $\alpha_{t-1}$  for which the invariants in

Definition 4 hold. We give a procedure that, given the next point  $z_t$ , computes  $c_t$ ,  $\mathcal{E}_t$ ,  $\alpha_t$  that still satisfy the invariants of Definition 4. Further, we prove that the resulting update satisfies an evolution condition (2.2.3)

$$\frac{1/\alpha_t - 1/\alpha_{t-1}}{\log \operatorname{vol}(\mathcal{E}_t) - \log \operatorname{vol}(\mathcal{E}_{t-1})} \le C,$$
(2.2.3)

where *C* is an absolute constant and vol $\mathcal{E}$  denotes the volume of the ellipsoid  $\mathcal{E}$ . While it is possible to find the optimal update using convex optimization (the update that satisfies the invariants and minimizes the ratio on the left of (2.2.3)), we instead provide an explicit formula for an update that readily satisfies (2.2.3) and as we show is monotone.

We now describe how we get the formula for the update rule. By applying an affine transformation, we may assume that  $\mathcal{E}_{t-1}$  is a unit ball and  $c_{t-1} = 0$ . Further, we may assume that  $z_t$  is colinear with  $e_1$  (the first basis vector):  $z_t = ||z_t||_2 e_1$ . Importantly, affine transformations preserve (a) the invariants in Definition 4 (if they hold for the original ellipsoids and points, then they also do for the transformed ones and vice versa) and (b) the value of the ratio in (2.2.3), since they preserve the value of  $vol(\mathcal{E}_t)/vol(\mathcal{E}_{t-1})$ .

Now consider the group  $G = \mathbb{O}(d)_{e_1} \cong \mathbb{O}(d-1)$  of orthogonal transformations that map  $e_1$  to itself: all of them map the unit ball  $\mathcal{E}_{t-1}$  to itself and  $z_t$  to itself. Thus, it is natural to search for an update  $(c_t, \mathcal{E}_t)$  that is symmetric with respect to all these transformations. It is easy to see that in this case  $\mathcal{E}_t$  is defined by equation  $(x_1/a)^2 + \sum_{i=2}^d (x_i/b)^2 = 1$  where a and b are some parameters (equal to the semiaxes of  $\mathcal{E}_t$ ) and  $c_t = ce_1$  for some c. Since all ellipsoids and points appearing in the invariant conditions are symmetric with respect to G, it is sufficient now to restrict our attention to their sections in the 2d-plane span  $(e_1, e_2)$  and prove that the invariants hold in this plane. Hence, the problem reduces to a statement in two-dimensional Euclidean geometry (however, when we analyze (2.2.3), we still use that the volume of  $\mathcal{E}_t$  is proportional to  $ab^{d-1}$  and not ab).

Let us denote the coordinates corresponding to basis vectors  $e_1$  and  $e_2$  by x and y. For brevity, let  $\mathcal{E} = \mathcal{E}_{t-1}$ ,  $z = z_t$ ,  $\mathcal{E}' = \mathcal{E}_t$ ,  $c = c_t = ce_1$ ,  $\alpha = \alpha_{t-1}$ , and  $\alpha' = \alpha_t$ . We now need to choose parameters a, b, and c so that invariants from Definition 4 and (2.2.3) hold. See Figure 2.1. As shown in that figure, the new outer ellipse  $c + \mathcal{E}'$  must contain the previous outer ellipse  $\mathcal{E}$  and the newly received point z. The new inner ellipse  $c + \alpha' \mathcal{E}'$  must be contained within the convex hull of the previous inner ellipse  $\alpha \mathcal{E}$  and z.

It is instructive to consider what happens when point z is at infinitesimal distance  $\Delta$  from  $\mathcal{E}$ :  $||z||_2 = 1 + \Delta$ . We consider a minimal axis-parallel outer ellipse  $\mathcal{E}'$  that contains  $\mathcal{E}$  and z. It must go through  $z = (1 + \Delta, 0)$  and touch  $\mathcal{E}$  at two points symmetric w.r.t. the *x*-axis, say,  $(-\sin \varphi, \pm \cos \varphi)$ . Angle  $\varphi$  uniquely determines  $\mathcal{E}'$ . Now we want to find the largest value of the scaling parameter  $\alpha'$  so that  $\alpha'\mathcal{E}'$  fits inside the convex hull of  $\mathcal{E}$  and z. When  $\Delta$  is infinitesimal, this condition splits into two lower bounds on  $\alpha'$  – loosely speaking, they say that  $\mathcal{E}$  does not extend out beyond the convex hull in the horizontal (one bound) and vertical directions (the other). The former bound becomes stronger (gives a smaller upper bound on  $\alpha'$ ) when  $\varphi$  increases, and the latter becomes stronger when  $\varphi$  decreases. When  $\varphi = \alpha/2 \pm O(\alpha^2)$ , then all terms linear in  $\alpha$  vanish in both bounds and then  $\alpha' = \alpha - \Theta(\alpha^2 \Delta)$  satisfies both of them; for other choices of  $\varphi$ , we have  $\alpha' \leq \alpha - \Omega(\alpha \Delta)$ . So we let  $\varphi = \alpha/2$  and from the formula for  $\alpha'$  get  $1/\alpha' = 1/\alpha + O(\Delta)$ . On the other hand,  $vol(\mathcal{E}') \geq (1 + \Delta/2)vol(\mathcal{E})$ , since  $\mathcal{E}'$  covers  $z = (1 + \Delta, 0)$ . It is easy to see now that the evolution condition (2.2.3) holds: the numerator is  $O(\Delta)$  and the denominator is  $\Omega(\Delta)$  in (2.2.3).

We remark that letting  $c + \mathcal{E}'$  be the minimum volume ellipsoid that contains  $\mathcal{E}$  and z is a

highly suboptimal choice (it corresponds to setting  $\varphi = \Theta(1/d)$ ). To derive our specific update formulas for arbitrary z, we, loosely speaking, represent an arbitrary update as a series of infinitesimal updates, get a differential equation on a, b, c, and  $\alpha'$ , solve it, and then simplify the solution (remove non-essential terms, etc). We get the following.

Our updates come from a family parameterized by  $\gamma \ge 0$ . Define  $\alpha'$  by  $1/\alpha' = 1/\alpha + 2\gamma$ . With this choice of  $\alpha'$ , define the new ellipses to be

$$\underbrace{\frac{1}{a^2}(x-c)^2 + \frac{1}{b^2}y^2 = 1}_{c+\mathcal{E}'}, \qquad \underbrace{\frac{1}{a^2}(x-c)^2 + \frac{1}{b^2}y^2 = \alpha'^2}_{c+\alpha'\mathcal{E}'}$$

where we use parameters

$$\left. \begin{array}{l} a = \exp\left(\gamma\right) \\ b = 1 + \frac{\alpha - \alpha'}{2} \\ c = -\alpha + \alpha' \cdot a \end{array} \right\}$$

Choose  $\gamma \approx \ln \|\mathbf{z}\|_2$  so that  $\mathbf{c} + \mathbf{\mathcal{E}'}$  covers point  $\mathbf{z}$ . We use two-dimensional geometry to prove that  $\mathbf{\mathcal{E}'}$ ,  $\mathbf{c}$ , and  $\alpha'$  satisfy the invariants (see Figure 2.1). Now to prove the evolution condition, we observe two key properties: (1) the increase in the approximation factor is given by  $\frac{1}{\alpha'} - \frac{1}{\alpha} = 2\gamma$  and (2) the length of the horizontal semiaxis of the new outer ellipse is  $\exp(\gamma)$ . The length of the vertical semiaxis is at least 1, so by the second property we have  $\log \operatorname{vol}(\mathbf{\mathcal{E}'}) - \log \operatorname{vol}(\mathbf{\mathcal{E}}) \ge \gamma$ . We combine this with the first property to prove that this update satisfies the evolution condition (2.2.3).

Finally, we obtain an upper bound on  $1/\alpha_n$  from the evolution equation. We have

$$\frac{1}{\alpha_n} = \frac{1}{\alpha_0} + \sum_{t=1}^n \left( \frac{1}{\alpha_t} - \frac{1}{\alpha_{t-1}} \right) \stackrel{\text{(by 2.2.3)}}{\leq} 1 + C \sum_{t=1}^n (\log \operatorname{vol}(\mathcal{E}_t) - \log \operatorname{vol}(\mathcal{E}_{t-1})) = 1 + C \log \frac{\operatorname{vol}(\mathcal{E}_n)}{\operatorname{vol}(\mathcal{E}_0)}.$$

It remains to get an upper bound on  $vol(\mathcal{E}_n)$ . We know that  $\mathcal{E}_n$  approximates Z, and Z, in turn, is contained in the ball of radius R(Z). Loosely speaking, we get  $vol(\mathcal{E}_n) \approx vol(Z) \leq R(Z)^d vol(B_2^d)$ . Since  $\mathcal{E}_0$  is the ball of radius r,  $vol\mathcal{E}_0 = r^d vol(B_2^d)$ . We conclude that the approximation factor is at most  $1/\alpha_n \leq 1 + C \log \frac{R(Z)^d}{r^d} = 1 + O(d \log \frac{R(Z)}{r})$ , as desired.

**Removing the initialization assumption.** Once we have a monotone update rule and guarantee on its approximation factor, we have to convert this to a guarantee where the algorithm does not have access to the initialization.

One natural approach is as follows. Let  $d' \leq d$  be the largest timestep for which points  $z_1, \ldots, z_{d'+1}$  are in general position. We can compute the John ellipsoid for conv ( $\{z_1, \ldots, z_{d'+1}\}$ ) and after that apply the monotone update rule guaranteed by Theorem 1 to obtain the rounding for every  $t \geq d' + 2$ , so long as for every such timestep we have  $z_t \in \text{Span}(z_1, \ldots, z_{t-1})$ .

The principal difficulty in this approach is designing an *irregular update step* that will handle points  $z_t$  outside of Span  $(z_1, \ldots, z_{t-1})$ ; when we add these points the dimensionality of the affine hull increases by 1. We consider the special case where the new point  $z_t$  is conveniently located with respect to our previous ellipsoid  $\mathcal{E}_{t-1}$  (see Figure 2.2 for a 2d-picture). Specifically,  $\mathcal{E}_{t-1}$  is the unit ball in span  $(e_1, \ldots, e_{d'})$ , and the new point  $z_t = (0, \ldots, 0, \sqrt{1+2\alpha}), 0, \ldots)$ . In  $z_t$ , only coordinate d' + 1 is nonzero. We show that we can design an irregular update step for this special case that makes the new approximation factor  $1/\alpha_t$  satisfy  $1/\alpha_t = 1/\alpha_{t-1} + 1$ .



Figure 2.2.: Irregular update step.  $\mathcal{E}_{t-1}$  and  $\alpha \cdot \mathcal{E}_{t-1}$  are, respectively, the light blue strip on the *x*-axis and the dark blue strip on the *x*-axis.  $z_t = (0, \sqrt{1+2\alpha})$  is the newly received point.

It turns out that it is sufficient to consider only this special case. To see this, note that we can choose an affine transformation that maps any new point  $z_t$  and previous ellipsoid  $\mathcal{E}_{t-1}$  to the setting shown in Figure 2.2. Next, observe that there are at most d - 1 irregular update steps. This means that the irregular update steps contribute at most an additive d - 1 to the final approximation factor.

Finally, observe that the inradius of conv  $(\{z_1, \ldots, z_t\})$  is not monotone in t. In particular, it can decrease after each irregular update step. Nonetheless, we can still give a bound on the radius of a ball that our convex body conv  $(z_1, \ldots, z_t)$  contains for all t. This will give us everything we need to apply Theorem 1 to this setting, and Theorem 3 follows.

**Improved bounds on lattices.** Finally, we briefly discuss how to remove the aspect ratio dependence in the setting where the input points  $z_t$  have coordinates in [-N, N]. At a high level, this improvement follows from carefully tracking how the approximation factors of our solutions change after an irregular update step. Following (2.2.3), recall that our goal is to analyze (where we write  $\alpha_0 = 1$ )

$$\sum_{t>1} \frac{1}{\alpha_t} - \frac{1}{\alpha_{t-1}}$$

By (2.2.3), we see that for all "regular" updates, we have

$$\frac{1}{\alpha_t} - \frac{1}{\alpha_{t-1}} \lesssim \log \left( \frac{\operatorname{vol}_{d_t}(\mathcal{E}_t)}{\operatorname{vol}_{d_t}(\mathcal{E}_{t-1})} \right),$$

where  $d_t = \dim (\text{Span}(z_1, \ldots, z_t))$ . Furthermore, as previously mentioned, in our irregular update step, we get

$$\frac{1}{\alpha_t} - \frac{1}{\alpha_{t-1}} = 1.$$

In order to control the sum of the  $1/\alpha_t - 1/\alpha_{t-1}$ , it remains to bound  $\operatorname{vol}_{d_t}(\mathcal{E}_t)/\operatorname{vol}_{d_{t-1}}(\mathcal{E}_{t-1})$  for an irregular update step *t*. We will then get a telescoping upper bound whose last term is the ratio of the volume of the final ellipsoid to the Euclidean ball in the same affine span.

Similarly to the improvements of Woodruff and Yasuda [WY22a] in the integer-valued case, it will turn out that we will be interested in the total product of these volume changes. By carefully tracking these, we will get that this product can be expressed as the determinant of a particular integer-valued matrix. Then, since this matrix has integer entries, the magnitude of its determinant must be at least 1. We then observe that the volume of  $\mathcal{E}_n$  after normalizing by the volume of vol $(B_2^{d_n})$  must be at most  $(N\sqrt{d})^{d_n}$ , since the length of any vector in this lattice is at most  $N\sqrt{d}$ . The desired result then follows.

#### 2.2.3. Coresets for convex hull (Theorem 5)

We now outline our proof strategy for Theorem 5. Our main task is to design an appropriate selection criterion for every new point – in other words, we must check whether a new point  $z_t$  is "important enough" to be added to our previous set of points  $S_{t-1}$ . We then have to show that this selection criterion yields the approximation guarantee promised by Theorem 5.

To design the selection criterion, we run an instance of the algorithm in Theorem 3 on the stream. For every new point  $z_t$ , we ask two questions – "Does  $z_t$  result in an irregular update step? Does it cause vol( $\mathcal{E}_t$ ) to be much larger than vol( $\mathcal{E}_{t-1}$ )?" If the answer to any of these questions is affirmative, we add  $z_t$  to the coreset. The first question is necessary to obtain even a bounded approximation factor (for example, imagine that the final point  $z_n$  results in an irregular update step – then, we must add it). The second question is quite natural, as it ensures that the algorithm adds "important points" – those that necessitate a significant update.

We now observe that at every irregular update step  $t_{d'}$  for  $d' \le d$  and subsequent timestep  $t \ge t_{d'}$  for which there are no irregular update steps in between  $t_{d'}$  and t, there exists a translation  $c_{d'}$  (which is the center for  $\mathcal{E}_{d'}$  that the algorithm maintains) and a value  $r_{d'}$  for which we know

$$\boldsymbol{c}_{d'} + \boldsymbol{r}_{d'} \cdot \left( B_2^d \cap \operatorname{span}\left(\boldsymbol{z}_1 - \boldsymbol{c}_{d'}, \dots, \boldsymbol{z}_{d'} - \boldsymbol{c}_{d'} \right) \right) \subseteq \operatorname{conv}\left(\boldsymbol{z}_1, \dots, \boldsymbol{z}_t\right) \subseteq \boldsymbol{c}_{\mathsf{C}} + \boldsymbol{R}_t \cdot B_2^d$$

where  $c_C$  is the circumcenter of conv ({ $z_1, ..., z_t$ }). The resulting bound on | $S_t$ | follows easily from the above observation and a simple volume argument.

Finally, we obtain the approximation guarantee from noting that for all t, the output of the algorithm from Theorem 3 given the first t points is the same as running it only on the points selected by  $S_t$ .

#### 2.2.4. Lower bound (Theorem 7)

Whereas in the upper bound we demonstrated a particular algorithm that satisfies the evolution condition (2.2.3), for the lower bound it suffices to show that for any monotone algorithm, there exists an instance of the problem (a sequence of  $z_1, ..., z_n$ ) where the algorithm must satisfy the "reverse evolution condition", i.e.

$$\frac{1/\alpha_t - 1/\alpha_{t-1}}{\log \operatorname{vol}(\mathcal{E}_t) - \log \operatorname{vol}(\mathcal{E}_{t-1})} \ge C$$
(2.2.4)

for some C > 0. In analogy to the argument of the upper bound, showing this reverse evolution condition yields a lower bound of the form  $\frac{1}{\alpha_n} \ge \widetilde{\Omega}(d \log(\kappa))$ . Given any monotone algorithm  $\mathcal{A}$ , the instance we use is produced by an adversary that repeatedly feeds  $\mathcal{A}$  a point that is a constant factor away from the previous ellipsoid.

In order to simplify showing this reverse evolution condition, we use a symmetrization argument. Specifically, by a particular sequence of Steiner symmetrizations, we see that the optimal response of  $\mathcal{A}$  can be completely described in two dimensions. Thus, it is sufficient to only show this reverse evolution condition in the two-dimensional case where the previous outer ellipsoid is the unit ball.

This transformed two-dimensional setting is significantly simpler to analyze. Specifically, we can assume that the point given by the adversary is always  $2e_1$ . The rest of the argument proceeds by cases, again using two-dimensional Euclidean geometry. On a high level, the constraints placed on the new outer and inner ellipsoid by the monotonicity condition force the update of  $\mathcal{A}$  to satisfy the reverse evolution condition.

#### 2.3. Preliminaries

#### 2.3.1. Notation

We denote the standard Euclidean norm of a vector v by ||v|| and the Frobenius norm of a matrix  $\mathbf{A}$  by  $||\mathbf{A}||_F$ . We denote the singular values of a matrix  $\mathbf{A} \in \mathbb{R}^{d \times d}$  by  $\sigma_1(\mathbf{A}), \ldots, \sigma_d(\mathbf{A})$ . Let  $\sigma_{\max}(\mathbf{A})$  and  $\sigma_{\min}(\mathbf{A})$  be the largest and smallest singular values of  $\mathbf{A}$ , respectively. We write diag  $(a_1, \ldots, a_d)$  to mean the  $d \times d$  diagonal matrix whose diagonal entries are  $a_1, \ldots, a_d$ . We use  $\mathbf{S}_{++}^d$  to denote the set of  $d \times d$  positive definite matrices. We use  $e_1, \ldots, e_d$  for the standard basis in  $\mathbb{R}^d$ .

Denote the  $\ell_2$ -unit ball by  $B_2^d = \{x \in \mathbb{R}^d : ||x|| \le 1\}$ , and  $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : ||x||_2 = 1\}$  the unit Euclidean sphere. We use  $\partial S$  for the boundary of an arbitrary set S. We use natural logarithms unless otherwise specified.

In this chapter, we will work extensively with ellipsoids. We will always assume that all ellipsoids and balls we consider are centered at the origin. We use the following representation of ellipsoids. For a non-singular matrix  $\mathbf{A} \in \mathbb{R}^{d \times d}$ , let  $\mathcal{E}_{\mathbf{A}} \coloneqq \{\mathbf{x} : \|\mathbf{A}\mathbf{x}\| \le 1\}$ . In other words, the matrix  $\mathbf{A}$  defines an bijective linear map satisfying  $\mathbf{A}\mathcal{E}_{\mathbf{A}} = B_2^d$ . Every full-dimensional ellipsoid (centered at the origin) has such a representation. We note that this representation is not unique as matrices  $\mathbf{A}$  and  $\mathbf{M}\mathbf{A}$  define the same ellipsoid if matrix  $\mathbf{M}$  is orthogonal (since  $\|\mathbf{A}\mathbf{v}\| = \|\mathbf{M}\mathbf{A}\mathbf{v}\|$  for every vector  $\mathbf{v}$ ). Sometimes, we will have to consider lower-dimensional ellipsoids within an ambient space of higher dimension; in this case, we will use the notation  $\mathcal{E} \cap H$  where H is some linear or affine subspace – note that  $\mathcal{E} \cap H$  is also an ellipsoid.

Now consider the singular value decomposition of **A**:  $\mathbf{A} = \mathbf{U}\Sigma^{-1}\mathbf{V}^T$  (it will be convenient for us to write  $\Sigma^{-1}$  instead of standard  $\Sigma$  in the decomposition). The diagonal entries of  $\Sigma$  are exactly the semi-axes of  $\mathcal{E}_{\mathbf{A}}$ . As mentioned above, matrices  $\mathbf{U}\Sigma^{-1}\mathbf{V}^T$  and  $\mathbf{U}'\Sigma^{-1}\mathbf{V}^T$  define the same ellipsoid for any orthogonal  $\mathbf{U}' \in \mathbb{R}^{d \times d}$ ; in particular, every ellipsoid can be represented by a matrix of the form  $\mathbf{A} = \Sigma^{-1}\mathbf{V}^T$ .

#### 2.3.2. Geometry

We restate the well-known result that five points determine an ellipse. This is usually phrased for conics, but for nondegenerate ellipses the usual condition that no three of the five points are collinear is vacuously true.
**Lemma 2.3.1** (Five points determine an ellipse). Let  $c_1 + \partial \mathcal{E}_1$ ,  $c_2 + \partial \mathcal{E}_2$  be two ellipses in  $\mathbb{R}^2$ . If they intersect at five distinct points, then  $c_1 + \partial \mathcal{E}_1$  and  $c_2 + \partial \mathcal{E}_2$  are the same.

The following claim, that every full-rank ellipsoid (i.e. an ellipsoid whose span has full dimension) can be represented by a positive definite matrix, follows from looking at the singular value decomposition of **A**.

**Lemma 2.3.2.** Let  $\mathcal{E} \subseteq \mathbb{R}^d$  be a full-rank ellipsoid. Then there exists  $\mathbf{A} > 0$  such that  $\mathcal{E} = \mathcal{E}_{\mathbf{A}}$ .

We also have the standard result relating volume and determinants, which follows from observing  $\mathbf{A}\mathcal{E}_{\mathbf{A}} = B_2^d$ .

**Lemma 2.3.3.** Let  $\mathbf{A} > 0$ . Then  $\operatorname{vol}(\mathcal{E}_{\mathbf{A}}) = \det(\mathbf{A}^{-1})\operatorname{vol}(\mathcal{B}_2^d)$ .

In order to give the reduction in the lower bound from the general case to the two-dimensional case, we use the technique of Steiner symmetrization (see e.g. [AGM15, Section 1.1.7]). Given some unit vector  $u \in \mathbb{R}^d$  and convex body  $K \subseteq \mathbb{R}^d$ , we write  $S_u(K)$  for the *Steiner symmetrization* in the direction of u. Recall that the Steiner symmetrization is defined so that for any  $x \perp u$ ,

$$\operatorname{vol}((x + \mathbb{R}u) \cap K) = \operatorname{vol}((x + \mathbb{R}u) \cap S_u(K)),$$

and so that  $(x + \mathbb{R}u) \cap S_u(K)$  is an interval centered at x. Note that we will overload notation slightly as we will allow you u to be a vector of any non-zero length while Steiner symmetrization is usually defined with u being a unit vector, but we will simply take  $S_u = S_{\frac{u}{\|u\|_2}}$ .

Importantly, Steiner symmetrization will preserve important properties of the update. We have the key facts that  $vol(S_u(K)) = vol(K)$ ,  $S_u(K') \subseteq S_u(K)$  if  $K \subseteq K'$ , and further the Steiner symmetrization preserves K being an ellipsoid:

**Lemma 2.3.4** ([BLM87, Lemma 2]). If  $c + \mathcal{E} \subseteq \mathbb{R}^d$  is an ellipsoid,  $S_u(c + \mathcal{E})$  is still an ellipsoid.

Further, if we apply Steiner symmetrization to a body that is a body of revolution about an axis, it does not change the body if u is perpendicular to the axis of revolution.

**Lemma 2.3.5.** Let  $K \subseteq \mathbb{R}^d$  be a body of revolution about the  $e_1$ -axis. Then if  $u \perp e_1$ ,  $S_u(K) = K$ .

# 2.4. Streaming ellipsoidal rounding

Our goal in this section is to prove Theorems 1 and 3.

## 2.4.1. Monotone algorithms solve Problem 2.1

To design algorithms to solve the streaming ellipsoidal rounding problem, we first show that any monotone algorithm gives a valid solution. We let  $c_0 \in \mathbb{R}^d$  and  $r_0 \ge 0$  be given so that

 $c_0 + r_0 \cdot B_2^d \subseteq Z$ , and denote the initial ellipsoid as  $\mathcal{E}_0 = r_0 \cdot B_2^d$ . Note that  $r_0$  need not be the inradius, although it is upper bounded by the inradius.

If we had for each intermediate step *t* that  $c_t + \alpha_t \cdot \mathcal{E}_t \subseteq \operatorname{conv}(z_1, \dots, z_t) \subseteq c_t + \mathcal{E}_t$ , then clearly any algorithm that satisfies this would give a valid final solution as well. However, in intermediate steps it is not clear that  $c_t + \alpha_t \cdot \mathcal{E}_t \subseteq \operatorname{conv}(z_1, \dots, z_t)$ , due to the initialization of  $c_0 + \mathcal{E}_0$  in our monotone algorithm framework. Instead, we relax this invariant to  $c_t + \alpha_t \cdot \mathcal{E}_t \subseteq \operatorname{conv}(\{z_1, \dots, z_t\} \cup (c_0 + \mathcal{E}_0)\})$ , which still suffices to produce a valid final solution.

**Lemma 2.4.1.** To solve Problem 2.1, it suffices for the sequence of ellipsoids  $c_i + \mathcal{E}_i$  and scalings  $\alpha_i$  to satisfy the invariants of Definition 4.

*Proof.* First, we argue that  $\operatorname{conv}(z_1, \ldots, z_n) \subseteq c_n + \mathcal{E}_n$ . As  $\mathcal{E}_n$  is an ellipsoid and therefore a convex set, it suffices to show  $\{z_1, \ldots, z_n\} \subseteq c_n + \mathcal{E}_n$ . We actually argue by induction that  $\{z_1, \ldots, z_t\} \subseteq c_t + \mathcal{E}_t$  for all  $0 \leq t \leq n$ . This is vacuously true for t = 0. At each step t > 0 the inductive hypothesis gives  $\{z_1, \ldots, z_{t-1}\} \subseteq c_{t-1} + \mathcal{E}_{t-1}$ , and thus by (2.2.1) we have  $\{z_1, \ldots, z_t\} \subseteq c_t + \mathcal{E}_t$ .

Now, we argue that  $c_n + \alpha_n \cdot \mathcal{E}_n \subseteq \operatorname{conv}(z_1, \ldots, z_n)$ . We show by induction that  $c_t + \alpha_t \cdot \mathcal{E}_t \subseteq \operatorname{conv}(\{z_1, \ldots, z_t\} \cup (c_0 + \mathcal{E}_0))$  for all  $0 \leq t \leq n$ . This is sufficient as  $\operatorname{conv}(\{z_1, \ldots, z_n\} \cup (c_0 + \mathcal{E}_0)) = Z$ . The case for t = 0 is trivial. For t > 0, the inductive hypothesis gives  $c_{t-1} + \alpha_{t-1} \cdot \mathcal{E}_{t-1} \subseteq \operatorname{conv}(\{z_1, \ldots, z_{t-1}\} \cup (c_0 + \mathcal{E}_0))$ , and by (2.2.2) we have

$$c_t + \alpha_t \cdot \mathcal{E}_t \subseteq \operatorname{conv}\left(\left(c_{t-1} + \alpha_{t-1} \cdot \mathcal{E}_{t-1}\right) \cup \{z_i\}\right) \subseteq \operatorname{conv}\left(\{z_1, \ldots, z_t\} \cup (c_0 + \mathcal{E}_0)\right),$$

as desired.

#### 2.4.2. Special case

In light of Lemma 2.4.1, our strategy is to design an algorithm that preserves the invariants given in Definition 4. This algorithm can be thought of as an *update rule* that, given the previous outer and inner ellipsoids  $c_{t-1} + \mathcal{E}_{t-1}$ ,  $c_{t-1} + \alpha_{t-1}\mathcal{E}_{t-1}$  and next point  $z_t$ , produces the next outer and inner ellipsoids  $c_t + \mathcal{E}_t$ ,  $c_t + \alpha_t \mathcal{E}_t$ .

It is in fact sufficient to consider the simplified case where the previous outer ellipsoid is the unit ball, and the previous inner ellipsoid is some scaling of the unit ball; we will show this in Section 2.4.3. We can further specialize by considering only the two-dimensional case d = 2. We will later show that the high-dimensional case is not much different, as all the relevant sets  $c_{t_1} + \mathcal{E}_{t-1}, c_t + \mathcal{E}_t$  and  $\operatorname{conv} (\alpha \cdot \mathcal{E}_{t-1} \cup \{z_t\})$  form bodies of revolution about the axis through  $c_{t-1}$  and  $z_t$ .

We now describe our two-dimensional update rule. In order to simplify notation, we will let  $\alpha$  be the previous scaling  $\alpha_{t-1}$ , and  $\alpha'$  be the next scaling  $\alpha_t$ . We will assume that  $\alpha \leq 1/2$  to simplify the analysis of our update rule; this will not affect the quality of our final approximation as this update rule will only be used in the "large approximation factor" regime. We will also overload notation by writing  $c + \mathcal{E}$  even when c is a scalar to mean  $(c, 0) + \mathcal{E}$ . We can describe the previous outer ellipsoid  $\mathcal{E}$  with the equation  $x^2 + y^2 \leq 1$ , and the previous inner ellipsoid  $\alpha \mathcal{E}$  with  $x^2 + y^2 \leq \alpha^2$ . We define the next outer and inner ellipsoids  $c + \mathcal{E}'$ ,  $c + \alpha' \mathcal{E}'$  as

$$\underbrace{\frac{1}{a^2}(x-c)^2 + \frac{1}{b^2}y^2 \le 1}_{c+\mathcal{E}'}, \qquad \underbrace{\frac{1}{a^2}(x-c)^2 + \frac{1}{b^2}y^2 \le \alpha'^2}_{c+\alpha'\mathcal{E}'}$$

where we use parameters

$$a = \exp(\gamma)$$

$$b = 1 + \frac{\alpha - \alpha'}{2}$$

$$c = -\alpha + \alpha' \cdot a$$

$$\alpha' = \frac{1}{\frac{1}{\alpha} + 2\gamma}$$
(2.4.1)

We will let z be the rightmost point of  $c + \mathcal{E}'$ , so that z = (c + a, 0). Eventually, we will choose  $\gamma$  so that z coincides with  $z_t$ , the point received in the next iteration. In Section 2.4.4, these parameters  $a(\gamma), b(\gamma), c(\gamma), \alpha'(\gamma)$  will be used as functions of the parameter  $\gamma \ge 0$ . However, we will not yet explicitly specify  $\gamma$ , so in this section these parameters can be thought of as constants for some fixed  $\gamma$ . This update rule is pictured in Figure 2.1.

We first collect a few straightforward properties of this update rule.

**Lemma 2.4.2.** *The parameters in the setup* (2.4.1) *satisfy the following.* 

- 1.  $\frac{1}{\alpha'} = \frac{1}{\alpha} + 2\gamma$ 2.  $b \ge 1$ 3.  $c \ge 0$
- 4.  $c + \alpha' \cdot a \ge \alpha$

Before proving these properties, we provide geometric interpretations. Intuitively, (1) means that  $\gamma$  is proportional to the increase in the approximation factor at this step, a fact that we will use when analyzing the general-case algorithm. (2) means that the outer ellipsoid grows on every axis; and (3) means that the centers of the next ellipsoids are to the right of the *y*-axis, i.e. the centers of the next ellipsoids are further towards *v* than those of the previous ellipsoids. The rightmost point of  $c + \alpha' \mathcal{E}'$  is  $c + \alpha' \cdot a$ , so (4) shows that this point is to the right of the rightmost point of  $\alpha \cdot \mathcal{E}$ .

We now prove Lemma 2.4.2.

*Proof of Lemma* 2.4.2. (1) is clear from rearranging the definition of  $\alpha'$ . From (1) we also have  $\alpha' \leq \alpha$ , so that (2) follows immediately.

For (3), observe that  $\frac{\alpha}{\alpha'} = 1 + 2\gamma\alpha$ . When  $\alpha \leq 1/2$ , this means

$$\frac{\alpha}{\alpha'} \le 1 + \gamma \le \exp\left(\gamma\right) = a \tag{2.4.2}$$

using  $1 + x \le e^x$ , Lemma 2.8.1-(1). By definition of c,  $\alpha/\alpha' \le a$  is equivalent to  $c \ge 0$ .

To show (4), by definition we have that  $c + \alpha' \cdot a = -\alpha + 2\alpha' a$ . Thus showing  $c + \alpha' \cdot a \ge \alpha$  is equivalent to showing that  $\alpha' a \ge \alpha$ , which is equivalent to the inequality in (2.4.2).

As Figure 2.1 depicts, the update step we defined satisfies the invariants in Definition 4 and so is monotone; in the rest of this section we make this picture formal. To start, we consider the

invariant concerning outer ellipsoids; we will show that  $\mathcal{E} \subseteq c + \mathcal{E}'$ . For now we can think of z as replacing  $z_t$ , and clearly  $z \in c + \mathcal{E}'$ , so if we show that  $\mathcal{E} \subseteq c + \mathcal{E}'$ , then conv  $(\mathcal{E} \cup \{z\}) \subseteq c + \mathcal{E}'$  as well since  $c + \mathcal{E}'$  is convex.

**Lemma 2.4.3.** We have  $\mathcal{E} \subseteq c + \mathcal{E}'$ .

*Proof.* First, observe that  $\mathcal{E} \subseteq \mathcal{E}'$  because both axes of  $\mathcal{E}'$  have greater length than those of  $\mathcal{E}$ :  $a \ge 1$  by definition, and  $b \ge 1$  from Lemma 2.4.2-(2). Now, we translate  $\mathcal{E}'$  to the right until it touches  $\mathcal{E}$  at two points. We call this translated ellipse  $c_r + \mathcal{E}'$ , as shown in Figure 2.3. Observe that as long as  $c \le c_r$ , we have  $\mathcal{E} \subseteq c + \mathcal{E}'$ . We now determine  $c_r$ .



Figure 2.3.: Outer ellipses of the update step. As before,  $\mathcal{E}$  is the black circle and  $c + \mathcal{E}'$  is the blue ellipse.  $c_r + \mathcal{E}'$  is the magenta ellipse, with its center at  $c_r$  and the dotted magenta line showing the position of  $c_r$  along the *x*-axis.  $c_r$  is defined so  $c_r + \mathcal{E}'$  and  $\mathcal{E}$  are tangent at two points. Q is one of these two tangent points.

First, note points on the boundary of  $c_r + \mathcal{E}'$  are described by the equation

$$\frac{(x-c_r)^2}{a^2} + \frac{y^2}{b^2} = 1$$
(2.4.3)

Let Q = (x', y') be the point of intersection between  $\mathcal{E}$  and  $c_r + \mathcal{E}'$  where y' > 0. Since Q is on the boundary of both ellipses, the vectors  $\left(\frac{2(x'-c_r)}{a^2}, \frac{2y'}{b^2}\right)$  and (2x', 2y'), which are the normal vectors at Q of  $c_r + \mathcal{E}'$  and  $\mathcal{E}$  respectively, must be parallel. Thus  $\frac{4(x'-c_r)}{a^2} \cdot y' = \frac{4y'x'}{b^2}$ , which simplifies to

$$x' = \frac{c_r}{1 - \frac{a^2}{h^2}}.$$
(2.4.4)

At this point we have a system of three equations relating (x', y') and  $c_r$ : (2.4.4), Q lying on  $\mathcal{E}$ , and Q satisfying (2.4.3). We now solve this system to find  $c_r$ . To start, we expand (2.4.3) into  $x'^2 - 2x'c_r + c_r^2 + y'^2 \frac{a^2}{b^2} = a^2$ , which we rewrite into  $x'^2 \frac{a^2}{b^2} + x'^2 \left(1 - \frac{a^2}{b^2}\right) - 2x'c_r + c_r^2 + y'^2 \frac{a^2}{b^2} = a^2$ . As Q lies on  $\mathcal{E}$ , this becomes  $x'^2 \left(1 - \frac{a^2}{b^2}\right) - 2x'c_r + c_r^2 + \frac{a^2}{b^2} = a^2$ . Substituting in (2.4.4), we get

$$\frac{c_r^2}{1-\frac{a^2}{b^2}} - 2\frac{c_r^2}{1-\frac{a^2}{b^2}} + c_r^2 + \frac{a^2}{b^2} = a^2.$$

Simplifying, we have  $c_r^2 \left(1 - \frac{b^2}{b^2 - a^2}\right) = a^2 \left(1 - \frac{1}{b^2}\right)$ , i.e.

$$c_r^2 = \frac{b^2 - 1}{b^2} (a^2 - b^2).$$

To complete the proof of Lemma 2.4.3, it suffices to show  $c^2 \leq \frac{b^2-1}{b^2}(a^2 - b^2)$ . This will follow from Lemma 2.8.5.

Now, we move on to the inner ellipsoid invariant of Definition 4. In particular, we will argue that  $c + \alpha' \mathcal{E}' \subseteq \operatorname{conv} (\alpha \mathcal{E} \cup \{z\})$ . On a high level, we show this by arguing that the boundary of  $c + \alpha' \mathcal{E}'$  does not intersect the boundary of  $\operatorname{conv} (\alpha \mathcal{E} \cup \{z\})$ , except at points of tangency.

We can split the boundary of  $\operatorname{conv}(\alpha \mathcal{E} \cup \{z\})$  into two pieces: the part that intersects with the boundary of  $\alpha \mathcal{E}$ , which is an arc of the boundary of  $\alpha \mathcal{E}$ ; and the remainder, which can described as two line segments connecting z to that arc. In particular, there are two lines that go through z and are tangent to  $\alpha \mathcal{E}$ , one of which we call line L, and the other line is the reflection of L across the x-axis. We define  $P_1$  and  $P_2$  as the tangent points of these lines to  $\alpha \mathcal{E}$ . Then, the boundary of  $\operatorname{conv}(\alpha \mathcal{E} \cup \{z\})$  consists of an arc  $P_1P_2$  and the line segments  $\overline{P_1z}, \overline{P_2z}$ . This is illustrated in Figure 2.4. Note that at this point it is possible a priori for the arc  $P_1P_2$  that coincides with the boundary of  $\operatorname{conv}(\alpha \mathcal{E} \cup \{z\})$  to be either the major or minor arc; we will later show it must be the major arc. We will take L to be the line whose tangent point to  $\alpha \mathcal{E}$ ,  $P_1$ , is above the x-axis, though this choice is arbitrary due to symmetry across the x-axis.



Figure 2.4.: Inner ellipses of the update step. As before,  $\alpha \mathcal{E}$  is the black circle and  $c + \alpha' \mathcal{E}$  is the blue ellipse.  $P_0$  is the shared leftmost point of  $\alpha \mathcal{E}$  and  $c + \alpha' \mathcal{E}'$ . There are two lines through v that are tangent to  $\alpha \mathcal{E}$ , one of which we call L and pictured in orange. We call the tangent points  $P_1$  and  $P_2$ . The line segments  $\overline{P_1 z}$ ,  $\overline{P_2 z}$  are the dotted black lines.  $P'_1$  and  $P'_2$  are the two points of intersection between  $c + \alpha' \mathcal{E}$  and the line segment  $\overline{P_1 P_2}$ .  $P''_1$  and  $P''_2$  are the two points of intersection between  $\partial(c + \alpha' \mathcal{E}')$  and  $\partial \alpha \mathcal{E}$  to the right of the y-axis. Note that  $P_2$ ,  $P'_2$ ,  $P''_2$  are the reflections of  $P_1$ ,  $P''_1$ ,  $P''_1$  across the x-axis.

We first show that  $c + \alpha' \mathcal{E}'$  does not intersect  $\overline{P_1 z}$  and  $\overline{P_2 z}$ , except possibly at points of tangency. In fact, we show a slightly stronger statement, in similar fashion to Lemma 2.4.3.

**Lemma 2.4.4.**  $c + \alpha' \mathcal{E}'$  lies inside the angle  $\angle P_1 z P_2$ .

*Proof.* We translate  $c + \alpha' \mathcal{E}'$  to the right until it touches L (and, by symmetry,  $\overline{P_2 z}$ ). We call this translated ellipse  $c_+ + \alpha' \mathcal{E}'$ , as shown in Figure 2.5. (Formally, the center  $c_+$  can be described not as a translation from some other ellipse, but as  $c_+$  such that  $c_+ + \alpha' \mathcal{E}'$  intersects L at one point). Observe that if  $c \leq c_+$ , then  $c + \alpha' \mathcal{E}'$  lies inside the angle  $\angle P_1 z P_2$ . We now determine  $c_+$ .



Figure 2.5.: Inner ellipses of the update step. As before,  $\alpha \mathcal{E}$  is the black circle,  $c + \alpha' \mathcal{E}$  is the blue ellipse, *L* is the orange line through *z* and tangent to  $\alpha \mathcal{E}$ ,  $P_1$  and  $P_2$  are the tangent points on the lines through *z* tangent to  $\alpha \mathcal{E}$ , and  $\overline{P_1 z}$ ,  $\overline{P_2 z}$  are the dotted black lines.  $c_+ + \alpha' \mathcal{E}'$  is the magenta ellipse, with its center at  $c_+$  and magenta dotted line showing its position on the *x*-axis.  $c_+$  is defined so that  $c_+ + \alpha' \mathcal{E}'$  is tangent to  $\overline{P_1 z}$  and  $\overline{P_2 z}$ , with *Q* as the tangent point of  $c_+ + \alpha' \mathcal{E}'$  and  $\overline{P_1 z}$ .

The equation of *L* is

$$\underbrace{\frac{1}{c+a}}_{\ell_1} \cdot x + \underbrace{\sqrt{\frac{1}{\alpha^2} - \frac{1}{(c+a)^2}}}_{\ell_2} \cdot y = 1$$

where we define  $\ell_1$ ,  $\ell_2$  as the coefficients for x and y. Observe that z is on L, and L is tangent to  $\alpha \mathcal{E}$  at  $P_1$ , which has coordinates

$$P_1 = \left(\frac{\alpha^2}{c+a}, \alpha^2 \sqrt{\frac{1}{\alpha^2} - \frac{1}{(c+a)^2}}\right).$$
 (2.4.5)

Tangency can be confirmed by checking that  $P_1$  is parallel to  $(\ell_1, \ell_2)$ , the normal vector definining *L*.

Let Q = (x', y') be the point of intersection of *L* and  $c_+ + \alpha' \mathcal{E}$ , there are three properties that define *Q*. First it lies on the boundary of  $c_+ + \alpha' \mathcal{E}$ , so it satisfies

$$\frac{(x'-c_+)^2}{a^2} + \frac{y'^2}{b^2} = \alpha'^2.$$
(2.4.6)

Second, at *Q* the normal vectors for the equations defining  $c_+ + \alpha' \mathcal{E}$  and *L* are parallel, i.e.  $(\frac{2(x-c_+)}{a^2}, \frac{2y}{b^2})$  is parallel to  $(\ell_1, \ell_2)$ . So

$$\frac{(x'-c_+)}{a^2}\ell_2 = \frac{y'}{b^2}\ell_1.$$
(2.4.7)

Finally, *Q* lies on *L*, so we have  $\ell_1 x' + \ell_2 y' = 1$ . Solving this for *y'*, we get

$$y' = \frac{1 - \ell_1 x'}{\ell_2}.$$
(2.4.8)

These three equations form a system for x', y' and  $c_+$ , which we now solve to find  $c_+$ . Taking the square of (2.4.7) and rearranging gives  $\frac{y'^2}{b^2} = \frac{b^2(x'-c_+)^2\ell_2^2}{a^4\ell_1^2}$ . Substituting this into (2.4.6), we get  $\frac{(x'-c_+)^2}{a^2} + \frac{b^2(x'-c_+)^2\ell_2^2}{a^4\ell_1^2} = \alpha'^2$ . Now, defining  $r \stackrel{\text{def}}{=} \frac{a^2\ell_1^2}{b^2\ell_2^2}$ , we group the terms of this equation into the form

$$(x' - c_{+})^{2} \cdot \frac{1}{a^{2}} \left( 1 + \frac{1}{r} \right) = \alpha'^{2}.$$
(2.4.9)

We substitute (2.4.8) into (2.4.7) to get  $\frac{x'-c_+}{a^2}\ell_2 = \frac{\ell_1}{b^2}\frac{1-x'\ell_1}{\ell_2}$ . Grouping for x' and rearranging yields

$$x' - c_{+} = \frac{r}{1+r} \left( \frac{1}{\ell_{1}} - c_{+} \right).$$
(2.4.10)

Next, we substitute (2.4.10) into (2.4.9), and get after some cancellation

$$\left(\frac{1}{\ell_1}-c_+\right)^2=\alpha'^2a^2\cdot\frac{1+r}{r}.$$

Observe on the left hand side that  $\frac{1}{\ell_1} - c_+ = c + a - c_+$ . Clearly the center  $c_+$  must be to the left of z, so this must be non-negative. Hence after taking the positive square root, we obtain

$$c_{+} = c + a - \alpha' \cdot a \sqrt{\frac{1+r}{r}}$$

It remains to show that  $c \leq c_+$ , or equivalently that

$$a - \alpha' \cdot a \sqrt{\frac{1+r}{r}} \ge 0,$$

which we do in Lemma 2.8.6. This completes the proof of Lemma 2.4.4.

Now, we build on the previous claim to show the inner ellipsoid invariant.

**Lemma 2.4.5.** We have  $c + \alpha' \cdot \mathcal{E}' \subseteq \operatorname{conv} (\alpha \cdot \mathcal{E} \cup \{z\})$ .

*Proof.* We will argue that the boundary of  $c + \alpha' \mathcal{E}'$  does not intersect the boundary of  $\operatorname{conv}(\alpha \mathcal{E} \cup \{z\})$ , except at points of tangency. This is sufficient to establish the claim, as Lemma 2.4.4 shows that  $c + \alpha' \mathcal{E}'$  is internal to  $\angle P_1 z P_2$ , and so if  $c + \alpha' \mathcal{E}'$  does not intersect the boundary of  $\operatorname{conv}(\alpha \mathcal{E} \cup \{z\})$ ,  $c + \alpha' \mathcal{E}'$  must lie inside of, or be disjoint from  $\operatorname{conv}(\alpha \mathcal{E} \cup \{z\})$ . Since the leftmost points of  $\alpha \mathcal{E}$  and  $c + \alpha' \mathcal{E}'$  coincide,  $c + \alpha' \mathcal{E}'$  must then lie inside of  $\operatorname{conv}(\alpha \mathcal{E} \cup \{z\})$ . Recall that the boundary of  $\operatorname{conv}(\alpha \mathcal{E} \cup \{z\})$  consists of the arc  $P_1 P_2$  and the line segments  $\overline{P_1 z}, \overline{P_2 z}$ . Lemma 2.4.4 already shows that the boundary of  $c + \alpha' \mathcal{E}'$  does not intersect the arc  $\overline{P_1 z}$  and  $\overline{P_2 z}$ , so we only need to show that the boundary of  $c + \alpha' \mathcal{E}'$  does not intersect the arc  $P_1 P_2$ .

To do this, we start by enumerating the points of intersection of  $\partial \alpha \mathcal{E}$  and  $\partial (c + \alpha' \mathcal{E}')$ , recalling that  $P_1P_2$  is an arc of  $\partial \alpha \mathcal{E}$ . Observe that the leftmost points of  $\alpha \mathcal{E}$  and  $c + \alpha' \mathcal{E}'$  coincide, as the leftmost point of  $c + \alpha' \mathcal{E}'$  is  $c - \alpha' \cdot a = -\alpha$  by definition; we call this point  $P_0$ .  $P_0$  is a point of

tangency and hence has intersection multiplicity 2, because the centers of  $\alpha \mathcal{E}$  and  $c + \alpha' \cdot \mathcal{E}'$  both lie on the *x*-axis.

Next, we argue for the existence of two more distinct intersection points  $P''_1, P''_2$  as depicted in Figure 2.4. The leftmost point of  $c + \alpha' \mathcal{E}'$  is  $(-\alpha, 0)$ , and the rightmost point is  $c + \alpha'$ , which by Lemma 2.4.2-(4) is to the right of  $(\alpha, 0)$ , the rightmost point of  $\alpha \mathcal{E}$ . Thus, by lying on  $\partial \alpha \mathcal{E}$ ,  $P_1, P_2$  lie between the leftmost and rightmost points of  $c + \alpha' \mathcal{E}'$ , and so  $c + \alpha' \mathcal{E}'$  intersects the line through  $P_1$  and  $P_2$ . Further, by Lemma 2.4.4, as  $c + \alpha' \mathcal{E}'$  lies in the angle  $\angle P_1 v P_2$ ,  $c + \alpha' \mathcal{E}'$  actually intersects the line segment  $\overline{P_1 P_2}$ . Observe that this intersection happens at two distinct points, which we call  $P'_1$  and  $P'_2$ . Both points are inside of  $\alpha \mathcal{E}$ , yet  $\partial(c + \alpha' \mathcal{E}')$  is a continuous path that connects both to the rightmost point of  $c + \alpha' \mathcal{E}'$ , which we call  $P''_1$  and  $P''_2$ .

Now, we argue that  $P_1''$  and  $P_2''$  lie on the minor arc  $P_1P_2$ . First, observe that the arc  $P_1P_2$  containing  $P_0$  is the major arc. This is because  $P_1$  lies to the right of the *y*-axis, as determined in (2.4.5); and by symmetry so does  $P_2$ . This also implies that major arc  $P_1P_2$  is the arc with which the boundary of conv ( $\alpha \mathcal{E} \cup \{z\}$ ) coincides.  $P_1'$  and  $P_2'$  are collinear with  $P_1$  and  $P_2$ , and as  $P_1''$  are to the right of  $P_1'$  and  $P_2'$ , this implies that they must lie on the minor arc  $P_1P_2$ .

Counting all the intersection points of  $\partial \alpha \mathcal{E}$  and  $\partial(c + \alpha' \mathcal{E}')$ , we have  $P_0$  (with multiplicity 2) and  $P_1''$  and  $P_2''$  (both with multiplicity 1); with total multiplicity 4. Using Lemma 2.3.1, it is impossible for them to have another intersection point without both ellipses being the same. Thus  $\partial(c + \alpha' \mathcal{E}')$  cannot intersect the major arc  $P_1P_2$  except at  $P_0$ , and so except at points of tangency the boundary of  $c + \alpha' \mathcal{E}'$  does not intersect the boundary of  $\operatorname{conv}(\alpha \mathcal{E} \cup \{z\})$ .

### 2.4.3. Generalizing to high dimension and arbitrary previous ellipsoids

Now that we have demonstrated the invariants of Definition 4 for the special two-dimensional case where the previous ellipsoid is the unit ball, we generalize slightly to higher dimensions. However, we first still assume the previous ellipsoid is the unit ball.

Using the parameters as defined in (2.4.1), we will let  $\mathcal{E} = B_2^d$ , and define the boundary of  $\mathcal{E}'$  as

$$\frac{1}{a^2}(x_1-c)^2+\frac{1}{b^2}x_2^2+\ldots+\frac{1}{b^2}x_d^2=1.$$

Observe that we can also write  $\mathcal{E}' = \mathcal{E}_{\mathbf{D}}$  where  $\mathbf{D} = \text{diag}\left(\frac{1}{a^2}, \frac{1}{b^2}, \dots, \frac{1}{b^2}\right)$ . Similarly to before, we let  $\mathbf{z} = (c + a, 0, 0, \dots, 0) \in \mathbb{R}^d$ , the furthest point of  $c + \mathcal{E}'$  in the positive direction of the  $x_1$ -axis.

Now, we argue that the invariants of Definition 4 still hold in this setting.

Lemma 2.4.6. The inner and outer ellipsoid invariants hold in this setting:

- 1.  $\mathcal{E} \subseteq c \cdot e_1 + \mathcal{E}'$
- 2.  $c \cdot e_1 + \alpha' \mathcal{E}' \subseteq \operatorname{conv} (\alpha \mathcal{E} \cup \{z\})$

*Proof.* Observe that  $\mathcal{E}$ ,  $c \cdot e_1 + \mathcal{E}'$ ,  $c \cdot e_1 + \alpha' \mathcal{E}'$ , and conv ( $\alpha \mathcal{E} \cup \{z\}$ ) are all bodies of revolution about the  $x_1$ -axis, with their cross-sections given by their counterparts in Section 2.4.2. As

Lemma 2.4.3 and Lemma 2.4.5 hold for these cross sections, the set containments hold for the bodies of revolution as well.  $\hfill \Box$ 

We further generalize to the case where the previous ellipsoid is arbitrary. In particular, let  $c^{\circ} + \mathcal{E}$  be the previous ellipsoid, with a vector  $c^{\circ} \in \mathbb{R}^d$  and  $\mathcal{E} = \mathcal{E}_A$  for non-singular matrix  $\mathbf{A} \in \mathbb{R}^{d \times d}$ . Let  $z^{\circ} \in \mathbb{R}^d$  be an arbitrary vector, representing the next point received. We let  $u = \mathbf{A}(z^{\circ} - c^{\circ})$ , and  $\mathbf{W} \in \mathbb{R}^{d \times d}$  be an orthogonal matrix with  $w = \frac{u}{\|u\|}$  as its first column (e.g. by using as its columns an orthonormal basis containing w). We define the next outer ellipsoid as  $c^{\circ} + c\mathbf{A}^{-1}w + \mathcal{E}'$  for  $\mathcal{E}' = \mathcal{E}_{\mathbf{W}\mathbf{D}\mathbf{W}^{\mathsf{T}}\mathbf{A}}$ , with  $\mathbf{D} = \operatorname{diag}\left(\frac{1}{a^2}, \frac{1}{b^2}, \ldots, \frac{1}{b^2}\right)$  as before. Observe that  $z = c^{\circ} + (c+a)\mathbf{A}^{-1}w$  is the furthest point of  $c^{\circ} + c\mathbf{A}^{-1}w + \mathcal{E}'$  from the previous center  $c^{\circ}$  towards  $z^{\circ}$ .

This setup works to preserve the key invariants, as we see in the next claim.

**Lemma 2.4.7.** *The inner and outer ellipsoid invariants hold in this setting:* 

1.  $c^{\circ} + \mathcal{E} \subseteq c^{\circ} + c\mathbf{A}^{-1}w + \mathcal{E}'$ 2.  $c^{\circ} + c\mathbf{A}^{-1}w + \alpha'\mathcal{E}' \subseteq \operatorname{conv}\left((c^{\circ} + \alpha\mathcal{E}) \cup \{z\}\right)$ 

*Proof.* We translate both set inclusions by  $-c^{\circ}$ , then apply the nonsingular linear transformation  $\mathbf{W}^{\top}\mathbf{A}$ . Observe that the set inclusions we wish to prove hold if and only if the transformed ones do. Noting that  $\mathbf{W}^{\top}\mathbf{A}\mathcal{E}' = \mathcal{E}_{\mathbf{WD}}$ , the transformed set inclusions are  $\mathcal{E}_{\mathbf{W}} \subseteq c \cdot e_1 + \mathcal{E}_{\mathbf{WD}}$  and  $c \cdot e_1 + \alpha' \mathcal{E}_{\mathbf{WD}} \subseteq \operatorname{conv}(\alpha \mathcal{E}_{\mathbf{W}} \cup \{(c + a) \cdot e_1\})$ . However, since **W** is an orthogonal matrix,  $\mathcal{E}_{\mathbf{W}} = B_2^d$  and  $\mathcal{E}_{\mathbf{WD}} = \mathcal{E}_{\mathbf{D}}$ , and so the inclusions are exactly those shown in Lemma 2.4.6.

Choosing  $\gamma$  correctly in (2.4.1) ensures that  $z \in c^{\circ} + c\mathbf{A}^{-1}w + \mathcal{E}'$  coincides with  $z^{\circ}$ , as stated in the upcoming claim. This can be seen by looking at the definition of z.

**Lemma 2.4.8.** If  $\gamma$  is chosen so that c + a = ||u||, then  $z = z^{\circ}$ .

## 2.4.4. General algorithm

The goal of this section is to give and analyze a full algorithm that solves the streaming ellipsoid approximation problem, building on the analysis of the update rule from the previous sections.

Before we describe the complete algorithm, we give pseudocode in Algorithm 1 for its primary primitive. It is an update step like the one we analyzed in the previous section, Section 2.4.3.

In Lines 3, 4 and 5, we use the definition of  $a(\gamma)$ ,  $b(\gamma)$ ,  $c(\gamma)$ ,  $\alpha'(\gamma)$  from (2.4.1), substituting  $\alpha_{t-1}$  for  $\alpha$ . Although the update step does not explicitly mention ellipsoids, we use  $\mathcal{E}_t = \mathcal{E}_{\mathbf{A}_t}$  so that at iteration t the next outer and inner ellipsoids are  $c_t + \mathcal{E}_{\mathbf{A}_t}$  and  $c_t + \alpha_t \mathcal{E}_{\mathbf{A}_t}$ , respectively. If at this iteration  $||\mathbf{u}||_2 \le 1$ , we will refer to this as the case where the ellipsoids are not updated, as is clear from Line 7.

Observe also that if in iteration t we let  $\mathbf{W} \in \mathbb{R}^{d \times d}$  be an orthogonal matrix with w as its first

### Algorithm 1 Full update step $\mathcal{A}^{\mathsf{full}}$

input:  $\mathbf{A}_{t-1} \in \mathbb{R}^{d \times d}$ ,  $c_{t-1} \in \mathbb{R}^{d}$ ,  $\alpha_{t-1} \in [0, \frac{1}{2}]$ ,  $z_t \in \mathbb{R}^{d}$ output:  $\mathbf{A}_t \in \mathbb{R}^{d \times d}$ ,  $c_t \in \mathbb{R}^{d}$ ,  $\alpha_t \in [0, \alpha_{t-1}]$ 1: Let  $u = \mathbf{A}_{t-1}(z_t - c_{t-1})$ ,  $w = \frac{u}{\|u\|}$ 2: if  $\|u\|_2 > 1$  then 3: | Let  $\gamma_t^{\star}$  be such that  $a(\gamma_t^{\star}) + c(\gamma_t^{\star}) = \|u\|$ 4:  $\hat{\mathbf{A}} = \frac{1}{b(\gamma_t^{\star})}\mathbf{I}_d + \left(\frac{1}{a(\gamma_t^{\star})} - \frac{1}{b(\gamma_t^{\star})}\right)ww^{\top}$ 5: | return  $\mathbf{A}_t = \hat{\mathbf{A}} \cdot \mathbf{A}_{t-1}$ ,  $c_t = c_{t-1} + c(\gamma_t^{\star})\mathbf{A}_{t-1}^{-1}w$ ,  $\alpha_t = \alpha'(\gamma_t^{\star})$ 6: else 7: | return  $\mathbf{A}_t = \mathbf{A}_{t-1}$ ,  $c_t = c_{t-1}$ ,  $\alpha_i = \alpha_{t-1}$ 

column, we can write

$$\hat{\mathbf{A}} = \mathbf{W} \cdot \operatorname{diag}\left(\frac{1}{a(\gamma_t^{\star})}, \frac{1}{b(\gamma_t^{\star})}, \cdots, \frac{1}{b(\gamma_t^{\star})}\right) \cdot \mathbf{W}^{\top}$$
(2.4.11)

Now, we argue that this algorithm satisfies the invariants defined in Definition 4. This argument is essentially the observation that the update step in the algorithm is the one analyzed in Lemma 2.4.7.

#### Lemma 2.4.9. Algorithm 1 is a monotone update; i.e., it satisfies the invariants in Definition 4.

*Proof.* If  $||u||_2 \le 1$ , then  $z_i \in c_n + \mathcal{E}_n$  and the inner and outer ellipsoids are not updated, so the invariants clearly hold. Otherwise, we apply Lemma 2.4.7 and Lemma 2.4.8 setting  $\mathbf{A} = \mathbf{A}_{t-1}, c^\circ = c_{t-1}, z^\circ = z_t, \alpha = \alpha_{t-1}$ . Using (2.4.11),  $\mathcal{E}_{\mathbf{A}_t}$  is the same as  $\mathcal{E}'$  in Lemma 2.4.7; and clearly  $\alpha_t = \alpha'$ . This establishes the inner ellipsoid invariant  $c_t + \alpha_t \mathcal{E}_t \subseteq \operatorname{conv}((c_{t-1} + \alpha_{t-1}\mathcal{E}_{t-1}) \cup \{z_t\})$  directly. To show  $\operatorname{conv}((c_{t-1} + \mathcal{E}_{t-1}) \cup \{z_t\}) \subseteq c_t + \mathcal{E}_t$ , observe that we have  $c_{t-1} + \mathcal{E}_{t-1} \subseteq c_t + \mathcal{E}_t$  from Lemma 2.4.7, and  $z_t \in c_t + \mathcal{E}_t$  from Lemma 2.4.8. Then the outer ellipsoid invariant follows as  $c_t + \mathcal{E}_t$  is a convex set.

Finally, we bound the relevant quantities that will be used in the analysis of the full algorithm's approximation factor. In particular, we show that  $\exp(\gamma_t^*)$  gives a lower bound on the increase in volume at each iteration *t*. If  $||u||_2 \le 1$ , and the ellipsoids are not updated, in that iteration we think of  $\gamma_t^* = 0$ .

**Lemma 2.4.10.** For any input given to Algorithm 1, we have  $\operatorname{vol}(\mathcal{E}_i) \ge \exp(\gamma_t^{\star})\operatorname{vol}(\mathcal{E}_{t-1})$ .

*Proof.* This formula is clearly true when the ellipsoids are not updated because  $\gamma_t^* = 0$ , so we consider the nontrivial case. Recall the formula  $vol(\mathcal{E}_{\mathbf{A}}) = det(\mathbf{A}^{-1})vol(B_2^d)$  from Lemma 2.3.3. Then we have

$$\mathsf{vol}(\mathcal{E}_{\mathbf{A}_i}) = \det(\mathbf{A}_i^{-1})\mathsf{vol}(B_2^d) = \det(\hat{\mathbf{A}}^{-1}) \cdot \det(\mathbf{A}_{t-1}^{-1}) \cdot \mathsf{vol}(B_2^d) = \det(\hat{\mathbf{A}}^{-1})\mathsf{vol}(\mathcal{E}_{\mathbf{A}_{t-1}})$$

where we use the definition of  $\hat{\mathbf{A}}$  from Line 4 on the *t*-th iteration. Then

$$\det(\hat{\mathbf{A}}^{-1}) = a(\gamma_t^{\star}) \cdot b(\gamma_t^{\star})^{d-1} \qquad \text{using (2.4.11)}$$
$$\geq a(\gamma_t^{\star}) \qquad \text{by Lemma 2.4.2-(2)}$$

$$= \exp(\gamma_t^{\star})$$
 by definition of *a* in (2.4.1)

and using  $vol(\mathcal{E}_{\mathbf{A}_i}) = det(\hat{\mathbf{A}}^{-1}) \cdot vol(\mathcal{E}_{\mathbf{A}_{t-1}})$  completes the proof.

We are now ready to present the complete algorithm in Algorithm 2. The algorithm is explicitly given  $c_0 + r_0 \cdot B_2^d \subseteq Z$ . For simplicity, here, we say  $r = r_0$ . Let R = R(Z). While the final approximation factor depends on this quantity, the algorithm is not given it. Note that  $\kappa(Z) \leq R/r$ , so the quality of the approximation depends not only on  $\kappa(Z)$ , but also on how well the given ball  $c_0 + r \cdot B_2^d$  is centered within *Z*.

This algorithm proceeds in two phases. It begins with a "local" first phase, where the inner ellipsoid is a ball kept at radius r, and the outer ellipsoid is a ball scaled to contain all the points. For readability, the variables of the algorithm in this phase are annotated with a superscript <sup>(l)</sup>. The second phase starts if the approximation factor of the first phase ever reaches  $\alpha^{(l)} \leq \frac{1}{d \log d}$ , at which point the algorithm uses the "full" update that was just described in Algorithm 1. We use two phases because while the full update reaches a near-optimal approximation factor when  $R/r \geq d \log d$ , the local phase using balls does better when  $R/r \leq d \log d$ . While we cannot tell when to switch phases exactly (this would require knowing R/r), we show that it is enough to approximate the aspect ratio during the first phase up to a constant factor.

Algorithm 2 Streaming ellipsoid rounding – complete algorithm input:  $c_0 + rB_2^d \subseteq Z$ output:  $c_n + \mathcal{E}_n, c_n + \alpha_n \cdot \mathcal{E}_n$ 1: Initialize  $\mathbf{A}_{0}^{(l)} = \frac{1}{r} \mathbf{I}_{d}, \mathbf{c}_{0}^{(l)} = \mathbf{c}_{0}, \alpha_{0}^{(l)} = 1$ 2:  $t^{(l)} = 0, R_0 = 0$ 3: while  $t^{(l)} \leq n$  do ▷ Phase I: Local update step that maintains a ball Receive point  $z_{t(l)}$ 4: if  $||z_{t^{(l)}} - c_0||_2 \le r \cdot d \log d$  then 5:  $\begin{aligned} \mathbf{if} \| \mathbf{z}_{t^{(l)}} - \mathbf{c}_0 \|_2 &> R_{t^{(l)-1}} \text{ then} \\ \| \mathbf{A}_{t^{(l)}}^{(l)} &= \frac{1}{\| \mathbf{z}_{t^{(l)}} - \mathbf{c}_0 \|_2} \cdot \mathbf{I}_d, \mathbf{c}_{t^{(l)}}^{(l)} = \mathbf{c}_{t^{(l)-1}}^{(l)}, \alpha_{t^{(l)}}^{(l)} = \frac{r}{\| \mathbf{z}_{t^{(l)}} - \mathbf{c}_0 \|_2} \\ R_{t^{(l)}} &= \frac{\| \mathbf{z}_{t^{(l)}} - \mathbf{c}_0 \|_2}{r} \end{aligned}$   $\triangleright \text{ Grow the ball to contain } \mathbf{z}_{t^{(l)}}$ 6: 7: 8: else 9:  $\begin{array}{l} \mathbf{A}_{t^{(l)}}^{(l)} = \mathbf{A}_{t^{(l)-1}}, \boldsymbol{c}_{i}^{(l)} = \boldsymbol{c}_{t^{(l)-1}}^{(l)}, \boldsymbol{\alpha}_{t^{(l)}}^{(l)} = \boldsymbol{\alpha}_{t^{(l)-1}} \\ \boldsymbol{R}_{t^{(l)}}^{(l)} = \boldsymbol{R}_{t^{(l)-1}} \end{array}$ 10: 11: else 12: break ▶ Break the loop and jump to Line 15 13:  $\overline{t}^{(l)} = t^{(l)} + 1$ 14: 15: **if**  $t^{(l)} > n$  **then** ▶ If we stayed in Phase I for the entire execution of the algorithm 16: **return**  $\mathbf{A}_{n}^{(l)}, \mathbf{c}_{n}^{(l)}, \alpha_{n}^{(l)}$ 17:  $t_{s} = t^{(l)}$  $\triangleright$  Point  $z_{t_s}$  has not yet been processed 18:  $\mathbf{A}_{t_s-1} = \frac{1}{rd \log d} \cdot \mathbf{I}_d, \mathbf{c}_{t_s-1} = \mathbf{c}_{t_s-1}^{(l)}, \alpha_{t_s-1} = \frac{1}{d \log d}$   $\triangleright$  Transition: grow the ball to maximium size 19: for  $t \in \{t_s, t_s + 1, ..., n\}$  do ▶ Phase II: full update for the remaining points Receive point  $z_i$ 20:  $\mathbf{A}_i, \boldsymbol{c}_i, \boldsymbol{\alpha}_i = \mathcal{A}^{\mathsf{full}}(\mathbf{A}_{t-1}, \boldsymbol{c}_{t-1}, \boldsymbol{\alpha}_{t-1}, \boldsymbol{z}_i)$ 21: 22: return  $\mathbf{A}_n, \mathbf{c}_n, \alpha_n$ 

Before Line 15, the algorithm executes the first phase that has the outer and inner ellipsoids as balls. In Line 15, we have  $t^{(l)} > n$  if the algorithm stayed in Phase I for every point, i.e. we had  $\max_{1 \le t^{(l)} \le n} ||_2 z_{t^{(l)}} - c_0 ||_2 \le r \cdot d \log d$ . In this case, the algorithm returns the

approximation maintained by Phase I. Otherwise we must have come across a point where  $||z_{t^{(l)}} - c_0||_2 > r \cdot d \log d$ , and the algorithm proceeds with Phase II. We let  $t_s$  in Line 17 mark the point received that causes the algorithm to proceed to Phase II. We then perform a "transition" on Line 18 that grows the ball of Phase I to its maximum size. This transition step makes the analysis of the complete algorithm easier, as then the starting approximation for the second phase is exactly  $\alpha_{t_s-1} = \frac{1}{d \log d}$ . Then the algorithm runs the full update  $\mathcal{A}^{\text{full}}$  for the rest of the points, including  $z_{t_s}$ . For simplicity, we write our algorithm so that it 'receives'  $z_{t_s}$  twice, once for each phase. However, the first phase does not commit to an update for this point, and the ellipsoids in Line 18 are not committed either; the algorithm does not commit to an update for this point this point until Line 21.

Recall the approximation guarantee stated in Theorem 1:

$$\frac{1}{\alpha_n} \le O(\min\{\frac{R}{r}, d\log(\frac{R}{r})\})$$
(2.4.12)

We can interpret the approximation guarantee (2.4.12) by cases depending on if  $R/r \ge d \log d$  (i.e. if the algorithm ever enters the second phase):

**Lemma 2.4.11.** We have for all  $d \ge 2$  that

$$\min\left\{\frac{R}{r}, d\log\left(\frac{R}{r}\right)\right\} = \Theta\left(\begin{cases} d\log\left(\frac{R}{r}\right) & \text{if } \frac{R}{r} > d\log d\\ \frac{R}{r} & \text{if } \frac{R}{r} \le d\log d \end{cases}\right).$$

Now, we claim a straightforward geometric fact – that the distance of the furthest  $z_t$  to  $c_0$  approximates the circumradius of Z up to a constant factor. We will use this to show that Line 5 will be able to properly detect when  $\frac{R}{r} > d \log d$  (again, up to a constant factor).

Lemma 2.4.12. Let  $c_0 + r_0 B_2^d \subseteq Z$ , and R = R(Z). Then,  $R \le \max_{1 \le t^{(l)} \le n} \|c_0 - z_{t^{(l)}}\|_2 \le 2 \cdot R.$ 

*Proof.* For the left inequality, observe that if we let 
$$r_{\max} = \max_{1 \le t^{(l)} \le n} ||c_0 - z_{t^{(l)}}||_2$$
, then  $Z \subseteq c_0 + r_{\max} \cdot B_2^d$ . For the right inequality, observe that for any containing ball  $c' + R' \cdot B_2^d \supseteq Z$ , its diameter is  $2R'$ . But as  $c' + R' \cdot B_2^d$  contains  $c_0$  and  $z_1, \ldots, z_n$ , we must have diam  $(c' + R' \cdot B_2^d) \ge diam(\{c_0\} \cup \{z_1, \ldots, z_n\})$  and so  $2R' \ge r_{\max}$ .

Next, we discuss the approximation guarantee that the algorithm achieves, depending on the phase that it terminates with. We start with if the algorithm only stays in the local phase, in which case we can readily apply the previous claim.

**Lemma 2.4.13.** If Algorithm 2 never enters Phase II, then its approximation guarantee satisfies  $\frac{1}{\alpha_n} \leq \frac{2R}{r}$ .

*Proof.* At the termination of Phase I, the algorithm produces approximation  $\alpha_n^{(l)} = \max_{1 \le t^{(l)} \le n} \frac{\|z_{t^{(l)}} - c_0\|_2}{r}$ . Using Lemma 2.4.12 we obtain

$$\frac{1}{\alpha_n^{(l)}} = \max_{1 \le t^{(l)} \le n} \frac{\|\boldsymbol{z}_{t^{(l)}} - \boldsymbol{c}_0\|_2}{r} \le \frac{2R}{r},$$

as desired.

The analysis in the case where the algorithm enters the full phase is more involved. We use Lemma 2.4.10, which shows that the increase in approximation factor each iteration is not too large compared to the increase in volume, to bound  $\frac{1}{\alpha_n}$ . We know that the volume of the final ellipsoid  $c_n + \mathcal{E}_n$  must be bounded relative to  $R \cdot B_2^d$ , as the algorithm produces  $c_n + \alpha_n \cdot \mathcal{E}_n \subseteq Z$ . However, this leads to an upper bound that is still a function of  $\frac{1}{\alpha_n}$ .

Lemma 2.4.14. If Algorithm 2 enters Phase II, the approximation guarantee satisfies

$$\frac{1}{\alpha_n} \le 2\left(d\log\left(\frac{1}{\alpha_n}\right) + d\log\left(\frac{R}{r}\right)\right).$$

*Proof.* The algorithm transitions to Phase II at Line 17, starting at iteration  $t_s$ . At each subsequent iteration, we claim that Algorithm 1 guarantees  $\frac{1}{\alpha_t} = \frac{1}{\alpha_{t-1}} + 2\gamma_t^*$ . By Lemma 2.4.2-(1), we have for all  $t_s \le t \le n-1$  where the ellipsoids were updated that  $\frac{1}{\alpha_t} = \frac{1}{\alpha_{t-1}} + 2\gamma_t^*$ . When the ellipsoids are not updated, this still holds, as in that case  $\gamma_t^* = 0$ .

As in Phase II the algorithm begins with  $\alpha_{t_s-1} = \frac{1}{d \log d}$ , we have

$$\frac{1}{\alpha_n} = d \log d + 2 \sum_{t=t_s}^{n-1} \gamma_t^{\star}.$$
 (2.4.13)

Now applying Lemma 2.4.10 for each t, we have  $\operatorname{vol}(\mathcal{E}_n) \ge \exp\left(\sum_{t=t_s}^{n-1} \gamma_t^{\star}\right) \cdot \operatorname{vol}(\mathcal{E}_{t_s})$ . Taking logarithms gives

$$\log\left(\frac{\operatorname{vol}\left(\mathcal{E}_{n}\right)}{\operatorname{vol}\left(\mathcal{E}_{t_{s}-1}\right)}\right) \geq \sum_{t=t_{s}}^{n-1} \gamma_{t}^{\star}.$$
(2.4.14)

Recall that  $c_0 + r \cdot B_2^d \subseteq Z$ , and by Definition 2,  $Z \subseteq c_c + R \cdot B_2^d$  for some center  $c_c$ . By Lemma 2.4.9, we have  $c_n + \alpha_n \cdot \mathcal{E}_n \subseteq Z$ , so that  $\operatorname{vol}(\mathcal{E}_n) \leq \frac{1}{\alpha_n^d} \cdot \operatorname{vol}(R \cdot B_2^d)$ . As in Phase II we start with  $\mathcal{E}_{t_s-1} = c_0 + rd \log d \cdot B_2^d$ , this yields

$$\sum_{t=t_s}^{n-1} \gamma_t^{\star} \leq \log\left(\frac{\operatorname{vol}(\mathcal{E}_n)}{\operatorname{vol}(\mathcal{E}_{t_s-1})}\right) \qquad \text{by (2.4.13)}$$
$$\leq d \log\left(\frac{1}{\alpha_n}\right) + \log\left(\frac{\operatorname{vol}(R \cdot B_2^d)}{\operatorname{vol}(rd \log d \cdot B_2^d)}\right) \qquad \text{by vol}(\mathcal{E}_n) \leq \frac{1}{\alpha_n^d} \operatorname{vol}(R \cdot B_2^d)$$
$$= d \log\left(\frac{1}{\alpha_n}\right) + d \log\left(\frac{R}{rd \log d}\right)$$
$$\leq d \log\left(\frac{1}{\alpha_n}\right) + d \log\left(\frac{R}{r}\right) - d \log d.$$

Plugging into (2.4.13) completes the proof of Lemma 2.4.14.

Intuitively,  $x \le a + b \cdot \log x$  for some constants a, b > 0 can only be true for bounded x, as  $x = \omega(\log x)$ . As we showed  $1/\alpha_n$  satisfies a relation like this in Lemma 2.4.14, we develop this intuition to give a quantitative upper bound on  $1/\alpha_n$ .

Lemma 2.4.15. If Algorithm 2 enters Phase II, then we have

$$\frac{1}{\alpha_n} \leq 8d(\log d + \log R/r).$$

*Proof.* Assume towards contradiction that  $\frac{1}{\alpha_n} > 8d(\log d + \log R/r)$ . Observe then that  $\frac{1}{\alpha_n} - \frac{3}{4} \cdot \frac{1}{\alpha_n} > 2d(\log d + \log R/r)$ . Using Lemma 2.4.14, we have

$$2(d\log 1/\alpha_n + d\log R/r) \geq \frac{1}{\alpha_n} > 2(d\log d + d\log R/r) + \frac{3}{4} \cdot \frac{1}{\alpha_n}$$

Simplifying the above inequality gives  $2d \log \frac{1}{d \cdot \alpha_n} > \frac{3}{4} \cdot \frac{1}{\alpha_n}$ , i.e.  $2 \log \frac{1}{d \cdot \alpha_n} > \frac{7}{8} \cdot \frac{1}{d \cdot \alpha_n}$ . It is clear that this is impossible by looking at the graph of the function  $x \mapsto 2 \log x - \frac{3}{4}x$ , which is concave with a maximum of  $2(\log(8/3) - 1) < 0$ .

Now we combine the previous claims to prove the guarantees of Algorithm 2 and obtain Theorem 1.

*Proof of Theorem 1.* We first discuss the approximation guarantee and correctness, then the memory and runtime complexity of Algorithm 2.

**Approximation guarantee** We break the analysis of the approximation guarantee by cases, depending on the aspect ratio. If  $R/r \le \frac{1}{2}d \log d$ , then by Lemma 2.4.12 we have  $\max_{1\le t^{(l)}\le n} ||c_0 - z_{t^{(l)}}||_2 \le rd \log d$ , and the algorithm never enters Phase II. By Lemma 2.4.13, the final approximation factor is 2R/r. If  $R/r > d \log d$ , then by Lemma 2.4.12 we have  $\max_{1\le t^{(l)}\le n} ||c_0 - z_{t^{(l)}}||_2 > rd \log d$ , and the algorithm must enter Phase II. Then Lemma 2.4.15 applies, and the final approximation factor is  $O(d(\log d + \log R/r) = O(d \log R/r))$ .

If  $\frac{1}{2}d \log d < R/r \le d \log d$ , then it is possible for the algorithm to never enter Phase II or for it to enter Phase II. Either way, we argue that the final approximation factor is  $\frac{1}{\alpha_n} \le O(R/r)$ . If it does not enter Phase II, then by Lemma 2.4.13, the approximation guarantee we get is  $\frac{1}{\alpha_n} \le O(R/r)$ . If it does enter Phase II, then by Lemma 2.4.15 we have

$$\frac{1}{\alpha_n} \le O(d\log d + d\log R/r)$$

Due to the assumption that  $\frac{1}{2}d \log d < \frac{R}{r} \le d \log d$ , we also have in this case that  $\frac{1}{\alpha_n} \le O(\frac{R}{r})$ .

**Correctness** By Lemma 2.4.1, to argue that the algorithm solves Problem 2.1 it is enough to show that it is monotone, i.e. it satisfies the invariants of Definition 4. It is clear that the local update in Phase I satisfies the invariants, as the outer ellipsoid is a ball of growing radius and the inner ellipsoid is kept to the ball of radius *r*. It is also clear that after the algorithm transitions to Phase II, all the full updates are monotone by Lemma 2.4.9 and the fact that the starting approximation factor for this phase is is  $\alpha_{t_s-1} = \frac{1}{d \log d} \leq \frac{1}{2}$ . As algorithm transitions to Phase II, observe that on Line 18 the radius of the outer ellipsoid grows again to  $rd \log d$  before applying the full update, so the first first full update of Phase II is also monotone.

**Memory and runtime complexity** The memory complexity of the algorithm is  $O(d^2)$ . Observe that Algorithm 1 only stores a constant number of matrices in  $\mathbb{R}^{d \times d}$ , vectors in  $\mathbb{R}^d$ , or constants, so its memory complexity is  $O(d^2)$ . It is only instantiated once for each point received in Phase II, so the memory complexity in this phase  $O(d^2)$ . Finally, the memory complexity in the first phase is also  $O(d^2)$  because it stores the same kind of quantities as Algorithm 1.

To show the runtime of the algorithm is  $\tilde{O}(nd^2)$ , we show that the runtime to process each next point is at most  $\tilde{O}(d^2)$ . This is clear in Phase I, and during the transition to Phase II. For the full update this is less clear, as Algorithm 1 uses both  $\mathbf{A}_{t-1}$  and  $\mathbf{A}_{t-1}^{-1}$  which naively would require inverting a matrix on each iteration. However, if we represent  $\mathbf{A}$  using the SVD (see the next section and Lemma 2.4.17), we can implement the update in  $\tilde{O}(d^2)$  time. This would require that  $\mathbf{A}_{t_s-1}$  be given in SVD form as well for the first full update, but it is already in that form as a scaled identity matrix.

Put together, these complete the proof of Theorem 1.

#### Efficient implementation of the full update step

In this section, we use a method similar to that in Algorithm 2 from [MMO22] to show that the full update step can be implemented in  $\tilde{O}(d^2)$  time. In particular, we use the same subroutine SVDRANKONEUPDATE with signature

$$(\mathbf{U}', \boldsymbol{\Sigma}', \mathbf{V}') = \text{SVDRankOneUpdate}((\mathbf{U}, \boldsymbol{\Sigma}, \mathbf{V}), \boldsymbol{y}_1, \boldsymbol{y}_2)$$
(2.4.15)

where the result  $\mathbf{U}' \mathbf{\Sigma}' (\mathbf{V}')^{\top}$  is the SVD of the matrix  $\mathbf{U} \mathbf{\Sigma} \mathbf{V}^{\top} + y_1 y_2^{\top}$ . Stange [Sta08] shows that this procedure be done in  $O(d^2 \log d)$  time. We rewrite Algorithm 1 in Algorithm 3 to make it clear how to use the SVD representation and the efficient rank-1 update to efficiently implement the full update. One can readily see that Algorithm 3 has the exact same behavior as Algorithm 1, and so gives the same approximation and correctness guarantees.

Algorithm 3 Efficient full update step  $\mathcal{A}^{\text{full}}$ input:  $(\mathbf{U}_{t-1}, \boldsymbol{\Sigma}_{t-1}, \mathbf{V}_{t-1}) \in \mathbb{R}^{d \times d}, c_{t-1} \in \mathbb{R}^{d}, \alpha_{t-1} \in [0, \frac{1}{2}], z_{t} \in \mathbb{R}^{d}$ output:  $(\mathbf{U}_{t}, \boldsymbol{\Sigma}_{t}, \mathbf{V}_{t}) \in \mathbb{R}^{d \times d}, c_{t} \in \mathbb{R}^{d}, \alpha_{t} \in [0, \alpha_{t}]$ 1: Let  $u = \mathbf{U}_{t-1}\boldsymbol{\Sigma}_{t-1}\mathbf{V}_{t-1}^{\top}(z_{t} - c_{t-1}), w = \frac{u}{\|u\|_{2}}$ 2: if  $\|u\|_{2} > 1$  then 3: Let  $\gamma_{t}^{\star}$  be such that  $a(\gamma_{t}^{\star}) + c(\gamma_{t}^{\star}) = \|u\|$ 4:  $y_{1} = \left(\frac{1}{a(\gamma_{t}^{\star})} - \frac{1}{b(\gamma_{t}^{\star})}\right) w, y_{2} = \mathbf{V}_{t-1}\boldsymbol{\Sigma}_{t-1}\mathbf{U}_{t-1}^{\top}w$ 5:  $(\mathbf{U}_{t}, \boldsymbol{\Sigma}_{t}, \mathbf{V}_{t}) = \text{SVDRANKONEUPDATE}((\mathbf{U}_{t-1}, \frac{1}{b(\gamma_{t}^{\star})}\boldsymbol{\Sigma}_{t-1}, \mathbf{V}_{t-1}), y_{1}, y_{2})$ 6: return  $(\mathbf{U}_{t}, \boldsymbol{\Sigma}_{t}, \mathbf{V}_{t}), c_{t} = c_{t-1} + c(\gamma_{t}^{\star})\mathbf{V}_{t-1}\boldsymbol{\Sigma}_{t-1}^{-1}\mathbf{U}_{t-1}^{\top}w, \alpha_{t} = \alpha'(\gamma_{t}^{\star})$ 7: else 8: return  $(\mathbf{U}_{t}, \boldsymbol{\Sigma}_{t}, \mathbf{V}_{t}) = (\mathbf{U}_{t-1}, \boldsymbol{\Sigma}_{t-1}, \mathbf{V}_{t-1}), c_{t} = c_{t-1}, \alpha_{t} = \alpha_{t-1}$ 

**Remark 2.4.16.** We briefly explain why Line 3, finding  $\gamma^*$  such that  $a(\gamma^*) + c(\gamma^*) = ||u||$  can be implemented efficiently. This is a one-dimensional optimization problem, and  $\gamma \mapsto a(\gamma) + c(\gamma)$  using a, c as defined in (2.4.1) is monotone increasing, so finding an approximate  $\gamma^*$  can be done efficiently with binary search. In particular, we can choose  $\gamma^*$  to be a slight overestimate so the update is still monotone after slightly increasing  $\alpha_t$ . This does not affect the final approximation guarantee beyond constant factors.

This algorithm performs a constant number of taking norms of vectors, matrix-vector products, and algebraic operations; as well as one rank-one SVD update. As explained in Remark 2.4.16, finding  $\gamma_i^*$  can also be done in effectively constant time. Thus for our runtime guarantee, we have:

**Lemma 2.4.17.** Algorithm 3 runs in time  $O(d^2 \log d)$ .

#### 2.4.5. Fully-online asymmetric ellipsoidal rounding algorithm

In this subsection, we prove Theorem 3. See Algorithm 4.

Algorithm 4 Fully online asymmetric ellipsoidal rounding

- 1: **Input:** Stream of points  $z_t$ ; monotone update rule  $\mathcal{A}$  (Definition 4) that takes as input the previous ellipsoid matrix **A**, center *c*, approximation factor  $\alpha$ , and update point *z* and outputs the next ellipsoid matrix **A**', center *c*', and approximation factor  $\alpha'$ .
- 2: **Output:** Ellipsoid  $\mathcal{E}$ , center c, and scale  $\alpha \in (0, 1)$  such that  $c + \alpha \cdot \mathcal{E} \subseteq \text{conv}(\{z_1, \dots, z_n\}) \subseteq c + \mathcal{E}$ .
- 3: Receive  $z_1$ ; set  $\mathbf{A} = \mathbf{I}_d$ ,  $d_1 = 1$ ,  $c_1 = z_1$ ,  $\alpha_1 = 1$ .
- 4: **for** t = 2, ..., n **do**
- 5: Receive  $z_t$ . if  $z_t - c_{t-1} \notin \text{Span}(z_1 - c_{t-1}, \dots, z_{t-1} - c_{t-1})$  then ▶ Irregular update step. 6: Let  $v_1, \ldots, v_{d_{t-1}}$  be the singular vectors of **A** corresponding to the semiaxes of  $\mathcal{E}_{t-1}$ . 7: Let  $d_t = d_{t-1} + \mathbf{1}$ . Let  $z'_{d_t} \coloneqq \frac{z_t - \sum_{i=1}^{d_{t-1}} v_i \langle v_i, z_t \rangle}{\left\| z_t - \sum_{i=1}^{d_{t-1}} v_i \langle v_i, z_t \rangle \right\|_2}$ . Let  $\mathbf{M} \coloneqq \mathbf{I}_d - \frac{1}{\left\langle v'_{d_t}, z \right\rangle} \cdot \left( z_t - \sqrt{1 + 2\alpha_{t-1}} \cdot v'_{d_t} \right) (v'_{d_t})^T$ .  $\mathbf{M} \coloneqq \mathbf{M} := \mathbf{V}_d - \mathbf{M}_{t-1} \mathbf{M}$ . Let  $d_t = d_{t-1} + 1$ 8: 9: 10: ▷ Use (2.4.15) of Stange [Sta08] to update  $v_1, \ldots, v_d$ . 11: Update  $c_t = \frac{\alpha_{t-1}}{1+2\alpha_{t-1}} \cdot z_t + \left(1 - \frac{\alpha_{t-1}}{1+2\alpha_{t-1}}\right) \cdot c_{t-1}.$ 12: Update  $1/\alpha_t \leftarrow 1/\alpha_{t-1} + 1$ . 13: else 14:  $\mathbf{A}_t, \mathbf{c}_t, \alpha_t = \mathcal{A}(\mathbf{A}_{t-1}, \mathbf{c}_{t-1}, \alpha_{t-1}, \mathbf{z}_t)$ 15:  $d_t \leftarrow d_{t-1}$ . 16: 17: **Output:**  $(c_n, \mathcal{E}_n, \alpha_n)$ .

To prove Theorem 3, we need to show that our *irregular update step* (a timestep *t* when we have to update the dimensionality of our ellipsoid  $\mathcal{E}_{t-1}$  – see Line 6 of Algorithm 4) still maintains the invariants we desire (Definition 4).

Our plan is to first consider the special case of the irregular update where the new point to cover is conveniently located with respect to our current ellipsoids. We will see later that this special case is nearly enough for us to conclude the proof.

**Lemma 2.4.18.** Let  $Z \subset \mathbb{R}^d$  be a convex body where Z lies in Span  $(v_1, \ldots, v_{d'})$  for d' < d. For  $0 < \alpha \le 1$ , suppose we have

$$\alpha \cdot \{ z \in \text{Span}(v_1, \dots, v_{d'}) : \|z\|_2 \le 1 \} \subseteq Z \subseteq \{ z \in \text{Span}(v_1, \dots, v_{d'}) : \|z\|_2 \le 1 \}.$$

Then, for any  $v_{d'+1}$  such that  $\langle v_i, v_{d'+1} \rangle = 0$  for all  $i \in [d']$  and for which

$$\mathcal{E}' \coloneqq \left\{ z \in \operatorname{Span}\left(v_1, \dots, v_{d'+1}\right) : \|z\|_2 \le \frac{1+\alpha}{\sqrt{1+2\alpha}} \right\}$$
$$c \coloneqq \frac{\alpha}{\sqrt{1+2\alpha}} \cdot v_{d'+1}$$

we have

$$c + \frac{1}{1 + 1/\alpha} \cdot \mathcal{E}' \subseteq \operatorname{conv}\left(Z \cup \left\{\sqrt{1 + 2\alpha} \cdot v_{d'+1}\right\}\right) \subseteq c + \mathcal{E}'.$$

*Proof of Lemma 2.4.18.* We will show that the pair of ellipsoids given below satisfy the conditions promised by the statement of Lemma 2.4.18.

$$\left\{ z \in \text{Span}(v_1, \dots, v_{d'+1}) : \|z\|_2 \le \frac{1+\alpha}{\sqrt{1+2\alpha}} \right\} + \frac{\alpha}{\sqrt{1+2\alpha}} \cdot v_{d'+1}$$
(2.4.16)

$$\left\{ z \in \operatorname{Span}\left(v_{1}, \ldots, v_{d'+1}\right) : \|z\|_{2} \leq \frac{1+\alpha}{\sqrt{1+2\alpha}} \right\} \cdot \frac{\alpha}{1+\alpha} + \frac{\alpha}{\sqrt{1+2\alpha}} \cdot v_{d'+1}$$
(2.4.17)

Clearly, the two ellipsoids given above are apart by a factor of  $1+\alpha/\alpha = 1/\alpha + 1$ , which means the approximation factor increases by exactly 1 as a result of this update. It now suffices to show that the ellipsoid described by (2.4.16) contains  $\operatorname{conv}\left(B_2^{d'} \cup \left\{\sqrt{1+2\alpha} \cdot v_{d'+1}\right\}\right)$  and that the ellipsoid described by (2.4.17) is contained by the cone whose base is  $\alpha \cdot B_2^{d'}$  and whose apex is  $\sqrt{1+2\alpha} \cdot v_{d'+1}$ .

For the first part, it suffices to verify that every point  $z \in Z$  and  $\sqrt{1 + 2\alpha} \cdot v_{d'+1}$  is contained by (2.4.16). We give both the calculations below, from which the result for (2.4.16) follows.

$$z \in Z: \qquad \left\| z - \frac{\alpha}{\sqrt{1+2\alpha}} \cdot v_{d'+1} \right\|_2 = \sqrt{\|z\|_2^2 + \frac{\alpha^2}{1+2\alpha}} \le \frac{1+\alpha}{\sqrt{1+2\alpha}}$$
$$z = \sqrt{1+2\alpha} \cdot v_{d'+1}: \qquad \left\| z - \frac{\alpha}{\sqrt{1+2\alpha}} \cdot v_{d'+1} \right\|_2 = \sqrt{1+2\alpha} - \frac{\alpha}{\sqrt{1+2\alpha}} = \frac{1+\alpha}{\sqrt{1+2\alpha}}$$

We now analyze (2.4.17). Our task is to show the below inclusion.

$$\left\{ z \in \operatorname{Span}\left(v_{1}, \dots, v_{d'+1}\right) : \left\| z - \frac{\alpha}{\sqrt{1+2\alpha}} \cdot v_{d'+1} \right\|_{2} \le \frac{\alpha}{\sqrt{1+2\alpha}} \right\}$$

$$\subseteq \operatorname{conv}\left(\alpha \cdot \left\{ z \in \operatorname{Span}\left(v_{1}, \dots, v_{d}\right) : \left\| z \right\|_{2} \le 1 \right\} \cup \left\{ \sqrt{1+2\alpha} \cdot v_{d'+1} \right\} \right)$$

Let w be an arbitrarily chosen unit vector in Span  $(v_1, \ldots, v_{d'})$ . Observe that it is enough to show

$$\left\{ z \in \operatorname{Span}\left(w, v_{d'+1}\right) : \left\| z - \frac{\alpha}{\sqrt{1+2\alpha}} \cdot v_{d'+1} \right\|_{2} \le \frac{\alpha}{\sqrt{1+2\alpha}} \right\} \subseteq \operatorname{conv}\left(\pm \alpha \cdot w, \sqrt{1+2\alpha} \cdot v_{d'+1}\right).$$

Since the above is a two-dimensional problem and that  $\langle w, v_{d'+1} \rangle = 0$ , it is equivalent to show that the inradius of the triangle with vertices  $(-\alpha, 0)$ ,  $(\alpha, 0)$ , and  $(0, \sqrt{1+2\alpha})$  is  $\alpha/\sqrt{1+2\alpha}$  and that its incenter is  $(0, \alpha/\sqrt{1+2\alpha})$ .

Recall that the inradius of a triangle can be written as K/s where K is the area of the triangle (in this case,  $\alpha\sqrt{1+2\alpha}$ ) and s is the semiperimeter of the triangle (in this case,  $1 + 2\alpha$ ). This

implies that the inradius is indeed  $\alpha/\sqrt{1+2\alpha}$ . Finally, since the triangle in question is isosceles with its apex being the *y*-axis, the *x*-coordinate of its incenter must be 0. These observations imply that the incenter is  $(0, \alpha/\sqrt{1+2\alpha})$ .

This is sufficient for us to conclude the proof of Lemma 2.4.18.  $\Box$ 

We will now see that the analysis for the convenient update that we gave in Lemma 2.4.18 is nearly enough for us to fully analyze the irregular update step. See Lemma 2.4.19, where we analyze the irregular update step in full generality (up to translating by  $c_{t-1}$ ).

**Lemma 2.4.19.** Let  $Z \subset \mathbb{R}^d$  be a convex body such that Z lies in a subspace H of dimension d' < d. Let  $\mathcal{E}$  be an ellipsoid and let  $0 < \alpha \le 1$  be such that

$$\alpha \cdot \mathcal{E} \subseteq Z \subseteq \mathcal{E}.$$

*Let*  $z \notin H$ . *Then, there exists a center* c *and an ellipsoid*  $\mathcal{E}'$  *such that* 

$$c + \frac{1}{1 + 1/\alpha} \cdot \mathcal{E}' \subseteq \operatorname{conv} (Z \cup \{z\}) \subseteq c + \mathcal{E}'.$$

*Proof of Lemma 2.4.19.* Recall that  $v_1, \ldots, v_{d'} \in \mathbb{R}^d$  are the unit vectors corresponding to the semiaxes of  $\mathcal{E}$ ; notice that these form a basis for H. Observe that  $v_{d'+1}$  is a unit vector orthogonal to  $v_1, \ldots, v_{d'}$  such that z can be expressed as  $\sum_{i=1}^{d'+1} v_i \langle v_i, z \rangle$ .

As stated in Algorithm 4, let

$$\mathbf{M} \coloneqq \mathbf{I}_d - \frac{1}{\left\langle \boldsymbol{v}_{d_t}^{\prime}, \boldsymbol{z}_t \right\rangle} \cdot \left( \boldsymbol{z}_t - \sqrt{1 + 2\alpha} \cdot \boldsymbol{v}_{d_t}^{\prime} \right) (\boldsymbol{v}_{d_t}^{\prime})^T.$$

We calculate

$$\mathbf{M}\boldsymbol{z}_{t} = \boldsymbol{z}_{t} - \frac{1}{\left\langle \boldsymbol{v}_{d_{t}}^{\prime}, \boldsymbol{z}_{t} \right\rangle} \cdot \left(\boldsymbol{z}_{t} - \sqrt{1 + 2\alpha} \cdot \boldsymbol{v}_{d_{t}}^{\prime}\right) (\boldsymbol{v}_{d_{t}}^{\prime})^{T} \boldsymbol{z}_{t} = \boldsymbol{z}_{t} - \boldsymbol{z}_{t} + \sqrt{1 + 2\alpha} \cdot \boldsymbol{v}_{d_{t}}^{\prime} = \sqrt{1 + 2\alpha} \cdot \boldsymbol{v}_{d_{t}}^{\prime}.$$

By the definition of  $A_{t-1}$ , we have

$$\mathbf{A}_{t-1}\mathbf{M}\mathbf{z}_t = \sqrt{1+2\alpha} \cdot \mathbf{A}_{t-1}\mathbf{v}'_{d_t} = \sqrt{1+2\alpha} \cdot \mathbf{v}'_{d_t}.$$

Next, for any  $z \in Z$ , we have  $z \in H_{t-1}$ . This means that

$$\mathbf{M}\boldsymbol{z} = \boldsymbol{z} - \frac{1}{\left\langle \boldsymbol{v}_{d_t}^{\prime}, \boldsymbol{z} \right\rangle} \cdot \left( \boldsymbol{z} - \sqrt{1 + 2\alpha} \cdot \boldsymbol{v}_{d_t}^{\prime} \right) (\boldsymbol{v}_{d_t}^{\prime})^T \boldsymbol{z} = \boldsymbol{z} - \boldsymbol{0} = \boldsymbol{z}.$$

By Lemma 2.4.18, we know for

$$\mathbf{A}_{t-1}\mathbf{M}\mathbf{c}_{t} = \frac{\alpha}{\sqrt{1+2\alpha}} \cdot \mathbf{v}'_{d_{t}}$$
$$\mathbf{A}_{t-1}\mathbf{M}\mathcal{E}_{t} = \left\{ z \in \operatorname{Span}\left(\mathbf{v}_{1}, \dots, \mathbf{v}_{d_{t-1}}, \mathbf{v}'_{d_{t}}\right) : \|z\|_{2} \leq 1 \right\}$$

that

$$\mathbf{A}_{t-1}\mathbf{M}\mathbf{c}_t + \frac{1}{1+1/\alpha_{t-1}} \cdot \mathbf{A}_{t-1}\mathbf{M}\mathbf{\mathcal{E}}_t \subseteq \operatorname{conv}\left(\mathbf{A}_{t-1}\mathbf{M} \cdot Z \cup \{\mathbf{A}_{t-1}\mathbf{M}\mathbf{z}_t\}\right) \subseteq \mathbf{A}_{t-1}\mathbf{M}\mathbf{c}_t + \mathbf{A}_{t-1}\mathbf{M}\mathbf{\mathcal{E}}_t$$

and, since  $A_{t-1}M$  is invertible (owing to the invertibility of  $A_{t-1}$  and M),

$$c_t + \frac{1}{1 + 1/\alpha_{t-1}} \cdot \mathcal{E}_t \subseteq \operatorname{conv} (Z \cup \{z_t\}) \subseteq c_t + \mathcal{E}_t.$$

Finally, note that

$$c_t = \frac{\alpha}{\sqrt{1+2\alpha}} \cdot \mathbf{M}^{-1} \mathbf{A}_{t-1}^{-1} v'_{d_t} = \frac{\alpha}{1+2\alpha} \cdot z_t$$
$$\mathcal{E}_t = \{ z \in \text{Span}(z_1, \dots, z_t) : \|\mathbf{A}_{t-1} \mathbf{M} z\|_2 \le 1 \}$$

and then translate by  $c_{t-1}$ , which concludes the proof of Lemma 2.4.19.

We are now ready to prove Theorem 3.

*Proof of Theorem 3.* Using Lemma 2.4.19, we have that the ellipsoids maintain our desired invariants (Definition 4) throughout the process. Hence, Algorithm 4 maintains an ellipsoidal approximation to  $conv(\{z_1, ..., z_t\})$  for all t.

It remains to verify the approximation factor  $\alpha_t$  of Algorithm 4.

Consider a timestep *t*. For every  $t' \le t$ , let  $H_{t'} = \text{Span}(z_1, \ldots, z_{t'})$ ,  $r_{t'} = r(Z_{t'})$  be the inradius of  $Z_{t'} = \text{conv}(z_1, \ldots, z_{t'})$ , and  $R_{t'} = R(Z_{t'})$  be the circumradius of  $Z_t$ . Let  $\hat{r} = \min_{t' \le t} r_{t'}$ . Consider the *d*-dimensional ellipsoid  $T(\mathcal{E}_{t'})$  which is exactly equal to  $\mathcal{E}_{t'}$  in the space  $H_{t'}$  and whose remaining semiaxes orthogonal to  $H_{t'}$  are equal and have length  $\hat{r}$ . Observe that for a regular update step t' (with  $d_{t'} = d_{t'-1}$ ), we have

$$\frac{\operatorname{vol}_{d_{t'}}(\mathcal{E}_{t'})}{\operatorname{vol}_{d_t}(\mathcal{E}_{t'-1})} = \frac{\operatorname{vol}_d(T(\mathcal{E}_{t'}))}{\operatorname{vol}_d(T(\mathcal{E}_{t'-1}))}.$$

Now applying the evolution condition (2.2.3) to the update restricted to  $H_{t'}$ , we get

$$\frac{1}{\alpha_{t'}} - \frac{1}{\alpha_{t'-1}} \le C \log \frac{\operatorname{vol}_{d_{t'}}(\mathcal{E}_t)}{\operatorname{vol}_{d_{t'}}(\mathcal{E}_{t'-1})} = C \log \frac{\operatorname{vol}_d(T(\mathcal{E}_{t'}))}{\operatorname{vol}_d(T(\mathcal{E}_{t'-1}))}.$$

We have obtained the following upper bound on the approximation-factor increase:

$$\frac{1}{\alpha_{t'}} - \frac{1}{\alpha_{t'-1}} \le \begin{cases} 1 & \text{if } t' \text{ is an irregular update step} \\ C \log \left( \frac{\operatorname{vol}_d(T(\mathcal{E}_{t'}))}{\operatorname{vol}_d(T(\mathcal{E}_{t'-1}))} \right) & \text{otherwise} \end{cases}$$
(2.4.18)

Let  $T_{reg}$  consist of all the timesteps  $t' \leq t$  where we perform a regular update. Then we have,

$$\frac{1}{\alpha_t} - \frac{1}{\alpha_0} = \alpha_0 + \sum_{t'=1}^t \left( \frac{1}{\alpha_{t'}} - \frac{1}{\alpha_{t'-1}} \right) \le d_t + C \sum_{t' \in T_{\mathsf{reg}}} \log \left( \frac{\mathsf{vol}_d(T(\mathcal{E}_{t'}))}{\mathsf{vol}_d(T(\mathcal{E}_{t'-1}))} \right).$$

Now we show that  $\log \left(\frac{\operatorname{vol}_d(T(\mathcal{E}_{t'}))}{\operatorname{vol}_{d_t}(T(\mathcal{E}_{t'-1}))}\right) \ge 0$  for an irregular step: let  $\sigma_1 \ge \cdots \ge \sigma_d$  and  $\sigma'_1 \ge \cdots \ge \sigma'_d$  be the lengths of semi-axes of  $T(\mathcal{E}_{t'})$  and  $T(\mathcal{E}_{t'-1})$ , respectively. Then  $\sigma_i \ge \sigma'_i$  for  $1 \le i \le d_{t'} - 1$ , since  $\mathcal{E}_{t'-1} \subset \mathcal{E}_{t'}$ ;  $\sigma_{d_{t'}} \ge r_{t'} \ge \hat{r} = \sigma'_{d_{t'}}$ ; and  $\sigma_i = \hat{r} = \sigma'_i$  for  $i > d_{t'}$ . Therefore,

$$\log\left(\frac{\operatorname{vol}_d(T(\mathcal{E}_{t'}))}{\operatorname{vol}_{d_t}(T(\mathcal{E}_{t'-1}))}\right) = \log\left(\frac{\sigma_1\cdot\ldots\cdot\sigma_d}{\sigma_1'\cdot\ldots\cdot\sigma_d'}\right) \ge \log 1 = 0.$$

Using this inequality and plugging in  $\alpha_0 = 1$ , we get

$$\begin{aligned} \frac{1}{\alpha_t} &= 1 + d_t + C \sum_{t' \in T_{\text{reg}}} \log \left( \frac{\text{vol}_d(T(\mathcal{E}_{t'}))}{\text{vol}_d(T(\mathcal{E}_{t'-1}))} \right) \leq 1 + d_t + C \sum_{t'=1}^t \log \left( \frac{\text{vol}_d(T(\mathcal{E}_{t'}))}{\text{vol}_d(T(\mathcal{E}_{t'-1}))} \right) \\ &\lesssim d_t + \log \left( \frac{\text{vol}_d(T(\mathcal{E}_t))}{\text{vol}_d(T(\mathcal{E}_0))} \right) \leq d_t + \log \left( \frac{(R_t/\alpha_t)^{d_t} \hat{r}^{d-d_t}}{\hat{r}^d} \right) \leq d_t + d_t \log \left( \frac{R_t}{\alpha_t \hat{r}} \right). \end{aligned}$$

We conclude that

$$\frac{1}{\alpha_t} \lesssim d_t + d_t \log\left(\frac{R_t}{\hat{r}}\right) + d_t \log d_t.$$

This concludes the proof of Theorem 3.

#### 2.4.6. Aspect ratio-independent bounds and proof of Theorem 4

To prove Theorem 4, we first establish Lemma 2.4.20.

Lemma 2.4.20. Let t be an iteration corresponding to an irregular update step in Algorithm 4. Then,

$$\frac{\operatorname{vol}_{d_{t-1}}\left(B_{2}^{d_{t-1}}\right)}{\operatorname{vol}_{d_{t}}\left(B_{2}^{d_{t}}\right)} \cdot \frac{\operatorname{vol}_{d_{t}}\left(\mathcal{E}_{t}\right)}{\operatorname{vol}_{d_{t}-1}\left(\mathcal{E}_{t-1}\right)} \geq \frac{\left\|\boldsymbol{z}_{t}^{\perp}\right\|_{2}}{2}$$

where  $||z_t^{\perp}||$  is the length of the component of  $z_t$  in the orthogonal complement of Span  $(z_1, \ldots, z_{t-1})$ .

*Proof of Lemma* 2.4.20. By affine invariance, we can apply an affine transformation to map  $z_t$  and  $\mathcal{E}_{t-1}$  to a convenient position. Hence, following the proof of Lemma 2.4.18, without loss of generality, suppose we have  $\mathcal{E}_{t-1} = B_2^{d_{t-1}}$  and  $z_t = \sqrt{1+2\alpha} \cdot e_{d_t}$ . By Lemma 2.4.18, the ellipsoid  $\mathcal{E}_t$  is a ball of radius  $(1+\alpha)/\sqrt{1+2\alpha}$ . Let  $z := \sqrt{1+2\alpha}$ . We now have

$$\frac{\operatorname{vol}_{d_{t-1}}\left(B_{2}^{d_{t-1}}\right)}{\operatorname{vol}_{d_{t}}\left(B_{2}^{d_{t}}\right)} \cdot \frac{\operatorname{vol}_{d_{t}}\left(\mathcal{E}_{t}\right)}{\operatorname{vol}_{d_{t-1}}\left(\mathcal{E}_{t-1}\right)} = \left(\frac{1+\alpha}{\sqrt{1+2\alpha}}\right)^{d_{t}} \ge 1 > \frac{\|\boldsymbol{z}_{t}\|_{2}}{2}$$

since  $||z_t||_2 = \sqrt{1 + 2\alpha} \le \sqrt{3} < 2$ . This concludes the Proof of Lemma 2.4.20.

We will also need Lemma 2.4.21, which we take from Gover and Krikorian [GK10].

**Lemma 2.4.21.** Let  $\mathbf{M} \in \mathbb{R}^{r \times d}$  have linearly independent rows  $m_1, \ldots, m_r$ . Then,

$$\prod_{i=1}^{r} \|\boldsymbol{m}_i\|_2 = \sqrt{\det\left(\mathbf{M}\mathbf{M}^T\right)},$$

We are now ready to prove Theorem 4.

*Proof of Theorem 4.* Our approach is reminiscent of that used in the proof of Theorem 1.5 in Woodruff and Yasuda [WY22a].

By applying a translation to all points, we may assume without loss of generality that  $z_1 = 0$ . We will prove the guarantee for the last timestamp t = n to simplify the notation. By replacing n with n', we can get a proof for any time stamp t = n'.

Let *S* be the set of timestamps of irregular update steps excluding the first step. Since the update rule satisfies the evolution condition (2.2.3), we have for all  $t \notin S$  (recall that  $d_t = d_{t-1}$  for  $t \notin S$ )

$$\frac{\operatorname{vol}_{d_t}\left(\mathcal{E}_t\right)}{\operatorname{vol}_{d_{t-1}}\left(\mathcal{E}_{t-1}\right)} \ge \exp\left(\frac{1}{\alpha_t} - \frac{1}{\alpha_{t-1}}\right).$$

Next, by Lemma 2.4.20, we have for every irregular update step t > 1

$$\frac{\operatorname{vol}_{d_{t-1}}\left(B_{2}^{d_{t-1}}\right)}{\operatorname{vol}_{d_{t}}\left(B_{2}^{d_{t}}\right)} \cdot \frac{\operatorname{vol}_{d_{t}}\left(\mathcal{E}_{t}\right)}{\operatorname{vol}_{d_{t-1}}\left(\mathcal{E}_{t-1}\right)} \geq \frac{\left\|\boldsymbol{z}_{t}^{\perp}\right\|_{2}}{2}.$$

Here, we assume that  $vol_0({0}) = 1$  and define  $||z_2^{\perp}|| = ||z_2||$ . Inductively combining the above for all t > 1 gives

$$\operatorname{vol}_{d_{n}}(\mathcal{E}_{n}) \geq \prod_{t \notin S} \exp\left(\frac{1}{\alpha_{t}} - \frac{1}{\alpha_{t-1}}\right) \cdot \prod_{t \in S} \frac{\|\boldsymbol{z}_{t}^{\perp}\|_{2}}{2} \cdot \prod_{j=1}^{d_{n}} \frac{\operatorname{vol}_{j}\left(\boldsymbol{B}_{2}^{j}\right)}{\operatorname{vol}_{j-1}\left(\boldsymbol{B}_{2}^{j-1}\right)}$$
$$= \prod_{t \notin S} \exp\left(\frac{1}{\alpha_{t}} - \frac{1}{\alpha_{t-1}}\right) \cdot \prod_{t \in S} \frac{\|\boldsymbol{z}_{t}^{\perp}\|_{2}}{2} \cdot \operatorname{vol}_{d_{n}}\left(\boldsymbol{B}_{2}^{d_{n}}\right)$$
(2.4.19)

Here we used that  $vol_0(\mathcal{E}_0) = vol_0(B_2^0) = 1$ . Now invoking Lemma 2.4.21, we get

$$\prod_{t \in S} \frac{\left\|\boldsymbol{z}_t^{\perp}\right\|_2}{2} \ge 2^{-|S|} \sqrt{\det\left(\mathbf{Z}|_S \mathbf{Z}|_S^T\right)} \ge 2^{-|S|} = 2^{-d_n},$$

where we used that det  $(\mathbf{Z}|_{S}\mathbf{Z}|_{S}^{T}) \geq 1$  because all the vectors  $\mathbf{z}_{t}$  have integer coordinates. Moreover, since all coordinates are at most N in absolute value, all the vectors  $\mathbf{z}_{t}$  have length at most  $N\sqrt{d}$ . Therefore,  $\frac{\operatorname{vol}(\mathcal{E}_{n})}{\operatorname{vol}(\mathcal{B}_{2}^{d_{n}})} \leq (N\sqrt{d})^{d_{n}}$ . We plug these bounds back into (2.4.19), rearrange, and take the logarithm of both sides, yielding

$$\sum_{t \notin S} \frac{1}{\alpha_t} - \frac{1}{\alpha_{t-1}} \lesssim d_n \log \left( dN \right).$$

Finally, by (2.4.18), we have  $\frac{1}{\alpha_t} - \frac{1}{\alpha_{t-1}} = 1$  for every  $t \in S$ . Combining everything gives

$$\sum_{t \leq n} \frac{1}{\alpha_t} - \frac{1}{\alpha_{t-1}} \leq d_n \log \left( dN \right) + |S| \leq d_n \log \left( dN \right),$$

thereby concluding the proof of Theorem 4.

# 2.5. Improved analysis for symmetric polytopes (Proof of Theorem 2)

In this section, we specialize the analysis framework developed in this chapter to the case when the polytope *Z* is symmetric – that is, when in each timestep *t*, we receive both  $z_t$  and  $-z_t$ . We then prove Theorem 2.

*Bibliographic notes.* The material in this section is derived from the paper [MMO22]. Although the algorithm is the same as the one in that paper, the analysis given here is considerably simpler and fits within the framework developed earlier in this chapter.

### 2.5.1. Monotone update rule for symmetric ellipsoidal approximation

The main result of this subsection is Lemma 2.5.1.

**Lemma 2.5.1.** *The update rule given by* 

$$\begin{split} \mathbf{A}_{t} &= \begin{cases} \mathbf{A}_{t-1} - \left(1 - \frac{1}{\|\mathbf{A}_{t-1}\boldsymbol{z}_{t}\|_{2}}\right) \left(\frac{(\mathbf{A}_{t-1}\boldsymbol{z}_{t})(\mathbf{A}_{t-1}\boldsymbol{z}_{t})^{\mathsf{T}}}{\|\mathbf{A}_{t-1}\boldsymbol{z}_{t}\|_{2}^{2}}\right) \mathbf{A}_{t-1} & \text{if } \|\mathbf{A}_{t-1}\boldsymbol{z}_{t}\|_{2} > 1\\ \mathbf{A}_{t-1} & \text{otherwise} \end{cases}\\ \boldsymbol{c}_{t} &= 0\\ \boldsymbol{\alpha}_{t} &= \begin{cases} \frac{\|\mathbf{A}_{t-1}\boldsymbol{z}_{t}\|_{2}\alpha_{t-1}}{\sqrt{\|\mathbf{A}_{t-1}\boldsymbol{z}_{t}\|_{2}^{2}-\alpha_{t-1}^{2}}} \cdot \frac{1}{\sqrt{1+\left(\frac{\|\mathbf{A}_{t-1}\boldsymbol{z}_{t}\|_{2}^{2}-\alpha_{t-1}^{2}}{\sqrt{\|\mathbf{A}_{t-1}\boldsymbol{z}_{t}\|_{2}^{2}-\alpha_{t-1}^{2}}}\right)^{2}} & \text{if } \|\mathbf{A}_{t-1}\boldsymbol{z}_{t}\|_{2} > 1\\ \boldsymbol{\alpha}_{t-1} & \text{otherwise} \end{cases} \end{split}$$

*is a monotone update rule when in each timestep the algorithm receives both points*  $\pm z_t$ *.* 

The goal of the rest of this subsection is to prove Lemma 2.5.1.

The monotone update rule we use in this section is much easier to describe than in the general case. Let  $0 = c_1 = \cdots = c_n$  – that is, we never shift the center. Let  $\mathbf{A}_{t-1}$  denote the matrix for the ellipsoid output at timestep t, so that we have  $\|\mathbf{A}_{t-1}\mathbf{z}_i\|_2 \leq 1$  for all  $i \leq t - 1$ . Given a new point pair  $\pm \mathbf{z}_t$ , we set  $\mathbf{A}_t$  according to the formula

$$\mathbf{A}_{t} = \mathbf{A}_{t-1} - \left(1 - \frac{1}{\|\mathbf{A}_{t-1}\mathbf{z}_{t}\|_{2}}\right) \left(\frac{(\mathbf{A}_{t-1}\mathbf{z}_{t})(\mathbf{A}_{t-1}\mathbf{z}_{t})^{\mathsf{T}}}{\|\mathbf{A}_{t-1}\mathbf{z}_{t}\|_{2}^{2}}\right) \mathbf{A}_{t-1}.$$

This formula has a natural interpretation – it is the minimum volume ellipsoid that covers both the ellipsoid  $\mathcal{E}_{t-1} = \{ x \in \mathbb{R}^d : \|\mathbf{A}_{t-1}x\|_2 \leq 1 \}$  and the new point pair  $\pm z_t$ . Specifically, see Lemma 2.5.2.

**Lemma 2.5.2.** Let  $\mathbf{A}_{t-1} \in \mathbb{R}^{d \times d}$  and let

$$\mathbf{A}_{t} = \mathbf{A}_{t-1} - \left(1 - \frac{1}{\|\mathbf{A}_{t-1}\mathbf{z}_{t}\|_{2}}\right) \left(\frac{(\mathbf{A}_{t-1}\mathbf{z}_{t})(\mathbf{A}_{t-1}\mathbf{z}_{t})^{\top}}{\|\mathbf{A}_{t-1}\mathbf{z}_{t}\|_{2}^{2}}\right) \mathbf{A}_{t-1}$$

If  $\mathcal{E}_{t-1} \coloneqq \{ x \in \mathbb{R}^d : \|\mathbf{A}_{t-1}x\|_2 \leq 1 \}$ , then the ellipsoid  $\mathcal{E}_t \coloneqq \{ x \in \mathbb{R}^d : \|\mathbf{A}_tx\|_2 \leq 1 \}$  is the minimum volume ellipsoid that contains both  $\mathcal{E}_{t-1}$  and  $\pm z_t$ .

Although we do not directly use Lemma 2.5.2, it raises an interesting conceptual point, and so we give a proof below.

*Proof of Lemma* 2.5.2. In this proof, in an abuse of notation, let  $M^{-1}$  denote the pseudoinverse of **M** and let det (**M**) of a singular matrix be the product of its nonzero singular values.

Note that the volume of the ellipsoid determined by  $\mathbf{A}_t$  is proportional to det  $(\mathbf{A}_t^{-1})$ . Therefore,  $\mathbf{A}_t$  is the solution to the following optimization problem, where we use that the volume of the ellipsoid determined by  $\mathbf{A}_t$  is proportional to det  $(\mathbf{A}_t^{-1})$ .

 $\max \det (\mathbf{A}_t)$  such that  $\mathbf{A}_t \leq \mathbf{A}_{t-1}$  and  $\|\mathbf{A}_t \mathbf{z}_t\|_2 \leq 1$ 

Additionally, since  $\det(AB) = \det(A) \cdot \det(B)$ , we have that this objective is invariant under linear transformations. It thus follows that our objective can be rewritten as

$\max \det \left( \mathbf{A}_{t}  ight)$	such that $\mathbf{A}_t \leq \mathbf{A}_{t-1}$ and $\ \mathbf{A}_t \mathbf{z}_t\ _2 \leq 1$
$\equiv \max \det \left( \mathbf{A}_t \cdot \mathbf{A}_{t-1}^{-1} \right)$	such that $\mathbf{A}_t \cdot \mathbf{A}_{t-1}^{-1} \leq \mathbf{I}$ and $\left\  \left( \mathbf{A}_t \cdot \mathbf{A}_{t-1}^{-1} \right) \mathbf{A}_{t-1} \mathbf{z}_t \right\ _2 \leq 1$
$\equiv \max \det \left( \widehat{\mathbf{A}} \right)$	such that $\widehat{\mathbf{A}} \leq \mathbf{I}$ and $\left\  \widehat{\mathbf{A}} \mathbf{A}_{t-1} \mathbf{z}_t \right\ _2 \leq 1$ ,

where the last line follows from using the intermediate variable  $\widehat{\mathbf{A}} = \mathbf{A}_t \cdot \mathbf{A}_{t-1}^{-1}$ .

In other words, after the transformation, the problem is equivalent to finding the minimum volume ellipsoid that contains (i) the unit ball and (ii) point  $\mathbf{A}_{t-1}\mathbf{z}_t$ . Geometrically, it is clear what the optimal ellipsoid for this problem is: one of its semi-axes is  $\mathbf{A}_{t-1}\mathbf{z}_t$ ; all others are orthogonal to  $\mathbf{A}_{t-1}\mathbf{z}_t$  and have length 1 (this can be formally proved using symmetrization). However, we do not use this observation and derive a formula for  $\hat{\mathbf{A}}$  using linear algebra.

We first give an upper bound on the objective value of the above optimization problem. Since  $\widehat{\mathbf{A}} \leq \mathbf{I}$ , we have that all its singular values must be at most 1. Additionally, since  $1 \geq \left\|\widehat{\mathbf{A}}\mathbf{A}_{t-1}\mathbf{z}_t\right\|_2 \geq \sigma_{\min}\left(\widehat{\mathbf{A}}\right) \cdot \|\mathbf{A}_{t-1}\mathbf{z}_t\|_2$ , we have that at least one singular value of  $\widehat{\mathbf{A}}$  must be  $\leq 1/\|\mathbf{A}_{t-1}\mathbf{z}_t\|_2$ . Putting everything together and using the fact that the determinant is the product of the singular values gives  $\det\left(\widehat{\mathbf{A}}\right) \leq 1/\|\mathbf{A}_{t-1}\mathbf{z}_t\|_2$ .

We now show that there exists a setting of  $\widehat{\mathbf{A}}$  that achieves this upper bound. Let  $v_1$  be a unit vector in the direction of  $\mathbf{A}_{t-1}\mathbf{z}_t$  and  $v_2, \ldots, v_d$  complete the orthonormal basis for  $\mathbb{R}^d$  from  $v_1$ , and write  $\widehat{\mathbf{A}} = \frac{1}{\|\mathbf{A}_{t-1}\mathbf{z}_t\|_2} v_1 v_1^\top + \sum_{i=2}^d v_i v_i^\top$ . We will show that  $\widehat{\mathbf{A}}$  satisfies the constraints imposed by the optimization problem. Since we have  $\|\mathbf{A}_{t-1}\mathbf{z}_t\|_2 \ge 1$  (as we impose that  $\mathbf{z}_t \notin \mathcal{E}_{\mathbf{A}_{t-1}}$ ), the fact that  $\widehat{\mathbf{A}} \le \mathbf{I}$  follows immediately. For the second constraint, we write

$$\left\|\widehat{\mathbf{A}}\mathbf{A}_{t-1}\boldsymbol{z}_{t}\right\|_{2} = \left\|\left(\frac{1}{\|\mathbf{A}_{t-1}\boldsymbol{z}_{t}\|_{2}}\boldsymbol{v}_{1}\boldsymbol{v}_{1}^{\top} + \sum_{i=2}^{d}\boldsymbol{v}_{i}\boldsymbol{v}_{i}^{\top}\right)\mathbf{A}_{t-1}\boldsymbol{z}_{t}\right\|_{2} = \left\|\frac{\mathbf{A}_{t-1}\boldsymbol{z}_{t}}{\|\mathbf{A}_{t-1}\boldsymbol{z}_{t}\|_{2}}\right\|_{2} = 1.$$

Furthermore, it is easy to see that  $det(\widehat{\mathbf{A}}) = \|\mathbf{A}_{t-1}\mathbf{z}_t\|_2^{-1}$ , which achieves our upper bound.

Finally, recall that we wrote  $\widehat{\mathbf{A}} = \mathbf{A}_t \cdot \mathbf{A}_{t-1}^{-1}$ ; rearranging this gives us the conclusion of Lemma 2.5.2.

We now prove Lemma 2.5.1.

*Proof of Lemma* 2.5.1. The only nontrivial cases we have to deal with are when  $\|\mathbf{A}_{t-1}\mathbf{z}_t\|_2 > 1 -$ in other words, when  $\mathbf{z}_t \notin \mathcal{E}_{t-1}$ . So, we assume this in the rest of the proof.

To validate  $z_t \in c_t + \mathcal{E}_t$ , it is enough to show  $\|\mathbf{A}_t z_t\|_2 \le 1$ . We have

$$\mathbf{A}_{t} \mathbf{z}_{t} = \mathbf{x} - \left(1 - \frac{1}{\|\mathbf{x}\|_{2}}\right) \left(\frac{\mathbf{x}\mathbf{x}^{\top}}{\|\mathbf{x}\|_{2}^{2}}\right) \mathbf{x} = \mathbf{x} - \left(1 - \frac{1}{\|\mathbf{x}\|_{2}}\right) \mathbf{x} = \frac{\mathbf{x}}{\|\mathbf{x}\|_{2}},$$

as desired.

The main challenge is to prove that  $c_t + \alpha_t \mathcal{E}_t \subseteq \operatorname{conv}((c_{t-1} + \alpha_{t-1}\mathcal{E}_{t-1}) \cup \{z_t\})$ , or in other words, that the "inner ellipsoid" is always feasible. Without loss of generality, by applying an affine transformation and considering the reduced 2-dimensional case in the same way as in the proof of Theorem 1, we let  $z_t = z \cdot e_1$  for z > 1 and  $\mathcal{E}_{t-1} = B_2^2$ . Notice that  $z = ||\mathbf{A}_{t-1}z_t||_2$ . Our goal is to show that the ellipse  $\alpha_t \cdot \mathcal{E}_t$  lies inside the shape  $\operatorname{conv}(\alpha_{t-1}B_2^2 \cup \{\pm z \cdot e_1\})$ .

First, we prove that the parallelogram *P* defined by the lines

$$y = -\frac{\alpha_{t-1}}{\sqrt{z^2 - \alpha_{t-1}^2}} (x - z)$$
$$y = \frac{\alpha_{t-1}}{\sqrt{z^2 - \alpha_{t-1}^2}} (x - z)$$

and their reflections over the *y*-axis (i) passes through  $\pm z \cdot e_1$  and (ii) has exactly four points of tangency with the circle  $\alpha_{t-1}B_2^2$ . Then, it will be enough to argue that  $\alpha_t \mathcal{E}_t \in P$ .

The first part is clear from the formulas so we handle the second. Here, it is enough to show that the slope of the line that passes through  $z \cdot e_1$  and is tangent to  $\alpha_{t-1}B_2^2$  has slope  $-\frac{\alpha_{t-1}}{\sqrt{z^2 - \alpha_{t-1}^2}}$ . The distance between  $z \cdot e_1$  and the point of tangency is  $\sqrt{z^2 - \alpha_{t-1}^2}$  by the Pythagorean Theorem and the result follows from similar triangles. Repeating this for all four lines defining *P* proves (ii).

Now, we are ready to prove that  $\alpha_t \mathcal{E}_t \in P$ . This is a morally similar calculation. In this reduced case, the equation that determines the (x, y) pairs lying on the surface of  $\alpha_t \mathcal{E}_t$  is given by

$$\frac{x^2}{z^2} + y^2 = \alpha_t^2.$$

Applying a linear transformation to our space does not affect tangency. So, we apply the linear transformation that maps  $z \cdot e_1$  to  $e_1$  (this is equivalent to multiplying every *x*-coordinate by 1/z and leaving all the *y*-coordinates unchanged). Now, our goal is to show that the circle  $\alpha_t B_2^2$  lies inside the parallelogram whose vertices in the nonnegative orthant are the pair of points

$$\left(0, \frac{z\alpha_{t-1}}{\sqrt{z^2 - \alpha_{t-1}^2}}\right) \text{ and } (1, 0).$$

By calculating the area of this triangle in two ways, we get that the radius of the incircle of this parallelogram is

$$\alpha_t = \frac{z\alpha_{t-1}}{\sqrt{z^2 - \alpha_{t-1}^2}} \cdot \frac{1}{\sqrt{1^2 + \left(\frac{z\alpha_{t-1}}{\sqrt{z^2 - \alpha_{t-1}^2}}\right)^2}}.$$

Undoing the affine transformations and recalling that doing so means that  $z = \|\mathbf{A}_{t-1}z_t\|_2$  completes the proof of Lemma 2.5.1.

#### 2.5.2. Approximation guarantee via stronger evolution condition

We now show that the monotone update rule given by Lemma 2.5.1 satisfies an evolution condition that is stronger than (2.2.3).

Lemma 2.5.3. Under the update rule given in the statement of Lemma 2.5.1, we have

$$\frac{1}{\alpha_t^2} - \frac{1}{\alpha_{t-1}^2} \leq \begin{cases} 2\log\left(\frac{\operatorname{vol}(\mathcal{E}_t)}{\operatorname{vol}(\mathcal{E}_{t-1})}\right) & \text{if } z_t \notin \mathcal{E}_{t-1} \\ 0 & \text{otherwise} \end{cases}.$$

*Proof of Lemma 2.5.3.* As usual, we only consider the case where  $z_t \notin \mathcal{E}_{t-1}$  so that there is an update.

We first prove

$$\frac{1}{\alpha_t^2} - \frac{1}{\alpha_{t-1}^2} \le 2\log(\|\mathbf{A}_{t-1} z_t\|_2).$$

Let  $z \coloneqq \|\mathbf{A}_{t-1}\mathbf{z}_t\|_2$  and let  $\Delta \coloneqq \frac{z\alpha_{t-1}}{\sqrt{z^2 - \alpha_{t-1}^2}}$ . Using the formula for  $\alpha_t$  given in Lemma 2.5.1, we have

$$\frac{1}{\alpha_t^2} - \frac{1}{\alpha_{t-1}^2} = \frac{\Delta^2 + 1}{\Delta^2} - \frac{1}{\alpha_{t-1}^2} = 1 + \frac{z^2 - \alpha_{t-1}^2}{z^2 \alpha_{t-1}^2} - \frac{z^2}{z^2 \alpha_{t-1}^2} = 1 - \frac{1}{z^2} \le 2\log z,$$

where the inequality follows from noting that  $1 - \frac{1}{z^2} = 2 \log z$  at z = 1 and the derivative of  $2 \log z$  is always at least as large as the derivative of  $1 - \frac{1}{z^2}$  whenever  $z \ge 1$ .

Next, we show vol  $(\mathcal{E}_t) = ||\mathbf{A}_{t-1}\mathbf{z}_t||_2 \cdot \text{vol}(\mathcal{E}_{t-1})$ . We write

$$\det \left( \mathbf{A}_{t} \right) = \det \left( \mathbf{I} - \left( 1 - \frac{1}{\|\mathbf{A}_{t-1}\mathbf{z}_{t}\|_{2}} \right) \left( \frac{\left( \mathbf{A}_{t-1}\mathbf{z}_{t} \right) \left( \mathbf{A}_{t-1}\mathbf{z}_{t} \right)^{\mathsf{T}}}{\|\mathbf{A}_{t-1}\mathbf{z}_{t}\|_{2}^{2}} \right) \right) \cdot \det \left( \mathbf{A}_{t-1} \right) = \frac{1}{\|\mathbf{A}_{t-1}\mathbf{z}_{t}\|_{2}} \cdot \det \left( \mathbf{A}_{t-1} \right),$$

and rearranging gives us what we need.

Combining these conclusions gives us the statement of Lemma 2.5.3.

We are now ready to prove Theorem 2.

*Proof of Theorem 2.* As in the asymmetric case (proof of Theorem 1), we consider two phases – name them Phase I and Phase II. In Phase II, we perform the monotone update rule as written in Lemma 2.5.1. In Phase I, we perform the trivial update

$$\mathbf{A}_{t} = \min\left\{1, \frac{1}{\|\mathbf{A}_{t-1}\boldsymbol{z}_{t}\|_{2}}\right\} \cdot \mathbf{A}_{t-1}$$
$$\boldsymbol{c}_{t} = 0$$

$$\alpha_t = \min\left\{1, \frac{1}{\|\mathbf{A}_{t-1}\mathbf{z}_t\|_2}\right\} \cdot \alpha_{t-1},$$

which can be visually described as applying the minimum scaling of the outer ellipsoid such that it contains the new points  $\pm z_t$  while leaving the inner ellipsoid unchanged. It is easy to see that by just applying the Phase I update, the approximation factor we get is  $R(Z)/r_0$ .

Next, we describe the switching condition we use to change from Phase I to Phase II. At iteration *t*, if  $R(Z_t)/r_0 \ge 2^{3/2}\sqrt{d \log (4d)}$ , then we change to Phase II. It is easy to see that  $R(Z_t)$  is monotonically increasing in *t*, so once this switching condition is satisifed, it will remain satisfied until the termination of the algorithm.

Applying Lemma 2.5.3 across all the Phase II iterates and recalling that in Phase I we never change the inner ellipsoid, we have

$$\frac{1}{\alpha_n^2} \le 2 \log \left( \frac{\operatorname{vol} \left( \mathcal{E}_n \right)}{\operatorname{vol} \left( \mathcal{E}_0 \right)} \right).$$

We know that the containment  $\alpha_n \mathcal{E}_n \subseteq Z_n \subseteq \mathcal{E}_n$  must hold. We also know that  $Z_n \subseteq R(Z_n) \cdot B_2^d$  must also hold. Thus, we know that  $\mathcal{E}_n \subseteq \frac{R(Z_n)}{\alpha_n} \cdot B_2^d$ . Hence, we have

$$\frac{1}{\alpha_n^2} \le 2\log\left(\frac{\operatorname{vol}\left(\mathcal{E}_n\right)}{\operatorname{vol}\left(\mathcal{E}_0\right)}\right) \le 2\log\left(\frac{\left(\frac{R(Z_n)}{\alpha_n}\right)^a}{r_0^d}\right) = d\log\left(\left(\frac{R(Z_n)}{r_0}\right)^2\right) + d\log\left(\frac{1}{\alpha_n^2}\right)$$

We now show that if we ever enter Phase II, then we have  $\alpha_n^{-2} \leq (R(Z_n)/r_0)^2$ . First, let us see how this gives us one of the cases of our proof. Substituting this gives us

$$\frac{1}{\alpha_n^2} \le 4d \log\left(\frac{R(Z_n)}{r_0}\right),\,$$

so taking square roots gives the result.

Now, for the sake of contradiction, suppose that  $\alpha_n^{-2} \ge (R(Z_n)/r_0)^2$ . This gives us

$$\frac{1}{\alpha_n^2} \le 4d \log\left(\frac{1}{\alpha_n^2}\right),\,$$

so solving for  $\alpha_n^{-1}$  gives

$$\frac{1}{\alpha_n^2} < 8d \log\left(4d\right).$$

This means that

$$\frac{R(Z_n)}{r_0} \le 2^{3/2} \sqrt{d \log{(4d)}},$$

which means we always stayed in Phase I and our approximation factor is just  $\frac{R(Z_n)}{r_0}$  (and the monotone update rule given by Lemma 2.5.1 never actually executed).

Combining the cases tells us that for some universal constant *C*, we have

$$\frac{1}{\alpha_n} \le C \min\left\{\frac{R(Z_n)}{r_0}, \sqrt{2d \log\left(\frac{R(Z_n)}{r_0}\right)}\right\}.$$

completing the proof of Theorem 2.

## 2.6. Forming small coresets for convex bodies (Proof of Theorem 5)

In this section, we prove Theorem 5. See Algorithm 5.

Algorithm 5 Streaming coreset for convex hull 1: **Input:** Stream of points  $z_t$ ; Update rule for Algorithm 4  $\mathcal{A}$ . 2: Output: Set  $S \subseteq [n]$ . 3: for t = 1, ..., n do Receive  $z_t$ . 4: Let  $\mathcal{E}_{\text{test}} = \mathcal{A}(c_{t-1}, \mathcal{E}_{t-1}, z_t).$ 5: Let  $d_t = \dim (\text{Span} (z_1 - c_{t-1}, \dots, z_t - c_{t-1})).$ 6: if  $d_t > d_{t-1}$  or  $\frac{\operatorname{Vol}_{d_t}(\mathcal{E}_{\text{test}})}{\operatorname{Vol}_{d_t}(\mathcal{E}_{t-1})} \ge e$  then 7: 8: Let  $c_t$ ,  $\mathcal{E}_t = \mathcal{A}(c_{t-1}, \mathcal{E}_{t-1}, z_t)$ . Update  $S_t = S_{t-1} \cup \{z_t\}$ . 9: 10: else 11: Let  $c_t$ ,  $\mathcal{E}_t = c_{t-1}$ ,  $\mathcal{E}_{t-1}$ . Let  $S_{t-1} = S_t$ . 12: 13: **Output:** *S<sub>n</sub>* 

For a sketch of the intuition and the argument we will use for the proof, see Section 2.2.3.

*Proof of Theorem 5.* We prove two properties of Algorithm 5. First, we show  $|S_t| \leq O(d_t \cdot \log(d_t \cdot \max_{t' \leq t} R_t/r_{t'}))$  and, further,  $|S_t| \leq O(d_t \cdot \log(dN))$  if points  $z_t$  have integer coordinates between -N and N. Second, we show that  $\operatorname{conv}(Z|_{S_t}) \subseteq \operatorname{conv}(Z|_{[t]}) \subseteq O(d_t \cdot \log(d_t \cdot \max_{t' \leq t} R_t/r_{t'})) \cdot \operatorname{conv}(Z|_{S_t})$  and  $\operatorname{conv}(Z|_{S_t}) \subseteq \operatorname{conv}(Z|_{[t]}) \subseteq O(d_t \cdot \log(dN)) \cdot \operatorname{conv}(Z|_{S_t})$ .

**Bounding**  $|S_t|$ . It is enough to count the number of steps *t* for which we have  $\frac{\operatorname{Vol}_{d_t}(\mathcal{E}_{test})}{\operatorname{Vol}_{d_t}(\mathcal{E}_{t-1})} \ge e$ .

It is easy to see that for all t, we have  $r(Z|_{[t]}) \cdot (B_2^d \cap \text{Span}(z_1 - c_t, \dots, z_t - c_t)) \subseteq c_t + \mathcal{E}_t$ . Additionally, by the definition of R(Z), we always have  $Z|_{[t]} \subseteq R(Z) \cdot (B_2^d \cap \text{Span}(z_1 - c_t, \dots, z_t - c_t))$ . These are enough to give volume lower and upper bounds in each step. Next, for each step in which we add an element to  $S_{t-1}$  to obtain  $S_t$ , the volume must increase by a factor of e. It easily follows that the number of elements in  $S_t$  satisfies

$$|S_t| \le \log\left(\max_{t' \le t} \frac{\prod_{i=1}^{d_t} R(Z|_{[t]})}{\prod_{i=1}^{d_t} r(Z|_{[t']})}\right) = d_t \log\left(\max_{t' \le t} \frac{R(Z|_{[t]})}{r(Z|_{[t']})}\right).$$

We now give an upper bound for the case when all coordinated of  $z_t$  are integers not exceeding N in absolute value. It is easy to see that the update rule in Algorithm 5 exactly corresponds to the steps where we have

$$\frac{1}{\alpha_t} - \frac{1}{\alpha_{t-1}} \gtrsim 1,$$

and in the same way as in the proof of Theorem 4, we have for all t that

$$\sum_{t\geq 1}\frac{1}{\alpha_t}-\frac{1}{\alpha_{t-1}}\lesssim d_t\log\left(dN\right).$$

It therefore follows that  $|S| \leq d_t \log(dN)$ , as desired.

**Bounding the distortion of the chosen points.** Consider some iteration  $t' \leq t$ . Without loss of generality, let  $c_{t'-1} = 0$ . Suppose  $z_{t'}$  does not result in an update to  $S_{t'-1}$ . This implies that  $z_{t'} \in 2e \cdot \mathcal{E}_{t'-1}$ . Next, observe that  $0 \in c_t + \mathcal{E}_t$ . Putting these together, we have  $z_{t'} \in (c_t + \mathcal{E}_t) + 2e \cdot \mathcal{E}_{t'-1}$ . Since  $\mathcal{A}$  is monotone, we must have  $2e \cdot \mathcal{E}_{t'-1} \subseteq c_t + e \cdot \mathcal{E}_t$ ; hence, we may write  $z_{t'} \in c_t + (2e + 1)\mathcal{E}_t$ .

The inner ellipsoid  $c_t + \alpha_t \cdot \mathcal{E}_t$  will still be an inner ellipsoid for the points determined by  $S_t$ . Stitching together all our inclusions, we have

$$\boldsymbol{c}_t + \boldsymbol{\alpha}_t \cdot \boldsymbol{\mathcal{E}}_t \subseteq Z|_{S_t} \subseteq Z \subseteq \boldsymbol{c}_t + (2e+1)\boldsymbol{\mathcal{E}}_t \subseteq \frac{2e+1}{\boldsymbol{\alpha}_t} \cdot Z|_{S_t}.$$
(2.6.1)

which means that

$$Z|_{S_t} \subseteq Z \subseteq O\left(d_t \cdot \log\left(d_t \cdot \max_{t' \leq t} \frac{R(Z|_{([t] \cap S_t)})}{r(Z|_{([t'] \cap S_t)})}\right)\right) \cdot Z|_{S_t}.$$

Notice that this is nearly what we want, except that the aspect ratio term is in terms of the subset body  $Z|_{S_t}$ . To obtain the final guarantee in terms of the aspect ratio of  $Z|_{[t]}$ , observe that the above guarantee readily implies that

$$O\left(d_t \cdot \log\left(d_t \cdot \max_{t' \le t} \frac{R(Z|_{([t] \cap S_t)})}{r(Z|_{([t'] \cap S_t)})}\right)\right) \le O\left(d_t \cdot \log\left(d_t \cdot \max_{t' \le t} \frac{R(Z|_{[t]})}{r(Z|_{[t']})}\right)\right).$$

We now give the corresponding improvement when the  $z_t$  are integer-valued. As before, (2.6.1) holds. From this, we get

$$Z|_{S_t} \subseteq Z \subseteq O\left(d_t \cdot \log\left(dN\right)\right) \cdot Z|_{S_t},$$

as desired. This concludes the proof of Theorem 5.

# 2.7. Approximation lower bound for monotone algorithms

In this section, we show Theorem 6 and Theorem 7.

#### 2.7.1. Inapproximability of John's ellipsoid

Before we begin, recall that a Hadamard basis is a set of vectors  $v_1, \ldots, v_d$  such that:

- $||v_i|| = 1$ , for all  $i \in [d]$ ;
- For all  $i \neq j$ ,  $\langle v_i, v_j \rangle = 0$ ;
- Every entry of  $v_i$  is in  $\{\pm 1/\sqrt{d}\}$ .

Our family of hard instances proceeds in two phases.

**Phase 1** Let *d* be such that there exists a Hadamard basis for  $\mathbb{R}^d$ . Consider a corresponding Hadamard basis  $v_1, \ldots, v_d$ . The adversary gives the algorithm the points  $v_1, \ldots, v_d$ .

**Phase 2** The adversary selects  $i \in [d]$  arbitrarily and  $\varepsilon \in (0, d - 1)$  arbitrarily. They then define the vectors  $w_i \coloneqq e_i \cdot 1/\sqrt{d-\varepsilon}$  and  $w_j \coloneqq e_j \cdot \sqrt{d-1/\varepsilon}$  for all  $j \neq i$ . The adversary gives the algorithm the points  $w_1, \ldots, w_d$ . Call the outcome here "Outcome (i)."

It is easy to see that at the end of Phase 1, the minimum volume outer ellipsoid is simply  $B_2^d$ . Furthermore, the algorithm's solution  $\widehat{\mathcal{E}}$  contains conv  $(\pm v_1, \ldots, \pm v_d)$ . On the other hand, consider the following claim.

**Lemma 2.7.1.** The following ellipsoid is the minimum-volume outer ellipsoid for Outcome (i):

$$\mathcal{E}_{OPT(i)} = \left\{ x : 1 \ge \frac{x_i^2}{\left(1/\sqrt{d-\varepsilon}\right)^2} + \sum_{j \ne i}^d \frac{x_j^2}{\left(\sqrt{d-1/\varepsilon}\right)^2} \right\}$$

*Proof.* Notice that all the points  $w_j$  are orthogonal. Thus, the minimum-volume outer ellipsoid containing all the  $w_j$  must be the one whose axes are along the directions of  $w_j$  and whose poles are located on  $w_j$ . Observe that  $\mathcal{E}_{OPT(i)}$  satisfies this, so it must be the minimum-volume outer ellipsoid for the convex body whose vertices are determined by the  $w_j$ .

It now remains to show that every Hadamard basis vector is on the surface of  $\mathcal{E}_{OPT(i)}$ :

$$\frac{(1/\sqrt{d})^2}{(1/\sqrt{d-\varepsilon})^2} + \sum_{j\neq i}^d \frac{(1/\sqrt{d})^2}{\left(\sqrt{d-1/\varepsilon}\right)^2} = \frac{1}{d} \left( (d-\varepsilon) + (d-1) \cdot \frac{\varepsilon}{d-1} \right) = 1$$

Since the minimum volume ellipsoid containing all the  $w_j$  also contains the Hadamard basis vectors, it (i.e.,  $\mathcal{E}_{OPT(i)}$ ) must be the minimum-volume outer ellipsoid for Outcome (i).

*Proof of Theorem 6.* We will now show that any that outputs an ellipsoid  $\widehat{\mathcal{E}}$  at the end of Phase 1 must have an approximation factor of at least  $\sqrt{d-\varepsilon}$  on at least one of Outcomes  $(1, \ldots, i)$ . Suppose that in each of Outcome (i), we obtain an ellipsoid  $\widehat{\mathcal{E}}_i$  that satisfies  $C \cdot \mathcal{E}_{OPT(i)} \supseteq \widehat{\mathcal{E}}_i$ . We now have:

$$\operatorname{conv}\left(\{\pm v_1,\ldots,v_d\}\right)\subseteq\widehat{\mathcal{E}}\subseteq\bigcap_{i=1}^d\widehat{\mathcal{E}}_i\subseteq C\cdot\bigcap_{i=1}^d\mathcal{E}_{OPT(i)}$$

We therefore want to argue about  $\widehat{\mathcal{E}}$  given that it must contain conv  $(\{\pm v_1, \ldots, v_d\})$  and be contained by  $C \cdot \bigcap_{i=1}^d \mathcal{E}_{OPT(i)}$ . Let *A* be a matrix mapping  $\widehat{\mathcal{E}}$  to the unit ball. Then, notice that we can write for all  $i \in [d]$ :

$$\|\mathbf{A}\boldsymbol{v}_i\|_2 \le 1 \qquad \qquad \left\|\mathbf{A} \cdot \frac{C\boldsymbol{e}_i}{\sqrt{d-\varepsilon}}\right\|_2 \ge 1$$

п

In particular, the rightmost exclusion follows from the fact that  $Ce_i/\sqrt{d-\varepsilon}$  lies on the boundary of  $C \cdot \bigcap_{i=1}^{d} \mathcal{E}_{OPT(b,i)}$ . Now, recall the well-known fact that for any unitary matrix **W**, we have  $\|\mathbf{AW}\|_F = \|\mathbf{A}\|_F$  (see, e.g., [HJ91]), and observe that we have

$$d \ge \sum_{i=1}^{d} \|\mathbf{A}\boldsymbol{v}_{i}\|^{2} = \|\mathbf{A}\mathbf{V}\|_{F}^{2} = \|\mathbf{A}\|_{F}^{2} = \|\mathbf{A}\mathbf{I}\|_{F}^{2} = \sum_{i=1}^{d} \|\mathbf{A}\boldsymbol{e}_{i}\|^{2} \ge \frac{d(d-\varepsilon)}{C^{2}}.$$

Rearranging gives  $C \ge \sqrt{d - \varepsilon}$ , as desired.

#### 2.7.2. Lower bound adversary

Our proof of Theorem 7 constructs an adversary, which given a monotone algorithm  $\mathcal{A}$  and  $\kappa \geq 1$ , constructs a sequence of points  $z_1, \ldots, z_n$  satisfying  $\kappa(\operatorname{conv}(z_1, \ldots, z_n)) \leq \kappa$  to witness that the algorithm does not produce an approximation better than  $\widetilde{\Omega}(d \log \kappa)$ . While by definition  $\kappa = \frac{R}{r}$ , our construction keeps r = 1 (notice that any lower bound construction must be scale-invariant), and for simplicity we use  $R = \kappa$ .

Let  $z_1^{\Delta}, z_2^{\Delta}, \ldots, z_{d+1}^{\Delta} \in \mathbb{R}^d$  be the d + 1 vertices of a regular simplex  $\Delta_d$  that circumscribes  $B_2^d$ . Our adversary is described in Algorithm 6. It uses a first phase that feeds  $\mathcal{A}$  the vertices of  $\Delta_d$ , then a second phase that repeatedly feeds  $\mathcal{A}$  points at a constant distance from the previous ellipsoid. Specifically, every new point  $z_t$  in the second phase is in  $c_{t-1} + 2 \cdot \mathcal{E}_{t-1}$ , i.e. its distance is 2 from  $c_{t-1}$  in the norm that is the gauge of  $\mathcal{E}_{t-1}$ .

Algorithm 6 Lower bound adversary **Input**: Monotone algorithm  $\mathcal{A}$ ,  $R \ge 1$ 1:  $(c_0 + \mathcal{E}_0, \alpha_0) = (0 + B_2^d, 1)$ ▶ Initialize to the unit ball 2: for  $t \in \{1, 2, ..., d + \tilde{1}\}$  do ▶ Phase I: feed A the vertices of a simplex  $(c_t + \mathcal{E}_t, \alpha_t) = \mathcal{A}(c_{t-1} + \mathcal{E}_{t-1}, \alpha_{t-1}, z_t^{\Delta})$ 3: 4:  $t \leftarrow d + 2$ 5: while  $\operatorname{vol}(\mathcal{E}_{t-1}) \leq \operatorname{vol}\left(\frac{R}{2} \cdot B_2^d\right) \mathbf{do}$ ▶ Phase II: feed A points outside the previous ellipsoid Let  $F_{t-1} = \partial (c_{t-1} + 2\mathcal{E}_{t-1}) \cap (R \cdot B_2^d)$ 6: 7: if  $F_{t-1} = \emptyset$  then stop 8: Let arbitrary  $z_t \in F_{t-1}$ 9:  $(\boldsymbol{c}_t + \boldsymbol{\mathcal{E}}_t, \boldsymbol{\alpha}_t) = \mathcal{A}(\boldsymbol{c}_{t-1} + \boldsymbol{\mathcal{E}}_{t-1}, \boldsymbol{\alpha}_{t-1}, \boldsymbol{z}_t)$ 10: 11:  $t \leftarrow t + 1$ 

**Remark 2.7.2.** This particular construction we give of the hard case is adaptive, meaning that the adversary's choice of points depend on the previous ellipsoids the algorithm outputs. However, this adversary can be made non-adaptive by taking an  $\varepsilon$ -net S of  $B_2^d$  for sufficiently small  $\varepsilon$ , then feeding  $\mathcal{A}$  the sequence of points in sets  $S, 2 \cdot S, 4 \cdot S, \ldots, 2^{\log_2 R-1}, 2^{\log_2 R} \cdot S$ . In consequence, this means that randomization on the part of the monotone algorithm does not help, unlike some other online settings.

Let *T* be the largest value of t - 1 before the adversary halts. We first show that the adversary only gives finitely many points before halting.

**Lemma 2.7.3.**  $T \le O(d \log R)$ 

*Proof.* We argue that the volume of  $\mathcal{E}_t$  increases by at least a constant factor on each iteration. This is sufficient to bound the number of iterations by  $O(d \log R)$ , as  $\mathcal{E}_0 = B_2^d$ , and Line 5 is no longer true when the volume of  $\mathcal{E}_t$  exceeds  $\left(\frac{R}{2}\right)^d \cdot \operatorname{vol}(B_2^d)$ .

We claim that for all  $t \ge d + 2$ ,  $\operatorname{vol}(\mathcal{E}_t) \ge \frac{3}{2} \cdot \operatorname{vol}(\mathcal{E}_{t-1})$ . By applying a nonsingular affine transformation, we can assume without loss of generality that  $\mathcal{E}_{t-1} = B_2^d$ . With a further rotation, we can assume the newly received point is  $z_t = 2e_1$ . From monotonicity of  $\mathcal{A}$  we must have that  $c_t + \mathcal{E}_t \supseteq B_2^d \cup \{2e_1\}$ . Clearly every semi-axis of  $\mathcal{E}_t$  must have length at least 1 in order to contain  $\mathcal{E}_{t-1}$ . Observe that  $\mathcal{E}_t$  must also contain the segment connecting  $-1e_1$  and  $2e_1$ , and so at least one semi-axis must have length at least  $\frac{3}{2}$  (if not, the diameter of  $\mathcal{E}_t$  would be strictly less than 3). Hence as  $\frac{\operatorname{vol}(\mathcal{E}_t)}{\operatorname{vol}(B_2^d)}$  equals the product of the length of the semi-axes of  $\mathcal{E}_t$ , we have  $\operatorname{vol}(\mathcal{E}_t) \ge \frac{3}{2}\operatorname{vol}(B_2^d)$ .

For the analysis we define quantities  $A_t$ ,  $P_t$  associated with the sequence of ellipsoids for  $1 \le t \le T$ :

$$A_t \stackrel{\text{def}}{=} \frac{1}{\alpha_t}, \quad P_t \stackrel{\text{def}}{=} \log\left(\frac{\operatorname{vol}(\mathcal{E}_t)}{\operatorname{vol}(\mathcal{B}_2^d)}\right)$$

By the monotonicity of  $\mathcal{A}$ , we have that  $A_t$  and  $P_t$  are both nondecreasing in t. We first observe that the adversary guarantees that the final volume of the ellipsoid output by  $\mathcal{A}$  is large:

Lemma 2.7.4. At the conclusion of Algorithm 6's execution, we have

$$P_T \ge d \log \frac{R}{2}$$

*Proof.* There are two ways that the adversary stops: if the condition in Line 5 is no longer true, or if Line 8 is reached. If the former occurs, then we have  $vol(\mathcal{E}_T) > vol(\frac{R}{2} \cdot B_2^d)$ , and clearly then  $P_T \ge d \log(\frac{R}{2})$ .

In the latter stopping condition, the algorithm halts at time *T* when  $\partial(c_T + 2\mathcal{E}_T) \cap R \cdot B_2^d = \emptyset$ . The sets  $\partial(c_T + 2\mathcal{E}_T)$  and  $R \cdot B_2^d$  can be disjoint in two cases:  $c_T + 2\mathcal{E}_T$  and  $R \cdot B_2^d$  are disjoint; or  $R \cdot B_2^d \subseteq c_T + 2\mathcal{E}_t$  with the boundaries of both ellipsoids disjoint. By the monotonicity of  $\mathcal{A}$ , we have  $1 \cdot B_2^d \subseteq c_T + 2\mathcal{E}_T$ , and so eliminate the former case. But then  $\operatorname{vol}(2 \cdot \mathcal{E}_T) \ge \operatorname{vol}(R \cdot B_2^d) = R^d \operatorname{vol}(B_2^d)$ , and taking logarithms on both sides yields the claim.

Now in contrast to the upper bound where we essentially gave an algorithm for which  $\frac{\Delta A}{\Delta P}$  was upper bounded by a constant, here we will show a constant lower bound on the same quantity for any monotone algorithm.

**Lemma 2.7.5.** There exists a constant  $C_{2.7.1} > 0$  such that if  $A_t \ge d$ , we have

$$A_{t+1} - A_t \ge C_{2.7.1}(P_{t+1} - P_t) \tag{2.7.1}$$

Observe that this lower bound requires  $A_t \ge d$ , hence necessitating a first phase using the simplex, whose optimal roundings show tightness for John's theorem for general convex bodies. In order to prove the lower bound we also need a second property, that  $A_t$  is large compared to  $P_t$ .

**Lemma 2.7.6.** Let  $0 \le \alpha \le 1$ ,  $c \in \mathbb{R}^d$ , and  $\mathcal{E}$  be an ellipsoid such that

$$\boldsymbol{c} + \boldsymbol{\alpha} \cdot \boldsymbol{\mathcal{E}} \subseteq \Delta_d \subseteq \boldsymbol{c} + \boldsymbol{\mathcal{E}}$$

then we have:

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1. 
$$\alpha \leq \frac{1}{d}$$
  
2.  $\log\left(\frac{\operatorname{vol}(\mathcal{E})}{\operatorname{vol}(\mathcal{B}_2^d)}\right) \leq O\left(\log(d) \cdot \frac{1}{\alpha}\right)$ 

With the statements of these claims in hand, we are ready to prove the lower bound.

*Proof of Theorem* 7. It is clear that  $\kappa(\operatorname{conv}(z_1, \ldots, z_T)) \leq R$ , as for every  $1 \leq t \leq T$  the adversary guarantees  $1 \leq ||z_t||_2 \leq R$ . So we focus on showing a lower bound on the quality of the approximation produced by  $\mathcal{A}$ .

As  $\mathcal{A}$  is monotone, after the end of Phase I we must have that

$$c_{d+1} + \alpha_{d+1} \cdot \mathcal{E}_{d+1} \subseteq \Delta_d \subseteq c_{d+1} + \mathcal{E}_{d+1} \tag{2.7.2}$$

Now because  $\mathcal{E}_{d+1}$  satisfies the conditions of Lemma 2.7.6, we get using the definition  $A_{d+1} = \frac{1}{\alpha_{d+1}}$  that  $A_t \ge A_{d+1} \ge d$  for any  $t \ge d+1$ . Then we can apply Lemma 2.7.5 for every  $t \ge d+1$  until termination of the algorithm:

$$A_{d+2} - A_{d+1} \ge C_{2.7.1} (P_{d+2} - P_{d+1})$$

$$A_{d+3} - A_{d+2} \ge C_{2.7.1} (P_{d+3} - P_{d+2})$$

$$\vdots$$

$$A_{T-1} - A_{T-2} \ge C_{2.7.1} (P_{T-1} - P_{T-2})$$

$$A_T - A_{T-1} \ge C_{2.7.1} (P_T - P_{T-1})$$

Summing these inequalities, we have

$$\sum_{t=d+1}^{T-1} A_{t+1} - A_t \ge C_{2.7.1} \left( \sum_{t=d+1}^{T-1} P_{t+1} - P_t \right)$$

Both sides of this inequality are telescoping sums, so simplifying we get

$$A_T \ge A_{d+1} + C_{2.7.1}(P_T - P_{d+1}) \tag{2.7.3}$$

Again because we can apply Lemma 2.7.6 for  $\mathcal{E}_{d+1}$ , we have  $P_{d+1} \leq O(\log(d) \cdot A_{d+1})$ , which along with (2.7.3) yields

$$A_T \ge A_{d+1} + \Omega(P_T - \log(d) \cdot A_{d+1}) \ge \Omega(P_T - \log(d) \cdot A_{d+1})$$

Thus we have

$$A_T \ge \Omega(\max(A_{d+1}, P_T - \log(d) \cdot A_{d+1})) \ge \Omega\left(\frac{P_T}{\log(d)}\right)$$

and we get the desired bound using Lemma 2.7.4.

Our proof of Lemma 2.7.5 relies on a symmetrization argument to a reduced case (essentially two-dimensional, like for our algorithms). We now define this reduced case, and related quantities.

**Definition 2.7.7.** In the reduced case, the previous outer and inner ellipsoids are given by  $B_2^d$ ,  $\alpha \cdot B_2^d$ , and the received point is  $z = 2e_1$ . The next outer and inner ellipsoids are given by  $c \cdot e_1 + \mathcal{E}_M$ ,  $c \cdot e_1 + \alpha' \cdot \mathcal{E}_M$  for  $c \in \mathbb{R}$ , and  $\mathbf{M} = \text{diag}(a, b, b, \dots, b, b)$  for  $a, b \ge 1$ . We let  $\Delta A = \frac{1}{\alpha'} - \frac{1}{\alpha}$  and  $\Delta P = \log\left(\frac{\text{vol}(\mathcal{E}_M)}{\text{vol}(B_2^d)}\right)$ .

Note that the update in this reduced case is monotone if  $B_2^d \cup \{2e_1\} \subseteq c \cdot e_1 + \mathcal{E}_M$  and  $c \cdot e_1 + \alpha' \cdot \mathcal{E}_M \subseteq conv((\alpha \cdot B_2^d) \cup \{2e_1\})$ .

Now we state the lower bound on  $\frac{\Delta A}{\Delta P}$  in this setting, which is established in Section 2.7.3. It is exactly the bound of Lemma 2.7.5 in this special case.

**Lemma 2.7.8.** In the reduced case, for any monotone update  $c \cdot e_1 + \mathcal{E}_M$ ,  $c \cdot e_1 + \alpha' \cdot \mathcal{E}_M$  when  $\alpha' \leq \frac{1}{d}$  we have

$$\frac{\Delta A}{\Delta P} \ge C_{2.7.1} \tag{2.7.4}$$

We now give the symmetrization argument that shows that the above bound in the special case implies the bound in the general case.

*Proof of Lemma* 2.7.5. By the monotonicity of  $\mathcal{A}$ , we have  $(c_t + \mathcal{E}_t) \cup \{z_{t+1}\} \subseteq c_{t+1} + \mathcal{E}_{t+1}$  and  $c_{t+1} + \alpha_{t+1} \cdot \mathcal{E}_{t+1} \subseteq \operatorname{conv}((c_t + \alpha_t \cdot \mathcal{E}_t) \cup \{z_{t+1}\})$ . Without loss of generality we assume that  $c_t + \mathcal{E}_t = B_2^d$  and  $z_{t+1} = 2 \cdot e_1$ ; we do this by applying a nonsingular affine transformation that maps  $c_t$  to the origin and  $\mathcal{E}_t$  to  $B_2^d$ , then apply a rotation that maps  $z_{t+1}$  to  $2 \cdot e_1$ . Let  $c = c_{t+1}$ ,  $\mathcal{E} = \mathcal{E}_{t+1}$ , and  $\alpha = \alpha_{t+1}$ . Summarizing the conditions guaranteed by the monotonicity of  $\mathcal{A}$ , we have that  $B_2^d \cup \{2e_1\} \subseteq c + \mathcal{E}$  and  $c + \alpha \cdot \mathcal{E} \subseteq \operatorname{conv}((\alpha_t \cdot B_2^d) \cup \{2e_1\})$ .

To perform the reduction to the two-dimensional case, we apply a sequence of volumepreserving symmetrizations to the new inner and outer ellipsoids; these symmetrizations will also ensure that the update remains monotone. We will first apply two Steiner symmetrizations. The first of these Steiner symmetrizations transforms the ellipsoids so that their center lies on the  $e_1$ -axis. The second ensures that the ellipsoids have a semi-axis that is parallel to  $e_1$ . Then, by a final symmetrization step we can transform the ellipsoids into bodies of revolution about  $e_1$ . At that point it will suffice to consider the two-dimensional reduced case.

Let c' be the projection of c onto the  $e_1$ -axis. The goal of the first symmetrization step is to transform  $c+\mathcal{E}$  to  $c'+\mathcal{E}'$  so that c' lies on the  $e_1$  axis. If c = c' then we do not need to do anything, otherwise we apply Steiner symmetrization and consider  $S_{c-c'}(c+\mathcal{E})$ . By Lemma 2.3.4 this is still an ellipsoid, and we also have that the center of  $S_{c-c'}(c+\mathcal{E})$  is actually c'; thus we may write  $S_{c-c'}(c+\mathcal{E}) = c'+\mathcal{E}'$  for some  $\mathcal{E}'$ . Further, we have that  $S_{c-c'}(c+\alpha \cdot \mathcal{E}) = c'+\alpha \cdot \mathcal{E}'$ , as the Steiner symmetrization acts similarly on the scaled version of  $\mathcal{E}$ . To show that the update is still monotone, we observe that  $c'+\mathcal{E}' = S_{c-c'}(c+\mathcal{E}) \subseteq S_{c-c'}(\operatorname{conv}((\alpha_t \cdot B_2^d) \cup \{2e_1\}))$ . But by Lemma 2.3.5 and that  $c-c' \perp e_1$ , conv  $((\alpha_t \cdot B_2^d) \cup \{2e_1\})$  is invariant under the symmetrization  $S_{c-c'}$  and so we still have the inclusion  $c'+\alpha \cdot \mathcal{E}' \subseteq \operatorname{conv}((\alpha_t \cdot B_2^d) \cup \{2e_1\})$ . The 'outer' inclusion  $B_2^d \cup \{2e_1\} \subseteq c' + \mathcal{E}'$  follows in the same way.

We now apply the second and final Steiner symmetrization. Let r be the rightmost point of  $c' + \mathcal{E}'$  along  $e_1$ ; i.e.  $r = \arg \max_{r \in c' + \mathcal{E}'} \langle r, e_1 \rangle$ . Also let r' be its projection along the  $e_1$ -axis; if r = r' we again do not need to perform this symmetrization step, otherwise the Steiner symmetrization we apply is  $S_{r-r'}(c' + \mathcal{E}')$ . Since c' is at the midpoint of  $c' + \mathbb{R}(r - r')$  the center of the new ellipsoid is still c', so we may write  $S_{r-r'}(c' + \mathcal{E}') = c' + \mathcal{E}''$  and similarly  $S_{r-r'}(c' + \alpha \cdot \mathcal{E}') = c' + \alpha \cdot \mathcal{E}''$ . Like for the previous symmetrization, the fact that  $r - r' \perp e_1$ 

means that both inclusions of the monotone update are preserved. Note finally that r' is the rightmost point of  $c' + \mathcal{E}''$  and that the tangent plane of  $c + \mathcal{E}''$  at r' is orthogonal to the line segment  $\overline{c'r'}$ , so  $e_1$  is a semi-axis of  $c' + \mathcal{E}''$ .

Our last transformation is a symmetrization of a different form, to turn  $c' + \mathcal{E}''$  into a body of revolution. Let  $\sigma_1$  be the length of the semi-axis  $e_1$  of  $\mathcal{E}''$ , and  $\sigma_2, \ldots, \sigma_d$  be the lengths of the other semi-axes of  $\mathcal{E}''$ . We let  $\mathcal{E}'''$  be the ellipsoid that has a  $e_1$  as a semi-axis of length  $\sigma_1$ , and where every other semi-axis of  $\mathcal{E}''$  has length  $\sigma' \stackrel{\text{def}}{=} \left(\prod_{i=2}^d \sigma_i\right)^{1/(d-1)}$ . Clearly  $c' + \mathcal{E}'''$ is now a body of revolution about  $e_1$  whose volume is the same as that of  $c' + \mathcal{E}''$  (and hence also of  $c + \mathcal{E}$ ). Note that  $c' + \alpha \mathcal{E}'''$  is also now a body of revolution. Since  $\sigma' \ge \min_{2 \le i \le d} \sigma_i$ we have  $B_2^d \cup \{2e_1\} \subseteq c' + \mathcal{E}'''$ , and correspondingly since  $\sigma' \le \max_{2 \le i \le d} \sigma_i$  we have that  $c' + \alpha \cdot \mathcal{E}''' \subseteq \operatorname{conv}\left((\alpha_t \cdot B_2^d) \cup \{2e_1\}\right)$ .

Clearly  $c' + \mathcal{E}'''$ ,  $c + \alpha \cdot \mathcal{E}'''$  now adhere to the reduced case of Definition 2.7.7. Since the update is monotone as well (and still  $\alpha \leq 1/d$ ) we can apply Lemma 2.7.8. As  $vol(\mathcal{E}''') = vol(\mathcal{E})$ , this means we have

$$A_{t+1} - A_t \ge C_{2.7.1} \cdot (P_{t+1} - P_t)$$

as desired.

*Proof of Lemma* 2.7.6. For the first property, this is exactly the well-known fact that the best ellipsoidal rounding for the simplex  $\Delta_d$  (see e.g. [How97, Remark 1.1]) has approximation factor *d*.

Now we show the second property. Again because the ball rounds the simplex  $\Delta_d$  with approximation factor *d*, we have

$$\frac{1}{d} \cdot \Delta_d \subseteq B_2^d \subseteq \Delta_d$$

As a result of this, we have

$$\log\left(\frac{\operatorname{vol}(c + \alpha \cdot \mathcal{E})}{\operatorname{vol}(B_2^d)}\right) \le \log\left(\frac{\operatorname{vol}(\Delta)}{\operatorname{vol}(B_2^d)}\right)$$
$$\le \log\left(\frac{\operatorname{vol}(d \cdot B_2^d)}{\operatorname{vol}(B_2^d)}\right)$$
$$\le O(d \log d)$$

And so

$$\log\left(\frac{\operatorname{vol}(\mathcal{E})}{\operatorname{vol}(B_2^d)}\right) = \log\left(\frac{\operatorname{vol}(c + \alpha \cdot \mathcal{E})}{\operatorname{vol}(B_2^d)}\right) + d\log\left(\frac{1}{\alpha}\right)$$
$$\leq O\left(d\log\left(\frac{1}{\alpha}\right)\right) \qquad \text{as } \alpha \leq \frac{1}{d}$$

To establish the second property, it remains to show  $d \log(1/\alpha) \le O((1/\alpha) \log(d))$ . Observe that  $x \mapsto \frac{x}{\log x}$  is increasing for  $x \ge e$ , so we have

$$\frac{d}{\log d} \le O\left(\frac{1/\alpha}{\log(1/\alpha)}\right)$$

for all  $d \ge 2$  as  $d \le 1/\alpha$ . Rearranging gives the desired inequality and thus the second property.

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#### 2.7.3. Analysis of the reduced case

In this section, we establish a lower bound on  $\frac{\Delta A}{\Delta P}$ , assuming we are in the 'reduced case' defined in Definition 2.7.7. Observe that in this case all relevant convex bodies  $\mathcal{E}$ ,  $\alpha \mathcal{E}$ ,  $c \cdot e_1 + \mathcal{E}'$ ,  $c \cdot e_1 + \alpha' \mathcal{E}'$ , conv ( $\alpha \mathcal{E} \cup \{z\}$ ) are all bodies of revolution about the  $x_1$ -axis, so to analyze the quantities involved we may instead look at any two-dimensional slice. Accordingly we talk about the ellipses  $\mathcal{E}$ ,  $\alpha \mathcal{E}$ ,  $c + \mathcal{E}'$ ,  $c + \alpha' \mathcal{E}'$  in this two-dimensional slice, where again  $\mathcal{E} = B_2^2$ , and  $c + \mathcal{E}'$ and  $c + \alpha' \mathcal{E}'$  are defined by

$$c + \mathcal{E}' = \left\{ (x, y) \in \mathbb{R}^d \left| \left( \frac{x - c}{a} \right)^2 + \left( \frac{y}{b} \right)^2 \le 1 \right\}$$
$$c + \alpha' \mathcal{E}' = \left\{ (x, y) \in \mathbb{R}^d \left| \left( \frac{x - c}{a} \right)^2 + \left( \frac{y}{b} \right)^2 \le \alpha'^2 \right\}$$

for  $a, b > 0, c \in \mathbb{R}$ . We also use for convenience  $A = \frac{1}{\alpha}$  and  $A' = \frac{1}{\alpha'}$  so that  $\Delta A = A' - A$ . Also note in this reduced case we have by symmetry that

$$\Delta P = \log\left(\frac{\operatorname{vol}(c \cdot e_1 + \alpha' \mathcal{E}')}{\operatorname{vol}(B_2^d)}\right) - \log\left(\frac{\operatorname{vol}(B_2^d)}{\operatorname{vol}(B_2^d)}\right) = \log(a \cdot b^{d-1})$$

Our lower bound in this reduced case is the following:

**Lemma 2.7.9.** There exists a fixed constant  $C_{2.7.12} > 0$  such that

$$\frac{\Delta A}{\Delta P} \geq \min\left\{C_{2.7.12}, \frac{1}{10}\frac{A}{d}\right\}$$

Clearly this claim yields Lemma 2.7.8 as a corollary, as by assumption in Lemma 2.7.8 we have  $A \ge d$  and so we get  $\frac{\Delta A}{\Delta P} \ge \Omega(1)$ .

The inner ellipses in this lower bound, and some relevant points used in the proof of this claim, are depicted in Figure 2.6.

*Proof.* We establish this claim through a geometric argument that we break down by cases (the logical tree of cases is visualized in Figure 2.7). First, as the new outer ellipse  $c + \mathcal{E}'$  contains  $\mathcal{E} = B_2^d$  we readily have that  $a, b \ge 1$ .

As the rightmost point of the new outer ellipse must be to the right of *z*, we have

$$c + a > 2$$
 (2.7.5)

As the leftmost point of the new inner ellipse must be to the right of the leftmost point of the previous inner ellipse, we have

$$c \ge \alpha' a - \alpha \tag{2.7.6}$$

**Lemma 2.7.10.** We have  $\alpha' \cdot b \leq \alpha$ , or equivalently  $A' \geq b \cdot A$ .



Figure 2.6.: The inner ellipses in the two-dimensional lower bound. *O* is the origin. The black solid circle is the previous inner ellipse  $\alpha \mathcal{E}$ , and the blue solid circle is the next inner ellipse  $c + \alpha' \mathcal{E}'$ . The vertical dotted blue line x = c through the center *c* marks the location of the next inner ellipse on the *x*-axis. The new point is  $z = 2e_1$ , and  $\overline{zQ}$  is one of the lines through *z* tangent to  $\alpha \mathcal{E}$ , with *Q* the point of tangency. *Q'* is the intersection of  $\alpha \mathcal{E}$  with the *y*-axis on the same side of the *x*-axis as *Q*. *P'* is the intersection of the line x = c with  $c + \alpha' \mathcal{E}'$  on the same side as *Q*, and *P* is the intersection of this line with  $\overline{zQ}$ . We denote the angle  $\angle Pzc$  with  $\varphi$ .

*Proof.* The geometry of this fact is visualized in Figure 2.6. We overload notation so that *c* will also denote the point (c, 0), the center of the new inner ellipse. We denote *O* as the origin. Let  $\overline{zQ}$  be one of the lines through *z* and tangent to  $\alpha \mathcal{E}$ , with *Q* the point of tangency (the choice of which line is arbitrary, in the figure we choose the one whose intersection with  $\alpha \mathcal{E}$  is above the *x*-axis). We let *P* be the intersection of the vertical line through (c, 0) with  $\overline{zQ}$ , and *P'* be the intersection of this line with the ellipse  $c + \alpha' \partial \mathcal{E}'$  on the same side of the *x*-axis as *Q*.

Observe that  $\alpha'b = \overline{cP'}$  as the vertical semi-axis of the ellipse  $c + \alpha'\mathcal{E}$ , and  $\alpha = \overline{OQ'}$ . Due to the fact that  $c + \alpha'\mathcal{E}' \subseteq \operatorname{conv}(\alpha\mathcal{E} \cup \{z\})$ , the projection of both sets onto the *y*-axis satisfies the same inclusion, and this gives the desired inequality.

Observe that as  $c + \mathcal{E}'$  must contain both the points (-1, 0) and (0, 2), we have

$$a \ge \frac{3}{2} \tag{2.7.7}$$

Observe that we can split  $\Delta P$  into two terms:

$$\Delta P = \underbrace{(d-1)\log b}_{\mathrm{I}} + \underbrace{\log a}_{\mathrm{II}}$$

First we show that if term I is larger, then we have a constant lower bound on  $\frac{\Delta A}{\Lambda P}$ .

**Lemma 2.7.11.** If  $(d-1)\log b \geq \log a$ , then  $\frac{\Delta A}{\Delta P} \geq \frac{1}{2}\frac{A}{d}$ .

*Proof.* Under the assumption, we have  $\Delta P \leq 2(d-1)\log b$ . Combining this with Lemma 2.7.10, we have

$$\frac{\Delta A}{\Delta P} \geq \frac{A'-A}{2(d-1)\log b}$$


Figure 2.7.: Tree of cases in the lower bound

$$\geq \frac{b \cdot A - A}{2(d-1)\log b}$$
$$= \frac{b-1}{2\log b} \frac{A}{d-1}$$
$$\geq \frac{1}{2} \frac{A}{d}$$

where the last line uses that  $\frac{x-1}{2\log(x)} > \frac{1}{2}$  when x > 1.

In light of Lemma 2.7.11, we can then assume in the sequel Term II is larger, meaning that

$$\Delta P \le 2 \log a \tag{2.7.8}$$

Case I (New ellipse is close to the previous one). Assume that

$$c + \alpha' \cdot a \le \frac{11}{10}\alpha\tag{2.7.9}$$

i.e. that the rightmost point of the new inner ellipse is to the left of  $\frac{11}{10}$ .

**Lemma 2.7.12.** In Case I, we have  $\Delta A \ge \frac{4}{11}A$ .

*Proof.* We prove this by cases. First, if we assume that  $\alpha' \leq \frac{\alpha}{2}$ , we get  $A' \geq 2A$  and  $\Delta A \geq A$ .

In the second case, we have  $\alpha' > \frac{\alpha}{2}$ . We first use this to show c > 0. By (2.7.6) and (2.7.5) we have  $\frac{c+\alpha}{\alpha'} \ge a > 2 - c$ , so  $c(1 + \alpha') > 2\alpha' - a > 0$  and so c > 0.

Using (2.7.9) and that c > 0, we have  $\alpha' a \le \frac{11}{10}\alpha$ . Thus  $A' \ge \frac{10}{11}aA$ . By (2.7.7) we get  $A' \ge \frac{15}{11}A$ , and finally  $\Delta A \ge \frac{4}{11}A$ .

**Lemma 2.7.13.** In Case I, we have  $A' \ge \frac{10}{21}a \cdot A$ .

*Proof.* From (2.7.6) we also get the weaker lower bound  $c \ge -\alpha$ . Combined with (2.7.9), this gives  $\alpha' a \le \frac{21}{10}\alpha$ , which is equivalent to the desired inequality.

We divide Case I into two sub-cases.

**Case Ia**  $(a, b \le 16)$ . First, assume that  $a, b \le 16$ . Then  $\Delta P = \log(a \cdot b^{d-1}) \le d \log(16)$ , and by Lemma 2.7.12 we get

$$\frac{\Delta A}{\Delta P} \ge \frac{4}{11\log(16)}\frac{A}{d} \ge \frac{1}{10}\frac{A}{d}$$

**Case Ib** (a > 16 or b > 16). Now assume that either a or b is greater than 16.

Combining Lemma 2.7.10 and Lemma 2.7.13 together, we get  $A'^2 \ge \frac{10}{21}abA^2$ , or  $A' \ge \sqrt{\frac{10}{21}ab} \cdot A$ . *A*. We have  $\Delta A = A' - A \ge \left(\sqrt{\frac{10}{21}ab} - 1\right)A$ . As  $ab \ge 16$  we get  $\sqrt{ab} \ge 2\sqrt{\frac{21}{10}}$ , so  $\Delta A = \left(\sqrt{\frac{10}{21}ab} - 1\right)A \ge \sqrt{\frac{5}{21}ab} \cdot A$ .

Using the analytic inequality that  $\log x \le \sqrt{x}$  for all x > 0, we get  $\Delta P = \log a + (d - 1)\log b \le \sqrt{a} + (d - 1)\sqrt{b} \le d \cdot \sqrt{ab}$ .

Combining these inequalites for  $\Delta A$  and  $\Delta P$ , we obtain

$$\frac{\Delta A}{\Delta P} \ge \sqrt{\frac{5}{21}} \frac{A}{d} \ge \frac{4}{10} \frac{A}{d}$$

**Case II** (New ellipse is far from the previous one). Assume that

$$c + \alpha' \cdot a > \frac{11}{10}\alpha \tag{2.7.10}$$

Lemma 2.7.14. We have

$$\alpha' b \le (2-c) \cdot \frac{\frac{\alpha}{2}}{\sqrt{1-(\frac{\alpha}{2})^2}}$$
 (2.7.11)

*Proof.* Again, the proof of this claim is pictured in Figure 2.6, where we construct the points in the same way as in the proof of Lemma 2.7.10. Let  $\angle Pzc$  be denoted by  $\varphi$ . Note that the angle  $\angle Pcz$  is a right angle, and so  $\tan \varphi = \frac{\overline{cP}}{\overline{cv}}$ . The line  $\overline{cz}$  has length 2-c, so we get  $\overline{cP} = (2-c) \tan \varphi$ . The segment  $\overline{cP'}$ , of length  $\alpha'b$ , is contained within the segment  $\overline{cP}$ , and so  $\alpha'b \leq (2-c) \tan \varphi$ .

Observe that the angle  $\angle OQz$  is also a right angle. Further, clearly the length of  $\overline{OQ}$  is  $\alpha$  and the length of  $\overline{0v}$  is 2. Since we have that  $\varphi$  is also the angle  $\angle QzO$ , we get  $\sin \varphi = \frac{\alpha}{2}$ . Now using the standard trigonometric identity that  $\tan \varphi = \frac{\sin \varphi}{\sqrt{1-\sin^2 \varphi}}$  for  $\varphi \in [-\pi/2, \pi/2]$ , we get the desired inequality.

We split Case II into several sub-cases, as for Case I.

**Case IIa** ( $a \le 16$ ). First, we look at the case where  $a \le 16$ .

**Case IIa-i** ( $A \ge 2$ ). Assume  $A \ge 2$ .

**Lemma 2.7.15.** When  $A \ge 2$ , we have  $\Delta A \ge \frac{1}{2}$ .

*Proof.* Adding (2.7.6) and (2.7.10) together gives  $c > \frac{1}{20}\alpha$ . Using this in (2.7.11) and rearranging using the definitions of *A* and *A'* yields

$$A' \ge bA \cdot \frac{1}{1 - \frac{1}{40}\frac{1}{A}} \left(1 - \frac{1}{A^2}\right)$$

and therefore we get the inequality

$$\Delta A \ge bA \cdot \frac{1}{1 - \frac{1}{40}\frac{1}{A}} \left(1 - \frac{1}{A^2}\right) - 1$$

and using  $b \ge 1$ , we obtain

$$\Delta A \ge A \cdot \frac{1}{1 - \frac{1}{40}\frac{1}{A}} \left( 1 - \frac{1}{A^2} \right) - 1$$

To prove the claim, it suffices to show the right hand side exceeds  $\frac{1}{2}$  when  $A \ge 2$ . Upon rearranging, this is equivalent to the inequality  $A - \frac{77}{80}\frac{1}{A} \ge \frac{3}{2}$  when  $A \ge 2$ .

Combining the assumption that  $a \le 16$  with (2.7.8) and Lemma 2.7.15 yields  $\frac{\Delta A}{\Delta P} \ge \frac{1}{4 \log 16} \ge \frac{1}{20}$ .

**Case IIa-ii** ( $A \le 2$ ). Next, we look at the other case where  $A \le 2$ .

**Case IIa-ii-A** ( $b \le 100$ ). Now we look at the case where  $b \le 100$ .

**Lemma 2.7.16.** If  $a \le 16$ ,  $A \le 2$ ,  $b \le 100$ , then there is  $C_{2.7.12} > 0$  such that

$$\frac{\Delta A}{\Delta P} \ge C_{2.7.12} \tag{2.7.12}$$

*Proof.* We show this by a compactness argument. By (2.7.8) and the assumption that  $a \le 16$  we have  $\frac{\Delta A}{\Delta P} \ge \frac{\Delta A}{2\log 16}$ . Now, observe that next outer and inner ellipsoids  $c + \mathcal{E}'$  and  $c + \alpha' \mathcal{E}'$  are fully determined by the parameters a, b, c, A, A'. Further, we assume without loss of generality that A' is a function of the other parameters. This is because when A' is decreased as much as possible while preserving the monotonicity of the update,  $\Delta A = A' - A$  only decreases. To show a lower bound on  $\Delta A$  it then suffices to only do so in this hardest case.

Note that we have  $1 \le a \le 16, 1 \le b \le 100, -1 \le c \le 2$ , and  $1 \le A \le 2$ ; thus all the parameters defining the next inner and outer ellipsoids are bounded. Observe that  $\Delta A$  is a continuous function of these parameters, and as a continuous function of a compact set it attains its minimum. Finally, we argue that it is impossible for the minimum of  $\Delta A$  to be zero, and so the minimum is some strictly positive constant  $\frac{C_{2,7,12}}{2\log 16}$ , which suffices to prove the claim.

The following argument only concerns the inner ellipsoids, and can be pictured in Figure 2.6. If c = 0, then as the leftmost point of  $c + \alpha' \mathcal{E}'$  must be to the right of the leftmost point of  $\alpha \mathcal{E}'$ , we have  $\alpha \ge \alpha' a$ . But by (2.7.7), we have  $\alpha \ge \frac{3}{2}\alpha'$ , so  $\alpha' < \alpha$  and  $\Delta A > 0$ . If  $c \ne 0$  then the vertical semi-axis of  $c + \alpha' \mathcal{E}'$  must have length strictly less than  $\alpha$ , and so  $\alpha > \alpha' b$ . As  $b \ge 1$ , this also gives  $\alpha > \alpha'$  and  $again \Delta A > 0$ .

**Case IIa-ii-B** (b > 100). Observe that the horizontal axis of the next inner ellipsoid  $c + \alpha' \mathcal{E}'$  must be contained within the interval  $[-\alpha, 2]$ , thus we have that  $2 + \alpha \ge 2b\alpha' > 200\alpha'$ . Using the definitions of A, A' this is equivalent to  $2 + \frac{1}{A} > \frac{200}{A'}$ , i.e.  $A' > \frac{200}{1 + \frac{1}{A}}$ . As  $A \ge 1$  we get  $2 + \frac{1}{A} \le 3$ , and so  $A' \ge \frac{200}{3}$ .

As  $A \le 2$ , we obtain  $\Delta A = A' - A \ge \frac{194}{3}$ . Now by (2.7.8) and that  $a \le 16$  we have

$$\frac{\Delta A}{\Delta P} \ge \frac{\Delta A}{2\log 16} \ge \frac{194}{6\log 16} \ge 11$$

**Case IIb** (*a* > 16). Now, we examine the case where *a* > 16. Scaling (2.7.6) by  $\frac{11}{10}$  and adding it to (2.7.10), we have  $\frac{21}{10}c - \frac{1}{10}\alpha' a \ge 0$ , i.e.  $c > \frac{1}{21}\alpha' a$ . Using this in (2.7.11), using  $b \ge 1$ , using the definitions of *A* and *A'* and rearranging, we obtain

$$A' \ge A \cdot \frac{1}{1 - \frac{1}{42} \cdot \frac{a}{A'}} \cdot \sqrt{1 - \left(\frac{\alpha}{2}\right)^2}$$

Using the inequalities  $\sqrt{1 - \left(\frac{x}{2}\right)^2} \ge 1 - \frac{x^2}{7}$  for  $0 \le x \le 1$  and  $\frac{1}{1-x} \ge 1 + x$  for  $0 \le x \le 1$ , we have

$$A' \ge A\left(1 - \frac{\alpha^2}{7}\right)\left(1 + \frac{1}{42}\frac{a}{A'}\right)$$

and thus  $A'^2 - A \cdot A'\left(1 - \frac{\alpha^2}{7}\right) - \frac{1}{42}\left(1 - \frac{\alpha^2}{7}\right) \ge 0$ , which implies by the quadratic formula that

$$A' \ge \frac{A\left(1 - \frac{\alpha^2}{7}\right) + \sqrt{A^2\left(1 - \frac{\alpha^2}{7}\right) + \frac{4}{42}aA\left(1 - \frac{\alpha^2}{7}\right)}}{2}$$
$$= A\left(1 - \frac{\alpha^2}{7}\right) \cdot \frac{1 + \sqrt{1 + \frac{2}{21}\frac{a}{A\left(1 - \frac{\alpha^2}{7}\right)}}}{2}$$
$$= A\left(1 - \frac{\alpha^2}{7}\right) \cdot \left(1 + \frac{\sqrt{1 + \frac{2}{21}\frac{a}{A\left(1 - \frac{\alpha^2}{7}\right)}} - 1}{2}\right)$$

Using the inequality  $\sqrt{1 + x} - 1 \ge \frac{2}{5} \min \{x, \sqrt{x}\}$  for all  $x \ge 0$ , we get that

$$A' \ge A\left(1 - \frac{\alpha^2}{7}\right)\left(1 + \frac{1}{5}\min\left\{\underbrace{\frac{2}{21}\frac{a}{A\left(1 - \frac{\alpha^2}{7}\right)}}_{\mathrm{I}}, \underbrace{\sqrt{\frac{2}{21}\frac{a}{A\left(1 - \frac{\alpha^2}{7}\right)}}_{\mathrm{II}}}_{\mathrm{II}}\right\}\right)$$
(2.7.13)

To finish this case, we show the lower bound in the case where either term in the  $\min$  of (2.7.13) is the smaller term.

Case IIb-i (Term I in (2.7.13) is smaller). In this case, (2.7.13) is equivalent to

$$A' \ge A\left(1 - \frac{\alpha^2}{7}\right) + \frac{2}{105}a$$

Using the definition of *A*, we have  $A' \ge A - \frac{1}{7A} + \frac{2}{105}a$ , and so  $\Delta A \ge -\frac{1}{7A} + \frac{2}{105}a$ . As  $A \ge 1$ , we have  $\Delta A \ge \frac{2}{105}a - \frac{1}{7}$ .

Now by (2.7.8), we get

$$\frac{\Delta A}{\Delta P} \ge \frac{\frac{2}{105}a - \frac{1}{7}}{2\log a}$$

Now, we complete this case by noticing the right hand side is at least  $\frac{1}{35}$  when a > 16.

Case IIb-ii (Term II in (2.7.13) is smaller). In this case, (2.7.13) is equivalent to

$$A' \ge A\left(1 - \frac{\alpha^2}{7}\right) + \sqrt{\frac{2}{525}aA\left(1 - \frac{\alpha^2}{7}\right)}$$
 (2.7.14)

Using the definition of *A* and that  $A \ge 1$ , we have  $A\left(1 - \frac{\alpha^2}{7}\right) = A - \frac{1}{7A} \ge -\frac{1}{7}$ . Using this and the definition of  $\Delta A$  in (2.7.14), we have

$$\Delta A \ge -\frac{1}{7} + \sqrt{\frac{2}{525}}aA\left(1 - \frac{\alpha^2}{7}\right)$$

Further, as  $0 \le \alpha \le 1$  we get  $A\left(1 - \frac{\alpha^2}{7}\right) = \frac{1}{\alpha} - \frac{\alpha}{7} \ge \frac{6}{7}$ , so

$$\Delta A \ge -\frac{1}{7} + \sqrt{\frac{12}{3675}a}$$

Now by (2.7.8), we get

$$\frac{\Delta A}{\Delta P} \ge \frac{-\frac{1}{7} + \sqrt{\frac{12}{3675}}a}{2\log a}$$

We finish with the fact that the right hand side is at least  $\frac{1}{65}$  when a > 16.

# 2.8. Details of analysis in Section 2.4.2

Here, we give the details for the outstanding claims in Section 2.4.2. We stay in the context from that section and reuse notation (specifically, the definition of parameters in (2.4.1)).

We begin with some well-known bounds on  $e^x$ .

**Lemma 2.8.1.** Recall the following well-known inequalities regarding  $e^x$ .

- 1.  $1 + x \leq e^x$  for all  $x \in \mathbb{R}$ ;
- 2.  $1 + x + \frac{x^2}{2} \le e^x$  for  $x \ge 0$ .

We will also use a more specialized upper bound on  $e^x$ .

**Lemma 2.8.2.** For  $0 \le x \le \frac{4}{3}$ , we have  $e^x \le 1 + x + \frac{x^2}{2} + \frac{x^3}{4}$ .

*Proof.* Using the Taylor series for  $e^x$  about 0, we get

$$\left(1+x+\frac{x^2}{2}+\frac{x^3}{4}\right)-e^x=\frac{x^3}{12}-\sum_{k=4}^{\infty}\frac{x^k}{k!}=x^3\left(\frac{1}{12}-\sum_{k=4}^{\infty}\frac{x^{k-3}}{k!}\right).$$

Clearly  $x^3 \ge 0$  for  $x \ge 0$ , so it remains to show  $\frac{1}{12} - \sum_{k=4}^{\infty} \frac{x^{k-3}}{k!} \ge 0$  for  $0 \le x \le \frac{4}{3}$ . Note that  $\sum_{k=4}^{\infty} \frac{x^{k-3}}{k!}$  is increasing (the derivative is clearly positive when  $x \ge 0$ ), and we finish by noting

$$\frac{1}{12} - \sum_{k=4}^{\infty} \frac{x^{k-3}}{k!} \bigg|_{x=\frac{4}{3}} = \frac{1+x+\frac{x^2}{2}+\frac{x^3}{4}-e^x}{x^3} \bigg|_{x=\frac{4}{3}} > 0.$$

Now we show some facts used in Lemma 2.8.5, which Lemma 2.4.3 reduces to. The proof of Lemma 2.8.5 will reduce to the following analytic inequality.

**Lemma 2.8.3.** For all  $\gamma \ge 0$ ,

$$\frac{(e^{\gamma}-1)^2}{e^{2\gamma}-(1+\frac{\gamma}{4})^2} \le \frac{3}{2}\gamma.$$

*Proof.* For the numerator of the left hand side, we have  $(e^{\gamma} - 1)^2 = e^{2\gamma} - 2e^{\gamma} + 1 \le e^{2\gamma} - 2\gamma - 1$ using Lemma 2.8.1-(1),  $1 + x \le e^x$ . Further,  $e^{\gamma} \ge 1 + \frac{\gamma}{4}$  implies  $e^{2\gamma} - (1 + \frac{\gamma}{4})^2 \ge 0$ , so after multiplying both sides by  $e^{2\gamma} - (1 + \frac{\gamma}{4})^2$  and rearranging, it suffices to show

$$\frac{3}{2}\gamma\left(1+\frac{\gamma}{4}\right)^2 - 1 - 2\gamma \le \left(\frac{3}{2}\gamma - 1\right)e^{2\gamma}.$$

We split this into two cases, based on the value of  $\gamma$ . If  $\gamma \geq \frac{2}{3}$ , then the right hand side is at least  $(\frac{3}{2}\gamma - 1)(1 + 2\gamma + 2\gamma^2)$  using Lemma 2.8.1-(2),  $1 + x + \frac{x^2}{2} \leq e^x$ , so it is sufficient to show  $(\frac{3}{2}\gamma - 1)(1 + 2\gamma + 2\gamma^2) \geq \frac{3}{2}\gamma(1 + \frac{\gamma}{4})^2 - 1 - 2\gamma$ . Expanding both sides, this is equivalent to showing  $\frac{\gamma^2}{4} + \frac{93}{32}\gamma^3 \geq 0$ , which is clearly true for  $\gamma \geq 0$ .

If  $\gamma < \frac{2}{3}$ , then we use Lemma 2.8.2 to lower bound the right hand side with  $(\frac{3}{2}\gamma - 1)(1 + 2\gamma + 2\gamma^2 + 2\gamma^3)$ , so it is sufficient to show  $(\frac{3}{2}\gamma - 1)(1 + 2\gamma + 2\gamma^2 + 2\gamma^3) \ge \frac{3}{2}\gamma(1 + \frac{\gamma}{4})^2 - 1 - 2\gamma$ . Similar to before, after expanding both sides this is equivalent to showing  $\frac{\gamma^2}{4} + \frac{93}{32}\gamma^3 + \frac{3}{2}\gamma^4 \ge 0$ , which is true for  $\gamma \ge 0$ .

We also show some relations between the parameters in the update step. Recall that we assumed  $\alpha \leq \frac{1}{2}$ .

Lemma 2.8.4. We have

1.  $b \le 1 + \frac{\gamma}{4}$ 2.  $b \le a$ 3.  $\frac{(a-1)^2}{a^2-b^2} \le 1$ 4.  $b^2 > 1 + \alpha - \alpha'$  *Proof.* We start by showing (1). As  $\frac{1}{\alpha'} = \frac{1}{\alpha} + 2\gamma$ , we have  $\alpha = \alpha' + 2\gamma\alpha\alpha'$ . Thus  $b = 1 + \gamma\alpha\alpha' \le 1 + \gamma\alpha' \le 1 + \gamma/4$  as  $\alpha' \le \alpha \le \frac{1}{2}$ .

For (2), observe that  $b \le 1 + \frac{\gamma}{4} \le 1 + \gamma \le e^{\gamma} = a$  using Lemma 2.8.1-(1),  $1 + x \le e^x$ , so  $a \ge b$ .

To show (3), we first argue it is sufficient to show  $1 + b^2 \le 2a$ . As a consequence  $-2a + 1 \le -b^2$ , so  $(a - 1)^2 \le a^2 - b^2$ . Because  $b \ge 1$  by Lemma 2.4.2-(2), from (2) we can say that  $a^2 - b^2 \ge 0$ , so that  $\frac{(a-1)^2}{a^2-b^2} \le 1$ .

Now to show  $1 + b^2 \le 2a$ , we write as a series in terms of  $\gamma$ . On the left hand side using (1), we have  $1 + b^2 \le 1 + (1 + \frac{\gamma}{4})^2 = 2 + \frac{\gamma}{2} + \frac{\gamma^2}{4}$ . Further, by Lemma 2.8.1-(2),  $e^x \ge 1 + x + \frac{x^2}{2}$ , we have that  $2a \ge 2 + 2\gamma + \gamma^2$ . Clearly  $2 + \frac{\gamma}{2} + \frac{\gamma^2}{4} \le 2 + 2\gamma + \gamma^2$  when  $\gamma \ge 0$ , so we are finished.

For (4), we have by definition that  $b^2 = 1 + \alpha - \alpha' + \frac{(\alpha - \alpha')^2}{4}$ , so  $b^2 \ge 1 + \alpha - \alpha'$ .

As the proof of Lemma 2.4.3 shows, that claim reduces to the following inequality.

Lemma 2.8.5. We have

$$c^{2} \leq \frac{b^{2} - 1}{b^{2}} \cdot (a^{2} - b^{2}).$$
 (2.8.1)

*Proof.* We first upper bound *c* to reduce the number of variables in (2.8.1). As  $b = 1 + \frac{\alpha - \alpha'}{2}$ , we have  $2(b - 1) = \alpha - \alpha'$  and so  $\alpha = \alpha' + 2(b - 1)$ . Thus

$$c = -\alpha + \alpha' \cdot a = -(\alpha' + 2(b-1)) + \alpha' \cdot a = \alpha' \cdot (a-1) + 2(1-b)a$$

As  $b \ge 1$  by Lemma 2.4.2-(2), we have that  $2(1 - b) \le 0$  and therefore

$$c \le \alpha' \cdot (a-1).$$

Using this in (2.8.1), it suffices to show  $\alpha'^2(a-1)^2 \leq \frac{b^2-1}{b^2} \cdot (a^2-b^2)$ , which rearranges to

$$\frac{b^2 - 1}{\alpha'^2} \ge \frac{(a - 1)^2 b^2}{a^2 - b^2}.$$

Using Lemma 2.8.4-(4), this reduces to

$$\frac{\alpha - \alpha'}{\alpha'^2} \ge \frac{(a-1)^2 b^2}{a^2 - b^2}.$$
(2.8.2)

The left hand side of (2.8.2) equals  $\frac{1}{\alpha'} \left(\frac{\alpha}{\alpha'} - 1\right)$ . Because  $\frac{1}{\alpha'} = \frac{1}{\alpha} + 2\gamma$ , we have  $\frac{\alpha}{\alpha'} - 1 = 2\gamma\alpha$ , so  $\frac{1}{\alpha'} \left(\frac{\alpha}{\alpha'} - 1\right) = 2\gamma \cdot \frac{\alpha}{\alpha'} = 2\gamma \cdot (1 + 2\gamma\alpha)$ . So it is sufficient to show

$$2\gamma(1+2\gamma\alpha) \ge \frac{(a-1)^2 b^2}{a^2 - b^2}.$$
(2.8.3)

Now, we will eliminate the other variables in this inequality to transform it into a statement involving only  $\gamma$ . We have

$$\frac{(a-1)^2 b^2}{a^2 - b^2} = \frac{(a-1)^2}{a^2 - b^2} \left( 1 + 2\alpha \alpha' \gamma + \alpha'^2 \alpha^2 \gamma^2 \right)$$

$$\leq rac{(a-1)^2}{a^2-b^2} + rac{\gamma}{2} + lpha rac{\gamma^2}{8},$$

where the first line uses that  $b = 1 + \alpha \alpha' \gamma$ , and the second line inequality follows from Lemma 2.8.4-(3) and the fact that  $\alpha' \le \alpha \le \frac{1}{2}$ . Thus we can reduce (2.8.3) to  $\frac{(a-1)^2}{a^2-b^2} \le \frac{3}{2}\gamma + \frac{31}{8}\alpha\gamma^2$ , or further to

$$\frac{(a-1)^2}{a^2 - b^2} \le \frac{3}{2}\gamma.$$
(2.8.4)

Using Lemma 2.8.4-(1) and that  $a = e^{\gamma}$ , we have  $\frac{(a-1)^2}{a^2-b^2} \le \frac{(e^{\gamma}-1)^2}{e^{2\gamma}-(1+\frac{\gamma}{4})^2}$ , so finally (2.8.4) reduces to

$$\frac{(e^{\gamma}-1)^2}{e^{2\gamma}-(1+\frac{\gamma}{4})^2} \le \frac{3}{2}\gamma,$$

which is proved in Lemma 2.8.3.

Recall in the proof of Lemma 2.4.4 we defined  $\ell_1 = \frac{1}{c+a}$ ,  $\ell_2 = \sqrt{\frac{1}{\alpha^2} - \frac{1}{(c+a)^2}}$ ,  $r = \frac{a^2 \ell_1^2}{b^2 \ell_2^2}$ . That claim reduces to the following.

Lemma 2.8.6. We have

$$a - \alpha' \cdot a \sqrt{\frac{1+r}{r}} \ge 0.$$

*Proof.* As by definition  $a \ge 0$ , it suffices to show

$$\alpha^{\prime 2} \cdot \left(\frac{1}{r} + 1\right) \le 1. \tag{2.8.5}$$

Observe that  $\ell_2^2 = \frac{1}{\alpha^2} - \ell_1^2$ , so we can write  $\frac{1}{r} = \frac{b^2}{a^2} \left( \frac{1}{\alpha^2 \ell_1^2} - 1 \right)$ , and hence rewrite (2.8.5) as

$$\alpha'^2 \left( 1 + \frac{b^2}{a^2} \left( \left( \frac{c+a}{\alpha} \right)^2 - 1 \right) \right) \le 1.$$

Multiplying both sides by  $\frac{\alpha^2}{\alpha'^2}$  and rearranging, this is equivalent to

$$\frac{b^2}{a^2} \left( (c+a)^2 - \alpha^2 \right) \le \frac{\alpha^2}{\alpha'^2} - \alpha^2.$$
(2.8.6)

Now, by definition of *c* we can write  $c + a = a(1 + \alpha') - \alpha$ , so that  $(c + a)^2 - \alpha^2 = a^2(1 + \alpha')^2 - 2\alpha a(1 + \alpha')$ . Thus, (2.8.6) is equivalent to

$$\frac{b^2}{a^2}(a^2(1+\alpha')^2 - 2\alpha a(1+\alpha')) \le \frac{\alpha^2}{\alpha'^2}(1+\alpha')(1-\alpha').$$

Dividing by  $1 + \alpha'$  and simplifying the left hand side, this is equivalent to

$$b^2\left(1+\alpha'-\frac{2\alpha}{a}\right)\leq \frac{\alpha^2}{\alpha'^2}(1-\alpha'),$$

which we show in Lemma 2.8.7.

Lemma 2.8.7. We have

$$b^2\left(1+\alpha'-\frac{2\alpha}{a}\right)\leq \frac{\alpha^2}{\alpha'^2}(1-\alpha').$$

*Proof.* Using Lemma 2.8.1-(1),  $e^{-x} \ge 1 - x$ ; and the fact that by definition  $\frac{1}{a} = e^{-\gamma}$ , it suffices to show

$$b^2 \left(1 + \alpha' - 2\alpha(1 - \gamma)\right) \le \frac{\alpha^2}{\alpha'^2} (1 - \alpha').$$

Using Lemma 2.8.4-(4), this reduces further to

$$(1 + \alpha - \alpha')(1 + \alpha' - 2\alpha(1 - \gamma)) \le \frac{\alpha^2}{{\alpha'}^2}(1 - \alpha').$$
(2.8.7)

We expand both sides of this inequality into polynomials involving  $\gamma$  and  $\alpha$ , and then analyze the resulting expression. Using the definition of  $\alpha'$ , we have  $\alpha' = \frac{\alpha}{1+2\gamma\alpha}$ , and thus  $1 + \alpha' = \frac{1+2\gamma\alpha+\alpha}{1+2\gamma\alpha}$  and  $1-\alpha' = \frac{1+2\gamma\alpha-\alpha}{1+2\gamma\alpha}$ . We also have  $\frac{\alpha}{\alpha'} = 1+2\gamma\alpha$ , and finally  $\alpha - \alpha' = \frac{2\gamma\alpha^2}{1+2\gamma\alpha}$ . Substituting these equalities into (2.8.7), we obtain the equivalent inequality

$$\left(\frac{1+2\gamma\alpha+\gamma\alpha^2}{1+2\gamma\alpha}\right)\left(\frac{1+2\gamma\alpha+\alpha}{1+2\gamma\alpha}-2\alpha(1-\gamma)\right) \le (1+2\gamma\alpha)^2\left(\frac{1+2\gamma\alpha-\alpha}{1+2\gamma\alpha}\right).$$

Multiplying both sides by  $(1 + 2\gamma \alpha)^2$  and rearranging the terms so that they are all on the same side, we get

$$(1+2\gamma\alpha)^3(1+2\gamma\alpha-\alpha) - (1+2\gamma\alpha+\gamma\alpha^2)(1+2\gamma\alpha+\alpha-2\alpha(1-\gamma)(1+2\gamma\alpha)) \ge 0.$$

Next, we expand this inequality:

$$16\alpha^4\gamma^4 - 16\alpha^4\gamma^3 + 8\alpha^4\gamma^2 + 24\alpha^3\gamma^3 - 12\alpha^3\gamma^2 + 12\alpha^2\gamma^2 + 2\alpha^3\gamma - 2\alpha^2\gamma + 2\alpha\gamma \ge 0$$

As  $\gamma \alpha \ge 0$ , we can divide both sides of this inequality by  $2\gamma \alpha$ . Grouping by powers of  $\alpha$ , we obtain

$$4\alpha^{3}\gamma \left(2\gamma^{2} - 2\gamma + 1\right) + \alpha^{2} \left(12\gamma^{2} - 6\gamma + 1\right) + \alpha(6\gamma - 1) + 1 \ge 0.$$

Upon inspection, both quadratics  $2\gamma^2 - 2\gamma + 1$  and  $12\gamma^2 - 6\gamma + 1$  are positive for all  $\gamma$ . Thus we only need to show  $\alpha(6\gamma - 1) + 1 \ge 0$ , but this is clear from writing it as  $1 - \alpha + 6\gamma\alpha \ge 0$  and using that  $\alpha \le 1$ .

# 3. Block Lewis weights for sparsification and minimizing sums of Euclidean norms

In this chapter, we introduce block Lewis weights in the context of sparsification (recall from the introduction that these weights give us an ellipsoidal approximation to a particular family of objectives that we are interested in sparsifying). The main goal is to define block Lewis weights and show how they enable near-optimal sparsification of matrix block norm objectives. We then apply this to get an improved algorithm for minimizing sums of Euclidean norms in moderate accuracy settings. The material in this chapter is based on a joint work with Max Ovsiankin [MO25].

## 3.1. Introduction

Suppose we are given a large dataset that is computationally inconvenient to work with in a downstream task. To alleviate this, we can try to randomly sample a small representative subset of the original dataset. The design and analysis of randomized sampling algorithms for this purpose is well-explored (for example, see [SS08; MMWY22; WY23b; WY23a] for preserving  $l_p$  objectives, [FL11] for preserving objectives for *k*-median, projective clustering, subspace approximation, and more, [SS08; KKTY22; JLS23; Lee23] for preserving graph and hypergraph  $l_2$ -energy, and [JLLS23] for sums (of powers) of general norms).

In order to design randomized sampling algorithms, we first need to understand the properties of the original dataset we want to preserve. To this end, we study the problem of preserving *block p-norm objectives*. Let  $\mathcal{G} = (\mathbf{A} \in \mathbb{R}^{n \times d}, S_1, \ldots, S_m, p_1, \ldots, p_m)$  be a dataset consisting of a matrix  $\mathbf{A} \in \mathbb{R}^{n \times d}$ . Consider a partitioning of [n] into groups  $S_1, \ldots, S_m$  and consider positive numbers  $p_1, \ldots, p_m$ . Let  $\mathbf{A}$  have rows  $a_1, \ldots, a_n$  and denote by  $\mathbf{A}_{S_i}$  the matrix in  $\mathbb{R}^{|S_i| \times d}$  whose rows are the rows of  $\mathbf{A}$  indexed by  $S_i$ . Consider the function  $\|\mathbf{A}\mathbf{x}\|_{\mathcal{G}_p}$  on some input vector  $\mathbf{x} \in \mathbb{R}^d$ :

$$\|\mathbf{A}\mathbf{x}\|_{\mathcal{G}_p}^p \coloneqq \sum_{i=1}^m \|\mathbf{A}_{S_i}\mathbf{x}\|_{p_i}^p$$
(3.1.1)

We use the norm notation because we can easily verify that for  $p \ge 1$  and  $p_i \ge 1$  for all i,  $\|\cdot\|_{\mathcal{G}_p}$  is a norm. We remark that objectives of the form of (3.1.1) are widely studied in geometric functional analysis, theoretical computer science, and data science. In Section 3.1.1, we go over one important application of the objective (3.1.1). We defer a broader discussion of more applications and connections to Section 3.1.4.

Our goal in this chapter is to design and analyze randomized sampling algorithms to output a weighted subset that preserves (3.1.1) for all  $x \in \mathbb{R}^d$ . We give a formal problem statement for the general problem we study in Problem 3.1.

**Problem 3.1** ( $\ell_p$  block norm sampling). We are given as input  $\mathcal{G} = (\mathbf{A} \in \mathbb{R}^{n \times d}, S_1, \dots, S_m, p_1, \dots, p_m)$ , p > 0, and an error parameter  $\varepsilon$ . For all  $i \in [m]$ , we must output a probability distribution  $\rho_1, \dots, \rho_m$  over [m] such that if we choose a collection of groups  $\mathcal{M} = (i_1, \dots, i_{\widetilde{m}})$  where each  $i_h$  is independently distributed according to  $\rho_i$ , then the following holds with probability  $\geq 1 - \delta$ :

for all 
$$\mathbf{x} \in \mathbb{R}^d$$
:  $(1-\varepsilon) \|\mathbf{A}\mathbf{x}\|_{\mathcal{G}_p}^p \le \frac{1}{\widetilde{m}} \sum_{i \in \mathcal{M}} \frac{1}{\rho_i} \cdot \|\mathbf{A}_{S_i}\mathbf{x}\|_{p_i}^p \le (1+\varepsilon) \|\mathbf{A}\mathbf{x}\|_{\mathcal{G}_p}^p$  (3.1.2)

We would like  $\tilde{m}$  to be small with probability  $1 - \delta$  (for example,  $\tilde{m}$  should not depend on m and the dependence on  $\delta^{-1}$  should be polylogarithmic).

Observe that the formulation of Problem 3.1 is an instantiation of an *importance sampling* framework. Specifically, we can think of the distribution  $\mathcal{D}$  as consisting of importance scores for each group. We form our sparse approximation by sampling group *i* with probability  $\rho_i$  and reweighting appropriately so that the function we return is an unbiased estimator of  $\|\mathbf{A}\mathbf{x}\|_{\mathcal{G}_p}$ . We call  $\tilde{m}$  the *sparsity* of the procedure described in Problem 3.1. Additionally, in the statement of our results, we will assume that *p* is a constant (and thus any function solely of *p* will treated as a constant in any  $O(\cdot)$  or  $\Omega(\cdot)$  terms).

In this chapter, we give new results for Problem 3.1 and show how these imply faster algorithms for commonly implemented optimization problems.

#### 3.1.1. Our results

For a quick summary of our existence results for the block norm sampling problem (Problem 3.1), see Section 3.1.4.

We begin with stating our main result<sup>1</sup>, Theorem 8.

**Theorem 8** (Block Lewis weight sampling). Let  $\mathcal{G} = (\mathbf{A} \in \mathbb{R}^{n \times d}, S_1, \dots, S_m, p_1, \dots, p_m)$  where  $S_1, \dots, S_m$  form a partition of [k]. Suppose at least one of the following holds:

- $1 \le p < \infty$  and  $p_1, ..., p_m \ge 2;$
- $1/\log d \le p_1 = \cdots = p_m = p < \infty;$
- $p_1 = \cdots = p_m = 2$  and  $1/\log d \le p < \infty$ .

Let  $P := \max(1, \max_{i \in [m]} \min(p_i, \log |S_i|))$ . Then, there exists a probability distribution  $\mathcal{D} = (\rho_1, \ldots, \rho_m)$  such that if

$$\widetilde{m} = \Omega\left(\log\left(1/\delta\right)\varepsilon^{-2}\left(\log d\right)^2\log\left(d/\varepsilon\right)P\cdot d^{\max(1,p/2)}\right),$$

and if we sample  $\mathcal{M} \sim \mathcal{D}^{\widetilde{m}}$ , then, with probability  $\geq 1 - \delta$ ,

for all 
$$\mathbf{x} \in \mathbb{R}^d$$
,  $(1-\varepsilon) \|\mathbf{A}\mathbf{x}\|_{\mathcal{G}_p}^p \le \frac{1}{\widetilde{m}} \sum_{i \in \mathcal{M}} \frac{1}{\rho_i} \cdot \|\mathbf{A}_{S_i}\mathbf{x}\|_{p_i}^p \le (1+\varepsilon) \|\mathbf{A}\mathbf{x}\|_{\mathcal{G}_p}^p$ 

<sup>&</sup>lt;sup>1</sup>In the statement of Theorem 8, writing the lower bound  $p \ge 1/\log d$  instead of p > 0 is somewhat arbitrary – we choose this lower bound to make our calculations easier later on.

We prove Theorem 8 in Section 3.5.1. It will follow from Theorem 11 (stated and proven in Section 3.4), which is a more general but more technical statement that also includes a description of the relevant distributions  $\mathcal{D}$ .

We remark that when  $p \ge 1$  and  $p_1, \ldots, p_m \ge 2$ , the sampling probabilities  $\rho$  mentioned in Theorem 8 can be found using the optimality conditions of a particular optimization problem that was stated and analyzed by Jambulapati, Lee, Liu, and Sidford [JLLS23, Section 4]. That problem itself can be viewed as the natural generalization of the determinant maximization problem that yields the existence of Lewis's measure (see [SZ01, Section 2] for details). However, [JLLS23] did not address the question of whether sparsification guarantees could be obtained with these weights beyond the case where the "outer norm" satisfies p = 2.

Additionally, although [JLLS23] study sparsification of sums of norms and sums of powers p > 1 of uniformly smooth norms, we obtain an improved sparsity in the case entailed by Problem 3.1 (by a factor of  $\psi_d \log (d/\epsilon)^{\min(p-1,2)}$ , where  $\psi_d$  is the KLS "constant" in d dimensions). We defer a more detailed comparison of our existence results to Section 3.1.4.

Furthermore, it is well-known that the polynomial terms in the sparsities in Theorem 8 are optimal. In particular, Li, Wang, and Woodruff [LWW21, Corollary 1.6 and Theorem 1.7] show that  $\Omega(d^{\max(1,p/2)} + \varepsilon^{-2} \operatorname{polylog}(\varepsilon^{-1})d)$  rows must be chosen in order to satisfy the requirement imposed by (3.1.2).

Finally, the setting where  $p = p_1 = \cdots = p_m$  is a particularly important case of Problem 3.1. Here, we see that  $\|\mathbf{A}\mathbf{x}\|_p = \|\mathbf{A}\mathbf{x}\|_{\mathcal{G}_p}$ , and so Problem 3.1 amounts to finding an  $\ell_p$  subspace embedding under a group constraint (that certain rows must be kept together in the subsample). This might be a useful notion in practice, where the  $S_i$  denote related observations that should be kept together for some downstream application. Moreover, this can be viewed as a higherrank analog of  $\ell_p$  row sampling, somewhat similarly to how the matrix Chernoff bound gives a higher-rank analog for the concentration of sums of bounded random matrices when compared to the rank-1 variant of Rudelson [Rud99].

**Computing sampling probabilities.** The previous results show the existence of sampling probabilities  $\rho_1, \ldots, \rho_m$  such that sampling using those probabilities gives a sparsifier in the setting of Problem 3.1. To get a sparsification *algorithm*, we need to also compute (or approximate) the sampling probabilities.

We give efficient algorithms to do so in natural cases.

**Theorem 9** (Computation of block Lewis weights). *Consider the setting of Theorem 8 and suppose at least one of the following holds:* 

- p = 2 and  $p_1, \ldots, p_m \ge 2;$
- $1/\log n \le p_1 = \cdots = p_m = p < \infty;$
- $p_1 = \cdots = p_m = 2$  and  $1/\log n \le p < \infty$ .

Let  $P = \max(1, \max_{i \in [m]} \min(p_i, \log |S_i|))$  and set

$$\widetilde{m} = O\left(\log\left(1/\delta\right)\varepsilon^{-2}\left(\log d\right)^2\log\left(d/\varepsilon\right)P\cdot d^{\max(1,p/2)}\right).$$

Then, there is an algorithm that outputs a probability distribution  $\mathcal{D} = (\rho_1, \dots, \rho_m)$  such that sampling a multiset  $\mathcal{M} \sim \mathcal{D}^{\widetilde{m}}$  satisfies, with probability  $1 - \delta$ ,

for all 
$$\mathbf{x} \in \mathbb{R}^d$$
,  $(1-\varepsilon) \|\mathbf{A}\mathbf{x}\|_{\mathcal{G}_p}^p \leq \frac{1}{\widetilde{m}} \sum_{i \in \mathcal{M}} \frac{1}{\rho_i} \cdot \|\mathbf{A}_{S_i}\mathbf{x}\|_{p_i}^p \leq (1+\varepsilon) \|\mathbf{A}\mathbf{x}\|_{\mathcal{G}_p}^p$ ,

*Further, the algorithm to find* D *performs at most polylog(k, n, m) leverage score overestimate computations or linear system solves.* 

We prove Theorem 9 in Section 3.5.2.

We formally define a *leverage score overestimate computation* in Definition 3.5.1. Alternately, these can be implemented using linear system solvers that solve systems of the form  $\mathbf{A}^{\mathsf{T}}\mathbf{D}\mathbf{A}\mathbf{y} = \mathbf{z}$  for diagonal **D** (see [LS19] for details). Although the runtime of this primitive depends on the structure of the input, each such iteration runs in  $\widetilde{O}(\mathsf{nnz}(\mathbf{A}) + d^{\omega})$  time. Moreover, in the special case where the matrix **A** is a graph edge-incidence matrix, the runtime improves to  $\widetilde{O}(\mathsf{nnz}(\mathbf{A}))$ .

Finally, we note that our algorithms are faster than the log-concave sampling-based routines given in [JLLS23] for calculating sparse approximations to sums (of powers of) more general norms, when the outer norm p satisfies  $1 \le p \le 2$  (they do not give algorithms for the case where p > 2). In particular, while their algorithm applies to a more general setting, the runtime is  $\tilde{O}(m + d^5)$ . In contrast, since our algorithms only depend on a polylogarithmic number of leverage score overestimate computations or linear system solves, we can obtain much faster runtimes (in particular improved powers of n). This means that we can apply our algorithms to downstream optimization tasks where the main computational primitive is a linear system solver (as is the case for many general frameworks for convex programming).

**Applications to minimizing sums of Euclidean norms.** A well-studied regression task is the *minimizing sums of Euclidean norms* (MSN) problem. We are given  $\mathbf{A} \in \mathbb{R}^{n \times d}$  and  $\mathbf{b} \in \mathbb{R}^{n}$ , and a partition  $S_1, \ldots, S_m$  of [n]. In this problem, we would like to find

$$\min_{\boldsymbol{x}\in\mathbb{R}^d}\sum_{i=1}^m \|\mathbf{A}_{S_i}\boldsymbol{x}-\boldsymbol{b}_{S_i}\|_2.$$
(3.1.3)

Solving the MSN objective (3.1.3) subsumes several widely implemented optimization problems such as variants of Euclidean single facility location, Euclidean multifacility location, Euclidean Steiner minimum tree under a given topology, and plastic collapse analysis. See the long line of work on this problem [And96; XY97; ACCO00; QSZ02] for a more detailed discussion. Additionally, observe that if all  $|S_i| = 1$ , then (3.1.3) is nothing but  $\ell_1$  regression (i.e.,  $\min_{x \in \mathbb{R}^d} ||\mathbf{A}x - \mathbf{b}||_1$ ). Thus, (3.1.3) is a generalization of  $\ell_1$  regression. Finally, notice that (3.1.3) subsumes the *stochastic robust approximation problem* when the norm in question is the Euclidean norm and the design **A** assumes a finite number of values – see [BV04, Section 6.4.1] for further discussion.

In this chapter, we will be interested in algorithms that return a  $(1 + \varepsilon)$ -multiplicative approximation to the objective – namely, we desire a point  $\widehat{x} \in \mathbb{R}^d$  such that

$$\sum_{i=1}^{m} \|\mathbf{A}_{S_{i}}\widehat{\mathbf{x}} - \mathbf{b}_{S_{i}}\|_{2} \leq (1+\varepsilon) \min_{\mathbf{x}\in\mathbb{R}^{d}} \sum_{i=1}^{m} \|\mathbf{A}_{S_{i}}\mathbf{x} - \mathbf{b}_{S_{i}}\|_{2}.$$

To our knowledge, the best known algorithms based on interior point methods output a  $(1 + \varepsilon)$ -approximate solution to (3.1.3)  $\tilde{O}(\sqrt{m} \log (1/\varepsilon))$  calls to a linear system solver [And96; XY97] for matrices of the form  $\mathbf{A}^{\mathsf{T}}\mathbf{D}\mathbf{A}$  for block-diagonal matrices **D**.

By applying Theorem 9 on the matrices  $[\mathbf{A}_{S_i}|\mathbf{b}_{S_i}] \in \mathbb{R}^{|S_i| \times (d+1)}$  with  $p_1 = \cdots = p_m = 2$  and p = 1, observe that within  $\widetilde{O}(1)$  linear system solves in matrices  $\mathbf{A}^\top \mathbf{D} \mathbf{A}$  for nonnegative diagonal  $\mathbf{D}$ , we obtain an objective with  $\widetilde{O}(\varepsilon^{-2} \cdot d)$  terms that approximates (3.1.3) up to a  $(1 \pm \varepsilon)$  multiplicative factor on all vectors  $\mathbf{x} \in \mathbb{R}^{d+1}$  whose last coordinate is 1. This immediately implies Theorem 10.

**Theorem 10** (Minimizing sums of Euclidean norms). Let  $\mathbf{A} \in \mathbb{R}^{n \times d}$  and  $\mathbf{b} \in \mathbb{R}^n$ , and  $S_1, \ldots, S_m$  be a partition of k. There exists an algorithm that, with probability  $\geq 1 - \delta$ , returns  $\hat{\mathbf{x}}$  such that

$$\sum_{i=1}^{m} \|\mathbf{A}_{S_{i}}\widehat{\boldsymbol{x}} - \boldsymbol{b}_{S_{i}}\|_{2} \leq (1+\varepsilon) \min_{\boldsymbol{x} \in \mathbb{R}^{d}} \sum_{i=1}^{m} \|\mathbf{A}_{S_{i}}\boldsymbol{x} - \boldsymbol{b}_{S_{i}}\|_{2}.$$

The algorithm runs in  $\widetilde{O}\left(\sqrt{d}/\varepsilon \cdot \sqrt{\log(1/\delta)}\right)$  calls to a linear system solver in matrices of the form  $\mathbf{A}^{\top}\mathbf{D}\mathbf{A}$  for block-diagonal matrices  $\mathbf{D}$ , where each block has size  $(|S_i| + 1) \times (|S_i| + 1)$ .

We prove Theorem 10 in Section 3.5.3.

Theorem 10 improves over the best-known iteration complexities for solving (3.1.3) when the number of summands is much larger than the input dimension, i.e.,  $m \gg d$ . Furthermore, the iteration complexity stated in Theorem 10 matches the iteration complexity for  $\ell_1$  regression up to the  $\varepsilon^{-1}$  term [vdBLLSSSW21]. It is an interesting (but probably challenging) open problem to design and analyze an algorithm for (3.1.3) with iteration complexity  $\widetilde{O}(\sqrt{d} \log (1/\varepsilon))$ , which would exactly match what is known for  $\ell_1$  regression.

Finally, we note that in the special case of the geometric median, where all the  $\mathbf{A}_{S_i} = \mathbf{I}_r$  for some fixed dimension r, an algorithm with runtime  $\widetilde{O}(\operatorname{nnz}(\boldsymbol{b}) \log (1/\varepsilon)^3)$  is known due to Cohen, Lee, Miller, Pachocki, and Sidford [CLMPS16]. The algorithm is a long-step interior point method with a custom analysis and follows from different techniques from ours.

**Outline.** The rest of this chapter is organized as follows. In the remainder of this section, we establish notation that we use throughout the rest of the chapter (Section 3.1.2), give an overview of our technical methods (Section 3.1.3), and discuss some prior and related works (Section 3.1.4). In Section 3.2, we give background from linear algebra, convex geometry, and probability that we rely on for the rest of the chapter. In Section 3.3, we prove bounds on geometric quantities known as *covering numbers*. These play a crucial role in our concentration arguments. In Section 3.4, we prove that our general sampling scheme concentrates and therefore preserves the original objective on all  $x \in \mathbb{R}^d$ , with high probability. In Section 3.5, we show how to apply our general sampling scheme to the problems we discuss in Section 3.1.1. Finally, in Section 3.5.2, we describe our algorithmic results.

### 3.1.2. Notation and definitions

**General notation.** For positive integer *N*, we let [N] denote the set  $\{i \in \mathbb{Z} : 1 \le i \le N\}$ . All logs are base 2; we use ln to denote the natural logarithm. We let  $e_1, \ldots, e_d$  denote the standard basis vectors in  $\mathbb{R}^d$ . When we write  $a \le b$ , we mean that  $a \le Cb$  for some universal constant C > 0.

**Linear algebra notation.** In this chapter, we work extensively with matrices and vectors. We always denote matrices with capital letters in boldface (e.g. **A**) and vectors with lowercase letters in boldface (e.g. *x*). With a few exceptions, we write the rows of a matrix using the lowercase boldface version of the same letter used to write the matrix along with a subscript denoting which index the row corresponds to. For example,  $a_i$  denotes the *i*th row of matrix **A**. In a slight abuse of notation, for a symmetric matrix **M**, we let  $\mathbf{M}^{-1} := \sum_{i=1}^{\operatorname{rank}(\mathbf{M})} \lambda_i^{-1} u_i u_i^{\mathsf{T}}$ , where  $u_i$  is the *i*th eigenvector of **M**. In other words, we write  $\mathbf{M}^{-1}$  to denote the pseudoinverse of **M** when **M** is symmetric. We will never use the inverse notation  $\mathbf{M}^{-1}$  for a non-symmetric matrix **M**.

## 3.1.3. Technical overview

In this subsection, we give a bird's eye view of the technical methods behind our proof of Theorem 8.

#### Concentration

We begin with an explanation of our concentration proof. This type of argument has become standard in the line of work on sparsification (particularly in [Lee23; JLLS23]), but we include a description for completeness.

Let  $B_p := \{ x \in \mathbb{R}^d : \|Ax\|_{\mathcal{G}_p} \le 1 \}$ . By a standard symmetrization reduction, it suffices to fix  $i_1, \ldots, i_{\widetilde{m}}$  (not necessarily distinct) and argue that for independent  $R_1, \ldots, R_{\widetilde{m}}$  where  $R_h \sim \text{Unif}(\pm 1)$  we have

$$\mathbb{E}_{R_{h}}\left[\sup_{\boldsymbol{x}\in B_{p}}\left|\sum_{h=1}^{\widetilde{m}}R_{h}\cdot\frac{\left\|\boldsymbol{A}_{S_{i_{h}}}\boldsymbol{x}\right\|_{p_{i_{h}}}^{p}}{\rho_{i_{h}}}\right|\right]\leq\widetilde{m}\cdot\varepsilon.$$
(3.1.4)

Intuitively, satisfying (3.1.4) means that for the rebalancing of the groups given by the  $\rho_{i_h}$ , a Rademacher average of the groups evaluated on every point in x is close to 0. It is straightforward to check that the above instantiation is a subgaussian process under an appropriately chosen distance function  $dist : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ . Thus, we will apply chaining [Tal21], which can be thought of as simultaneously controlling (3.1.4) on  $\varepsilon$ -nets of  $B_p$  using the metric dist, for all  $\varepsilon > 0$ .

To apply chaining, the main technical task is to understand the *entropy numbers*  $e_N(B_p, dist)$ . The entropy numbers  $e_N(B_p, dist)$  are the values  $\eta$  that answer the question, "what is the smallest  $\eta$  such that  $B_p$  can be covered by at most  $2^{2^N}$  balls of *dist*-radius  $\eta$ ?" (or see Definition 3.2.5).

#### **Covering numbers**

In this subsection, we explain how to control the entropy numbers as required by Section 3.1.3. We first define the sampling body (Definition 3.1.1).

**Definition 3.1.1** (Sampling body). Let *S* be some subset of [m] and  $\rho_1, \ldots, \rho_m$  be a probability distribution. Define the norm  $\|\mathbf{x}\|_{\mathcal{G},\rho,\infty,S} \coloneqq \max_{i \in S} \rho_i^{-1/p} \|\mathbf{A}_{S_i}\mathbf{x}\|_{p_i}$ . We call the unit norm ball of  $\|\mathbf{x}\|_{\mathcal{G},\rho,\infty,S}$  the sampling body.

Recall that the covering number  $N(K_1, K_2)$  for two symmetric convex bodies  $K_1$  and  $K_2$  is the minimum number of translates of  $K_2$  required to cover  $K_1$ . Additionally, recall from the previous subsection the notion of *entropy numbers* (which we will define in Definition 3.2.5). We will reduce controlling (3.1.4) to bounding the entropy numbers

$$e_N\left(\left\{\boldsymbol{x}\in\mathbb{R}^d : \|\mathbf{A}\boldsymbol{x}\|_{\mathcal{G}_p}\leq 1\right\}, \left\{\boldsymbol{x}\in\mathbb{R}^d : \|\boldsymbol{x}\|_{\mathcal{G},\rho,\infty,S}\leq 1\right\}\right)$$

when N is small. This places us in the setting where a simple volume-based argument becomes suboptimal. In this range, the dual Sudakov inequality (Fact 3.2.6) is the technical workhorse that allows us to get sharper bounds than what we would get if we applied just a volume-based bound. It states that if B is the Euclidean ball in d dimensions and K is some symmetric convex body in d dimensions, then we have

$$\log \mathcal{N}(B,\eta K) \lesssim \eta^{-2} \mathop{\mathbb{E}}_{\boldsymbol{g} \sim \mathcal{N}(0,\mathbf{I}_d)} \left[ \|\boldsymbol{g}\|_K \right]^2,$$

where  $\|\cdot\|_K$  is the *gauge norm* for *K*, defined by  $\|\mathbf{x}\|_K \coloneqq \inf \{t > 0 : \mathbf{x}/t \in K\}$ .

However, applying the dual Sudakov inequality requires that we analyze covering numbers of the form  $\log N(B, K)$  where *B* is the Euclidean ball in *d* dimensions and *K* is some symmetric convex body. Denoting  $\{x \in \mathbb{R}^d : \|x\|_{\mathcal{G},\rho,\infty,S} \leq 1\}$  by *K*, we see that we cannot immediately apply the dual Sudakov inequality to bound  $\log N(B_p, \eta K)$ . This is because  $B_p$  is not a (linear transformation of a) Euclidean ball. The work of [JLLS23] resolve this by generalizing the dual Sudakov inequality to cover arbitrary symmetric convex bodies. Unfortunately, this approach is not optimal in every setting. One source of the loss arises from exploiting the concentration of Lipschitz functionals of isotropic log-concave random vectors – improving the bounds on this concentration depends on further progress on the KLS conjecture. Another is that the one-dimensional conditionals of isotropic log-concave random variables, without any further assumptions, are only subexponential.

To escape these inefficiencies, we will want to try to find a way to apply the dual Sudakov inequality as-is. We may then exploit the concentration of Lipschitz functionals of Gaussian random vectors, which we do have a tight understanding of (for a precise statement, see Fact 3.2.11). A natural attempt is to first observe that for any t > 0,

$$\log \mathcal{N}(B_p, \eta K) \le \log \mathcal{N}(B_p, t\widehat{B_2}) + \log \mathcal{N}(t\widehat{B_2}, \eta K) = \log \mathcal{N}(B_p, t\widehat{B_2}) + \log \mathcal{N}(\widehat{B_2}, \frac{\eta}{t} \cdot K).$$
(3.1.5)

We will choose  $\widehat{B_2}$  to be a linear transformation of a Euclidean ball so that we can control  $\log \mathcal{N}\left(\widehat{B_2}, \eta/t \cdot K\right)$  using the dual Sudakov inequality.

Here, we will split our argument based on whether  $p \ge 2$ . When  $p \ge 2$ , it will become clear later on that it will be sufficient to choose  $\widehat{B_2}$  so that  $B_p \subseteq \widehat{B_2}$ . Then, it is easy to see that when t = 1, we get  $\log \mathcal{N}(B_p, t\widehat{B_2}) = 0$ . Hence, we have  $\log \mathcal{N}(B_p, \eta K) \le \log \mathcal{N}(\widehat{B_2}, \eta K)$ , and the required bound will follow from exploiting the concentration of Lipschitz functionals of Gaussian random vectors and then applying the dual Sudakov inequality. However, when p < 2, we are still left with a pesky  $\log \mathcal{N}(B_p, tB_2)$  term. Loosely, this is almost dual to the statement of the dual Sudakov inequality. Now, because it is known that covering number duality does hold when one of the bodies in question is the Euclidean ball, it may be tempting to simply write  $\log \mathcal{N}(B_p, tB_2) = \log \mathcal{N}(B_2, tB_q)$  where  $B_q$  is the dual ball to  $B_p$  after applying some linear transformation to map  $\widehat{B}_2$  to B. The challenge here is that we do not believe that the gauge of the resulting  $B_q$  has a form that is amenable to analysis. We will therefore need to be more careful, and we describe our alternative approach in Section 3.1.3.

#### The change-of-measure principle and norm interpolation

Recall from the previous part that our goal is to bound  $\log \mathcal{N}(B_p, t\widehat{B_2})$  when p < 2.

We are now ready to introduce our main conceptual message – by changing the measure under which we take norms, we can almost automatically identify a linear transformation of a Euclidean ball  $\widehat{B_2}$  that is a good approximation to  $B_p$ . This sort of idea has already been used by Bourgain, Lindenstrauss, and Milman [BLM89] and Schechtman and Zvavitch [SZ01] to obtain the required  $\widehat{B_2}$  in the special case where all the  $S_i$  are singletons. We will generalize this machinery to give similar results for the block norm sampling problem.

Let us describe this idea further. Let  $\lambda = [\lambda_1, ..., \lambda_m]^{\top}$  denote a probability measure over the groups. Let  $\Lambda \in \mathbb{R}^{n \times n}$  be the diagonal matrix such that if  $j \in S_i$ , then  $\Lambda_{jj} = \lambda_i$ . Finally, for any r > 0 and  $y \in \mathbb{R}^n$ , let  $\|y\|_{\mathcal{G}_r(\lambda)} = (\sum_{i \le m} \lambda_i \|y_{S_i}\|_{p_i}^r)^{1/r}$  and  $B_r := \{x \in \mathbb{R}^d : \|\Lambda^{-1/p} Ax\|_{\mathcal{G}_r(\lambda)} \le 1\}$ . Notice that under this definition, we still have  $B_p$  as before. We will first describe the argument when the groups are singletons, then explain how to move onto the general case. We will take  $\widehat{B_2} = B_2$ ; it is easy to see that this is a linear transformation of a Euclidean ball.

Next, notice that by log-convexity of norms, if we choose  $0 < \theta < p$  and r > 2 for which  $1/2 = (\theta/2)/p + (1 - \theta/2)/r$ , we have

$$\|\boldsymbol{y}\|_{\mathcal{G}_{2}(\lambda)}^{2} \leq \|\boldsymbol{y}\|_{\mathcal{G}_{p}(\lambda)}^{\theta} \cdot \|\boldsymbol{y}\|_{\mathcal{G}_{r}(\lambda)}^{2-\theta}.$$
(3.1.6)

We will exploit this observation as follows. For all integers  $h \ge 0$ , we will show that there exists a set  $\mathcal{L}_h$  that is (a subset of) the unit ball of  $B_2$  such that every pair of points in  $\mathcal{L}_h$  is  $\delta_h$ -separated according to  $\|\cdot\|_{\mathcal{G}_r(\lambda)}$ . We will find  $\delta_h$  according to the interpolation inequality (3.1.6). Furthermore, we will generate  $\mathcal{L}_h$  using a sort of compactness argument arising from a  $B_2$ -maximally separated subset of  $B_p$ . This means we get, for every  $h \ge 0$ ,

$$\log \mathcal{N}(\widehat{B_2}, \delta_h B_r) \ge \log |\mathcal{L}_h| \ge \log \left( \frac{\mathcal{N}(B_p, 8^h t \widehat{B_2})}{\mathcal{N}(B_p, 8^{h+1} t \widehat{B_2})} \right)$$

Then, summing over  $h \ge 0$  (noting that once *h* is sufficiently large,  $\mathcal{N}(B_p, 8^h t \widehat{B_2}) = 1$ ), we have

$$\log \mathcal{N}(B_p, t\widehat{B_2}) \leq \sum_{h \geq 0} \log \mathcal{N}(\widehat{B_2}, \delta_h B_r).$$

Notice that the right hand side can be evaluated using the dual Sudakov inequality<sup>2</sup> (recall the previous section), so it suffices to show that  $\log \mathcal{N}(\widehat{B_2}, \eta B_r)$  is small.

 $<sup>^{2}</sup>$ For technical reasons that will be clearer in Section 3.3, we will have to do this after another interpolation step.

This is where the choice of measure becomes crucial. Since both  $B_2$  and  $B_r$  are dependent on our choice of measure  $\lambda$ , we will need to carefully choose the measure so that our covering numbers are well-behaved. A classical result of Lewis [Lew78] establishes the existence of a change-of-measure under which we simultaneously get:

$$d^{1/2-1/r}B_r \subset \widehat{B_2} \subset B_r \qquad \text{for all } r < 2$$

$$B_r \subset \widehat{B_2} \subset d^{1/2-1/r}B_r \quad \text{for all } r > 2$$
(3.1.7)

This change-of-measure corresponds to the " $\ell_p$  Lewis weights" of **A** (in particular, if  $w_i$  is the *i*th  $\ell_p$  Lewis weight, then we set  $\lambda_i = w_i/n$ ). It will turn out that this choice of  $\lambda$  is enough for us to ensure that  $\mathcal{N}(\widehat{B}_2, \eta B_r)$  is sufficiently small for our purposes, which eventually follows from (3.1.7).

**Handling general**  $S_i$ . The main challenge with directly porting this argument to the block norm sampling problem is that  $B_2$  is not a linear transformation of a Euclidean ball unless  $p_1 = \cdots = p_m = 2$ . We will therefore have to choose  $\widehat{B}_2$  to be a "rounding" of  $B_2$  such that  $B_2 \subseteq \widehat{B}_2$ . Observe that the interpolation step (3.1.6) will continue to hold here, as we will get  $\|y\|_{\widehat{B}_2} \leq \|y\|_{\mathcal{G}_2(\lambda)}$ . However, if  $\widehat{B}_2$  is chosen suboptimally, then there could a large loss in the interpolation step (3.1.6).

To understand what we need from our measure and rounding, let us try to derive a version of (3.1.7) for general  $S_i$ . We show an example of this calculation for  $r = p \le 2$ ; the other cases follow similarly. Let  $\lambda \in \mathbb{R}^m_{\ge 0}$  denote a probability measure. Let **W** be a diagonal "rounding matrix" so that for all  $x \in \mathbb{R}^d$ , we have

$$\left\|\mathbf{W}^{1/2}\mathbf{\Lambda}^{1/2-1/p}\mathbf{A}\mathbf{x}\right\|_{2} \leq \left\|\mathbf{\Lambda}^{1/2-1/p}\mathbf{A}\mathbf{x}\right\|_{\mathcal{G}_{2}} = \left\|\mathbf{\Lambda}^{-1/p}\mathbf{A}\mathbf{x}\right\|_{\mathcal{G}_{2}(\lambda)}.$$

Letting  $\widehat{B_2} = \{ \mathbf{x} \in \mathbb{R}^d : \|\mathbf{W}^{1/2} \mathbf{\Lambda}^{1/2-1/p} \mathbf{A} \mathbf{x}\|_2 \le 1 \}$ , the above inequality gives  $B_2 \subseteq \widehat{B_2}$ , as desired. Next, observe that since  $\lambda$  is a probability measure, we get  $B_2 \subseteq B_p$  for free. For the other direction, we write

$$\begin{split} \left\| \mathbf{\Lambda}^{-1/p} \mathbf{A} \mathbf{x} \right\|_{\mathcal{G}_{2}(\lambda)}^{2} &= \left\| \mathbf{\Lambda}^{1/2 - 1/p} \mathbf{A} \mathbf{x} \right\|_{\mathcal{G}_{2}}^{2} = \sum_{i=1}^{m} \lambda_{i} \left\| \lambda_{i}^{-1/p} \mathbf{A} \mathbf{x} \right\|_{p_{i}}^{2} = \sum_{i=1}^{m} \lambda_{i} \left\| \lambda_{i}^{-1/p} \mathbf{A} \mathbf{x} \right\|_{p_{i}}^{p} \left\| \lambda_{i}^{-1/p} \mathbf{A} \mathbf{x} \right\|_{p_{i}}^{2-p} \\ &\leq \sum_{i=1}^{m} \lambda_{i} \left\| \lambda_{i}^{-1/p} \mathbf{A} \mathbf{x} \right\|_{p_{i}}^{p} \cdot \max_{i \in [m]} \left\| \lambda_{i}^{-1/p} \mathbf{A} \mathbf{x} \right\|_{p_{i}}^{2-p} = \left\| \mathbf{A} \mathbf{x} \right\|_{\mathcal{G}_{p}}^{p} \cdot \max_{i \in [m]} \left\| \lambda_{i}^{-1/p} \mathbf{A} \mathbf{x} \right\|_{p_{i}}^{2-p} \\ &\leq \left\| \mathbf{A} \mathbf{x} \right\|_{\mathcal{G}_{p}}^{p} \cdot \max_{i \in [m]} \left( \max_{\mathbf{x} \in \mathbb{R}^{d}} \frac{\left\| \lambda_{i}^{-1/p} \mathbf{A} \mathbf{x} \right\|_{p_{i}}^{2}}{\left\| \mathbf{A}^{1/2 - 1/p} \mathbf{A} \mathbf{x} \right\|_{\mathcal{G}_{2}}^{2}} \right)^{1-p/2} \cdot \left\| \mathbf{A}^{1/2 - 1/p} \mathbf{A} \mathbf{x} \right\|_{\mathcal{G}_{2}}^{2-p}. \end{split}$$

We combine the  $\|\mathbf{\Lambda}^{1/2-1/p}\mathbf{A}\mathbf{x}\|_{\mathcal{G}_2}$  terms and take the *p*th root of both sides, giving

$$\left\| \mathbf{\Lambda}^{1/2-1/p} \mathbf{A} \mathbf{x} \right\|_{\mathcal{G}_2} \le \| \mathbf{A} \mathbf{x} \|_{\mathcal{G}_p} \cdot \max_{i \in [m]} \left( \max_{\mathbf{x} \in \mathbb{R}^d} \frac{\left\| \lambda_i^{-1/p} \mathbf{A} \mathbf{x} \right\|_{p_i}^2}{\left\| \mathbf{\Lambda}^{1/2-1/p} \mathbf{A} \mathbf{x} \right\|_{\mathcal{G}_2}^2} \right)^{1/p-1/2}$$

$$= \|\mathbf{A}\mathbf{x}\|_{\mathcal{G}_{p}} \cdot \max_{i \in [m]} \left( \frac{1}{\lambda_{i}} \cdot \max_{\mathbf{x} \in \mathbb{R}^{d}} \frac{\left\|\lambda_{i}^{1/2-1/p} \mathbf{A}\mathbf{x}\right\|_{p_{i}}^{2}}{\left\|\mathbf{\Lambda}^{1/2-1/p} \mathbf{A}\mathbf{x}\right\|_{\mathcal{G}_{2}}^{2}}}{\widehat{\tau}_{i}(\mathbf{\Lambda}^{1/2-1/p} \mathbf{A})} \right)^{1/p-1/2}$$

We may think of the quantity  $\hat{\tau}_i$  as a generalized *leverage score*. Specifically, it upper bounds the contribution of the term  $\|\lambda_i^{1/2-1/p} \mathbf{A} \mathbf{x}\|_{p_i}^2$  to the objective  $\|\mathbf{\Lambda}^{1/2-1/p} \mathbf{A} \mathbf{x}\|_{\mathcal{G}_2}^2$ . The above calculation shows us that if we make  $\hat{\tau}_i / \lambda_i$  small for all *i*, then we can get a tight relationship between  $B_2$  and  $B_p$ . A slight weakening of the definition of the  $\hat{\tau}_i$  motivates the notion of a *block Lewis overestimate* that we use in the remainder of the chapter.

**Definition 3.1.2** (Block Lewis overestimate). Let  $\tau_j(\mathbf{M})$  denote the leverage score of the *j*th row of  $\mathbf{M}$ . Let  $F^* > 0$ . For p > 0 and  $p_i > 0$ , we say the probability measure  $\lambda$  and rounding  $\mathbf{W}$  form an  $F^*$ -block Lewis overestimate if for all  $i \in [m]$ , we have

$$\frac{1}{\lambda_i} \left( \sum_{j \in S_i} \left( \frac{\tau_j(\mathbf{W}^{1/2} \mathbf{\Lambda}^{1/2 - 1/p} \mathbf{A})}{w_j} \right)^{p_i/2} \right)^{2/p_i} \le F^{\star}.$$

Following the above argument, establishing a probability measure  $\lambda$  and a rounding matrix **W** that form an *F*\*-block Lewis overestimate will imply

$$B_2 \subseteq \widehat{B_2} \quad \text{and} \quad \frac{(F^\star)^{1/2-1/r} B_r \subset B_2 \subset B_r}{B_r \subset B_2 \subset (F^\star)^{1/2-1/r} B_r} \quad \text{for all } r < 2.$$
(3.1.8)

With (3.1.8) in hand, we at least have enough reason to believe that establishing  $(\lambda, \mathbf{W})$  that form an  $F^*$ -block Lewis overestimate may yield the requisite control over  $\log \mathcal{N}(\widehat{B}_2, \eta B_r)$ . To actually get this, by the dual Sudakov inequality, we estimate  $\|g\|_{B_r}$  for  $g \sim \mathcal{N}(0, \mathbf{I}_d)$ . Doing so is a matter of applying again the fact that the concentration of Lipschitz functionals of Gaussian vectors is determined entirely by the Lipschitz parameter of the functional.

To actually find  $\lambda$  and  $\mathbf{W}$  with a small value of  $F^*$ , we split into cases. When  $p \ge 1$  and  $p_1 = \cdots = p_m \ge 2$ , we extract the relevant  $\lambda$  and  $\mathbf{W}$  from the analysis in the proof of [JLLS23, Lemma 4.2], which yields  $F^* = d$ . When  $p_1 = \cdots = p_m = p \ge 1/\log d$  or  $p_1 = \cdots = p_m = 2$  and  $p \ge 1/\log d$ , we separately prove that we can find  $\lambda$  and  $\mathbf{W}$  satisfying Definition 3.1.2, again with  $F^* = d$ . Hence, in all cases, the control we get over  $\log \mathcal{N}(\widehat{B_2}, \eta B_r)$  is essentially as good as what we get in the case where all the  $S_i$  are singletons.

**Change-of-measures in functional analysis.** We note that other change-of-measure arguments are used throughout the study of finite-dimensional subspaces of  $L_p$ , as they are a very useful way to compare the  $L_p$  norm to some other norm of interest. See the survey by Johnson and Schechtman [JS00, Section 1.2] for more information.

#### 3.1.4. Prior results, related works, and connections

**Relevance of matrix block norms.** We discuss the importance of the matrix block norm objective (3.1.1) to functional analysis, theoretical computer science, and data science, beyond

our previous discussion of the MSN problem (3.1.3).

In the special case where all the  $p_i$  are equal to one another (call this value q), the set of x for which  $\|\mathbf{A}x\|_{\mathcal{G}_p} < \infty$  yields a subspace of a mixed p, q norm space (sometimes notated as  $\ell_p(\ell_q)$ ). Observe that Theorem 8 implies a finite-dimensional subspace embedding result for finite-dimensional subspaces of infinite-dimensional  $L_p(\ell_q^r)$  (where r is finite). Spaces of the form  $\ell_p(\ell_q)$  are widely studied in the geometric functional analysis and approximation theory communities; see, e.g., [PS12; KV17; MU21; JKP22] and the references therein.

As mentioned in those works, a central motivation for studying  $\ell_p(\ell_q)$  is that they are natural testbeds with which to evaluate and further our understanding of the geometry of symmetric convex bodies in high dimensions. Consequently, studying block norm subspace embedding problems (Problem 3.1) is a fruitful direction through which to improve our geometric handle of subspaces of  $\ell_p(\ell_q)$  and symmetric convex bodies in general. We note that understanding the correct polylog dependencies in *d* for this problem typically requires new geometric insights. For instance, the necessity of additional polylogarithmic dependencies on the dimension *d* is not even totally understood when  $|S_i| = 1$  and  $p \neq 2$ , and resolving them likely requires significant new geometric ideas [BRR23, Conjecture 2].

The matrix block norms are also ubiquitous in both theoretical computer science and data science. For example, the block norm objective has been studied in the context of hypergraph Laplacians. One recovers this by choosing p = 2 and  $p_1 = \cdots = p_m = \infty$ ; see the discussion in [JLS23, Section 1.2] to see how to rewrite the hypergraph Laplacian in the form of (3.1.1). Within data science, the block norms are used to encourage structured solutions to underdetermined linear systems (i.e., in a noiseless setting, we can set up and solve the convex optimization problem<sup>3</sup>, "find a vector y in the affine space By = b minimizing  $||y||_{\mathcal{G}_p}^p$ "); see, e.g., [YL06; Bac08; NHCD10; SFHT13] and other applications mentioned in [Sra12]. As a concrete candidate application of our results to such settings, inspired by [CD21; MMWY22], we believe that our results can be used as subroutines to give runtime and query-efficient algorithms for active  $|| \cdot ||_{\mathcal{G}_p}$  regression when p > 0 and  $p_1 = \cdots = p_m = 2$  (generalizing the basis pursuit equivalent of the group Lasso objective) or when p = 2 and  $p_1, \ldots, p_m \ge 2$ .

On a more conceptual level, we are optimistic that some of our results will be useful for designing faster algorithms for norm-constrained optimization problems. This is partly motivated by our discussion around the MSN problem (3.1.3) and is similar to how an improved geometric understanding of Lewis weights improved the iteration complexities for linear programming and  $\ell_p$  regression [LS19; JLS22].

**Lewis weights for**  $\ell_p$  **row sampling.** When each group  $S_i$  has size 1, notice that we have  $\|\mathbf{A}\mathbf{x}\|_{\mathcal{G}_p}^p = \sum_{i=1}^m |\langle a_i, \mathbf{x} \rangle|^p = \|\mathbf{A}\mathbf{x}\|_p^p$ . Consequently, in this special case, satisfying (3.1.2) is exactly equivalent to computing an  $\ell_p$  subspace embedding for **A**. There is a long line of work studying computing  $\ell_p$  subspace embeddings using Lewis weights, starting with that of Bourgain, Lindenstrauss, and Milman [BLM89]. For the details of this argument, see [BLM89, Section 7] and [SZ01].

**Sparsifying sums of norms.** The work perhaps most closely related to ours is [JLLS23]. There, the authors give existence results for sparse approximations to sums (of powers) of norms. It is

<sup>&</sup>lt;sup>3</sup>This is similar to how basis pursuit can be seen as encouraging sparsity in a noiseless setting, while LASSO does so in the presence of noise [Wai19, Section 7.2].

easy to see that this is a more general problem than the one we study. However, this generality comes at a cost. In particular, the sparsity given by our Theorem 8 improves over theirs by a factor of  $\psi_d \log (d/\varepsilon)^{\min(p-1,2)}$ , where  $\psi_d$  denotes the KLS "constant" in *d* dimensions (which is currently  $\sqrt{\log d}$ , due to Klartag [Kla23]). See their Theorem 1.3 for more details. And, as mentioned earlier, we believe understanding Problem 3.1 down to the correct polylogarithmic dependencies in *d* is an important geometric question.

The authors also define the block Lewis weights as the natural generalization of the determinant-maximization program that Schechtman and Zvavitch [SZ01] use to prove the existence of Lewis's measure for all p > 0. They use this to give results for sparsification of sums of certain powers of arbitrary norms (obtaining a sparsity of  $\sim d^{2-1/p}$  when  $1 \le p \le 2$ ) and for Problem 3.1 when the outer norm p = 2 (obtaining a sparsity of  $\sim d$ ). We note that the result they obtain when p = 2 provides logarithmic-factor improvements over ours, as the main technical primitive they use is a chaining estimate developed by Lee [Lee23] that meaningfully exploits the fact that the space of events is a subset of a 2-uniformly convexity set. However, they did not address whether the block Lewis weights could yield to sparsification guarantees for Problem 3.1. Additionally, their construction of the block Lewis weights does not yield a change-of-measure that allows for sparsification when the inner norms  $p_i \le 2$  or when the outer norm  $p \le 1$ .

**Summary for sparsifying sums of norms.** See Section 3.1.4 for a comparison between our new results and a selection of the most relevant prior work on sparsifying sums of norms. We focus on results concerning  $\ell_p$ -norms specifically, although [JLLS23] has results for more general classes of norms. By "sampling", we mean the work provides an analysis that shows how sampling according to some sampling probabilities gives a sparsifier of size  $\tilde{O}(\varepsilon^{-2}d^{\max(1,p/2)})$  with good probability. By "fast computation", we mean the work provides an algorithm to compute sampling probabilities with polylog(k, m, d) leverage score computations or linear system solves (or some other primitive that can be implemented in time  $\tilde{O}(\operatorname{nnz}(\mathbf{A}) + d^{\omega})$ ). For works that only explicitly handle  $|S_i| = 1$ , we leave  $p_1, \ldots, p_m$  blank because the choice of inner norms does not affect the objective.

Block size	р	$p_1,\ldots,p_m$	Sampling	Fast computation	
1	$1 \le p < \infty$		$\checkmark$		[BLM89]
1	$0$		$\checkmark$		[SZ01]
1	$0$			$\checkmark$	[CP15]
$\geq 1$	p = 2	$p_1 = \cdots = p_m = \infty$	$\checkmark$		[Lee23]
$\geq 1$	p = 2	$p_1 = \cdots = p_m = \infty$	$\checkmark$	$\checkmark$	[JLS23]
$\geq 1$	$1 \le p < \infty$	$p_1,\ldots,p_m\geq 2$	$\checkmark$		[JLLS23]
$\geq 1$	$1 \le p < \infty$	$p_1,\ldots,p_m\geq 2$	$\checkmark$		This work
$\geq 1$	$1/\log d \le p < \infty$	$p_1,\ldots,p_m=p$	$\checkmark$	$\checkmark$	This work
$\geq 1$	$1/\log d \le p < \infty$	$p_1,\ldots,p_m=2$	$\checkmark$	$\checkmark$	This work
$\geq 1$	p = 2	$p_1,\ldots,p_m\geq 2$	$\checkmark$	$\checkmark$	This work

# 3.2. Preliminaries

In this section, we set up and review definitions and existing facts that will play crucial roles in our analyses. In Section 3.2.1, we review material from linear algebra, and in Section 3.2.2, we review material from convex geometry.

### 3.2.1. Linear algebra background

We introduce a few definitions concerning *leverage scores* (Definition 3.2.1).

**Definition 3.2.1.** For a matrix  $\mathbf{A} \in \mathbb{R}^{n \times d}$ , we let  $\tau_i(\mathbf{A}) \coloneqq \mathbf{a}_i^\top (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{a}_i$  denote the leverage score of row  $\mathbf{a}_i$  with respect to the matrix  $\mathbf{A}$ . When  $\mathbf{A}$  is clear from context, we omit it and simply write  $\tau_i$  in place of  $\tau_i(\mathbf{A})$ .

The following are well-known properties of leverage scores.

**Fact 3.2.2.** *For a matrix*  $\mathbf{A} \in \mathbb{R}^{n \times d}$ *, we have:* 

- $\sum_{i=1}^{n} \tau_i(\mathbf{A}) = \operatorname{rank}(\mathbf{A});$
- $\tau_i(\mathbf{A}) = \max_{\mathbf{x} \in \mathbb{R}^d \setminus \{0\}} \frac{|\langle \mathbf{a}_i, \mathbf{x} \rangle|^2}{\|\mathbf{A}\mathbf{x}\|_2^2}$  for all i;
- $0 \le \tau_i(\mathbf{A}) \le 1;$
- For any positive constant *C*, we have  $\tau_i(C\mathbf{A}) = \tau_i(\mathbf{A})$  for all *i*.

We will also need the following fact relating the leverage scores of a matrix **A** to its singular value decomposition.

**Fact 3.2.3.** Let  $\mathbf{A} \in \mathbb{R}^{n \times d}$  and  $\mathbf{U} \Sigma \mathbf{V}^{\top}$  be a singular value decomposition for  $\mathbf{A}$ , where  $\mathbf{U} \in \mathbb{R}^{n \times d}$  and  $\Sigma, \mathbf{V} \in \mathbb{R}^{d \times d}$ . Then,  $\tau_i(\mathbf{A}) = \|\mathbf{u}_i\|_2^2$ .

*Proof of Fact 3.2.3.* To understand why this equality might hold, observe that we can think of **U** as the resulting matrix from applying a statistical whitening transform to **A**. More precisely, recall that

$$\tau_i(\mathbf{A}) = \mathbf{a}_i^{\mathsf{T}} \left( \mathbf{V} \mathbf{\Sigma}^2 \mathbf{V}^{\mathsf{T}} \right)^{-1} \mathbf{a}_i^{\mathsf{T}} = \mathbf{a}_i^{\mathsf{T}} \left( \sum_{j=1}^d \frac{1}{\sigma_i^2} \cdot \mathbf{v}_j \mathbf{v}_j^{\mathsf{T}} \right) \mathbf{a}_i^{\mathsf{T}} = \sum_{j=1}^d \frac{1}{\sigma_j^2} \left\langle \mathbf{a}_i, \mathbf{v}_j \right\rangle^2.$$

We now calculate  $\langle a_i, v_j \rangle$ . Notice that

$$\langle a_i, v_j \rangle = \left\langle v_j, e_i^\top \sum_{j'=1}^d \sigma_{j'} \mathbf{U} e_{j'} v_{j'}^\top \right\rangle = \left\langle v_j, \sum_{j'=1}^d \sigma_{j'} \left\langle e_i, \mathbf{U} e_{j'} \right\rangle v_{j'}^\top \right\rangle = \sigma_j \left\langle e_i, \mathbf{U} e_j \right\rangle.$$

Substituting this back in gives

$$\tau_i(\mathbf{A}) = \sum_{j=1}^d \left\langle \boldsymbol{e}_i, \mathbf{U} \boldsymbol{e}_j \right\rangle^2 = \|\boldsymbol{u}_i\|_2^2.$$

This concludes the proof of Fact 3.2.3.

### 3.2.2. Convex geometry background

In this subsection, we review foundational facts regarding convex geometry we use throughout the remainder of this chapter.

We will need the notions of covering and entropy numbers.

**Definition 3.2.4** (Covering numbers [Rot23, p. 69]). Let  $X, Y \subset \mathbb{R}^d$ . The covering number  $\mathcal{N}(X, Y)$  is the minimum number of translates of Y required to cover X. Formally, we have

$$\mathcal{N}(X,Y) := \min \left\{ N \in \mathbb{N} : \text{ there exists } \mathbf{x}_1, \dots, \mathbf{x}_N \in \mathbb{R}^d \text{ such that } X \subseteq \bigcup_{i=1}^N (\mathbf{x}_i + Y) \right\}$$

**Definition 3.2.5** (Entropy numbers [vHan18, Definition 2.1]). Let  $X, Y \subset \mathbb{R}^d$ . The entropy number  $e_N(X, Y)$  is the minimum radius  $\eta$  such that  $\log \mathcal{N}(X, \eta \cdot Y) \leq 2^N$ .

Sometimes, when writing  $e_N$ , we will write  $e_N(X, \|\cdot\|)$  for some quasi-norm  $\|\cdot\|$ . Here, we take *Y* to be the object formed by the unit ball of  $\|\cdot\|$ .

Finally, we state the dual Sudakov inequality.

**Fact 3.2.6** (Dual Sudakov inequality, due to Pajor and Tomczak-Jaegermann [PT86]). For a symmetric convex body  $K \subset \mathbb{R}^d$ , define

$$\|x\|_{K} \coloneqq \inf \{t > 0 : x/t \in K\}.$$

Let K be a symmetric convex body in  $\mathbb{R}^d$ . We have the below.

$$\log \mathcal{N}(B_2^d, \eta \cdot K) \leq \eta^{-2} \cdot \mathbb{E}_{\boldsymbol{g} \sim \mathcal{N}(0, \mathbf{I}_d)} \left[ \|\boldsymbol{g}\|_K \right]^2.$$
(3.2.1)

#### 3.2.3. Probability background

In this subsection, we review a few facts about subgaussian random variables. These are mostly derived from the presentation of Vershynin [Ver18].

**Definition 3.2.7** ( $\|\cdot\|_{\psi_2}$  and subgaussian random variable [Ver18, Definition 2.5.6]). Let X be a random variable. Define  $\|X\|_{\psi_2} := \inf \{t > 0 : \mathbb{E} [\exp(X^2/t^2)] \le 2\}$ . If  $\|X\|_{\psi_2} < \infty$ , we say X is subgaussian.

**Fact 3.2.8** (Properties of subgaussian random variables [Ver18, Proposition 2.5.2]). *The following properties equivalently characterize a subgaussian random variable X up to constants:* 

- For all  $t \ge 0$ ,  $\Pr[|X| \ge t] \le 2\exp\left(-\frac{t^2}{\|X\|_{\psi_2}^2 K_1^2}\right);$
- For all  $r \geq 1$ ,  $\mathbb{E}\left[|X|^r\right] \leq r^{r/2}$ ;

• For all  $\lambda$  such that  $|\lambda| \leq ||X||_{\psi_2}^{-1}$ , we have  $\mathbb{E}\left[\exp\left(\lambda^2 X^2\right)\right] \leq \exp\left(||X||_{\psi_2}^2 \lambda^2\right)$ .

**Fact 3.2.9** (Maximum of subgaussian random variables [Ver18, Exercise 2.5.10]). Let  $X_1, \ldots, X_N$  be a sequence of (not necessarily independent) subgaussian random variables. Then

$$\mathbb{E}\left[\max_{i\in[N]}|X_i|\right] \lesssim \max_{i\in[N]} \|X_i\|_{\psi_2} \sqrt{\log N}.$$

Fact 3.2.10 (Decentering). We have

$$||X||_{\psi_2} \leq ||X - \mathbb{E}[X]||_{\psi_2} + \mathbb{E}[|X|].$$

*Proof of Fact 3.2.10.* By the triangle inequality, we get

$$\|X\|_{\psi_{2}} \leq \|X - \mathbb{E}[X]\|_{\psi_{2}} + \|\mathbb{E}[X]\|_{\psi_{2}} \leq \|X - \mathbb{E}[X]\|_{\psi_{2}} + \mathbb{E}[|X|],$$

which is exactly the statement of Fact 3.2.10.

**Fact 3.2.11** (Lipschitz functionals of Gaussians are subgaussian [Ver18, Theorem 5.2.2]). If  $g \sim \mathcal{N}(0, \mathbf{I}_d)$  and if  $f : \mathbb{R}^d \to \mathbb{R}$ , then

$$\|f(\boldsymbol{g}) - \mathbb{E}[f(\boldsymbol{g})]\|_{\psi_2} \lesssim \|f\|_{\operatorname{Lip}}.$$

## 3.3. Covering number estimates

In this section, we develop our metric entropy estimates. It will be helpful to keep in mind the context and outline from Section 3.1.3.

#### 3.3.1. Notation and general formula

We begin with some definitions that are necessary for our results.

**Definition 3.3.1** (Block-constant diagonal matrix). We say that a vector  $v \in \mathbb{R}^n$  and corresponding diagonal matrix  $\mathbf{V} \in \mathbb{R}^{n \times n}$  is block-constant or "constant down the blocks" if for every  $j_1, j_2 \in S_i$ , we have  $v_{j_1} = v_{j_2}$ .

In particular, when we define a probability measure  $\lambda \in \mathbb{R}^m$  over [m], we will find it useful to extend it to a block-constant diagonal matrix  $\Lambda \in \mathbb{R}^{n \times n}$ .

**Definition 3.3.2** (Rounding matrix). For a probability measure  $\lambda$  over [m], we say that a positive diagonal matrix  $\mathbf{W} \in \mathbb{R}^{k \times k}$  rounds the measure matrix  $\Lambda \in \mathbb{R}^{n \times n}$  if for all  $\mathbf{x} \in \mathbb{R}^d$  we have  $\|\mathbf{W}^{1/2} \mathbf{\Lambda}^{1/2-1/p} \mathbf{A} \mathbf{x}\|_2 \leq \|\mathbf{\Lambda}^{1/2-1/p} \mathbf{A} \mathbf{x}\|_{G_2}$ . We also denote

$$\widehat{B_2} \coloneqq \left\{ \boldsymbol{x} \in \mathbb{R}^d : \left\| \mathbf{W}^{1/2} \mathbf{\Lambda}^{1/2 - 1/p} \mathbf{A} \boldsymbol{x} \right\|_2 \le 1 \right\}.$$

The  $\widehat{B_2}$  defined in Definition 3.3.2 is the linear transformation of the Euclidean ball that we will "pass through" to get our covering number estimates (recall (3.1.5)).

Next, recall our notion of measure overestimates. This is a generalization of prior definitions of Lewis measure overestimates (see e.g. [JLS22, Definition 2.4], [WY22b, Definition 2.3]) and group leverage score overestimates ([JLS23, Definition 1.1]).

**Definition 3.1.2** (Block Lewis overestimate). Let  $\tau_j(\mathbf{M})$  denote the leverage score of the *j*th row of  $\mathbf{M}$ . Let  $F^* > 0$ . For p > 0 and  $p_i > 0$ , we say the probability measure  $\lambda$  and rounding  $\mathbf{W}$  form an  $F^*$ -block Lewis overestimate if for all  $i \in [m]$ , we have

$$\frac{1}{\lambda_i} \left( \sum_{j \in S_i} \left( \frac{\tau_j(\mathbf{W}^{1/2} \mathbf{\Lambda}^{1/2 - 1/p} \mathbf{A})}{w_j} \right)^{p_i/2} \right)^{2/p_i} \le F^{\star}.$$

For example, observe that when all the  $S_i$  have size 1, then Definition 3.1.2 corresponds to standard definitions of Lewis weight overestimates, and there exist weights such that  $F^* \leq d$ . Furthermore, we will see that there exist a  $\lambda$  and w such that  $F^* = d$  (it will follow from Lemma 3.3.8).

Next, we define the vector  $\alpha$ , whose entries capture a notion of group importance.

**Definition 3.3.3.** Let  $\lambda$  be a probability measure over [m] and let  $\mathbf{W}$  be a rounding matrix for  $\Lambda$  (Definition 3.3.2). If  $p_1, \ldots, p_m \ge 2$ , then let  $\alpha \in \mathbb{R}^m$  be the vector such that for all  $i \in [m]$ , we have

$$\alpha_i^p \coloneqq \lambda_i^{1-p/2} \left( \sum_{j \in S_i} \left( \frac{\tau_j \left( \mathbf{W}^{1/2} \mathbf{\Lambda}^{1/2 - 1/p} \mathbf{A} \right)}{w_j} \right)^{p_i/2} \right)^{p/p_i}$$

Equivalently, if **U** is a matrix whose columns consist of the left singular vectors of  $\mathbf{W}^{1/2} \mathbf{\Lambda}^{1/2-1/p} \mathbf{A}$ , and if we denote by  $\mathbf{u}_j$  the jth row of **U** and let  $\mathbf{f}_j \coloneqq \lambda_i^{-1/2} w_j^{-1/2} \mathbf{u}_j$ , then by Fact 3.2.3, we may also write

$$\alpha_i^p \coloneqq \lambda_i \left( \sum_{j \in S_i} \left\| f_j \right\|_2^{p_i} \right)^{p/p_i}$$

On the other hand, if  $p_1 = \cdots = p_m = p < 2$ , then let  $\widehat{\lambda}$  be a probability measure over [n] and let  $\widehat{\alpha} \in \mathbb{R}^n$  be defined as above accordingly. Finally, let  $\alpha \in \mathbb{R}^m$  be such that

$$\alpha_i^p \coloneqq \left(\sum_{j \in S_i} \widehat{\alpha}_j^p\right)^{1/p}.$$

To help ground Definition 3.3.3, notice that combining Definition 3.1.2 with Definition 3.3.3 gives us, for  $p_1, \ldots, p_m \ge 2$  and  $p_1, \ldots, p_m = p < 2$ , respectively,

$$\alpha_{i}^{p} \leq \lambda_{i}^{1-p/2} \left(\lambda_{i} F^{\star}\right)^{p/2} = \lambda_{i} \left(F^{\star}\right)^{p/2}$$

$$\alpha_{i}^{p} = \sum_{j \in S_{i}} \widehat{\alpha}_{j}^{p} = \sum_{j \in S_{i}} \widehat{\lambda}_{j}^{1-p/2} \tau_{j} \left(\widehat{\mathbf{\Lambda}}^{1/2-1/p} \mathbf{A}\right)^{p/2} \leq \sum_{j \in S_{i}} \widehat{\lambda}_{j} \left(F^{\star}\right)^{p/2} = \left(\sum_{j \in S_{i}} \widehat{\lambda}_{j}\right) \left(F^{\star}\right)^{p/2}, \qquad (3.3.1)$$

and that when  $F^* \leq 2d$  (say), we get  $\|\boldsymbol{\alpha}\|_p^p \leq d^{p/2}$ . Thus, at least when  $p \geq 2$ , we can think of  $\|\boldsymbol{\alpha}\|_p^p$  as giving the sparsity we should expect when we sample with probabilities proportional to the  $\alpha_i$ . Although this does not quite work when p < 2, a minor modification of it will.

**Definition 3.3.4** (Notation for unit balls and norms under change-of-measure). Let  $\lambda$  be a probability measure over [m] and  $\Lambda \in \mathbb{R}^{n \times n}$  be its corresponding block-constant diagonal matrix (Definition 3.3.1). For any r > 0, and  $y \in \mathbb{R}^n$  we define

$$\|\boldsymbol{y}\|_{\boldsymbol{\mathcal{G}}_{r}(\boldsymbol{\lambda})} \coloneqq \left(\sum_{i=1}^{m} \lambda_{i} \left(\sum_{j \in S_{i}} \left|\boldsymbol{y}_{j}\right|^{p_{i}}\right)^{r/p_{i}}\right)^{1/r}.$$

We also define

$$B_r := \left\{ \boldsymbol{x} \in \mathbb{R}^d : \left\| \boldsymbol{\Lambda}^{-1/p} \mathbf{A} \boldsymbol{x} \right\|_{\mathcal{G}_r(\lambda)} \leq 1 \right\}.$$

From Definition 3.3.4, it is easy to verify that  $\|\mathbf{\Lambda}^{-1/p}\mathbf{A}\mathbf{x}\|_{\mathcal{G}_p(\lambda)} = \|\mathbf{A}\mathbf{x}\|_{\mathcal{G}_p}$ . Indeed, we have

$$\left\|\mathbf{\Lambda}^{-1/p}\mathbf{A}\mathbf{x}\right\|_{\mathcal{G}_p(\lambda)}^p = \sum_{i=1}^m \lambda_i \left\|\mathbf{\Lambda}_{S_i}^{-1/p}\mathbf{A}_{S_i}\mathbf{x}\right\|_{p_i}^p = \sum_{i=1}^m \left\|\mathbf{A}_{S_i}\mathbf{x}\right\|_{p_i}^p = \left\|\mathbf{A}\mathbf{x}\right\|_{\mathcal{G}_p}^p.$$

We will also require the crucial property that  $\|\boldsymbol{y}\|_{\mathcal{G}_r(\lambda)}$  is log-convex in 1/r. To see this, note that the vector in  $\mathbb{R}^m$  formed by calculating all the inner norms  $p_1, \ldots, p_m$  is constant regardless of the outer norm, and then we can use the fact that for a fixed measure  $\boldsymbol{\mu}$ , the  $\ell_r^m(\boldsymbol{\mu})$  norms are log-convex in 1/r.

We now have the language to state the main result of this section, Theorem 3.3.5.

**Theorem 3.3.5.** Let  $\lambda$  be a probability measure and **W** be a rounding matrix (Definition 3.3.2) so that  $\lambda$  and **W** form an F\*-block Lewis overestimate (Definition 3.1.2). Suppose at least one of the following holds:

- $p > \frac{1}{\log d}$  and  $|S_1| = \cdots = |S_m| = 1;$
- $p = p_1 = \cdots = p_m$  and p < 2;
- $p \ge 1$  and  $p_1, ..., p_m \ge 2;$
- $p_1 = \cdots = p_m = 2$  and  $1/\log d \le p < \infty$ .

If  $H \ge 1$  is such that the sampling probabilities  $\rho_i$  satisfy  $H\rho_i \ge \alpha_i^p / \|\boldsymbol{\alpha}\|_p^p$  for all  $i \in [m]$ , and if we write  $p^* := \max\{1, \max_i \min\{p_i, \log|S_i|\}\}$ , then (recall Definition 3.1.1 for the definition of  $\|\cdot\|_{\mathcal{G},\rho,\infty,S}$ )

$$\log \mathcal{N}\left(B_p, \eta\left\{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\|_{\mathcal{G}, \rho, \infty, S} \leq 1\right\}\right) \lesssim \eta^{-\min(2, p)} \cdot H^{2/\max(p, 2)} \cdot C(p) p^{\star} F^{\star} \log \max\left\{\widetilde{m}, F^{\star}\right\},$$

where C(p) is a constant that only depends on p.

Although Theorem 3.3.5 is stated abstractly, we will see that there exists a convenient instantiation for all the parameters stated. **Corollary 3.3.6.** *In the same cases as in Theorem 3.3.5, there exists a probability measure*  $\lambda$  *over* [m] *and a rounding* **W** *for which in the same setting as Theorem 3.3.5, we have* 

$$\log \mathcal{N}\left(B_p, \eta\left\{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\|_{\mathcal{G},\rho,\infty,S} \le 1\right\}\right) \lesssim_p \eta^{-\min(2,p)} \cdot \max_{i \in S} \min\left\{p_i, \log|S_i|\right\} d\log \widetilde{m}.$$

### 3.3.2. Block Lewis weights

For the sake of motivation, let us first prove Corollary 3.3.6 given Theorem 3.3.5. We first need Lemma 3.3.7, which is derived from the *block Lewis weights* of Jambulapati, Lee, Liu, and Sidford [JLLS23].

For a nonnegative diagonal matrix **V**, let  $\beta_i(\mathbf{V}) \coloneqq \left(\sum_{j \in S_i} \left( \mathbf{a}_j^\top (\mathbf{A}^\top \mathbf{V} \mathbf{A})^{-1} \mathbf{a}_j \right)^{p_i/2} \right)^{1/p_i}$ . We call the  $\beta_i(\mathbf{V})^p$  the *block Lewis weights*.

**Lemma 3.3.7.** If  $p_i \in [2, \infty]$  and  $p \in [1, \infty)$ , then there exist diagonal  $\mathbf{V}, \mathbf{\Lambda} \in \mathbb{R}^{n \times n}$  such that  $\mathbf{\lambda}$  is a probability measure over [m] and the corresponding  $\mathbf{\Lambda} \in \mathbb{R}^{n \times n}$  is constant on the blocks, then  $\sum_{i=1}^{m} \beta_i(\mathbf{V})^p = d$  and for all  $\mathbf{x} \in \mathbb{R}^d$ ,

$$\left\|\mathbf{V}^{1/2}\mathbf{A}\mathbf{x}\right\|_{2} \le d^{1/2-1/p} \left\|\mathbf{\Lambda}^{1/2-1/p}\mathbf{A}\mathbf{x}\right\|_{\mathcal{G}_{2}} \le d^{\max(0,1/2-1/p)} \left\|\mathbf{A}\mathbf{x}\right\|_{\mathcal{G}_{p}}$$

*Proof of Lemma 3.3.7.* The reader familiar with the work of Jambulapati, Lee, Liu, and Sidford [JLLS23] will notice that Lemma 3.3.7 is a strengthened variant of Lemma 4.2 from that work.

Indeed, consider the context of the proof of Lemma 4.2 from [JLLS23]. There, notice that **W** is initially chosen so that  $\sum_{i=1}^{m} \beta_i(\mathbf{W})^p = d$  and  $\mathbf{U} = (\mathbf{A}^\top \mathbf{W} \mathbf{A})^{-1/2}$ . We choose **V** in the same way. Next, using their choice of  $\boldsymbol{u}$ , we have  $\|\boldsymbol{u}_{S_i}\|_{p_i}^{p-2} = (\beta_i(\mathbf{V})^p)^{1-2/p}$ .

Restating (4.8) from [JLLS23] in our notation, we have for all  $x \in \mathbb{R}^d$  that

$$\left\|\mathbf{V}^{1/2}\mathbf{A}\mathbf{x}\right\|_{2}^{2} \leq \sum_{i=1}^{m} \left\|\boldsymbol{u}_{S_{i}}\right\|_{p_{i}}^{p-2} \left\|\mathbf{A}_{S_{i}}\mathbf{x}\right\|_{p_{i}}^{2}$$

For each  $i \in [m]$  let  $\lambda_i = \frac{\beta_i(\mathbf{V})^p}{d}$ , so that  $\lambda$  is a probability measure. Then

$$\left\| \mathbf{V}^{1/2} \mathbf{A} \mathbf{x} \right\|_{2}^{2} \le d^{1-2/p} \sum_{i=1}^{m} \lambda_{i}^{1-2/p} \left\| \mathbf{A}_{S_{i}} \mathbf{x} \right\|_{p_{i}}^{2}.$$

Let  $\mathbf{\Lambda}$  be a  $n \times n$  diagonal matrix, where for every  $i \in [m]$  and  $j \in S_i$ , we define  $\mathbf{\Lambda}_{jj} = \lambda_i$ . Because  $\lambda_i^{1-2/p} \|\mathbf{A}_{S_i} \mathbf{x}\|_{p_i}^2 = \|(\mathbf{\Lambda}^{1/2-1/p} \mathbf{A})_{S_i} \mathbf{x}\|_{p_i}^2$ , we obtain

$$\left\|\mathbf{V}^{1/2}\mathbf{A}\mathbf{x}\right\|_{2} \le d^{1/2-1/p} \left\|\mathbf{\Lambda}^{1/2-1/p}\mathbf{A}\mathbf{x}\right\|_{\mathcal{G}_{2}}.$$
(3.3.2)

Since *p*-norms taken with respect to a probability measure are increasing in *p* we immediately get for all  $p \ge 2$  that

$$\left\|\mathbf{V}^{1/2}\mathbf{A}\mathbf{x}\right\|_{2} \stackrel{(3.3.2)}{\leq} d^{1/2-1/p} \left\|\mathbf{\Lambda}^{1/2-1/p}\mathbf{A}\mathbf{x}\right\|_{\mathcal{G}_{2}} = d^{1/2-1/p} \left\|\mathbf{\Lambda}^{-1/p}\mathbf{A}\mathbf{x}\right\|_{\mathcal{G}_{2}(\lambda)}$$

$$\leq d^{1/2-1/p} \left\| \mathbf{\Lambda}^{-1/p} \mathbf{A} \mathbf{x} \right\|_{\mathcal{G}_p(\lambda)} = d^{1/2-1/p} \left\| \mathbf{A} \mathbf{x} \right\|_{\mathcal{G}_p}$$

The case where  $p \le 2$  follows from the " $1 \le q \le 2$ " subcase of the proof of Lemma 4.2 from [JLLS23], which yields

$$\left\|\mathbf{V}^{1/2}\mathbf{A}\mathbf{x}\right\|_{2} \stackrel{(3.3.2)}{\leq} d^{1/2-1/p} \left\|\mathbf{\Lambda}^{1/2-1/p}\mathbf{A}\mathbf{x}\right\|_{\mathcal{G}_{2}} \leq \|\mathbf{A}\mathbf{x}\|_{\mathcal{G}_{p}}.$$

We therefore conclude the proof of Lemma 3.3.7.

We use Lemma 3.3.7 to give an instantiation for the parameters in Theorem 3.3.5.

**Lemma 3.3.8.** Let  $\mathbf{V}$ ,  $\mathbf{\Lambda}$  be the matrices from Lemma 3.3.7 and let  $\mathbf{U}$  and  $f_j$  be as defined in Definition 3.3.3. Let  $p \ge 1$  and  $p_i \ge 2$  for all  $i \in [m]$ . If we choose  $\mathbf{W}$  such that

$$\frac{\mathbf{V}^{1/2}}{d^{1/2-1/p}} = \mathbf{W}^{1/2} \mathbf{\Lambda}^{1/2-1/p},$$

then:

- $\left\|\mathbf{W}^{1/2}\mathbf{\Lambda}^{1/2-1/p}\mathbf{A}\mathbf{x}\right\|_{2} \leq \left\|\mathbf{\Lambda}^{1/2-1/p}\mathbf{A}\mathbf{x}\right\|_{\mathcal{G}_{2}};$
- for all i,  $\frac{\alpha_i}{d^{1/2-1/p}} = \beta_i(\mathbf{V});$
- $\|\boldsymbol{\alpha}\|_{p}^{p} = d^{p/2};$
- for all i,  $\left(\sum_{j \in S_i} \left\| f_j \right\|_2^{p_i} \right)^{1/p_i} = \| \alpha \|_p = d^{1/2}$ .
- The rounding matrix **W** and measure  $\lambda$  are an F\*-block Lewis overestimate (Definition 3.1.2) with  $F^* = d$ .

*Proof of Lemma 3.3.8.* The first property follows immediately from Lemma 3.3.7. Using Fact 3.2.3, notice that

$$\boldsymbol{a}_{j}^{\top}(\mathbf{A}^{\top}\mathbf{V}\mathbf{A})^{-1}\boldsymbol{a}_{j} = \frac{\tau_{j}(\mathbf{V}^{1/2}\mathbf{A})}{v_{j}} = \frac{\tau_{j}(\mathbf{W}^{1/2}\mathbf{\Lambda}^{1/2-1/p}\mathbf{A})}{d^{1-2/p}w_{j}\lambda_{i}^{1-2/p}} = \frac{\left\|\boldsymbol{u}_{j}\right\|_{2}^{2}}{d^{1-2/p}w_{j}\lambda_{i}^{1-2/p}} = \frac{\lambda_{i}^{2/p}\left\|\boldsymbol{f}_{j}\right\|_{2}^{2}}{d^{1-2/p}},$$

so after substituting into the formula for  $\beta_i(\mathbf{V})$ ,

$$\begin{split} \beta_i(\mathbf{V}) &= \left( \sum_{j \in S_i} \left( \boldsymbol{a}_j^\top (\mathbf{A}^\top \mathbf{V} \mathbf{A})^{-1} \boldsymbol{a}_j \right)^{p_i/2} \right)^{1/p_i} = \left( \sum_{j \in S_i} \left( \frac{\lambda_i^{2/p} \left\| \boldsymbol{f}_j \right\|_2^2}{d^{1-2/p}} \right)^{p_i/2} \right)^{1/p_i} \\ &= \frac{\lambda_i^{1/p} \left( \sum_{j \in S_i} \left\| \boldsymbol{f}_j \right\|_2^{p_i} \right)^{1/p_i}}{d^{1/2-1/p}} = \frac{\alpha_i}{d^{1/2-1/p}}, \end{split}$$

where the last equality follows from the formula for  $\alpha$  stated in Theorem 3.3.5. This also implies that

$$\|\boldsymbol{\alpha}\|_{p}^{p} = \sum_{i=1}^{m} \alpha_{i}^{p} = \sum_{i=1}^{p} \beta_{i}(\mathbf{V})^{p} d^{p/2-1} = d^{p/2}.$$

Finally, observe that the above calculation shows that  $\lambda_i \propto \alpha_i^p$ , since we have defined  $\lambda_i \propto \beta_i(\mathbf{V})^p$ and we have just seen that  $\beta_i(\mathbf{V})^p \propto \alpha_i^p$ . This means we can write  $\lambda_i = \alpha_i^p / \|\boldsymbol{\alpha}\|_p^p$ . Using this, we have

$$\alpha_i = \lambda_i^{1/p} \left( \sum_{j \in S_i} \left\| f_j \right\|_2^{p_i} \right)^{1/p_i} = \frac{\alpha_i}{\|\boldsymbol{\alpha}\|_p} \cdot \left( \sum_{j \in S_i} \left\| f_j \right\|_2^{p_i} \right)^{1/p_i}$$

After rearranging, we have

$$\left(\sum_{j\in S_i} \left\|f_j\right\|_2^{p_i}\right)^{1/p_i} = \|\boldsymbol{\alpha}\|_p = d^{1/2},$$

and so we may take  $F^* = d$ . This concludes the proof of Lemma 3.3.8.

We now handle the cases that are not covered by the block Lewis weight construction of [JLLS23].

**Lemma 3.3.9.** If  $0 < p_1 = \cdots = p_m = p < 2$  or if  $p_1 = \cdots = p_m = 2$  and  $1/\log d \le p < \infty$ , then there exists a probability measure  $\widehat{\lambda}$  over [k] and corresponding  $\widehat{\alpha} \in \mathbb{R}^n$  such that  $\widehat{\lambda}$  is an F\*-block Lewis overestimate for F\* = n.

*Proof.* For the case where  $0 < p_1 = \cdots = p_m = p < 2$ , we simply use the fact that Lewis's measure tells us that there exists a measure  $\hat{\lambda}$  such that

$$\frac{\tau_j\left(\widehat{\mathbf{\Lambda}}^{1/2-1/p}\mathbf{A}\right)}{\widehat{\lambda}_j} \leq d.$$

In the other case, we will see later that the guarantee of a natural contraction mapping (Algorithm 7 and Lemma 3.5.4) imply that  $\mathbf{W} = \mathbf{I}_n$  and the resulting  $\lambda$  form an *d*-block Lewis overestimate, thereby concluding the proof of Lemma 3.3.9.

Lemma 3.3.8 and Lemma 3.3.9 easily imply Corollary 3.3.6.

*Proof of Corollary* 3.3.6. We combine Theorem 3.3.5 with the instantiations in Lemma 3.3.8 and Lemma 3.3.9, directly yielding Corollary 3.3.6.

In light of Corollary 3.3.6, the goal of the remainder of this section is to prove Theorem 3.3.5.

It will be useful to consider a corresponding change-of-basis that arises from our setting of  $\lambda$ . Let  $\mathbf{U}\Sigma\mathbf{V}^{\top}$  be a singular value decomposition of  $\mathbf{W}^{1/2}\mathbf{\Lambda}^{1/2-1/p}\mathbf{A}$  where  $\mathbf{U} \in \mathbb{R}^{m \times d}$  and  $\Sigma, \mathbf{V} \in \mathbb{R}^{d \times d}$ . Let  $\mathbf{R}$  be the invertible matrix  $\mathbf{V}\Sigma^{-1}$  (we assume without loss of generality that rank ( $\mathbf{A}$ ) = d, and it is easy to extend the results of this section to the case where rank ( $\mathbf{A}$ ) < d). We take  $\mathbf{R}$  as our change-of-basis matrix. Using this, it is easy to see that  $\mathbf{W}^{1/2}\mathbf{\Lambda}^{1/2-1/p}\mathbf{A}\mathbf{R} = \mathbf{U}$  consists of orthonormal columns. Furthermore, we have  $\mathbf{\Lambda}^{-1/p}\mathbf{A}\mathbf{R} = \mathbf{W}^{-1/2}\mathbf{\Lambda}^{-1/2}\mathbf{U}$ .

#### **3.3.3. Covering numbers for** 0 < *p* < 2

The goal of this section is to prove Lemma 3.3.10 under the notion of overestimate given by Definition 3.1.2.

We are now ready to state the main result of this subsection.

**Lemma 3.3.10.** Let  $\lambda$  and w be such that they form an  $F^*$ -block Lewis overestimate. Then,

$$\log \mathcal{N}(B_p, \eta \widehat{B_2}) \lesssim \eta^{-\frac{2p}{2-p}} \cdot C(p) \max_i \min(p_i, \log |S_i|) F^* \log F^*,$$

where C(p) is a constant that only depends on p.

The goal of the rest of this subsection is to prove Lemma 3.3.10. We follow the outline detailed in Section 3.1.3. In short, our plan is the following:

- 1. We first reduce bounding  $\log \mathcal{N}(B_p, \eta \widehat{B_2})$  to bounding  $\log \mathcal{N}(\widehat{B_2}, \delta_h B_r)$  for all  $h \ge 0$  and appropriate choices of r and  $\delta_h$ .
- 2. We then control each term  $\log \mathcal{N}(B_2, \delta_h B_r)$ . To do so, we will apply the dual Sudakov inequality (Fact 3.2.6, (3.2.1)). To actually estimate  $\mathbb{E} \| \boldsymbol{g} \|_{B_r}$  where  $\boldsymbol{g} \sim \mathcal{N}(0, \mathbf{I}_d)$ , we need to prove that every resulting summand of the form  $\| \cdot \|_{p_i}$  is subgaussian with a parameter that only depends on  $p_i$ . To do so, we exploit the fact that these summands are Lipschitz and then apply Fact 3.2.11.
- 3. We finally assemble all the previous pieces together to get the desired handle on  $\log \mathcal{N}(B_p, \eta \widehat{B_2})$ .

**Reduction to bounding**  $\log \mathcal{N}(\widehat{B_2}, \eta B_r)$ 

As stated in Section 3.1.3, we begin with reducing the calculation of  $\log \mathcal{N}(B_p, \eta B_2)$  to calculating  $\log \mathcal{N}(\widehat{B_2}, \eta B_r)$  (for a different  $\eta$ ).

**Lemma 3.3.11.** Let  $\theta$  and r be such that  $r = (2 - \theta)p/(p - \theta)$ . Define

$$\delta_h \coloneqq \left(\frac{8^{h+1}\eta}{2\cdot 8^{2/\theta}}\right)^{\frac{\theta}{2-\theta}} = \eta^{\frac{\theta}{2-\theta}} \cdot 8^{(h+1)\cdot\frac{\theta}{2-\theta}} \cdot \left(2\cdot 8^{2/\theta}\right)^{-\frac{\theta}{2-\theta}}$$

Then, we have

$$\log \mathcal{N}(B_p, \eta \widehat{B_2}) \leq \sum_{h \geq 0} \log \mathcal{N}(\widehat{B_2}, \delta_h B_r).$$

*Proof of Lemma* 3.3.11. For  $h \in \mathbb{N}_{\geq 0}$ , let  $\mathcal{N}_h$  be a maximal subset of  $B_p$  such that for any two distinct elements  $z_1, z_2 \in B_p$ , we have  $\|\mathbf{W}^{1/2} \mathbf{\Lambda}^{-1/p} \mathbf{A}(z_1 - z_2)\|_{2(\lambda)} \geq 8^h \eta$  (where by  $\|\cdot\|_{p(\lambda)}$  we mean the  $\ell_p$  norm taken with respect to the measure given by  $\lambda$ ). This yields  $|\mathcal{N}_h| \geq \mathcal{N}(B_p, 8^h \eta \widehat{B_2})$ .

Next, since for every *h* there are  $z_i \in B_p$  for which  $B_p \subseteq \bigcup_{i=1}^{\mathcal{N}(B_p, 8^{h+1}\eta \widehat{B_2})} \{z_i + 8^{h+1}\eta \widehat{B_2}\}$ , for every *h* there must exist a  $z_h^* \in B_p$  for which

$$\left|\left\{\boldsymbol{z}_{h}^{\star}+8^{h+1}\eta\widehat{B_{2}}\right\}\cap\mathcal{N}_{h}\right|\geq\frac{|\mathcal{N}_{h}|}{\mathcal{N}(B_{p},8^{h+1}\eta\widehat{B_{2}})}\geq\frac{\mathcal{N}(B_{p},8^{h}\eta\widehat{B_{2}})}{\mathcal{N}(B_{p},8^{h+1}\eta\widehat{B_{2}})}.$$

Let

$$\mathcal{L}_h \coloneqq \left\{ \frac{z - z_h^{\star}}{8^{h+1}\eta} : z \in \left\{ z_h^{\star} + 8^{h+1}\eta \widehat{B_2} \right\} \cap \mathcal{N}_h \right\}$$

from which we get by the sub-triangle inequality that

$$\left\|\mathbf{W}^{1/2}\mathbf{\Lambda}^{-1/p}\mathbf{A}\mathbf{z}\right\|_{2(\lambda)} \le 1 \text{ and } \left\|\mathbf{\Lambda}^{-1/p}\mathbf{A}\mathbf{z}\right\|_{\mathcal{G}_p(\lambda)} \le \frac{\max\left\{2^{1/p},2\right\}}{8^{h+1}\eta} \text{ for any } \mathbf{z} \in \mathcal{L}_h$$

and

$$\left\| \mathbf{W}^{1/2} \mathbf{\Lambda}^{-1/p} \mathbf{A} \left( \boldsymbol{z}_1 - \boldsymbol{z}_2 \right) \right\|_{2(\lambda)} \ge \frac{1}{8} \text{ for any distinct } \boldsymbol{z}_1, \boldsymbol{z}_2 \in \mathcal{L}_h$$

We now apply an interpolation estimate. Let  $z_1$  and  $z_2$  be distinct elements from  $\mathcal{L}_h$ , set  $0 < \theta < 2$  and  $r = (2 - \theta)p/(p - \theta)$ , and observe that  $\theta = p(r - 2)/(r - p)$  and

$$\begin{split} \frac{1}{8^2} &\leq \left\| \mathbf{W}^{1/2} \mathbf{\Lambda}^{-1/p} \mathbf{A} \left( z_1 - z_2 \right) \right\|_{2(\lambda)}^2 \\ &\leq \left\| \mathbf{\Lambda}^{-1/p} \mathbf{A} \left( z_1 - z_2 \right) \right\|_{\mathcal{G}_2(\lambda)}^2 \\ &\leq \left\| \mathbf{\Lambda}^{-1/p} \mathbf{A} \left( z_1 - z_2 \right) \right\|_{\mathcal{G}_p(\lambda)}^{\theta} \cdot \left\| \mathbf{\Lambda}^{-1/p} \mathbf{A} \left( z_1 - z_2 \right) \right\|_{\mathcal{G}_r(\lambda)}^{2-\theta} \\ &\leq \left( \frac{\max\left\{ 2^{1/p}, 2 \right\}}{8^{h+1}\eta} \right)^{\theta} \left\| \mathbf{\Lambda}^{-1/p} \mathbf{A} \left( z_1 - z_2 \right) \right\|_{\mathcal{G}_r(\lambda)}^{2-\theta} \end{split}$$

which means that after rearranging we have

$$\left\|\mathbf{\Lambda}^{-1/p}\mathbf{A}\left(\mathbf{z}_{1}-\mathbf{z}_{2}\right)\right\|_{\mathcal{G}_{r}(\lambda)} \geq \left(\left(\frac{8^{h+1}\eta}{\max\left\{2^{1/p},2\right\}}\right)^{\theta}\cdot\frac{1}{8^{2}}\right)^{\frac{1}{2-\theta}} \geq \delta_{h}.$$

The above argument gives

$$\log \mathcal{N}(\widehat{B_2}, \delta_h B_r) \geq \log |\mathcal{L}_h| \geq \log \mathcal{N}\left(B_p, 8^h \eta \widehat{B_2}\right) - \log \mathcal{N}\left(B_p, 8^{h+1} \eta \widehat{B_2}\right).$$

We sum these inequalities over all  $h \ge 0$  (noting that when h is sufficiently large, we have  $\log \mathcal{N}(B_p, 8^{h+1}\eta \widehat{B_2}) = 0$ ), and get

$$\log \mathcal{N}(B_p, \eta \widehat{B_2}) \leq \sum_{h \geq 0} \log \mathcal{N}(\widehat{B_2}, \delta_h B_r).$$

This concludes the proof of Lemma 3.3.11.

**Bounding**  $\log \mathcal{N}(\widehat{B_2}, \eta B_r)$ 

As we saw in Lemma 3.3.11, it will be enough to understand the behavior of  $\log N(\widehat{B}_2, \eta B_r)$ . Since  $\widehat{B}_2$  is a linear transformation of a Euclidean ball, we will be able to apply the dual Sudakov inequality (Fact 3.2.6, (3.2.1)).

To prepare for an application of the dual Sudakov inequality, we bound the Gaussian width of the ball  $\{x \in \mathbb{R}^d : \|\mathbf{\Lambda}^{-1/p}\mathbf{A}\mathbf{R}x\|_{\mathcal{G}_r(\lambda)} \leq 1\}$ . As we will see in a moment, the relevance of this ball arises from the fact that it is the *r*-ball with respect to the  $\lambda$  measure after a suitable linear transformation of the underlying space. In particular, it is under the invertible mapping  $x \mapsto \mathbf{R}x$  that we get  $\mathbf{W}^{1/2}\mathbf{\Lambda}^{1/2-1/p}\mathbf{A}\mathbf{R}x \mapsto \mathbf{W}^{1/2}\mathbf{\Lambda}^{1/2-1/p}\mathbf{A}\mathbf{R}x = \mathbf{U}x$ .

**Lemma 3.3.12.** Let  $g \sim \mathcal{N}(0, \mathbf{I}_d)$ . We have

$$\left\|\left(\sum_{j\in S_i} \left|\left\langle f_j, g\right\rangle\right|^{p_i}\right)^{1/p_i}\right\|_{\psi_2} \lesssim (1+\sqrt{p_i}) \left(\sum_{j\in S_i} \left\|f_j\right\|_2^{p_i}\right)^{1/p_i}.$$

*Proof of Lemma* 3.3.12. Observe the following Lipschitzness bound, i.e., for any *x*, by Cauchy-Schwarz, we have

$$\left(\sum_{j\in S_i} \left|\left\langle \boldsymbol{f}_j, \boldsymbol{x}\right\rangle\right|^{p_i}\right)^{1/p_i} \leq \left(\sum_{j\in S_i} \left\|\boldsymbol{f}_j\right\|_2^{p_i}\right)^{1/p_i} \|\boldsymbol{x}\|_2$$

which means by Fact 3.2.11, we get

$$\left\| \left( \sum_{j \in S_i} \left| \left\langle f_j, g \right\rangle \right|^{p_i} \right)^{1/p_i} - \mathbb{E}_{g \sim \mathcal{N}(0, \mathbf{I}_d)} \left[ \left( \sum_{j \in S_i} \left| \left\langle f_j, g \right\rangle \right|^{p_i} \right)^{1/p_i} \right] \right\|_{\psi_2} \lesssim \left( \sum_{j \in S_i} \left\| f_j \right\|_2^{p_i} \right)^{1/p_i}$$

Now, observe that

$$\mathbb{E}_{\boldsymbol{g} \sim \mathcal{N}(0,\mathbf{I}_d}\left[\left(\sum_{j \in S_i} \left|\left\langle f_j, \boldsymbol{g} \right\rangle\right|^{p_i}\right)^{1/p_i}\right] \leq \left(\sum_{j \in S_i} \left\|f_j\right\|_{2}^{p_i} \mathbb{E}_{\boldsymbol{g} \sim \mathcal{N}(0,\mathbf{I}_d)}\left[\left|\left\langle\frac{f_j}{\left\|f_j\right\|_2}, \boldsymbol{g} \right\rangle\right|^{p_i}\right]\right)^{1/p_i} \asymp p_i^{1/2} \left(\sum_{j \in S_i} \left\|f_j\right\|_2^{p_i}\right)^{1/p_i},$$

and by Fact 3.2.10,

$$\left\| \left( \sum_{j \in S_i} \left| \left\langle f_j, g \right\rangle \right|^{p_i} \right)^{1/p_i} \right\|_{\psi_2} \lesssim (1 + \sqrt{p_i}) \left( \sum_{j \in S_i} \left\| f_j \right\|_2^{p_i} \right)^{1/p_i}$$

completing the proof of Lemma 3.3.12.

Next, we estimate the expected norm of a Gaussian random vector under the norm given by  $\left\{ \boldsymbol{x} \in \mathbb{R}^d : \| \boldsymbol{\Lambda}^{-1/p} \mathbf{A} \mathbf{R} \boldsymbol{x} \|_{\mathcal{G}_r(\boldsymbol{\lambda})} \leq 1 \right\}$ .

**Lemma 3.3.13.** *For*  $r \ge 2$ *, we have* 

$$\mathbb{E}_{\boldsymbol{g} \sim \mathcal{N}(0, \mathbf{I}_d)} \left[ \left\| \mathbf{\Lambda}^{-1/p} \mathbf{A} \mathbf{R} \boldsymbol{g} \right\|_{\mathcal{G}_r(\lambda)} \right] \lesssim \begin{cases} r^{1/2} \left( 1 + \sqrt{\max_i p_i} \right) \left( \sum_{i=1}^m \lambda_i \left( \sum_{j \in S_i} \left\| \boldsymbol{f}_j \right\|_2^{p_i} \right)^{r/p_i} \right)^{1/r} & \text{if } r \le \log m \\ \sqrt{\log m} \cdot \max_i \left( 1 + \sqrt{p_i} \right) \left( \sum_{j \in S_i} \left\| \boldsymbol{f}_j \right\|_2^{p_i} \right)^{1/p_i} & \text{otherwise} \end{cases}$$

*Proof of Lemma 3.3.13.* Let  $u_i$  denote the rows of U. Note that by Fact 3.2.3, we have

$$\left\|\boldsymbol{u}_{j}\right\|_{2}^{2} = \tau_{j}(\mathbf{W}^{1/2}\boldsymbol{\Lambda}^{1/2-1/p}\mathbf{A}).$$

Now, observe that  $\mathbf{\Lambda}^{-1/p} \mathbf{A} \mathbf{R} = \mathbf{W}^{-1/2} \mathbf{\Lambda}^{-1/2} \mathbf{U}$ . By Lemma 3.3.12, we know that

$$\mathbb{E}_{\substack{\boldsymbol{g} \sim \mathcal{N}(0, \mathbf{I}_d)}} \left[ \left( \sum_{j \in S_i} \left| \left\langle \boldsymbol{f}_j, \boldsymbol{g} \right\rangle \right|^{p_i} \right)^{r/p_i} \right] \lesssim r^{r/2} \left( 1 + \sqrt{p_i} \right)^r \left( \sum_{j \in S_i} \left\| \boldsymbol{f}_j \right\|_2^{p_i} \right)^{r/p_i}$$

We first handle the case where  $r \leq \log m$ . Notice that

$$\begin{split} \mathbb{E}_{\boldsymbol{g} \sim \mathcal{N}(0,\mathbf{I}_d)} \left[ \left\| \mathbf{\Lambda}^{-1/p} \mathbf{A} \mathbf{R} \boldsymbol{g} \right\|_{\mathcal{G}_r(\lambda)} \right] &= \mathbb{E}_{\boldsymbol{g} \sim \mathcal{N}(0,\mathbf{I}_d)} \left[ \left( \sum_{i=1}^m \lambda_i \left( \sum_{j \in S_i} \left| \left\langle w_j^{-1/2} \lambda_i^{-1/2} \boldsymbol{u}_j, \boldsymbol{g} \right\rangle \right|^{p_i} \right)^{r/p_i} \right] \right]^{1/r} \right] \\ &\leq \left( \sum_{i=1}^m \lambda_i \mathbb{E}_{\boldsymbol{g} \sim \mathcal{N}(0,\mathbf{I}_d)} \left[ \left( \sum_{j \in S_i} \left| \left\langle w_j^{-1/2} \lambda_i^{-1/2} \boldsymbol{u}_j, \boldsymbol{g} \right\rangle \right|^{p_i} \right)^{r/p_i} \right] \right]^{1/r} \\ &\lesssim \left( \sum_{i=1}^m \lambda_i \left( r^{r/2} \left( 1 + \sqrt{p_i} \right)^r \left( \sum_{j \in S_i} \left\| f_j \right\|_2^{p_i} \right)^{r/p_i} \right) \right)^{1/r} \\ &\leq r^{1/2} \left( 1 + \sqrt{\max_i p_i} \right) \left( \sum_{i=1}^m \lambda_i \left( \sum_{j \in S_i} \left\| f_j \right\|_2^{p_i} \right)^{r/p_i} \right)^{1/r} . \end{split}$$

We now handle the case where  $r \gtrsim \log m$ . We have

$$\begin{split} \mathbb{E}_{\boldsymbol{g} \sim \mathcal{N}(0, \mathbf{I}_d)} \left[ \left\| \mathbf{\Lambda}^{-1/p} \mathbf{A} \mathbf{R} \boldsymbol{g} \right\|_{\mathcal{G}_r(\boldsymbol{\lambda})} \right] &= \mathbb{E}_{\boldsymbol{g} \sim \mathcal{N}(0, \mathbf{I}_d))} \left[ \left( \sum_{i=1}^m \lambda_i \left( \sum_{j \in S_i} \left| \left\langle w_j^{-1/2} \lambda_i^{-1/2} \boldsymbol{u}_j, \boldsymbol{g} \right\rangle \right|^{p_i} \right)^{1/p_i} \right] \\ &\lesssim \mathbb{E}_{\boldsymbol{g} \sim \mathcal{N}(0, \mathbf{I}_d))} \left[ \max_i \left( \sum_{j \in S_i} \left| \left\langle w_j^{-1/2} \lambda_i^{-1/2} \boldsymbol{u}_j, \boldsymbol{g} \right\rangle \right|^{p_i} \right)^{1/p_i} \right] \\ &\lesssim \sqrt{\log m} \cdot \max_i \left( 1 + \sqrt{p_i} \right) \left( \sum_{j \in S_i} \left\| \boldsymbol{f}_j \right\|_2^{p_i} \right)^{1/p_i} \end{split}$$

and conclude the proof of Lemma 3.3.13 (the last line follows from Fact 3.2.9).

Now, we show how to relate  $(\sum_{j \in S_i} \|f_j\|_2^{p_i})^{1/p_i}$  to  $F^*$ .

**Lemma 3.3.14.** For all  $i \in [m]$ , we have

$$\left(\sum_{j\in S_i} \left\|f_j\right\|_2^{p_i}\right)^{2/p_i} \leq F^\star.$$

Proof of Lemma 3.3.14. Recall Fact 3.2.3; this gives us

$$\left(\sum_{j\in S_i} \left\|f_j\right\|_2^{p_i}\right)^{2/p_i} = \left(\sum_{j\in S_i} \left(\frac{\tau_j\left(\mathbf{W}^{1/2}\mathbf{\Lambda}^{1/2-1/p}\mathbf{A}\right)}{w_j\lambda_i}\right)^{p_i/2}\right)^{2/p_i} = \frac{1}{\lambda_i} \left(\sum_{j\in S_i} \left(\frac{\tau_j\left(\mathbf{W}^{1/2}\mathbf{\Lambda}^{1/2-1/p}\mathbf{A}\right)}{w_j}\right)^{p_i/2}\right)^{2/p_i}.$$

We recall that  $F^*$  satisfies Definition 3.1.2 and conclude the proof of Lemma 3.3.14.

We now have enough tools to build a naïve estimate of  $\log N(\widehat{B}_2, \eta B_r)$  via directly applying the dual Sudakov inequality.

Lemma 3.3.15. We have

$$\log \mathcal{N}\left(\widehat{B_{2}}, \eta B_{r}\right) \lesssim \eta^{-2} \cdot \begin{cases} r \left(1 + \sqrt{\max_{i} p_{i}}\right)^{2} \left(\sum_{i=1}^{m} \lambda_{i} \left(\sum_{j \in S_{i}} \left\|f_{j}\right\|_{2}^{p_{i}}\right)^{r/p_{i}}\right)^{2/r} & \text{if } r \leq \log m \\ \log m \cdot \max_{i} \left(1 + \sqrt{p_{i}}\right)^{2} \left(\sum_{j \in S_{i}} \left\|f_{j}\right\|_{2}^{p_{i}}\right)^{2/p_{i}} & \text{otherwise} \end{cases}$$

Simply put, we may also write

$$\log \mathcal{N}\left(\widehat{B_2}, \eta B_r\right) \lesssim \eta^{-2} \cdot r \max_i \min\left(p_i, \log|S_i|\right) F^{\star}$$

Proof of Lemma 3.3.15. Since R is invertible, it will be enough to bound the covering number

$$\mathcal{N} \coloneqq \mathcal{N} \left\{ \left\{ \boldsymbol{x} \in \mathbb{R}^d : \| \mathbf{U} \boldsymbol{x} \|_2 \le 1 \right\}, \eta \left\{ \boldsymbol{x} \in \mathbb{R}^d : \| \boldsymbol{\Lambda}^{-1/p} \mathbf{A} \mathbf{R} \boldsymbol{x} \|_{\mathcal{G}_r(\boldsymbol{\lambda})} \le 1 \right\} \right\}.$$

Because  $||\mathbf{U}\mathbf{x}||_2 = ||\mathbf{x}||_2$ , we can apply the dual Sudakov Inequality (Fact 3.2.6, (3.2.1)). This means we get

$$\log \mathcal{N} \lesssim \eta^{-2} \left( \mathbb{E}_{g \sim \mathcal{N}(0, \mathbf{I}_d)} \left[ \left\| \mathbf{\Lambda}^{-1/p} \mathbf{A} \mathbf{R} g \right\|_{\mathcal{G}_r(\boldsymbol{\lambda})} \right] \right)^2$$

We plug in the result from Lemma 3.3.13 and conclude the proof of Lemma 3.3.15. The statement after the "simply put" follows from Lemma 3.3.14.  $\hfill \Box$ 

Although the calculation in Lemma 3.3.15 works pretty well for small r, this degrades quite rapidly once r is large (say, larger than  $\log d$ ).

To resolve this, we build another estimate for  $\log N(\widehat{B_2}, \eta B_r)$  that performs better when *r* is larger than  $\log d$  or so. We will be able to do this after an interpolation step and a simple geometric observation relating  $\widehat{B_2}$  and  $B_r$ .

**Lemma 3.3.16.** Let  $\Delta_i$  be defined such that

$$\Delta_i^{1/2} \coloneqq \max_{\boldsymbol{x} \in \mathbb{R}^d \setminus \{0\}} \frac{\lambda_i^{-1/p} \|\mathbf{A}_{S_i} \boldsymbol{x}\|_{p_i}}{\|\mathbf{W}^{1/2} \mathbf{\Lambda}^{1/2 - 1/p} \mathbf{A} \boldsymbol{x}\|_2}$$

and let  $\Delta \coloneqq \max_{i \in [m]} \Delta_i$ .

For all  $x \in \mathbb{R}^d$  and r > 2, if  $|S_i| = 1$  for all *i*, then we have

$$\left\|\mathbf{\Lambda}^{-1/2}\mathbf{U}\mathbf{x}\right\|_{r(\boldsymbol{\lambda})} \leq \Delta^{1/2-1/r} \cdot \|\mathbf{U}\mathbf{x}\|_{2} \leq \Delta^{1/2-1/r} \cdot \left\|\mathbf{\Lambda}^{-1/2}\mathbf{U}\mathbf{x}\right\|_{r(\boldsymbol{\lambda})}.$$

Moreover, if there exists at least one  $S_i$  for which  $|S_i| > 1$ , then we have

$$\left\|\mathbf{\Lambda}^{-1/p}\mathbf{A}\mathbf{x}\right\|_{\mathcal{G}_{r}(\lambda)} \leq \Delta^{1/2} \left\|\mathbf{W}^{1/2}\mathbf{\Lambda}^{1/2-1/p}\mathbf{A}\mathbf{x}\right\|_{2}.$$

*Proof of Lemma 3.3.16.* For the sake of intuition and an interpretation, the reader may think  $\Delta \approx d$ .

Note that for the case where all the  $S_i$  are singletons, we may assume  $\mathbf{W} = \mathbf{I}_m$ .

Since  $\lambda$  is a probability measure, we have for any  $r \ge 2$  and for all  $x \in \mathbb{R}^d$  that

$$\|\mathbf{U}\boldsymbol{x}\|_{2} = \left\|\boldsymbol{\Lambda}^{-1/2}\mathbf{U}\boldsymbol{x}\right\|_{2(\boldsymbol{\lambda})} \leq \left\|\boldsymbol{\Lambda}^{-1/2}\mathbf{U}\boldsymbol{x}\right\|_{r(\boldsymbol{\lambda})}.$$

We now prove the lower bound. We have

$$\begin{split} \left\| \mathbf{\Lambda}^{-1/2} \mathbf{U} \mathbf{x} \right\|_{r(\lambda)} &= \left( \sum_{i=1}^{m} \lambda_i \left| \left\langle \lambda_i^{-1/2} \mathbf{u}_i, \mathbf{x} \right\rangle \right|^r \right)^{1/r} = \left( \sum_{i=1}^{m} \lambda_i \left| \left\langle \lambda_i^{-1/2} \mathbf{u}_i, \mathbf{x} \right\rangle \right|^{2} \cdot \left| \left\langle \lambda_i^{-1/2} \mathbf{u}_i, \mathbf{x} \right\rangle \right|^{r-2} \right)^{1/r} \\ &\leq \left( \| \mathbf{U} \mathbf{x} \|_2^2 \cdot \max_i \left| \left\langle \lambda_i^{-1/2} \mathbf{u}_i, \mathbf{x} \right\rangle \right|^{r-2} \right)^{1/r} \leq \left( \| \mathbf{U} \mathbf{x} \|_2^2 \cdot \left( \Delta^{1/2} \| \mathbf{x} \|_2 \right)^{r-2} \right)^{1/r} \\ &= \Delta^{1/2 - 1/r} \| \mathbf{x} \|_2 \,. \end{split}$$

We now move onto the more general case where the  $S_i$  are allowed to have multiple elements. We write

$$\begin{split} \left\| \mathbf{\Lambda}^{-1/p} \mathbf{A} \mathbf{x} \right\|_{\mathcal{G}_{r}(\lambda)} &= \left( \sum_{i=1}^{m} \lambda_{i} \left\| \mathbf{\Lambda}_{S_{i}}^{-1/p} \mathbf{A}_{S_{i}} \mathbf{x} \right\|_{p_{i}}^{r} \right)^{1/r} \\ &\leq \left( \sum_{i=1}^{m} \lambda_{i} \Delta_{i}^{r/2} \left\| \mathbf{W}^{1/2} \mathbf{\Lambda}^{1/2-1/p} \mathbf{A} \mathbf{x} \right\|_{2}^{r} \right)^{1/r} \leq \Delta^{1/2} \left\| \mathbf{W}^{1/2} \mathbf{\Lambda}^{1/2-1/p} \mathbf{A} \mathbf{x} \right\|_{2}^{r}, \end{split}$$

which concludes the proof of Lemma 3.3.16.

At last, we have the tools we need to give a characterization of  $\log \mathcal{N}(\widehat{B_2}, \eta B_r)$  when  $r \ge \log d$ .

**Lemma 3.3.17.** If all the  $S_i$  have size 1, then

$$\log \mathcal{N}\left(\widehat{B_2}, \eta B_r\right) \lesssim \left(\frac{\eta}{2}\right)^{-\frac{2r}{r-2}} \cdot rF^* \log F^*,$$

and if there is at least one  $S_i$  larger than 1, then

$$\log \mathcal{N}\left(\widehat{B_2}, \eta B_r\right) \lesssim \left(\frac{\eta}{2}\right)^{-\frac{2r}{r-2}} \cdot \frac{2r^2}{r-2} \max_i \min(p_i + 1, \log|S_i|) F^* \log F^*.$$

*Proof of Lemma 3.3.17.* The reader familiar with the work of Bourgain, Lindenstrauss, and Milman [BLM89] can think of the present Lemma as a generalization of (7.13) of Proposition 7.2 of that work.

Define

$$M(\mathbf{F}, r) \coloneqq \begin{cases} r \left(1 + \sqrt{\max_i p_i}\right)^2 \left(\sum_{i=1}^m \lambda_i \left(\sum_{j \in S_i} \left\| f_j \right\|_2^{p_i} \right)^{r/p_i} \right)^{2/r} & \text{if } r \le \log m \\ \log m \cdot \max_i \left(1 + \sqrt{p_i}\right)^2 \left(\sum_{j \in S_i} \left\| f_j \right\|_2^{p_i} \right)^{2/p_i} & \text{otherwise} \end{cases}$$

Let q > r and  $0 < \theta < 1$  be such that

$$\frac{1}{r} = \frac{1-\theta}{2} + \frac{\theta}{q}.$$

By interpolation, observe that we have

$$\begin{split} \left\| \mathbf{\Lambda}^{-1/p} \mathbf{A}(\mathbf{x}_1 - \mathbf{x}_2) \right\|_{\mathcal{G}_r(\lambda)} &\leq \left\| \mathbf{\Lambda}^{-1/p} \mathbf{A}(\mathbf{x}_1 - \mathbf{x}_2) \right\|_{\mathcal{G}_2(\lambda)}^{1-\theta} \cdot \left\| \mathbf{\Lambda}^{-1/p} \mathbf{A}(\mathbf{x}_1 - \mathbf{x}_2) \right\|_{\mathcal{G}_q(\lambda)}^{\theta} \\ &\leq 2 \left\| \mathbf{\Lambda}^{-1/p} \mathbf{A}(\mathbf{x}_1 - \mathbf{x}_2) \right\|_{\mathcal{G}_q(\lambda)}^{\theta} \end{split}$$

which means that

$$\log \mathcal{N}(B_2, \eta B_r) \leq \log \mathcal{N}\left(B_2, (\eta/2)^{1/\theta} B_q\right) \leq \left(\frac{\eta}{2}\right)^{-2/\theta} \mathcal{M}(\mathbf{F}, q).$$

Let us set  $q = r \log D$ , where we will choose *D* in a moment. Then, notice that

$$\left(\frac{\eta}{2}\right)^{-2/\theta} = \left(\frac{\eta}{2}\right)^{-2r(q-2)/(q(r-2))} = \left(\frac{\eta}{2}\right)^{-\frac{2r}{r-2}\left(1-\frac{2}{r\log D}\right)} = \left(\frac{\eta}{2}\right)^{-\frac{2r}{r-2}+\frac{4}{(r-2)\log D}}$$

It is now sufficient to identify *D* such that whenever  $\eta$  is small enough to have  $\log N > 0$ , we have

$$\left(\frac{\eta}{2}\right)^{\frac{4}{(r-2)\log D}} \lesssim 1.$$

To identify this *D*, notice that Lemma 3.3.16 implies that if all the  $S_i$  have size 1, then only values of  $\eta$  such that  $\eta \leq \Delta^{1/2-1/r}$  contribute to  $\log N$ . In the more general setting, observe only  $\eta \leq \Delta^{1/2}$  counts.

Hence, if all the  $S_i$ s are singletons, we choose  $D = \Delta$ . For any  $\eta \le 2\Delta^{1/2-1/r} = 2D^{1/2-1/r}$ , we see that

$$\left(\frac{\eta}{2}\right)^{\frac{4}{(r-2)\log D}} \le \Delta^{\frac{r-2}{2r} \cdot \frac{4}{(r-2)\log(\Delta)}} = \Delta^{\frac{2}{r\log\Delta}} = 2^{2/r} \le 2.$$
Similarly, for the case where the  $S_i$  are more generally sized, we choose  $D = \Delta^{2r/(r-2)}$ . Now, for any  $\eta \leq 2\Delta^{1/2}$ , we get

$$\left(\frac{\eta}{2}\right)^{\frac{4}{(r-2)\log D}} \leq \Delta^{\frac{4}{(r-2)\log\left(\Delta^{(2r/(r-2))}\right)}} = \Delta^{\frac{2}{r\log\Delta}} = 2^{2/r} \leq 2.$$

Putting everything together, if all the  $S_i$ s are singletons, we get

$$\log \mathcal{N}\left(\widehat{B_2}, \eta B_r\right) \lesssim \left(\frac{\eta}{2}\right)^{-\frac{2r}{r-2}} \cdot M(\mathbf{F}, q) \lesssim \left(\frac{\eta}{2}\right)^{-\frac{2r}{r-2}} \cdot rF^{\star} \log F^{\star},$$

and in the more general case,

$$\log \mathcal{N}\left(\widehat{B_2}, \eta B_r\right) \lesssim \left(\frac{\eta}{2}\right)^{-\frac{2r}{r-2}} \cdot M(\mathbf{F}, q) \lesssim \left(\frac{\eta}{2}\right)^{-\frac{2r}{r-2}} \cdot \frac{2r^2}{r-2} \max_i \min(p_i + 1, \log|S_i|) F^{\star} \log F^{\star}.$$

This concludes the proof of Lemma 3.3.17.

# Putting everything together

We are finally ready to combine all the tools we have built in the last few subsections to prove our entropy estimate when 0 .

Below, we state Lemma 3.3.18, which more precisely characterizes the behavior of the dependence on p referred to by Lemma 3.3.10.

Lemma 3.3.18. We have

$$\log \mathcal{N}(B_p, \eta \widehat{B_2}) \lesssim \eta^{-\frac{2p}{2-p}} \cdot C(p) \max_i \min \left( p_i + 1, \log |S_i| \right) F^\star \log F^\star,$$

where C(p) is a constant that only depends on p. The constant C(p) is defined as follows. For  $0 < \theta < p$ , let  $r = (2 - \theta)p/(p - \theta)$ . Then, we define

$$\widehat{C}(p,\theta) \coloneqq \left(\frac{\left(2 \cdot 8^{2/\theta}\right)^{-\frac{\theta}{2-\theta}}}{2}\right)^{-\frac{2r}{r-2}}$$

$$\widehat{\widetilde{C}}(p) \coloneqq \left\{\min \left\{\widehat{C}(p,p/2), \widehat{C}(p,1)\right\} \quad if \ 1 \le p < 2 \\
\widehat{C}(p,p/2) \quad if \ 0 < p < 1$$
(3.3.3)

$$C(p) \coloneqq \widehat{\widehat{C}}(p) \begin{cases} r & \text{if } |S_i| = 1 \text{ for all } i \\ 2r + \frac{4r}{r-2} & \text{otherwise} \end{cases},$$
(3.3.4)

where, in an abuse of notation, r in (3.3.4) is chosen according to the value of  $\theta$  that is selected by  $\widehat{C}(p)$  in (3.3.3).

Proof of Lemma 3.3.18 and Lemma 3.3.10. This time, following Lemma 3.3.17, we define

$$M(\mathbf{F}, r) \coloneqq \begin{cases} rF^{\star} \log F^{\star} & \text{if } |S_i| = 1\\ (2r + \frac{4r}{r-2}) \left( \max_i \min(p_i + 1, \log |S_i|) \right) F^{\star} \log F^{\star} & \text{otherwise} \end{cases}.$$

Use Lemma 3.3.11 and Lemma 3.3.15 to write

$$\log \mathcal{N}(B_{p}, \eta \widehat{B_{2}}) \leq \sum_{h \geq 0} \log \mathcal{N}(\widehat{B_{2}}, \delta_{h}B_{r}) \leq M(\mathbf{F}, r) \sum_{h \geq 0} \left(\frac{\delta_{h}}{2}\right)^{-\frac{2r}{r-2}}$$

$$= M(\mathbf{F}, r) \sum_{h \geq 0} \left(\eta^{\frac{\theta}{2-\theta}} \cdot 8^{(h+1) \cdot \left(\frac{\theta}{2-\theta}\right)} \cdot \frac{\left(2 \cdot 8^{2/\theta}\right)^{-\frac{2r}{2-\theta}}}{2}\right)^{-\frac{2r}{r-2}}$$

$$= \eta^{-\frac{\theta}{2-\theta} \cdot \frac{2r}{r-2}} \cdot \left(\underbrace{\left(\frac{2 \cdot 8^{2/\theta}}{2}\right)^{-\frac{\theta}{2-\theta}}}_{=\widehat{C}(p,\theta)}\right)^{-\frac{2r}{r-2}} M(\mathbf{F}, r) \sum_{h \geq 0} 8^{(h+1) \cdot \left(-\frac{\theta}{2-\theta}\right) \cdot \frac{2r}{r-2}}.$$
(3.3.5)

0...

We now make the substitution  $\theta = p/2$ . By the formula in Lemma 3.3.11, this means that r = 4 - p and  $-\theta/(2 - \theta) \cdot 2r/(r - 2) = -2p/(2 - p)$ . We continue.

$$\begin{split} \log \mathcal{N}(B_p, \eta B_2) &\leq \eta^{-\frac{\theta}{2-\theta} \cdot \frac{2r}{r-2}} \cdot \widehat{C}(p, \theta) M(\mathbf{F}, r) \sum_{h \geq 0} 8^{(h+1) \cdot -\frac{\theta}{2-\theta} \cdot \frac{2r}{r-2}} \\ &= \eta^{-\frac{2p}{2-p}} \cdot \widehat{C}(p, p/2) M(\mathbf{F}, 4-p) \sum_{h \geq 0} 8^{(h+1) \cdot -\frac{2p}{2-p}} \\ &\leq \eta^{-\frac{2p}{2-p}} \cdot \widehat{C}(p, p/2) M(\mathbf{F}, 4-p). \end{split}$$

A regrettable consequence of the above calculation is that the "constant"  $C(p, \theta) = C(p, p/2)$  explodes as  $p \to 2$ , as observed by Schechtman and Zvavitch [SZ01]. To fix this, we perform a slightly different variant of this calculation in the regime where  $1 . We resume from (3.3.5) except we use <math>\theta = 1$ . Here, again using Lemma 3.3.11, we check that r = p/(p - 1) and  $-\theta/(2 - \theta) \cdot 2r/(r - 2) = -2p/(2 - p)$  (note that now r is the conjugate exponent of p). This means that

$$\begin{split} \log \mathcal{N}(B_p, \eta \widehat{B_2}) &\leq \eta^{-\frac{\theta}{2-\theta} \cdot \frac{2r}{r-2}} \cdot \widehat{C}(p, \theta) M(\mathbf{F}, r) \sum_{h \geq 0} 8^{(h+1) \cdot -\frac{\theta}{2-\theta} \cdot \frac{2r}{r-2}} \\ &\lesssim \eta^{-\frac{2p}{2-p}} \cdot \widehat{C}(p, 1) \sum_{h \geq 0} 8^{(h+1) \cdot -\frac{2p}{2-p}} \lesssim \eta^{-\frac{2p}{2-p}} M\left(\mathbf{F}, \frac{p}{p-1}\right) \end{split}$$

Taking the minimum over all the cases (of course where applicable) and expanding out the definition of  $M(\mathbf{F}, r)$  concludes the proof of Lemma 3.3.18 and Lemma 3.3.10.

# **3.3.4.** Covering numbers for $p \ge 2$

We will see that compared to the previous section, our task when  $p \ge 2$  is far easier. The main technical lemma we need is Lemma 3.3.19, which we need for both regimes of p.

We first state and prove Lemma 3.3.19.

**Lemma 3.3.19.** Let **W** and  $\Lambda$  be chosen according to Theorem 3.3.5. Suppose that  $H \ge 1$  is such that  $H\rho_i \ge \alpha_i^p / \|\alpha\|_p^p$  for all  $i \in [m]$ . Then,

$$\log \mathcal{N}\left(\widehat{B_2}, \eta \left\{ \boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\|_{\mathcal{G}, \rho, \infty, S} \le 1 \right\} \right) \lesssim \eta^{-2} \cdot \max_{i \in S} \min \left\{ p_i, \log |S_i| \right\} H^{2/p} \|\boldsymbol{\alpha}\|_p^2 \log \widetilde{m}.$$

*Proof of Lemma 3.3.19.* Following the proof of Lemma 3.3.15 and the references therein, our goal here is to analyze the quantity

$$\log \mathcal{N}\left(\left\{\boldsymbol{x} \in \mathbb{R}^{d} : \left\|\boldsymbol{W}^{1/2}\boldsymbol{\Lambda}^{1/2-1/p}\boldsymbol{A}\boldsymbol{x}\right\|_{2} \leq 1\right\}, \eta\left\{\boldsymbol{x} \in \mathbb{R}^{d} : \left\|\boldsymbol{x}\right\|_{\mathcal{G},\rho,\infty,S} \leq 1\right\}\right)$$

Using the same type of linear transformation argument as in Lemma 3.3.15 (so, replacing every x above with  $\mathbf{R}x$ ), we find that it is in fact sufficient to analyze

$$\log \mathcal{N} \coloneqq \log \mathcal{N} \left\{ \left\{ \boldsymbol{x} : \| \mathbf{U} \boldsymbol{x} \|_{2} \leq 1 \right\}, \eta \left\{ \boldsymbol{x} : \max_{i \in S} \rho_{i}^{-1/p} \left\| \mathbf{W}_{S_{i}}^{-1/2} \mathbf{\Lambda}_{S_{i}}^{1/p-1/2} \mathbf{U} \boldsymbol{x} \right\|_{p_{i}} \leq 1 \right\} \right\}.$$

Recall that  $||\mathbf{U}\mathbf{x}||_2 = ||\mathbf{x}||_2$ , so a natural plan is to apply the dual Sudakov inequality (Fact 3.2.6, (3.2.1)). We first consider the quantity (when  $2 \le p_i \le \log |S_i|$ )

$$\begin{split} \mathbb{E}_{g \sim \mathcal{N}(0, \mathbf{I}_{d})} \left[ \left\| \mathbf{W}_{S_{i}}^{-1/2} \mathbf{\Lambda}_{S_{i}}^{1/p-1/2} \mathbf{U}g \right\|_{p_{i}} \right] &= \mathbb{E}_{g \sim \mathcal{N}(0, \mathbf{I}_{d})} \left[ \left( \sum_{j \in S_{i}} \left| \lambda_{i}^{1/p} \left\langle w_{j}^{-1/2} \lambda_{i}^{-1/2} u_{j} \right\rangle_{p_{i}} \right)^{p_{i}} \right)^{1/p_{i}} \right] \\ &\leq \left( \sum_{j \in S_{i}} \left( \lambda_{i}^{1/p} \left\| w_{j}^{-1/2} \lambda_{i}^{-1/2} u_{j} \right\|_{2} \right)^{p_{i}} \mathbb{E}_{g \sim \mathcal{N}(0, 1)} \left[ |g|^{p_{i}} \right] \right)^{1/p_{i}} \\ &\leq p_{i}^{1/2} \left( \sum_{j \in S_{i}} \left( \lambda_{i}^{1/p} \left\| w_{j}^{-1/2} \lambda_{i}^{-1/2} u_{j} \right\|_{2} \right)^{p_{i}} \right)^{1/p_{i}} \\ &= p_{i}^{1/2} \left( \sum_{j \in S_{i}} \left( \lambda_{i}^{1/p} \left\| f_{j} \right\|_{2} \right)^{p_{i}} \right)^{1/p_{i}} = p_{i}^{1/2} \lambda_{i}^{1/p} \left( \sum_{j \in S_{i}} \left\| f_{j} \right\|_{2}^{p_{i}} \right)^{1/p_{i}} = p_{i}^{1/2} \alpha_{i} \end{split}$$

On the other hand, if  $p_i \ge \log |S_i|$ , then we get

$$\begin{split} \mathbb{E}_{\boldsymbol{g} \sim \mathcal{N}(0, \mathbf{I}_d)} \left[ \left\| \mathbf{W}_{S_i}^{-1/2} \mathbf{\Lambda}_{S_i}^{1/p - 1/2} \mathbf{U} \boldsymbol{g} \right\|_{p_i} \right] &\lesssim \mathbb{E}_{\boldsymbol{g} \sim \mathcal{N}(0, \mathbf{I}_d)} \left[ \left\| \mathbf{W}_{S_i}^{-1/2} \mathbf{\Lambda}_{S_i}^{1/p - 1/2} \mathbf{U} \boldsymbol{g} \right\|_{\infty} \right] \\ &\lesssim \lambda_i^{1/p} \max_{j \in S_i} \left\| \boldsymbol{f}_j \right\|_2 \sqrt{\log |S_i|} \asymp \alpha_i \sqrt{\log |S_i|}. \end{split}$$

Finally, when  $p_1 = \cdots = p_m = p < 2$ , we have

$$\begin{split} \mathbb{E}_{\boldsymbol{g} \sim \mathcal{N}(0, \mathbf{I}_d)} \left[ \left\| \widehat{\boldsymbol{\Lambda}}_{S_i}^{1/p - 1/2} \mathbf{U} \boldsymbol{g} \right\|_p \right] &= \mathbb{E}_{\boldsymbol{g} \sim \mathcal{N}(0, \mathbf{I}_d)} \left[ \left( \sum_{j \in S_i} \widehat{\lambda}_j \left| \left\langle \widehat{\lambda}_j^{-1/2} \boldsymbol{u}_j, \boldsymbol{g} \right\rangle \right|^p \right)^{1/p} \right] \\ &\leq \left( \sum_{j \in S_i} \widehat{\lambda}_j \left\| \widehat{\lambda}_j^{-1/2} \boldsymbol{u}_j \right\|_{2 \ \boldsymbol{g} \sim \mathcal{N}(0, 1)}^p \left[ |\boldsymbol{g}|^p \right] \right)^{1/p} \lesssim p^{1/2} \left( \sum_{j \in S_i} \widehat{\alpha}_j^p \right)^{1/p} = p^{1/2} \alpha_i. \end{split}$$

This means that throughout the rest of the proof, we assume without loss of generality that  $p_i \leq \log |S_i|$ . Next, observe that for any  $x \in \mathbb{R}^d$  and when  $p_1, \ldots, p_m \geq 2$ ,

$$\left\|\mathbf{W}_{S_{i}}^{-1/2}\mathbf{\Lambda}_{S_{i}}^{1/p-1/2}\mathbf{U}\mathbf{x}\right\|_{p_{i}} = \left(\sum_{j\in S_{i}}\left|\lambda_{i}^{1/p}\left\langle f_{j},\mathbf{x}\right\rangle\right|^{p_{i}}\right)^{1/p_{i}} \leq \left(\sum_{j\in S_{i}}\left|\lambda_{i}^{1/p}\left\|f_{j}\right\|_{2}\|\mathbf{x}\|_{2}\right)^{1/p_{i}} = \alpha_{i}\|\mathbf{x}\|_{2}.$$

Similarly, when  $p_1 = \cdots = p_m = p$ , we first observe

$$\left| \left\langle \widehat{\lambda}_{j}^{1/p-1/2} \boldsymbol{u}_{j}, \boldsymbol{x} \right\rangle \right| \leq \widehat{\alpha}_{j} \|\boldsymbol{x}\|_{2}$$

We therefore get

$$\left\|\widehat{\mathbf{A}}_{S_i}^{1/p-1/2}\mathbf{U}\mathbf{x}\right\|_p \le \left(\sum_{j\in S_i} \widehat{\alpha}_j^p\right)^{1/p} \|\mathbf{x}\|_2 = \alpha_i \|\mathbf{x}\|_2.$$
(3.3.6)

Hence, after applying Fact 3.2.11 and Fact 3.2.10, we notice that  $\alpha_i(H\rho_i)^{-1/p} \leq \|\boldsymbol{\alpha}\|_p$  and get

$$\left\|\rho_i^{-1/p} \left\|\mathbf{\Lambda}_{S_i}^{1/p-1/2} \mathbf{U} \boldsymbol{g}\right\|_{p_i}\right\|_{\psi_2} \lesssim \left(1 + \sqrt{p_i}\right) H^{1/p} \left\|\boldsymbol{\alpha}\right\|_p.$$

All of this implies that for any subset *S* of size  $\tilde{m}$  (see Exercise 2.5.10 of [Ver18]),

$$\mathbb{E}_{\substack{\boldsymbol{g} \sim \mathcal{N}(0,\mathbf{I}_d)}} \left[ \max_{i \in S} \rho_i^{-1/p} \left\| \mathbf{\Lambda}_{S_i}^{1/p-1/2} \mathbf{U} \boldsymbol{g} \right\|_{p_i} \right] \lesssim \max_{i \in S} p_i^{1/2} H^{1/p} \left\| \boldsymbol{\alpha} \right\|_p \sqrt{\log \widetilde{m}}.$$

Thus,

$$\log \mathcal{N} \lesssim \eta^{-2} \left( \left( \max_{i \in S} p_i + 1 \right) H^{2/p} \|\boldsymbol{\alpha}\|_p^2 \log \widetilde{m} \right).$$

This concludes the proof of Lemma 3.3.19.

We are finally ready to prove Theorem 3.3.5.

Proof of Theorem 3.3.5. For notational simplicity in this proof, write

$$K := \left\{ \boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\|_{\mathcal{G},\rho,\infty,S} \le 1 \right\}.$$

Let us first handle the case where  $p \ge 2$ . By our choice of **W**, we have for all  $x \in \mathbb{R}^d$  that

$$\left\|\mathbf{W}^{1/2}\mathbf{\Lambda}^{1/2-1/p}\mathbf{A}\mathbf{x}\right\|_{2} \leq \left\|\mathbf{\Lambda}^{-1/p}\mathbf{A}\mathbf{x}\right\|_{\mathcal{G}_{p}(\lambda)} = \left\|\mathbf{A}\mathbf{x}\right\|_{\mathcal{G}_{p}}.$$

This implies the containment  $B_p \subseteq \widehat{B_2}$ , and thus

$$\log \mathcal{N}(B_p, \eta K) \leq \log \mathcal{N}(\widehat{B_2}, \eta K) \leq \eta^{-2} \max_i \min \{p_i, \log |S_i|\} H^{2/p} F^{\star} \log \widetilde{m}.$$

The desired result now follows immediately from Lemma 3.3.19.

For the case where p < 2 and  $p_i \ge 2$  for all i, we require a bit more work. By our choice of **W**, we have

$$\left\|\mathbf{W}^{1/2}\mathbf{\Lambda}^{1/2-1/p}\mathbf{A}\mathbf{x}\right\|_{2} \leq \left\|\mathbf{\Lambda}^{-1/p}\mathbf{A}\mathbf{x}\right\|_{\mathcal{G}_{2}(\lambda)} = \left\|\mathbf{\Lambda}^{1/2-1/p}\mathbf{A}\mathbf{x}\right\|_{2}$$

and for any t > 0 (after remembering (3.3.1) which tells us  $\|\boldsymbol{\alpha}\|_p \leq \sqrt{F^{\star}}$ ),

$$\log \mathcal{N}(B_p, \eta K) \leq \log \mathcal{N}(B_p, t\widehat{B_2}) + \log \mathcal{N}(t\widehat{B_2}, \eta K) = \log \mathcal{N}(B_p, t\widehat{B_2}) + \log \mathcal{N}(\widehat{B_2}, \frac{\eta}{t} \cdot K)$$

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$$\leq t^{-\frac{2p}{2-p}} \cdot C(p) \max_{i} \min \left\{ p_i + 1, \log |S_i| \right\} F^{\star} \log F^{\star} \\ + \left(\frac{\eta}{t}\right)^{-2} \max_{i} \min \left\{ p_i + 1, \log |S_i| \right\} H^{2/p} F^{\star} \log \widetilde{m},$$

where the last line follows from Lemma 3.3.10 and Lemma 3.3.19. Choose  $t = \eta^{1-p/2} \cdot H^{1/2-1/p}$ , and for simplicity let  $p^* := \max_i \min \{p_i + 1, \log |S_i|\}$ . We write

$$\log \mathcal{N}(B_p, \eta K) \leq t^{-\frac{2p}{2-p}} \cdot C(p) p^* F^* \log F^* + \left(\frac{\eta}{t}\right)^{-2} p^* H^{2/p} F^* \log \widetilde{m}$$
$$\lesssim \eta^{-p} \cdot H \cdot C(p) \left(p^* F^* \log \max\left\{\widetilde{m}, F^*\right\}\right).$$

Finally, we need to address the case where  $p_1 = \cdots = p_m = p < 2$ , as this is not covered by the construction of the block Lewis weights (Lemma 3.3.7).

To do so, notice that  $\|\mathbf{A}x\|_{\mathcal{G}_p} = \|\mathbf{A}x\|_p$  for all  $x \in \mathbb{R}^d$ . So, we reduce to the case where all the  $S_i$  have size 1. In particular, let  $\hat{\lambda}$  be a probability measure over [k] such that for all  $j \in [k]$ ,

$$F^{\star} \geq \frac{\tau_j\left(\widehat{\mathbf{\Lambda}}^{1/2-1/p}\mathbf{A}\right)}{\widehat{\lambda}_j},$$

and let

$$\widehat{B_2} = \left\{ \boldsymbol{x} \in \mathbb{R}^d : \left\| \widehat{\boldsymbol{\Lambda}}^{1/2 - 1/p} \mathbf{A} \boldsymbol{x} \right\|_2 \le 1 \right\}.$$

Notice that this setup is in accordance with Definition 3.3.3. By Lemma 3.3.10, we get

$$\log \mathcal{N}(B_p, t\widehat{B_2}) \leq t^{-\frac{2p}{2-p}} \cdot C(p)F^* \log F^*.$$

It remains to bound  $\log N(\widehat{B}_2, (\eta/t)K)$ . Let  $\lambda$  be a probability measure over [m], where  $\lambda_i = \sum_{j \in S_i} \widehat{\lambda}_j$ . Define  $\widehat{\alpha} \in \mathbb{R}^n$  analogously to  $\widehat{\lambda}$ . Notice that, for these choices of  $\widehat{\alpha}$  and  $\widehat{\lambda}$ , we see that the conclusion of Lemma 3.3.19 still holds, and we have

$$\log \mathcal{N}\left(\widehat{B_2}, \left(\frac{\eta}{t}\right) \cdot K\right) \lesssim \left(\frac{\eta}{t}\right)^{-2} H^{2/p} F^{\star} \log \widetilde{m}.$$

Now, the calculation is the same as before, and we conclude the proof of Theorem 3.3.5.

# 3.3.5. Volume-based metric entropy

In this subsection, we prove Lemma 3.3.21, which is an easy consequence of a volume-based argument to obtain a covering number guarantee.

We start with Lemma 3.3.20.

**Lemma 3.3.20.** Let  $S \subseteq [m]$  have size  $\widetilde{m}$ . Let  $H \ge 1$  be such that  $H\rho_i \ge \alpha_i^p / \|\alpha\|_p^p$  for all  $i \in [m]$ . For all  $x \in \mathbb{R}^d$ , we have

$$\max_{i \in S} \frac{\|\mathbf{A}_{S_i} \mathbf{x}\|_{p_i}}{(F^{\star})^{\max(1/2, 1/p)} \rho_i^{1/p} H^{1/p}} \leq \|\mathbf{A} \mathbf{x}\|_{\mathcal{G}_p}.$$

*Proof of Lemma* 3.3.20. Consider the invertible mapping  $x \mapsto \mathbf{R}x$ , and write

$$\left\|\mathbf{W}_{S_{i}}^{-1/2}\mathbf{\Lambda}_{S_{i}}^{1/p-1/2}\mathbf{U}\mathbf{x}\right\|_{p_{i}} = \left(\sum_{j\in S_{i}}\left|\lambda_{i}^{1/p}\left\langle f_{j},\mathbf{x}\right\rangle\right|^{p_{i}}\right)^{1/p_{i}} \le \left(\sum_{j\in S_{i}}\left|\lambda_{i}^{1/p}\left\|f_{j}\right\|_{2}\|\mathbf{x}\|_{2}\right|^{p_{i}}\right)^{1/p_{i}} = \alpha_{i}\|\mathbf{x}\|_{2}.$$
(3.3.7)

This means that when  $p \ge 2$ ,

$$\|\mathbf{A}_{S_i} \mathbf{x}\|_{p_i} \leq \alpha_i \left\|\mathbf{W}^{1/2} \mathbf{\Lambda}^{1/2 - 1/p} \mathbf{A} \mathbf{x}\right\|_2 \leq \alpha_i \left\|\mathbf{\Lambda}^{1/2 - 1/p} \mathbf{A} \mathbf{x}\right\|_{\mathcal{G}_2} \leq \alpha_i \|\mathbf{A} \mathbf{x}\|_{\mathcal{G}_p} \leq \|\mathbf{\alpha}\|_p \rho_i^{1/p} H^{1/p} \|\mathbf{A} \mathbf{x}\|_{\mathcal{G}_p}.$$

Dividing both sides by  $\|\boldsymbol{\alpha}\|_p \rho_i^{1/p} H^{1/p}$  and then recalling (3.3.1) yields the desired conclusion (in particular, we see that  $\|\boldsymbol{\alpha}\|_p \leq (F^{\star})^{1/2}$ ).

We now analyze what happens when  $p \leq 2$  and  $p_1, \ldots, p_m \geq 2$ . Let

$$\Delta_i^{1/2} \coloneqq \max_{\boldsymbol{x} \in \mathbb{R}^d \setminus \{0\}} \frac{\lambda_i^{-1/p} \|\mathbf{A}_{S_i} \boldsymbol{x}\|_{p_i}}{\|\mathbf{\Lambda}^{1/2 - 1/p} \mathbf{A} \boldsymbol{x}\|_{\mathcal{G}_2}}.$$

Let  $\Delta \coloneqq \max_{i \in [m]} \Delta_i$ . We now see that

$$\begin{split} \left\| \mathbf{\Lambda}^{1/2-1/p} \mathbf{A} \mathbf{x} \right\|_{\mathcal{G}_{2}} &= \left( \sum_{i=1}^{m} \lambda_{i} \left\| \mathbf{\Lambda}_{S_{i}}^{-1/p} \mathbf{A}_{S_{i}} \mathbf{x} \right\|_{p_{i}}^{2} \right)^{1/2} = \left( \sum_{i=1}^{m} \lambda_{i} \left\| \mathbf{\Lambda}_{S_{i}}^{-1/p} \mathbf{A}_{S_{i}} \mathbf{x} \right\|_{p_{i}}^{p} \cdot \left\| \mathbf{\Lambda}_{S_{i}}^{-1/p} \mathbf{A}_{S_{i}} \mathbf{x} \right\|_{p_{i}}^{2-p} \right)^{1/2} \\ &\leq \left( \left\| \mathbf{A} \mathbf{x} \right\|_{\mathcal{G}_{p}}^{p} \left( \Delta^{1/2} \cdot \left\| \mathbf{\Lambda}^{1/2-1/p} \mathbf{A} \mathbf{x} \right\|_{\mathcal{G}_{2}} \right)^{2-p} \right)^{1/2} = \left\| \mathbf{A} \mathbf{x} \right\|_{\mathcal{G}_{p}}^{p/2} \left( \Delta^{1/2} \cdot \left\| \mathbf{\Lambda}^{1/2-1/p} \mathbf{A} \mathbf{x} \right\|_{\mathcal{G}_{2}} \right)^{1-p/2}. \end{split}$$

Rearranging and taking the 2/p-power gives

$$\left\|\mathbf{\Lambda}^{1/2-1/p}\mathbf{A}\mathbf{x}\right\|_{\mathcal{G}_2} \leq \|\mathbf{A}\mathbf{x}\|_{\mathcal{G}_p}\,\Delta^{1/p-1/2}.$$

Observe that this yields the inequalities

$$\begin{split} \frac{\|\mathbf{A}_{S_i} \boldsymbol{x}\|_{p_i}}{\Delta^{1/p-1/2}} &\leq \frac{\alpha_i}{\Delta^{1/p-1/2}} \left\| \mathbf{W}^{1/2} \mathbf{\Lambda}^{1/2-1/p} \mathbf{A} \boldsymbol{x} \right\|_2 \leq \frac{\alpha_i}{\Delta^{1/p-1/2}} \left\| \mathbf{\Lambda}^{1/2-1/p} \mathbf{A} \boldsymbol{x} \right\|_{\mathcal{G}_2} \\ &\leq \alpha_i \left\| \mathbf{A} \boldsymbol{x} \right\|_{\mathcal{G}_p} \leq \|\boldsymbol{\alpha}\|_p \, \rho_i^{1/p} H^{1/p} \left\| \mathbf{A} \boldsymbol{x} \right\|_{\mathcal{G}_p}. \end{split}$$

It remains to bound the  $\Delta_i$ . By the same sort of argument from the  $p \ge 2$  case (i.e., (3.3.7)), we get

$$\lambda_i^{1/p} \Delta_i^{1/2} \le \alpha_i = \lambda_i^{1/p} \left( \sum_{j \in S_i} \left\| f_j \right\|_2^{p_i} \right)^{1/p_i}$$

which means that

$$\Delta \leq \max_{i \in [m]} \left( \sum_{j \in S_i} \left\| f_j \right\|_2^{p_i} \right)^{2/p_i} \leq F^{\star}.$$

Now, again using the fact that  $\|\boldsymbol{\alpha}\|_p \leq (F^{\star})^{1/2}$ , we see that  $\Delta^{1/p-1/2} \|\boldsymbol{\alpha}\|_p \leq (F^{\star})^{1/p}$ .

Finally, we analyze the case where  $p_1 = \cdots = p_m = p < 2$ . Using (3.3.6), and the same sort of argument from above, we have

$$\left\|\mathbf{A}_{S_{i}}\boldsymbol{x}\right\|_{p} \leq \alpha_{i} \left\|\widehat{\boldsymbol{\Lambda}}^{1/2-1/p}\mathbf{A}\boldsymbol{x}\right\|_{2} \leq \alpha_{i} \left(F^{\star}\right)^{1/p-1/2} \left\|\mathbf{A}\boldsymbol{x}\right\|_{\mathcal{G}_{p}} \leq \left\|\boldsymbol{\alpha}\right\|_{p} H^{1/p} \rho_{i}^{1/p} \left(F^{\star}\right)^{1/p-1/2} \left\|\mathbf{A}\boldsymbol{x}\right\|_{\mathcal{G}_{p}}$$

Once again, we use  $\|\boldsymbol{\alpha}\|_p \leq (F^{\star})^{1/2}$ .

We have covered all our cases and may conclude the proof of Lemma 3.3.20.

Lemma 3.3.20 also suggests a useful sanity check, as the denominator points to a sparsity of  $\sum_{i \le m} \left( (F^{\star})^{\max(1/2,1/p)} \rho_i^{1/p} H^{1/p} \right)^p = H(F^{\star})^{\max(1,p/2)}$ . And, recall that we should be able to set  $F^{\star} \sim d$ , which indeed gives us the dependence on d we see in Theorem 8.

**Lemma 3.3.21.** Let  $S \subseteq [m]$  have size  $\widetilde{m}$ . We have

$$\log \mathcal{N}\left(B_{p}, \eta\left\{\boldsymbol{x} \in \mathbb{R}^{d} : \|\boldsymbol{x}\|_{\mathcal{G},\rho,\infty,S} \leq 1\right\}\right) \leq n \log\left(\frac{4H^{1/p}\left(F^{\star}\right)^{\max(1/p,1/2)}}{\eta}\right)$$

Proof of Lemma 3.3.21. Define

$$K := \left\{ \boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\|_{\mathcal{G},\rho,\infty,S} \le 1 \right\}.$$

Let *C* be a value such that for all  $i \in S$  and  $x \in \mathbb{R}^d$ , we have  $(C\rho_i)^{-1/p} \|\mathbf{A}_{S_i} x\|_{p_i} \leq \|\mathbf{A} x\|_{\mathcal{G}_p}$ . This means that  $B_p \subseteq C^{1/p} K$ . Then,

$$\log \mathcal{N}(B_p, \eta K) \leq \log \mathcal{N}\left(C^{1/p}K, \eta K\right) = \log \mathcal{N}\left(K, \frac{\eta}{C^{1/p}} \cdot K\right) \leq n \log\left(\frac{4C^{1/p}}{\eta}\right).$$

By Lemma 3.3.20, when  $p \ge 2$ , we can choose  $C = H(F^{\star})^{\max(1/p,1/2)}$ . This concludes the proof of Lemma 3.3.21.

# 3.4. Concentration analysis

In this section, we prove Theorem 11. Theorem 11 states our main result in its fullest generality. Theorem 8 follows easily from this, as we show in Section 3.5.

We first state Theorem 11.

**Theorem 11** (General concentration result). Let  $\mathcal{G} = (\mathbf{A} \in \mathbb{R}^{n \times d}, S_1, \dots, S_m, p_1, \dots, p_m)$  where  $S_1, \dots, S_m$  form a partition of [n]. Suppose at least one of the following holds:

- $1 \le p < \infty$  and  $p_1, ..., p_m \ge 2;$
- $1/\log d \le p_1 = \cdots = p_m = p < \infty;$
- $p_1 = \cdots = p_m = 2$  and  $1/\log n \le p < \infty$ .

Let  $P := \max(1, \max_{i \in [m]} \min(p_i, \log |S_i|))$ . Suppose that  $\lambda \in \mathbb{R}^m$  is a probability measure over [m], let  $\mathbf{W}$  be a rounding matrix (Definition 3.3.2) such that we get an  $F^*$ -block Lewis overestimate (Definition 3.1.2), and define  $\boldsymbol{\alpha}$  according to Definition 3.3.3. Let  $\mathcal{D} = (\rho_1, \ldots, \rho_m)$  be a probability distribution over [m] and  $H \ge 1$  be such that  $H\rho_i \ge \alpha_i^p / \|\boldsymbol{\alpha}\|_p^p$ .

If

$$\widetilde{m} = \Omega\left(\log\left(1/\delta\right)\varepsilon^{-2}\left(\log d\right)^2\log\left(d/\varepsilon\right)\cdot H\cdot P\left(F^\star\right)^{\max(1,p/2)}\right)$$

and if we sample  $\mathcal{M} \sim \mathcal{D}^{\widetilde{m}}$ , then, with probability  $\geq 1 - \delta$ , we have:

for all 
$$\mathbf{x} \in \mathbb{R}^d$$
,  $(1-\varepsilon) \|\mathbf{A}\mathbf{x}\|_{\mathcal{G}_p}^p \leq \frac{1}{\widetilde{m}} \sum_{i \in \mathcal{M}} \frac{1}{\rho_i} \cdot \|\mathbf{A}_{S_i}\mathbf{x}\|_{p_i}^p \leq (1+\varepsilon) \|\mathbf{A}\mathbf{x}\|_{\mathcal{G}_p}^p$ 

The goal of the rest of this section is to prove Theorem 11. It may be helpful to recall the argument sketch given in Section 3.1.3.

To formalize the idea given there, we first introduce the following notation (recall that  $\rho_i$  is the probability that we choose group *i* in a round of sampling and that Definition 3.2.5 defines the  $e_N$ ).

$$g_i(\boldsymbol{x}) \coloneqq \|\mathbf{A}_{S_i}\boldsymbol{x}\|_{p_i} \tag{3.4.1}$$

$$dist(\boldsymbol{x}, \widehat{\boldsymbol{x}}) \coloneqq \left(\sum_{h=1}^{\widetilde{m}} \left(\frac{g_{i_h}(\boldsymbol{x})^p}{\rho_{i_h}} - \frac{g_{i_h}(\widehat{\boldsymbol{x}})^p}{\rho_{i_h}}\right)^2\right)^{1/2}$$
(3.4.2)

$$\gamma_2(B_p, dist) := \inf_{\substack{|T_N| \le 2^{2^N}; \, \mathbf{x} \in B_p \\ T_N \subset B_n}} \sup_{X \in B_p} \sum_{N \ge 0} 2^{N/2} \cdot d_2(\mathbf{x}, T_N)$$
(3.4.3)

The goal is to control  $\gamma_2(B_p, dist)$ . This quantity represents the worst-case approximation error that one incurs by using the discretization scheme given by the  $T_N$ , where the discretization is taken with respect to the *dist* metric.

Towards this goal, we first apply a standard symmetrization reduction. Informally, this reduction (Lemma 3.4.1) states that it is enough to analyze the average fluctuations of a Rademacher average of any set of  $\tilde{m}$  (not necessarily distinct) reweighted groups.

**Lemma 3.4.1** (Symmetrization reduction). Let  $R_1, \ldots, R_{\tilde{m}}$  be independent Rademacher random variables (i.e., Unif  $(\pm 1)$ ). We have

$$\mathbb{E}_{\mathcal{G}'}\left[\left\|\left\|\mathbf{A}\boldsymbol{x}\right\|_{\mathcal{G}_{p}}^{p}-\left\|\boldsymbol{x}\right\|_{\mathcal{G}'}^{p}\right]\right] \leq 2\mathbb{E}_{\mathcal{G}'R_{1},\dots,R_{\widetilde{m}}}\mathbb{E}\left[\frac{1}{\widetilde{m}}\left|\sum_{h=1}^{\widetilde{m}}R_{h}\frac{g_{i_{h}'}(\boldsymbol{x})^{p}}{\rho_{i_{h}'}}\right|\right].$$

*Proof of Lemma* 3.4.1. We follow the proof of Lemma 3.1 due to Lee [Lee23]. Let  $\tilde{\mathcal{G}}$  be an independent copy of  $\mathcal{G}'$ . Fixing  $\mathcal{G}'$ , we have by Jensen's inequality that

$$\left|\left\|\mathbf{A}\mathbf{x}\right\|_{\mathcal{G}_{p}}^{p}-\left\|\mathbf{x}\right\|_{\mathcal{G}'}^{p}\right|=\left|\mathbb{E}\left[\left\|\mathbf{x}\right\|_{\tilde{\mathcal{G}}}^{p}\right]-\left\|\mathbf{x}\right\|_{\mathcal{G}'}^{p}\right|\leq\mathbb{E}\left[\left|\left\|\mathbf{x}\right\|_{\tilde{\mathcal{G}}}^{p}-\left\|\mathbf{x}\right\|_{\mathcal{G}'}^{p}\right|\right].$$

Thus, taking expectation over G',

$$\mathbb{E}_{\mathcal{G}'}\left[\left\|\|\mathbf{A}\mathbf{x}\|_{\mathcal{G}_{p}}^{p}-\|\mathbf{x}\|_{\mathcal{G}'}^{p}\right\|\right] \leq \mathbb{E}_{\mathcal{G}',\tilde{\mathcal{G}}}\left[\left\|\|\mathbf{x}\|_{\tilde{\mathcal{G}}}^{p}-\|\mathbf{x}\|_{\mathcal{G}'}^{p}\right\|\right] = \mathbb{E}_{\mathcal{G}',\tilde{\mathcal{G}}}\left[\frac{1}{\widetilde{m}}\left|\sum_{h=1}^{\widetilde{m}}\frac{g_{\tilde{i}_{h}}(\mathbf{x})^{p}}{\rho_{\tilde{i}_{h}}}-\frac{g_{i_{h}'}(\mathbf{x})^{p}}{\rho_{i_{h}'}}\right|\right].$$

Observe that  $\frac{g_{\tilde{i}_h}(\mathbf{x})^p}{\rho_{\tilde{i}_h}} - \frac{g_{i'_h}(\mathbf{x})^p}{\rho_{i'_h}}$  is symmetric, and so is distributed the same as  $R_h\left(\frac{g_{\tilde{i}_h}(\mathbf{x})^p}{\rho_{\tilde{i}_h}} - \frac{g_{i'_h}(\mathbf{x})^p}{\rho_{i'_h}}\right)$  where  $R_h$  is an independent Rademacher variable. Then,

$$\mathbb{E}_{\mathcal{G}',\tilde{\mathcal{G}}}\left[\left|\frac{1}{\widetilde{m}}\sum_{h=1}^{\widetilde{m}}\left(\frac{g_{\tilde{i}_{h}}(\boldsymbol{x})^{p}}{p_{\tilde{i}_{h}}}-\frac{g_{i_{h}'}(\boldsymbol{x})^{p}}{p_{i_{h}'}}\right)\right|\right] = \mathbb{E}_{R_{1},\dots,R_{\widetilde{m}}\mathcal{G}',\tilde{\mathcal{G}}}\left[\frac{1}{\widetilde{m}}\left|\sum_{h=1}^{\widetilde{m}}R_{h}\left(\frac{g_{\tilde{i}_{h}}(\boldsymbol{x})^{p}}{\rho_{\tilde{i}_{h}}}-\frac{g_{i_{h}'}(\boldsymbol{x})^{p}}{\rho_{i_{h}'}}\right)\right|\right] \\ \leq 2\mathbb{E}_{\mathcal{G}'R_{1},\dots,R_{M}}\left[\frac{1}{\widetilde{m}}\left|\sum_{h=1}^{\widetilde{m}}R_{h}\frac{g_{i_{h}'}(\boldsymbol{x})^{p}}{\rho_{i_{h}'}}\right|\right].$$

This concludes the proof of Lemma 3.4.1.

With Lemma 3.4.1 in hand, we set up our chaining argument in Lemma 3.4.2. We first confirm that our random process is subgaussian with respect to our choice of *dist*.

**Lemma 3.4.2** (Choosing the distance). The random process  $\sum_{h=1}^{\tilde{m}} R_{i_h} \cdot g_{i_h}(x)^p / \rho_{i_h}$  is subgaussian with respect to dist as defined in (3.4.2).

Proof of Lemma 3.4.2. Let

$$P := \left| \sum_{h=1}^{\widetilde{m}} R_h \left( \frac{g_{i_h}(\boldsymbol{x})^p}{\rho_{i_h}} - \frac{g_{i_h}(\widehat{\boldsymbol{x}})^p}{\rho_{i_h}} \right) \right|.$$

Let us first calculate  $||P||_{\psi_2}$ . Using the fact that every term in this sum is independent and Fact 3.2.8, we get

$$\|P\|_{\psi_2}^2 = \sum_{h=1}^{\widetilde{m}} \left\| R_h \left( \frac{g_{i_h}(\boldsymbol{x})^p}{\rho_{i_h}} - \frac{g_{i_h}(\widehat{\boldsymbol{x}})^p}{\rho_{i_h}} \right) \right\|_{\psi_2}^2 \le \sum_{h=1}^{\widetilde{m}} 2 \left( \frac{g_{i_h}(\boldsymbol{x})^p}{\rho_{i_h}} - \frac{g_{i_h}(\widehat{\boldsymbol{x}})^p}{\rho_{i_h}} \right)^2$$

By Fact 3.2.8, we have

$$\Pr_{R_h}\left[P \ge v dist(\boldsymbol{x}, \widehat{\boldsymbol{x}})\right] \le 2 \exp\left(-\frac{v^2 dist(\boldsymbol{x}, \widehat{\boldsymbol{x}})^2}{4\sum_{h=1}^{\widetilde{m}} \left(\frac{g_{i_h}(\boldsymbol{x})^p}{\rho_{i_h}} - \frac{g_{i_h}(\widehat{\boldsymbol{x}})^p}{\rho_{i_h}}\right)^2}\right) = 2 \exp\left(-\frac{v^2}{2}\right).$$

This concludes the proof of Lemma 3.4.2.

Lemma 3.4.2 tells us that *dist* is a choice of distance on  $B_p$  that allows us to use the subgaussian form of chaining to analyze our random process. Along with the way we have set up our sampling process, we have enough to apply Theorem 3.4.3. This is simply a restatement of Lemma 2.6 of [JLLS23] for our setting.

**Theorem 3.4.3** (Restatement of Lemma 2.6 from [JLLS23],  $\alpha = 2$ ). Recall dist (3.4.2) and  $\gamma_2$  (3.4.3). Suppose that for some D and for every choice of  $i_1, \ldots, i_{\widetilde{m}}$ , we have

$$\gamma_2\left(B_p, \frac{dist}{\widetilde{m}}\right) \lesssim D\left(\max_{\boldsymbol{x}\in B_p} \|\boldsymbol{x}\|_{\mathcal{G}'}^p\right)^{1/2},$$

	-	

where

$$\|\boldsymbol{x}\|_{\mathcal{G}'}^p = \frac{1}{\widetilde{m}} \sum_{h=1}^{\widetilde{m}} \frac{g_{i_h}(\boldsymbol{x})^p}{\rho_{i_h}}.$$

Then, we have the following.

$$\mathbb{E}_{\mathcal{D}}\left[\sup_{\boldsymbol{x}\in B_{p}}\left|\left\|\mathbf{A}\boldsymbol{x}\right\|_{\mathcal{G}_{p}}-\left\|\boldsymbol{x}\right\|_{\mathcal{G}'}^{p}\right|\right]\lesssim D.$$

*If we also have for all choices*  $i_1, \ldots, i_{\widetilde{m}}$  *and for some*  $\widehat{D}$  *that* 

diam 
$$\left(B_p, \frac{dist}{\widetilde{m}}\right) \lesssim \widehat{D}\left(\max_{\boldsymbol{x}\in B_p} \|\boldsymbol{x}\|_{\boldsymbol{\mathcal{G}}'}^p\right)^{1/2}$$
,

then there exists a universal constant C > 0 such that for all  $0 \le t \le 1/2K\widehat{D}$ ,

$$\Pr_{\mathcal{D}}\left[\sup_{\boldsymbol{x}\in B_{p}}\left|\left\|\mathbf{A}\boldsymbol{x}\right\|_{\mathcal{G}_{p}}^{p}-\left\|\boldsymbol{x}\right\|_{\mathcal{G}'}^{p}\right|\geq C(D+t\widehat{D})\right]\leq \exp\left(-\frac{Kt^{2}}{4}\right).$$

With Theorem 3.4.3 in our arsenal, our task becomes to compute  $\gamma_2(B_p, dist)$  (which we will then divide by  $\tilde{m}$  so that we can apply Theorem 3.4.3). Hereafter, we will simply abbreviate  $\gamma_2(B_p, dist)$  as  $\gamma_2$ . We will first weaken the definition of  $\gamma_2$ , which is essentially equivalent to Dudley's integral. Recall the definition of the entropy numbers  $e_N$  (Definition 3.2.5) and notice that

$$\gamma_2 \le \sum_{N\ge 0} 2^{N/2} e_N(B_p, dist).$$

We now rewrite *dist* in a form that will be more convenient for us.

Lemma 3.4.4. We have

$$dist(\boldsymbol{x}, \widehat{\boldsymbol{x}}) \leq \max(p, 2) \left(F^{\star}\right)^{\max(0, p/4 - 1/2)} \widetilde{m}^{1/2} \cdot \|\boldsymbol{x} - \widehat{\boldsymbol{x}}\|_{\mathcal{G}, \rho, \infty, S}^{\min(p/2, 1)} \cdot \left(\max_{\boldsymbol{x} \in B_p} \|\boldsymbol{x}\|_{\mathcal{G}'}^p\right)^{1/2}$$

and therefore

$$\gamma_{2} \leq \widetilde{m}^{1/2} \max(p,2) \left( H^{1/p} \left( F^{\star} \right)^{1/2} \right)^{\max(0,p/2-1)} \left( \max_{\boldsymbol{x} \in B_{p}} \|\boldsymbol{x}\|_{\mathcal{G}'}^{p} \right)^{1/2} \sum_{N \geq 0} 2^{N/2} e_{N} \left( B_{p}, \|\cdot\|_{\mathcal{G},\rho,\infty,S} \right)^{\min(p/2,1)}$$

*Proof of Lemma* 3.4.4. We have two cases. We first address the case  $0 . Recall that in this regime, we have <math>|a^{p/2} - b^{p/2}| \le |a - b|^{p/2}$ . Since  $g_{i_h}(x)$  is a norm, the triangle inequality tells us that  $|g_{i_h}(x) - g_{i_h}(\widehat{x})| \le g_{i_h}(x - \widehat{x})$ . We use these and write

$$dist(\mathbf{x}, \widehat{\mathbf{x}}) = \left(\sum_{h=1}^{\widetilde{m}} \left(\frac{g_{i_h}(\mathbf{x})^{p/2}}{\sqrt{\rho_{i_h}}} - \frac{g_{i_h}(\widehat{\mathbf{x}})^{p/2}}{\sqrt{\rho_{i_h}}}\right)^2 \left(\frac{g_{i_h}(\mathbf{x})^{p/2}}{\sqrt{\rho_{i_h}}} + \frac{g_{i_h}(\widehat{\mathbf{x}})^{p/2}}{\sqrt{\rho_{i_h}}}\right)^2\right)^{1/2} \\ \le \left(\sum_{h=1}^{\widetilde{m}} \left(\frac{g_{i_h}(\mathbf{x} - \widehat{\mathbf{x}})^{p/2}}{\sqrt{\rho_{i_h}}}\right)^2 \left(\frac{g_{i_h}(\mathbf{x})^{p/2}}{\sqrt{\rho_{i_h}}} + \frac{g_{i_h}(\widehat{\mathbf{x}})^{p/2}}{\sqrt{\rho_{i_h}}}\right)^2\right)^{1/2}$$

$$\leq \|\boldsymbol{x} - \widehat{\boldsymbol{x}}\|_{\mathcal{G},\rho,\infty,S}^{p/2} \cdot \left(\sum_{h=1}^{\widetilde{m}} \left(\frac{g_{i_h}(\boldsymbol{x})^{p/2}}{\sqrt{\rho_{i_h}}} + \frac{g_{i_h}(\widehat{\boldsymbol{x}})^{p/2}}{\sqrt{\rho_{i_h}}}\right)^2\right)^{1/2}$$
  
$$\leq 2\widetilde{m}^{1/2} \cdot \|\boldsymbol{x} - \widehat{\boldsymbol{x}}\|_{\mathcal{G},\rho,\infty,S}^{p/2} \left(\max_{\boldsymbol{x}\in B_p} \|\boldsymbol{x}\|_{\mathcal{G}'}^p\right)^{1/2}$$

which concludes the proof in the range 0 .

We now move onto the case where  $p \ge 2$ . Recall that by Lipschitzness, we get

$$\left|a^{p/2} - b^{p/2}\right| \le \frac{p}{2} \cdot \max(a, b)^{p/2-1} |a - b|$$

Next, by Lemma 3.3.20, we know that for all *i*,

$$\|\mathbf{A}_{S_i} \mathbf{x}\|_{p_i} \leq (F^{\star})^{1/2} \rho_i^{1/p} H^{1/p} \|\mathbf{A} \mathbf{x}\|_{\mathcal{G}_p} \leq (F^{\star})^{1/2} \rho_i^{1/p} H^{1/p}.$$

We use these to rewrite  $dist(x, \hat{x})$ .

$$dist(\mathbf{x},\widehat{\mathbf{x}}) = \left(\sum_{h=1}^{\widetilde{m}} \left(\frac{g_{i_h}(\mathbf{x})^{p/2}}{\sqrt{\rho_{i_h}}} - \frac{g_{i_h}(\widehat{\mathbf{x}})^{p/2}}{\sqrt{\rho_{i_h}}}\right)^2 \left(\frac{g_{i_h}(\mathbf{x})^{p/2}}{\sqrt{\rho_{i_h}}} + \frac{g_{i_h}(\widehat{\mathbf{x}})^{p/2}}{\sqrt{\rho_{i_h}}}\right)^2\right)^{1/2}$$

$$\leq \frac{p}{2} \cdot H^{1/2 - 1/p} \left(F^{\star}\right)^{p/4 - 1/2} \left(\sum_{h=1}^{\widetilde{m}} \left(\rho_{i_h}^{-1/p} g_{i_h}(\mathbf{x} - \widehat{\mathbf{x}})\right)^2 \left(\frac{g_{i_h}(\mathbf{x})^{p/2}}{\sqrt{\rho_{i_h}}} + \frac{g_{i_h}(\widehat{\mathbf{x}})^{p/2}}{\sqrt{\rho_{i_h}}}\right)^2\right)^{1/2}$$

$$\leq \frac{p}{2} \cdot H^{1/2 - 1/p} \left(F^{\star}\right)^{p/4 - 1/2} \|\mathbf{x} - \widehat{\mathbf{x}}\|_{\mathcal{G}, \rho, \infty, S} \cdot \left(\sum_{h=1}^{\widetilde{m}} \left(\frac{g_{i_h}(\mathbf{x})^{p/2}}{\sqrt{\rho_{i_h}}} + \frac{g_{i_h}(\widehat{\mathbf{x}})^{p/2}}{\sqrt{\rho_{i_h}}}\right)^2\right)^{1/2}$$

$$\leq p \cdot H^{1/2 - 1/p} \left(F^{\star}\right)^{p/4 - 1/2} \widetilde{m}^{1/2} \cdot \|\mathbf{x} - \widehat{\mathbf{x}}\|_{\mathcal{G}, \rho, \infty, S} \left(\max_{\mathbf{x} \in B_p} \|\mathbf{x}\|_{\mathcal{G}'}^p\right)^{1/2}$$

We conclude the proof of Lemma 3.4.4.

In light of Lemma 3.4.4, observe that it is enough to calculate each term of the sum  $\sum_{N\geq 0} 2^{N/2} e_N(B_p, \|\cdot\|_{\mathcal{G},\rho,\infty,S})^{\min(p/2,1)}$ . We begin this analysis with Lemma 3.4.5.

**Lemma 3.4.5.** For all  $N \ge 0$ , we have

$$2^{N/2} e_N \left( B_p, \|\cdot\|_{\mathcal{G}, \rho, \infty, S} \right)^{\min(p/2, 1)} \leq 2^{N/2 + \min(p/2, 1)(2 + (1/p)\log C - 2^N/d)},$$

where *C* is such that for all  $i \in S$  and  $\mathbf{x} \in \mathbb{R}^d$ , we have  $(C\rho_i)^{-1/p} \|\mathbf{A}_{S_i}\mathbf{x}\|_{p_i} \leq \|\mathbf{A}\mathbf{x}\|_{\mathcal{G}_p}$ .

Proof of Lemma 3.4.5. Recall that by Lemma 3.3.21, we have

$$\log \mathcal{N}\left(B_p, \eta \left\{ \boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\|_{\mathcal{G}, \rho, \infty, S} \leq 1 \right\} \right) \leq d \log \left(\frac{4C^{1/p}}{\eta}\right).$$

We set  $\eta = 2^{2+(1/p)\log C - 2^N/d}$  so that

$$\log \mathcal{N}\left(B_p, \eta\left\{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\|_{\mathcal{G},\rho,\infty,S} \leq 1\right\}\right) \leq 2^N.$$

Then,

$$2^{N/2} e_N\left(B_p, \|\cdot\|_{\mathcal{G}, \rho, \infty, S}\right)^{\min(p/2, 1)} \le 2^{N/2 + \min(p/2, 1)(2 + (1/p)\log C - 2^N/d)},$$

concluding the proof of Lemma 3.4.5.

Using Lemma 3.4.5, we get a rapidly converging tail in our summation for large values of *N*. See Lemma 3.4.6.

**Lemma 3.4.6.** Let  $N_S \coloneqq \log(6d \log d)$ . We have

$$\begin{split} \sum_{N \ge 0} 2^{N/2} e_N \left( B_p, \|\cdot\|_{\mathcal{G}, \rho, \infty, S} \right)^{\min(p/2, 1)} &\lesssim \sqrt{n H^{1/\max(2, p)} \left( F^{\star} \right)^{1/2} \log d} \\ &+ \sum_{N \le N_S} 2^{N/2} e_N \left( B_p, \|\cdot\|_{\mathcal{G}, \rho, \infty, S} \right)^{\min(p/2, 1)}. \end{split}$$

*Proof of Lemma* 3.4.6. For now, let *C* be such that for all  $i \in S$  and  $\mathbf{x} \in \mathbb{R}^d$ , we have  $(C\rho_i)^{-1/p} \|\mathbf{A}_{S_i}\mathbf{x}\|_{p_i} \leq \|\mathbf{A}\mathbf{x}\|_{\mathcal{G}_v}$ .

Let  $N_S$  be a threshold such that for all  $N \ge N_S$ , we use the entropy number bound given by Lemma 3.4.5 (as the volume-based covering number bound is much better for small values of  $e_N$ ). Let us enforce the constraint  $N_S \ge \lceil \log (3d/\min(p,2)) \rceil$ , so that the entropy number bound in Lemma 3.4.5 is decreasing in N and is dominated from above by a geometric series with common ratio 1/2.

We now set  $N_S = \log (6d \log d)$ . Since  $p \ge 1/\log d$ , we know that  $N_S \ge \lceil \log (3d/\min(p,2)) \rceil$ . Now, since  $2^{N/2}e_N$  is bounded above by a geometric series with common ratio 1/2, the summation for all  $N \ge N_S$  is dominated by the first term. Let us evaluate this. We first observe that

$$\frac{N_S}{2} + \min\left(\frac{p}{2}, 1\right) \left(-\frac{2^{N_S}}{d} + 2 + \frac{\log C}{p}\right) = \frac{\log\left(6d\log d\right)}{2} + \min\left(\frac{p}{2}, 1\right) \left(-\frac{2^{\log(6d\log d)}}{d} + 2 + \frac{\log C}{p}\right)$$
$$= \frac{\log\left(6d\log d\right)}{2} + \min\left(\frac{p}{2}, 1\right) \left(-6\log d + 2 + \frac{\log C}{p}\right)$$
$$\leq \frac{\log\left(6d\log d\right)}{2} + \min\left(\frac{p}{2}, 1\right) \left(-6\log d + 2 + \frac{\log C}{p}\right)$$
$$\leq \frac{\log\left(6d\log d\right)}{2} + \frac{\log C}{\max(2, p)} \leq \frac{6Cd\log d}{2}$$

By Lemma 3.4.5, we see that

$$2^{N_S/2} e_{N_S} \left( B_p, \|\cdot\|_{\mathcal{G},\rho,\infty,S} \right)^{\min(p/2,1)} \lesssim \sqrt{dC^{1/\max(2,p)}\log d},$$

and by Lemma 3.3.20, we can choose

$$C = H\left(F^{\star}\right)^{\max(1,p/2)}$$

We plug this in, account for the remaining terms in the summation, and conclude the proof of Lemma 3.4.6.

We now give another way to evaluate the terms of our summation when the indices *N* are such that the entropy numbers are rather large. See Lemma 3.4.7.

**Lemma 3.4.7.** For all  $N \ge 0$ , we have

$$2^{N/2} e_N\left(B_p, \left\|\cdot\right\|_{\mathcal{G},\rho,\infty,S}\right)^{\min(p/2,1)} \leq \left(C(p) \cdot \max_i \min\left\{p_i, \log\left|S_i\right|\right\} H^{2/\max(2,p)} F^{\star} \log\left(F^{\star} + \widetilde{m}\right)\right)^{1/2},$$

where C(p) is a constant that only depends on p.

Proof of Lemma 3.4.7. For now, let

$$f(F^{\star}, \mathcal{G}) \coloneqq C(p) \cdot \max_{i} \min \left\{ p_{i}, \log |S_{i}| \right\} H^{2/\max(2,p)} F^{\star} \log \left( F^{\star} + \widetilde{m} \right).$$

By Theorem 3.3.5, we have

$$\log \mathcal{N}\left(B_{p}, \eta \cdot \left\{\boldsymbol{y} \in \mathbb{R}^{d} : \|\boldsymbol{x}\|_{\mathcal{G}, \rho, \infty, S} \leq 1\right\}\right) \lesssim \eta^{-\min(2, p)} \cdot f(F^{\star}, \mathcal{G}).$$

so if we choose, for some universal constant  $C_0$ ,

$$\eta = C_0 \cdot 2^{-N/\min(p,2)} \left( f(F^\star, \mathcal{G}) \right)^{\max(1/2,1/p)}$$

then we get

$$\log \mathcal{N}\left(B_p, \eta \cdot \left\{ \boldsymbol{y} \in \mathbb{R}^d : \|\boldsymbol{x}\|_{\mathcal{G},\rho,\infty,S} \le 1 \right\} \right) \le 2^N.$$

Thus,  $e_N(B_p, \|\cdot\|_{\mathcal{G},\rho,\infty,S}) \leq \eta$ . Exponentiating and substituting the definition of  $f(F^*, \mathcal{G})$  concludes the proof of Lemma 3.4.7.

We now show how to complete the sum by combining Lemma 3.4.7 and Lemma 3.4.6.

Lemma 3.4.8. We have

$$\begin{split} \sum_{N \geq 0} 2^{N/2} e_N \left( B_p, \|\cdot\|_{\mathcal{G}, \rho, \infty, S} \right)^{\min(p/2, 1)} \lesssim \sqrt{n H^{1/\max(2, p)} \left( F^\star \right)^{1/2} \log n} \\ + \log n \left( C(p) \cdot \max_i \min \left\{ p_i, \log |S_i| \right\} HF^\star \log \left( F^\star + \widetilde{m} \right) \right)^{1/2} \end{split}$$

*Proof of Lemma 3.4.8.* Noting that  $N_5 = \log (6d \log d) \leq \log d$ , we combine the conclusions of Lemma 3.4.7 and Lemma 3.4.6 to obtain the statement of Lemma 3.4.8.

Finally, we translate Lemma 3.4.8 into an upper bound on the process we started with by using Theorem 3.4.3. We then use this to complete the proof of Theorem 11.

Proof of Theorem 11. As we have done in previous proofs, as a shorthand, we define

$$p^{\star} \coloneqq \max_{i} \min \left\{ p_i, \log |S_i| \right\}.$$

We first weaken the statement of Lemma 3.4.8 to read

$$\begin{split} &\sum_{N\geq 0} 2^{N/2} e_N \left( B_p, \|\cdot\|_{\mathcal{G},\rho,\infty,S} \right)^{\min(p/2,1)} \\ &\lesssim \max\left\{ d,F^\star \right\} H^{1/2\max(2,p)} \sqrt{\log d} \\ &+ \log d \left( C(p) \cdot H^{2/\max(2,p)} p^\star \max\left\{ d,F^\star \right\} \log \left( d+F^\star + \widetilde{m} \right) \right)^{1/2} \\ &\lesssim \log d \left( C(p) \cdot H^{2/\max(2,p)} p^\star \max\left\{ d,F^\star \right\} \log \left( d+F^\star + \widetilde{m} \right) \right)^{1/2}. \end{split}$$

Let *V* denote the right hand side of the above. Combining this rewrite with Lemma 3.4.4, we get

$$\gamma_{2} \lesssim \widetilde{m}^{1/2} \max(p,2) \left( H^{1/p} \left( F^{\star} \right)^{1/2} \right)^{\max(0,p/2-1)} V \left( \max_{\boldsymbol{x} \in B_{p}} \left\| \boldsymbol{x} \right\|_{\mathcal{G}'}^{p} \right)^{1/2}.$$

By the symmetrization reduction (Lemma 3.4.1) and Theorem 3.4.3, we have

$$\mathbb{E}\left[\sup_{\boldsymbol{x}\in B_p}\left|\|\mathbf{A}\boldsymbol{x}\|_{\mathcal{G}_p}-\|\boldsymbol{x}\|_{\mathcal{G}'}^p\right|\right] \lesssim \frac{\max(p,2)\left(H^{1/p}\|\boldsymbol{\alpha}\|_p\right)^{\max(0,p/2-1)}V}{\widetilde{m}^{1/2}}$$

and so to make the RHS upper-bounded by  $\varepsilon$ , it is sufficient to set  $\tilde{m}$  according to

$$\widetilde{m} \asymp \frac{\max(p,2)^2 \left( H^{1/p} \left( F^{\star} \right)^{1/2} \right)^{\max(0,p-2)} V^2}{\varepsilon^2}.$$

This means that when p < 2, we have

$$\widetilde{m} \approx \frac{(\log d)^2 \cdot C(p) \cdot Hp^{\star} \max\left\{d, F^{\star}\right\} \log\left(d + F^{\star} + \widetilde{m}\right)}{\varepsilon^2} \\ \approx \frac{(\log d)^2 \log\left(\max\left\{d, F^{\star}\right\} / \varepsilon\right) \cdot C(p) \cdot Hp^{\star} \max\left\{d, F^{\star}\right\}}{\varepsilon^2},$$

which is what we desired.

For  $p \ge 2$ , we have

$$\widetilde{m} \approx \frac{p^2 H^{1-2/p} \left(F^\star\right)^{p/2-1} \left( (\log d)^2 H^{2/p} p^\star \max\left\{d, F^\star\right\} \log\left(d + F^\star + \widetilde{m}\right) \right)}{\varepsilon^2}$$
$$\approx \frac{(\log d)^2 \log\left(\max\left\{d, F^\star\right\} / \varepsilon\right) \cdot p^2 \cdot Hp^\star \max\left\{d, F^\star\right\}^{p/2}}{\varepsilon^2}$$

To bound diam( $B_p$ , dist), by the triangle inequality, it is enough to estimate dist(x, 0) for all  $x \in B_p$ . Recalling Lemma 3.3.20, we have

$$dist(\mathbf{x}, 0) = \left(\sum_{h=1}^{\widetilde{m}} \left(\frac{g_{i_h}(\mathbf{x})^p}{\rho_{i_h}} - \frac{g_{i_h}(0)^p}{\rho_{i_h}}\right)^2\right)^{1/2} = \left(\sum_{h=1}^{\widetilde{m}} \left(\frac{g_{i_h}(\mathbf{x})^p}{\rho_{i_h}}\right) \cdot \left(\frac{g_{i_h}(\mathbf{x})^p}{\rho_{i_h}}\right)\right)^{1/2}$$
$$\leq \left(\sum_{h=1}^{\widetilde{m}} H\left(F^{\star}\right)^{\max(1,p/2)} \cdot \left(\frac{g_{i_h}(\mathbf{x})^p}{\rho_{i_h}}\right)\right)^{1/2} \leq \widetilde{m}^{1/2} H^{1/2} \left(F^{\star}\right)^{\max(1/2,p/4)} \max_{\mathbf{x} \in B_p} \left(\|\mathbf{x}\|_{\mathcal{G}'}^p\right)^{1/2},$$

which means that we may set  $\widehat{D}$  in Theorem 3.4.3 according to

$$\widehat{D} \asymp \frac{H^{1/2} \left(F^{\star}\right)^{\max\left(1/2, p/4\right)} \max_{\boldsymbol{x} \in B_p} \left(\|\boldsymbol{x}\|_{\mathcal{G}'}^p\right)^{1/2}}{\widetilde{m}^{1/2}}.$$

We now verify that if we choose

$$\widetilde{m} \asymp \varepsilon^{-2} \left(\log d\right)^2 \log \left(\frac{d}{\varepsilon}\right) \cdot H \max_{i \in [m]} \min(p_i, \log |S_i|) \left(F^{\star}\right)^{\max(1, p/2)} \log \left(\frac{1}{\delta}\right),$$

that we indeed get for some universal constant C that

$$\Pr_{\mathcal{D}}\left[\max_{\boldsymbol{x}\in B_{p}}\left|\left\|\mathbf{A}\boldsymbol{x}\right\|_{\mathcal{G}_{p}}^{p}-\left\|\boldsymbol{x}\right\|_{\mathcal{G}'}^{p}\right|\geq C\varepsilon\right]\lesssim\delta.$$

We rescale  $\varepsilon$  appropriately and conclude the proof of Theorem 11.

# 3.5. Applications and algorithms

At this point in the chapter, we are ready to prove our main results (Theorem 8 and Theorem 9).

# 3.5.1. Block norm approximations via block Lewis weights (Proof of Theorem 8)

We restate and prove the main result of the chapter.

**Theorem 8** (Block Lewis weight sampling). Let  $\mathcal{G} = (\mathbf{A} \in \mathbb{R}^{n \times d}, S_1, \dots, S_m, p_1, \dots, p_m)$  where  $S_1, \dots, S_m$  form a partition of [k]. Suppose at least one of the following holds:

- $1 \le p < \infty$  and  $p_1, ..., p_m \ge 2;$
- $1/\log d \le p_1 = \cdots = p_m = p < \infty;$
- $p_1 = \cdots = p_m = 2$  and  $1/\log d \le p < \infty$ .

Let  $P := \max(1, \max_{i \in [m]} \min(p_i, \log |S_i|))$ . Then, there exists a probability distribution  $\mathcal{D} = (\rho_1, \ldots, \rho_m)$  such that if

$$\widetilde{m} = \Omega\left(\log\left(1/\delta\right)\varepsilon^{-2}\left(\log d\right)^2\log\left(d/\varepsilon\right)P\cdot d^{\max(1,p/2)}\right),$$

and if we sample  $\mathcal{M} \sim \mathcal{D}^{\widetilde{m}}$ , then, with probability  $\geq 1 - \delta$ ,

for all 
$$\mathbf{x} \in \mathbb{R}^d$$
,  $(1-\varepsilon) \|\mathbf{A}\mathbf{x}\|_{\mathcal{G}_p}^p \leq \frac{1}{\widetilde{m}} \sum_{i \in \mathcal{M}} \frac{1}{\rho_i} \cdot \|\mathbf{A}_{S_i}\mathbf{x}\|_{p_i}^p \leq (1+\varepsilon) \|\mathbf{A}\mathbf{x}\|_{\mathcal{G}_p}^p$ .

*Proof of Theorem 8.* Observe that Lemma 3.3.8 proves the existence of a probability measure  $\lambda$  over [m] and a rounding **W** that are  $F^*$ -block Lewis overestimates for  $F^* = d$  if  $p_i \ge 2$ , and Lemma 3.3.9 proves the existence of a probability measure  $\hat{\lambda}$  over [n] and corresponding  $\hat{\alpha} \in \mathbb{R}^n$  such that we get an  $F^* = d$ -Lewis overestimate.

We now apply Theorem 11 and conclude the proof of Theorem 8.

# 3.5.2. Efficient computation of block Lewis weight overestimates (Proof of Theorem 9)

In this subsection, we restate and prove Theorem 9.

**Theorem 9** (Computation of block Lewis weights). *Consider the setting of Theorem 8 and suppose at least one of the following holds:* 

- p = 2 and  $p_1, \ldots, p_m \ge 2;$
- $1/\log n \le p_1 = \cdots = p_m = p < \infty;$
- $p_1 = \cdots = p_m = 2$  and  $1/\log n \le p < \infty$ .

Let  $P = \max(1, \max_{i \in [m]} \min(p_i, \log |S_i|))$  and set

$$\widetilde{m} = O\left(\log\left(1/\delta\right)\varepsilon^{-2}\left(\log d\right)^2\log\left(d/\varepsilon\right)P\cdot d^{\max(1,p/2)}\right).$$

Then, there is an algorithm that outputs a probability distribution  $\mathcal{D} = (\rho_1, \dots, \rho_m)$  such that sampling a multiset  $\mathcal{M} \sim \mathcal{D}^{\widetilde{m}}$  satisfies, with probability  $1 - \delta$ ,

for all 
$$\mathbf{x} \in \mathbb{R}^d$$
,  $(1-\varepsilon) \|\mathbf{A}\mathbf{x}\|_{\mathcal{G}_p}^p \leq \frac{1}{\widetilde{m}} \sum_{i \in \mathcal{M}} \frac{1}{\rho_i} \cdot \|\mathbf{A}_{S_i}\mathbf{x}\|_{p_i}^p \leq (1+\varepsilon) \|\mathbf{A}\mathbf{x}\|_{\mathcal{G}_p}^p$ ,

*Further, the algorithm to find* D *performs at most polylog(k, n, m) leverage score overestimate computations or linear system solves.* 

We break up the proof into two sections – one where  $p_1 = \cdots = p_m = 2$  and p > 0 and another where p = 2 and  $p_1, \ldots, p_m \ge 2$ .

**Special case –**  $p > 0, p_1 = \cdots = p_m = 2$ 

Following Definition 3.1.2 and the discussion in Section 3.3.2, observe that when  $p_1 = \cdots = p_m = 2$ , we have

$$\beta_i(\mathbf{V}) \coloneqq \left(\sum_{j \in S_i} \boldsymbol{a}_j^\top (\mathbf{A}^\top \mathbf{V} \mathbf{A})^{-1} \boldsymbol{a}_j\right)^{1/2}.$$

As before, we call  $\beta_i(\mathbf{V})^p$  block Lewis weights. Given  $\mathbf{b} \in \mathbb{R}^m$ , we let  $\mathbf{B} \in \mathbb{R}^n$  be a diagonal matrix given by  $\mathbf{B}_{jj} = \mathbf{b}_i$  for all  $i \in [m]$ ,  $j \in S_i$ . First, let us specialize the definition of block Lewis weight overestimates (recall Definition 3.1.2) to this special case.

**Definition 3.5.1.** For  $v \ge 0$ , we say  $b \in \mathbb{R}^m_{\ge 0}$  is a vector of *v*-bounded block Lewis weight overestimates for **A** if

$$\|\boldsymbol{b}\|_1 \leq \boldsymbol{\nu},$$

and for all  $i \in [m]$ ,

$$b_i \ge \beta_i (\mathbf{B}^{1-2/p})^p.$$

We can think of the definition of block Lewis weight overestimates as being a relaxation of the fixed point condition for block Lewis weights that is described in [JLLS23, Page 31, Proof of Lemma 4.2].

As a primitive, our algorithm will use *leverage score overestimates* (see [JLS23, Definition 2.2]). They are approximate forms of leverage scores  $\tau_i(\mathbf{M})$ .

**Definition 3.5.2.** For  $v \ge 0$ , we say  $\tilde{\tau} \in \mathbb{R}^m_{\ge 0}$  is a vector of v-bounded leverage score overestimates for  $\mathbf{M} \in \mathbb{R}^{m \times d}$  if

 $\|\widetilde{\tau}\|_1 \leq \nu$ 

 $\widetilde{\tau}_i \geq \tau_i(\mathbf{M}).$ 

and for all  $i \in [k]$ ,

There are known efficient algorithms for computing leverage score overestimates or reducing leverage score computations to linear system solves.

**Theorem 3.5.3** ([JLS23, Theorem 3]). There is an algorithm OverLev that, given  $\mathbf{M} \in \mathbb{R}^{n \times d}$ , produces O(n)-bounded leverage score overestimates for  $\mathbf{M}$  in  $\widetilde{O}(\operatorname{nnz}(\mathbf{M}) + d^{\omega})$  time, where  $\omega$  is the matrix multiplication exponent.

We will use two different algorithms depending on the value of p. If  $p \le 2$ , then we present a contractive scheme reminiscent of the algorithm of Cohen and Peng [CP15]. If p > 2, we present an algorithm similar to those of Cohen, Cousins, Lee, and Yang [CCLY19] and Jambulapati, Liu, and Sidford [JLS22].

We begin with the case where 0 (in fact, we will see that this algorithm yields guarantees where <math>p < 4). The main objects of interest here are Algorithm 7 and Lemma 3.5.4.

**Algorithm 7** Algorithm to compute block Lewis weight overestimates,  $0 and <math>p_1 = \cdots = p_m = 2$ .

1: Input:  $\mathbf{A} \in \mathbb{R}^{n \times d}$ , outer norm  $0 , group structure <math>(S_1, \ldots, S_m, 2, \ldots, 2)$ .

2: Initialize  $\boldsymbol{b}^{(0)} = \frac{d}{m} \cdot \mathbb{1}_m$ .

3: Define  $\psi$  such that

$$\psi_i(\boldsymbol{b}) \coloneqq \left( \left( \sum_{j \in S_i} \left( \boldsymbol{a}_j^\top \left( \mathbf{A}^\top \mathbf{B}^{1-2/p} \mathbf{A} \right)^{-1} \boldsymbol{a}_j \right)^{p_i/2} \right)^{2/p_i} \right)^{p/2}$$

4: for t = 1, ..., T do 5:  $b^{(t)} = \max(\psi(b^{(t-1)}), \mathbb{1}_m \cdot 1/m)$ 6: return  $1.1b^{(T)}$ 

▶ The max is taken elementwise here.

Lemma 3.5.4. Let

$$T \geq \frac{\ln \left(\frac{\ln \left(\frac{m}{d}\right)}{\ln (1+\varepsilon)}\right)}{\ln \left(\left|\frac{2}{2-p}\right|\right)}.$$

Then, the weights  $1.1b^{(T)}$  output by Algorithm 7 are a 1.1*n*-block Lewis overestimate (Definition 3.1.2). Furthermore, computing  $b^{(T)}$  requires at most T computations of the vector whose entries are the  $a_j^{T} (\mathbf{A}^{T} \mathbf{D} \mathbf{A})^{-1} a_j$  for all *j*, where **D** is a diagonal matrix.

To prove Lemma 3.5.4, we first state and prove Lemma 3.5.5.

**Lemma 3.5.5.** Let  $\psi$  be as defined in Line 3 in Algorithm 7. For all  $u \in \mathbb{R}^m_{>0}$  and  $v \in \mathbb{R}^m_{>0}$ , we have

$$\max_{1 \le i \le m} \left| \ln \left( \frac{\psi_i(\boldsymbol{u})}{\psi_i(\boldsymbol{v})} \right) \right| \le \left| \frac{p}{2} - 1 \right| \max_{1 \le i \le m} \left| \ln \left( \frac{u_i}{v_i} \right) \right|,$$

and therefore  $\psi(\mathbf{u})$  is a contraction whenever 0 .

*Proof of Lemma* 3.5.5. Fix some index  $i \le m$ . For notational simplicity in this proof, let  $\alpha$  be such that  $\ln(\alpha) \coloneqq \Delta(u, v)$ .

This easily implies that

$$\frac{1}{\alpha^{|1-2/p|}} \cdot \boldsymbol{a}_j^{\top} \left( \mathbf{A}^{\top} \mathbf{V}^{1-2/p} \mathbf{A} \right)^{-1} \boldsymbol{a}_j \leq \boldsymbol{a}_j^{\top} \left( \mathbf{A}^{\top} \mathbf{U}^{1-2/p} \mathbf{A} \right)^{-1} \boldsymbol{a}_j \leq \alpha^{|1-2/p|} \cdot \boldsymbol{a}_j^{\top} \left( \mathbf{A}^{\top} \mathbf{V}^{1-2/p} \mathbf{A} \right)^{-1} \boldsymbol{a}_j.$$

We take the  $p_i/2$ -norm and take the p/2 power, which tells us that for all  $1 \le i \le m$ ,

$$\frac{1}{\alpha^{|p/2-1|}} \cdot \psi_i(\boldsymbol{v}) \le \psi_i(\boldsymbol{u}) \le \alpha^{|p/2-1|} \cdot \psi_i(\boldsymbol{v})$$

Hence,

$$\max_{1 \le i \le m} \left| \ln \left( \frac{\psi_i(\boldsymbol{u})}{\psi_i(\boldsymbol{v})} \right) \right| \le \left| \frac{p}{2} - 1 \right| \ln \left( \alpha \right) = \left| \frac{p}{2} - 1 \right| \max_{1 \le i \le m} \left| \ln \left( \frac{u_i}{v_i} \right) \right|,$$

completing the proof of Lemma 3.5.5.

We are now ready to complete the proof of Lemma 3.5.4.

*Proof of Lemma 3.5.4.* The computational complexity guarantee is immediate, so we focus on the approximation guarantee.

By Lemma 3.5.5 and the Banach fixed point theorem, we know that  $\psi$  has a unique fixed point. Denote this fixed point by  $b^*$ . We would like to argue that since the convergence in the ln-metric is linear, it takes roughly  $\log \log(1 + \varepsilon)/\log |2/(2 - p)|$  applications of  $\psi$  to reach a  $(1 + \varepsilon)$ -multiplicative approximation to  $b^*$ . An annoying technicality is that if  $b^*$  has some elements arbitrarily close to 0, then the convergence rate could be very slow. To fix this, we simply enforce that the coordinates of the iterates never drop below 1/m. It is easy to see that this only overestimates the true weights and therefore does not affect our sampling guarantees (in particular,  $\|\max(b^*, \mathbb{1}_m \cdot 1/m)\|_1 \le d + 1$ ).

More precisely, let  $\boldsymbol{b} \coloneqq \max(\boldsymbol{b}^{\star}, \mathbb{1}_m \cdot 1/m)$ . Notice that for all  $1 \le i \le m$ ,  $\left| \ln \left( \frac{b_i^{(0)}}{b_i} \right) \right| \le \ln \left( \frac{m}{d} \right)$ . This means that after *T* iterations, we have

$$\max_{1 \le i \le m} \left| \ln \left( \frac{b_i^{(T)}}{b_i} \right) \right| \le \left| \frac{p}{2} - 1 \right|^T \max_{1 \le i \le m} \left| \ln \left( \frac{b_i^{(0)}}{b_i} \right) \right| \le \left| \frac{p}{2} - 1 \right|^T \ln \left( \frac{m}{d} \right).$$

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Choosing

$$T \ge \frac{\ln\left(\frac{\ln\left(\frac{m}{d}\right)}{\ln(1+\varepsilon)}\right)}{\ln\left(\left|\frac{2}{2-p}\right|\right)}$$

and observing that for sampling that it is sufficient to choose  $\varepsilon = 0.1$  implies that  $b^{(T)}$  is an entrywise 1.1-approximation to b. As this is sufficient to get the concentration in the setting of Theorem 11, we may complete the proof of Lemma 3.5.4.

Now, we move onto the case where  $p \ge 2$ . This covers the cases of p where Algorithm 7 is not a contraction (whenever  $p \ge 4$ ). In this setting, we have Algorithm 8. At a high level, observe that Line 7 of Algorithm 8 performs a fixed point iteration on the stationary condition  $b_i = \beta_i (\mathbf{B}^{1-2/p})^p$  that holds for the optimal choice of block Lewis weights (see [JLLS23, proof of Lemma 4.2 and (4.5)]).

**Algorithm 8** Algorithm to compute block Lewis weight overestimates,  $p \ge 2$ 

1: Input:  $\mathbf{A} \in \mathbb{R}^{n \times d}$ 2: Output: 3: Initialize  $\mathbf{b}^{(1)} = \frac{d}{m} \cdot \mathbb{1}$ 4: for  $t = 1, \dots, T - 1$  do 5:  $\mathbf{B}_{jj}^{(t)} = b_i^{(t)}$  for all  $i \in [m], j \in S_i$ 6:  $\widetilde{\tau}^{(t)} = \text{OverLev}((\mathbf{B}^{(t)})^{1/2 - 1/p} \mathbf{A})$ 7:  $\mathbf{b}_i^{(t+1)} = \sum_{j \in S_i} \widetilde{\tau}_j$  for all  $i \in [m]$ 8:  $\overline{\mathbf{b}} = \frac{1}{T} \sum_{t=1}^T \mathbf{b}^{(t)}$ 9: return  $\mathbf{b} = \frac{3}{2}\overline{\mathbf{b}}$ 

The guarantee we obtain for Algorithm 8 is captured by Lemma 3.5.6.

**Lemma 3.5.6.** The return value **b** of Algorithm 8 is a vector of O(d)-bounded block Lewis weight overestimates. Further, this vector of overestimates is found in polylog(k, m, d) leverage score overestimate computations.

The goal of the rest of this section is to analyze Algorithm 8 and to prove Lemma 3.5.6. Let us briefly describe the analysis of Algorithm 8. We first give a collection of potential functions  $\varphi_i(b)$ , with the goal of showing the potential of  $\varphi_i(\overline{b})$  decreases with *T*. A low potential will also imply that  $\overline{b}$  is nearly a vector of block Lewis weight overestimates. For each  $i \in [m]$  define  $\varphi_i : \mathbb{R}^m \to \mathbb{R}$  by

$$\varphi_i(\boldsymbol{b}) \coloneqq \ln\left(\frac{1}{\boldsymbol{b}_i}\sum_{j\in S_i}\tau_j(\mathbf{B}^{1/2-1/p}\mathbf{A})\right).$$

The key property of this potential is convexity, which we now show.

**Lemma 3.5.7.** For each  $i \in [m]$ ,  $\varphi_i$  is convex.

*Proof of Lemma 3.5.7.* Our argument for the convexity of this function is similar to the one given in [JLS22, Lemma A.2]. First, notice that by the definition of  $\tau_i$ ,  $\varphi(\mathbf{b})$  is equal to

$$\varphi_i(\boldsymbol{b}) = \ln\left(\frac{1}{\boldsymbol{b}_i^{2/p}}\sum_{j\in S_i}\boldsymbol{a}_j^{\mathsf{T}}\left(\mathbf{A}^{\mathsf{T}}\mathbf{B}^{1-2/p}\mathbf{A}\right)^{-1}\boldsymbol{a}_j\right).$$

Since  $-\frac{2}{p}\ln(b_i)$  is convex, it suffices to show the convexity of

$$f(\boldsymbol{b}) \coloneqq \ln\left(\sum_{j \in S_i} \boldsymbol{a}_j^\top \left(\mathbf{A}^\top \mathbf{B}^{1-2/p} \mathbf{A}\right)^{-1} \boldsymbol{a}_j\right).$$

Now we define for each  $j \in [k]$  the function

$$h_j(\boldsymbol{b}) \coloneqq \ln\left(\boldsymbol{a}_j^{\mathsf{T}}(\mathbf{A}^{\mathsf{T}}\mathbf{B}^{1-2/p}\mathbf{A})^{-1}\boldsymbol{a}_j\right).$$

A version of this function without repeated entries in **B** was shown to be convex in [JLS22, Lemma A.2], but  $h_i(b)$  is still convex. Next, notice that we may write

$$f(\boldsymbol{b}) = \ln\left(\sum_{j \in S_i} \exp(h_j(\boldsymbol{b}))\right).$$

Now taking  $\boldsymbol{b}, \boldsymbol{b}' \in \mathbb{R}^m$  and  $\lambda \in [0, 1]$  we have

$$f(\lambda \boldsymbol{b} + (1 - \lambda)\boldsymbol{b}') = \ln\left(\sum_{j \in S_i} \exp(h_j(\lambda \boldsymbol{b} + (1 - \lambda)\boldsymbol{b}'))\right)$$
$$\leq \ln\left(\sum_{j \in S_i} \exp(\lambda h_j(\boldsymbol{b}) + (1 - \lambda)h_j(\boldsymbol{b}'))\right) \tag{3.5.1}$$

$$\leq \lambda \ln\left(\sum_{j \in S_i} \exp(h_j(\boldsymbol{b}))\right) + (1 - \lambda) \ln\left(\sum_{j \in S_i} \exp(h_j(\boldsymbol{b}'))\right)$$
(3.5.2)  
=  $\lambda f(\boldsymbol{b}) + (1 - \lambda) f(\boldsymbol{b}'),$ 

where (3.5.1) follows from the convexity of  $h_j$  and the monotonicity of log-sum-exp, and (3.5.2) is due to the convexity of log-sum-exp (see e.g. [BV04, Section 3.1.5]). Hence, we may conclude the proof of Lemma 3.5.7.

We now give an argument that  $\varphi_i(\overline{b}) = \widetilde{O}(1/T)$  using the convexity of  $\varphi_i$ .

**Lemma 3.5.8.** Assume that OVERLEV returns v-bounded leverage score overestimates. Then after Algorithm 8 runs for T iterations, we have for all  $i \in [m]$  that

$$\varphi_i(\overline{\boldsymbol{b}}) \leq \frac{1}{T} \ln\left(\frac{m\nu}{d}\right).$$

Proof of Lemma 3.5.8. We have

$$\varphi_i(\overline{\boldsymbol{b}}) \leq \frac{1}{T} \sum_{t=1}^T \varphi_i(\boldsymbol{b}^{(t)})$$

Jensen's inequality

$$= \frac{1}{T} \sum_{t=1}^{T} \ln\left(\frac{1}{b_i^{(t)}} \sum_{j \in S_i} \tau_j((\mathbf{B}^{(t)})^{1/2 - 1/p} \mathbf{A})\right) \qquad \text{Definition of } \varphi_i$$

$$\leq \frac{1}{T} \sum_{t=1}^{T} \ln\left(\frac{1}{b_i^{(t)}} \sum_{j \in S_i} \tilde{\tau}_j^{(t)}\right) \qquad \text{Definition of } \tau_j$$

$$= \frac{1}{T} \sum_{t=1}^{T} \ln\left(\frac{b_i^{(t+1)}}{b_i^{(t)}}\right) \qquad \text{Line 7}$$

$$= \frac{1}{T} \ln\left(\frac{b_i^{(T+1)}}{b_i^{(1)}}\right)$$

$$= \frac{1}{T} \ln(m/n) + \frac{1}{T} \ln(b_i^{(T+1)}). \qquad \text{Line 3}$$

In the above, for the sake of analysis we define  $\boldsymbol{b}_i^{(T+1)}$  to be as if the algorithm executed T + 1 iterations. Now,  $\boldsymbol{b}_i^{(T+1)} \leq \|\tilde{\tau}\|_1 \leq v$  by the fact that OverLev returns *v*-bounded leverage score overestimates. We therefore conclude the proof of Lemma 3.5.8.

Now, we show that polylog(k, m, d) leverage score overestimate computations suffice to find the  $(3/2 \cdot n)$ -bounded block Lewis weight overestimates. This proves Lemma 3.5.6.

*Proof of Lemma* 3.5.6. By Theorem 3.5.3, OVERLEV returns O(d)-bounded leverage score overestimates. We take  $T = O(\ln(m))$ ; clearly this satisfies the desired runtime guarantee. Further, using Lemma 3.5.8 and taking the constant in T large enough, we have for each i that  $\varphi_i(\overline{b}) \leq \ln(\frac{3}{2})$ . Thus we have

$$\frac{3}{2} \cdot \overline{b_i} \ge \sum_{j \in S_i} \tau_j(\overline{\mathbf{B}}^{1/2 - 1/p} \mathbf{A}).$$
(3.5.3)

And so

$$b_i \ge \frac{3}{2}\overline{b_i}$$
  

$$\ge \sum_{j \in S_i} \tau_j(\overline{\mathbf{B}}^{1/2-1/p} \mathbf{A}) \qquad \text{by (3.5.3)}$$
  

$$= \sum_{i \in S_i} \tau_j(\mathbf{B}^{1/2-1/p} \mathbf{A}). \qquad \text{Fact 3.2.2}$$

We now manipulate this guarantee into the desired guarantee of block Lewis weight overestimates by some algebra. Splitting powers on the left hand side of the above, we get

$$b_i^{2/p} \ge \frac{1}{b_i^{1-2/p}} \sum_{j \in S_i} \tau_j(\mathbf{B}^{1/2-1/p}\mathbf{A}).$$

Taking this to the p/2th power, we obtain

$$\boldsymbol{b}_i \geq \left(\frac{\sum_{j \in S_i} \tau_j(\mathbf{B}^{1/2-1/p}\mathbf{A})}{b_i^{1-2/p}}\right)^{p/2} = \beta_i(\mathbf{B}^{1-2/p})^p,$$

as desired. To bound  $||b||_1$ , notice

$$\|\boldsymbol{b}\|_{1} = \frac{3}{2} \|\overline{\boldsymbol{b}}\|_{1} \le \frac{3}{2} \frac{1}{T} \sum_{t=1}^{T} \|\boldsymbol{b}^{(t)}\|_{1} \le \frac{3}{2} \nu \le O(n).$$

Finally, to obtain the concentration statement, we observe that the above implies that we get a measure  $\lambda$  and a rounding **W** that are *F*\*-block Lewis estimates for *F*\* = *O*(*d*). This concludes the proof of Lemma 3.5.6.

**Special case –**  $p = 2, p_1, ..., p_m \ge 2$ 

9: return  $(b, \overline{u})$ 

Finally, we are ready to introduce and analyze the algorithm for the case where p = 2 and  $p_1, \ldots, p_m \ge 2$ . See Algorithm 9. The main property of Algorithm 9 is given in Lemma 3.5.9.

Algorithm 9 Algorithm to compute block Lewis weight overestimates, p = 2 and  $p_1, \ldots, p_m \ge 2$ 1: Input:  $\mathbf{A} \in \mathbb{R}^{n \times d}$ , group structure  $(S_1, \ldots, S_m, p_1, \ldots, p_m)$ . 2: Initialize  $\mathbf{b}^{(0)} = \frac{d}{m} \cdot \mathbb{1}$  and  $\mathbf{u}^{(0)}$  such that for all  $1 \le i \le m$ ,  $u_j^{(0)} = 1/|S_i|$  for all  $j \in S_i$ 3: for  $t = 1, \ldots, T - 1$  do

 $\begin{array}{l} \text{S. IOF } t = 1, \dots, T \quad \text{Full} \\ \text{Final } \mathbf{f}^{(t)} = \mathsf{OVERLev}(\mathbf{V}(\boldsymbol{u}^{(t-1)})^{1/2}\mathbf{A}) \quad \triangleright \mathbf{V}(\boldsymbol{u}) \text{ is such that } \boldsymbol{v}_j = u_j^{1-2/p_i} \text{ for all } 1 \leq i \leq m \text{ and } j \in S_i \\ \text{S:} \quad b_i^{(t)} = \sum_{j \in S_i} \widetilde{\tau}_j^{(t)} \text{ for all } 1 \leq i \leq m \\ \text{e:} \quad u_j^{(t)} = \widetilde{\tau}_j^{(t)} / (\sum_{j' \in S_i} \widetilde{\tau}_{j'}^{(t)}) \text{ for all } 1 \leq i \leq m \text{ and all } j \in S_i \\ \text{Final } \overline{\boldsymbol{b}} = |S_i|^{1/T} \cdot \frac{1}{T} \sum_{t=1}^T \boldsymbol{b}^{(t)} \\ \text{S:} \quad \overline{\boldsymbol{u}} = \frac{1}{T} \sum_{t=0}^{T-1} \boldsymbol{u}^{(t)} \end{array}$ 

**Lemma 3.5.9.** If OVERLEV is a routine that returns leverage score overestimates whose sum is at most v, then Algorithm 9 returns  $F^*$ -block Lewis overestimates (Definition 3.1.2) with  $F^* = \max_{1 \le i \le m} |S_i|^{1/T} \cdot v$ .

In particular, if  $T = \max_{1 \le i \le m} \log |S_i|$  and  $\nu \le (4/e)d$ , then we get  $F^* = 4d$ .

As with the analysis of Algorithm 8, we first prove that a relevant potential is convex and then show that we can control it effectively.

**Lemma 3.5.10.** Let  $\mathbf{U} \in \mathbb{R}^{k \times k}$  be a nonnegative diagonal matrix. Let  $\mathbf{V}(\boldsymbol{u})$  denote the matrix such that for all *i* and  $j \in S_i$ , we have  $v_j = \boldsymbol{u}_i^{1-2/p_i}$ . Then,  $\varphi$  as defined below is log-convex in  $\boldsymbol{u}$ .

for all 
$$j \in S_1 \cup \cdots \cup S_m$$
:  $\varphi_j(\boldsymbol{u}) = \ln\left(\frac{\tau_j\left((\mathbf{V}(\boldsymbol{u}))^{1/2}\mathbf{A}\right)}{u_j}\right)$ 

*Proof of Lemma 3.5.10.* We expand the above definition after taking the ln of both sides.

$$\varphi_j(\boldsymbol{u}) = \ln\left(\frac{\tau_j\left((\mathbf{V}(\boldsymbol{u}))^{1/2}\mathbf{A}\right)}{u_j}\right) = \ln\left(\boldsymbol{a}_j^\top\left(\mathbf{A}^\top\mathbf{V}(\boldsymbol{u})\mathbf{A}\right)^{-1}\boldsymbol{a}_j\right) + \ln\left(\frac{v_j}{u_j}\right)$$

$$= \ln \left( \boldsymbol{a}_j^{\mathsf{T}} \left( \mathbf{A}^{\mathsf{T}} \mathbf{V}(\boldsymbol{u}) \mathbf{A} \right)^{-1} \boldsymbol{a}_j \right) - \frac{2}{p_i} \ln \left( u_j \right).$$

The last term is convex, so it is sufficient to argue that the first term is convex. To see this, first observe that by concavity of  $x^{1-2/p_i}$ , we have

$$\lambda \mathbf{V}(\boldsymbol{u}^{(1)}) + (1-\lambda)\mathbf{V}(\boldsymbol{u}^{(2)}) \leq \mathbf{V}(\lambda \boldsymbol{u}^{(1)} + (1-\lambda)\boldsymbol{u}^{(2)}),$$

which implies

$$\begin{split} \ln\left(\boldsymbol{a}_{j}^{\top}\left(\boldsymbol{\mathbf{A}}^{\top}\boldsymbol{\mathbf{V}}(\lambda\boldsymbol{u}^{(1)}+(1-\lambda)\boldsymbol{u}^{(2)})\boldsymbol{\mathbf{A}}\right)^{-1}\boldsymbol{a}_{j}\right) &\leq \ln\left(\boldsymbol{a}_{j}^{\top}\left(\lambda\boldsymbol{\mathbf{A}}^{\top}\boldsymbol{\mathbf{V}}(\boldsymbol{u}^{(1)})\boldsymbol{\mathbf{A}}+(1-\lambda)\boldsymbol{\mathbf{A}}^{\top}\boldsymbol{\mathbf{V}}(\boldsymbol{u}^{(2)})\boldsymbol{\mathbf{A}}\right)^{-1}\boldsymbol{a}_{j}\right) \\ &\leq \lambda\ln\left(\boldsymbol{a}_{j}^{\top}\left(\boldsymbol{\mathbf{A}}^{\top}\boldsymbol{\mathbf{V}}(\boldsymbol{u}^{(1)})\boldsymbol{\mathbf{A}}\right)^{-1}\boldsymbol{a}_{j}\right) \\ &+(1-\lambda)\ln\left(\boldsymbol{a}_{j}^{\top}\left(\boldsymbol{\mathbf{A}}^{\top}\boldsymbol{\mathbf{V}}(\boldsymbol{u}^{(2)})\boldsymbol{\mathbf{A}}\right)^{-1}\boldsymbol{a}_{j}\right). \end{split}$$

Above, the last line follows from the well-known (see, e.g., [CCLY19, Lemma 3.4]) fact that  $\ln \left( a_j^{\top} \mathbf{M}^{-1} a_j \right)$  is convex in **M** where **M** is symmetric positive-semidefinite. This completes the proof of Lemma 3.5.10.

Next, we will show that any choice of nonnegative u such that  $\sum_{j \in S_i} u_j = b_i^{\left(1-\frac{2}{p}\right) \cdot \left(\frac{p_i}{p_i-2}\right)}$  can be used to find a rounding matrix **W**.

**Lemma 3.5.11.** For all  $x \in \mathbb{R}^d$  and all nonnegative u such that  $\sum_{j \in S_i} u_j = b_i^{\left(1-\frac{2}{p}\right)\cdot \left(\frac{p_i}{p_i-2}\right)}$ , we have

$$\frac{\left\|\mathbf{V}(\boldsymbol{u})^{1/2}\mathbf{A}\boldsymbol{x}\right\|_{2}}{\left(\sum_{i'\leq m}b_{i'}\right)^{1/2-1/p}}\leq \left\|\boldsymbol{\Lambda}^{1/2-1/p}\mathbf{A}\boldsymbol{x}\right\|_{\mathcal{G}_{2}}.$$

*Proof of Lemma* 3.5.11. We start with the LHS.

$$\begin{aligned} \left\| \mathbf{V}(\boldsymbol{u})^{1/2} \mathbf{A} \boldsymbol{x} \right\|_{2}^{2} &= \sum_{i=1}^{m} \sum_{j \in S_{i}} u_{j}^{1-2/p_{i}} \left| \left\langle \boldsymbol{a}_{j}, \boldsymbol{x} \right\rangle \right|^{2} \\ &\leq \sum_{i=1}^{m} b_{i}^{1-2/p} \left\| \mathbf{A}_{S_{i}} \boldsymbol{x} \right\|_{p_{i}}^{2} \end{aligned}$$
by Hölder's Inequality with powers  $\frac{p_{i}}{p_{i}-2}, \frac{p_{i}}{2}$ 

Normalizing concludes the proof of Lemma 3.5.11.

Now, we show that finding uniform overestimates  $\tau_j/u_j \leq b_i^{\left(\frac{2}{p}-\frac{2}{p_i}\right)\cdot\frac{p_i}{p_i-2}}$  is enough to satisfy Definition 3.1.2 with  $F^* = \sum_{i \leq m} b_i$ .

**Lemma 3.5.12.** If we have  $\boldsymbol{u}$  such that  $\tau_j(\mathbf{V}(\boldsymbol{u})^{1/2}\mathbf{A})/u_j \leq b_i^{\left(\frac{2}{p}-\frac{2}{p_i}\right)\cdot\frac{p_i}{p_i-2}}$  for all i, then the rounding matrix  $\mathbf{W} = \mathbf{V}(\boldsymbol{u})\mathbf{B}^{2/p-1}$  and measure  $\lambda_i = b_i/(\sum_{i' \leq m} b_{i'})$  is an  $F^*$ -block Lewis overestimate (Definition 3.1.2) with  $F^* = \sum_{i' \leq m} b_{i'}$ .

Proof of Lemma 3.5.12. Observe that we must have

$$\frac{\tau_j\left(\mathbf{V}^{1/2}\mathbf{A}\right)}{v_j} \le v_j^{\frac{2}{p_i-2}} \cdot b_i^{\left(\frac{2}{p}-\frac{2}{p_i}\right)\cdot\frac{p_i}{p_i-2}}.$$

Let  $T = \sum_{i' \le m} b_{i'}$ . Following Lemma 3.3.7, let  $\lambda_i = b_i/T$  and **W** be such that

$$\frac{\mathbf{V}^{1/2}}{T^{1/2-1/p}} = \mathbf{W}^{1/2} \mathbf{\Lambda}^{1/2-1/p}.$$

In particular, this means that

$$w_j = \frac{v_j}{(T\lambda_i)^{1-2/p}} = \frac{v_j}{b_i^{1-2/p}}.$$

Hence,

$$\begin{split} \left(\sum_{j\in S_{i}} \left(\frac{\tau_{j}\left(\mathbf{V}^{1/2}\mathbf{A}\right)}{w_{j}}\right)^{p_{i}/2}\right)^{2/p_{i}} &= b_{i}^{1-2/p} \left(\sum_{j\in S_{i}} \left(\frac{\tau_{j}\left(\mathbf{V}^{1/2}\mathbf{A}\right)}{v_{j}}\right)^{p_{i}/2}\right)^{2/p_{i}} \\ &\leq b_{i}^{1-2/p} \cdot b_{i}^{\left(\frac{2}{p}-\frac{2}{p_{i}}\right) \cdot \frac{p_{i}}{p_{i}-2}} \cdot \left(\sum_{j\in S_{i}} v_{j}^{\frac{2}{p_{i}-2} \cdot \frac{p_{j}}{2}}\right)^{2/p_{i}} \\ &= b_{i}^{1-2/p} \cdot b_{i}^{\left(\frac{2}{p}-\frac{2}{p_{i}}\right) \cdot \frac{p_{i}}{p_{i}-2}} \cdot \left(\sum_{j\in S_{i}} u_{j}\right)^{2/p_{i}} \\ &= b_{i}^{1-2/p} \cdot b_{i}^{\left(\frac{2}{p}-\frac{2}{p_{i}}\right) \cdot \frac{p_{i}}{p_{i}-2}} \cdot \left(\sum_{j\in S_{i}} u_{j}\right)^{2/p_{i}} \\ &= b_{i}^{1-2/p} \cdot b_{i}^{\left(\frac{2}{p}-\frac{2}{p_{i}}\right) \cdot \frac{p_{i}}{p_{i}-2}} \cdot \left(b_{i}^{\left(1-\frac{2}{p}\right) \cdot \left(\frac{p_{i}}{p_{i}-2}\right)}\right)^{2/p_{i}} = b_{i} = \lambda_{i} \left(\sum_{i' \leq m} b_{i'}\right), \end{split}$$

which is exactly the statement of Lemma 3.5.12.

We now have the tools we need to prove Lemma 3.5.9.

*Proof of Lemma 3.5.9.* Note that Algorithm 9 and Lemma 3.5.9 are reminiscent of [JLS23, Algorithm 2 and Theorem 4].

Using Lemma 3.5.10, we begin with using the convexity of the potential  $\varphi$ . For all  $j \in S_i$ , we have

$$\begin{split} \varphi_{j}(\overline{\boldsymbol{u}}) &\leq \frac{1}{T} \sum_{t=0}^{T-1} \varphi\left(\boldsymbol{u}^{(t)}\right) = \frac{1}{T} \sum_{t=0}^{T-1} \ln\left(\frac{\tau_{j}\left(\mathbf{V}(\boldsymbol{u}^{(t)})^{1/2}\mathbf{A}\right)}{u_{j}^{(t)}}\right) &\leq \frac{1}{T} \sum_{t=0}^{T-1} \ln\left(\frac{\widetilde{\tau}_{j}^{(t+1)}}{u_{j}^{(t)}}\right) \\ &= \frac{1}{T} \sum_{t=0}^{T-1} \left(\ln\left(\frac{u_{j}^{(t+1)}}{u_{j}^{(t)}}\right) + \ln\left(\sum_{j' \in S_{i}} \widetilde{\tau}_{j'}^{(t+1)}\right)\right) = \frac{1}{T} \ln\left(\frac{u_{j}^{(T)}}{u_{j}^{(0)}}\right) + \frac{1}{T} \sum_{t=0}^{T-1} \ln\left(\sum_{j' \in S_{i}} \widetilde{\tau}_{j'}^{(t+1)}\right) \end{split}$$

$$\leq \frac{1}{T} \ln \left( \frac{u_j^{(T)}}{u_j^{(0)}} \right) + \ln \left( \frac{1}{T} \sum_{t=0}^{T-1} \sum_{j' \in S_i} \tilde{\tau}_{j'}^{(t+1)} \right) = \frac{1}{T} \ln \left( \frac{u_j^{(T)}}{u_j^{(0)}} \right) + \ln \left( |S_i|^{-1/T} b_i \right) \leq \ln \left( b_i \right) + \ln \left( |S_i|^{-1/T} b_i \right) \leq \ln \left( b_i \right) + \ln \left( |S_i|^{-1/T} b_i \right) \leq \ln \left( b_i \right) + \ln \left( |S_i|^{-1/T} b_i \right) \leq \ln \left( b_i \right) + \ln \left( |S_i|^{-1/T} b_i \right) \leq \ln \left( b_i \right) + \ln \left( |S_i|^{-1/T} b_i \right) \leq \ln \left( b_i \right) + \ln \left( |S_i|^{-1/T} b_i \right) \leq \ln \left( b_i \right) + \ln \left( |S_i|^{-1/T} b_i \right) \leq \ln \left( b_i \right) + \ln \left( |S_i|^{-1/T} b_i \right) \leq \ln \left( b_i \right) + \ln \left( |S_i|^{-1/T} b_i \right) \leq \ln \left( b_i \right) + \ln \left( |S_i|^{-1/T} b_i \right) \leq \ln \left( b_i \right) + \ln \left( |S_i|^{-1/T} b_i \right) \leq \ln \left( b_i \right) + \ln \left( |S_i|^{-1/T} b_i \right) \leq \ln \left( b_i \right) + \ln \left( |S_i|^{-1/T} b_i \right) \leq \ln \left( b_i \right) + \ln \left( |S_i|^{-1/T} b_i \right) \leq \ln \left( b_i \right) + \ln \left( |S_i|^{-1/T} b_i \right) \leq \ln \left( b_i \right) + \ln \left( |S_i|^{-1/T} b_i \right) \leq \ln \left( b_i \right) + \ln \left( |S_i|^{-1/T} b_i \right) \leq \ln \left( b_i \right) + \ln \left( |S_i|^{-1/T} b_i \right) \leq \ln \left( b_i \right) + \ln \left( |S_i|^{-1/T} b_i \right) \leq \ln \left( b_i \right) + \ln \left( |S_i|^{-1/T} b_i \right) \leq \ln \left( b_i \right) + \ln \left( |S_i|^{-1/T} b_i \right) \leq \ln \left( b_i \right) + \ln \left( |S_i|^{-1/T} b_i \right) \leq \ln \left( b_i \right) + \ln \left( |S_i|^{-1/T} b_i \right) \leq \ln \left( b_i \right) + \ln \left( |S_i|^{-1/T} b_i \right) \leq \ln \left( b_i \right) + \ln \left( |S_i|^{-1/T} b_i \right) \leq \ln \left( b_i \right) + \ln \left$$

We now apply Lemma 3.5.12 and see that the measure  $\lambda_i = b_i / \|\overline{b}\|_1$  is a *F*\*-block Lewis overestimate (Definition 3.1.2) with  $F^* = \|\overline{b}\|_1$ .

Now, observe that

$$\left\|\overline{\boldsymbol{b}}\right\|_{1} \leq \max_{1 \leq i \leq m} \left|S_{i}\right|^{1/T} \cdot \frac{1}{T} \sum_{t=1}^{T} \left\|\boldsymbol{b}^{(t)}\right\|_{1} \leq \max_{1 \leq i \leq m} \left|S_{i}\right|^{1/T} \cdot \nu \leq 4d,$$

where we use our setting of *T* and the fact that OVERLEV returns leverage score estimates whose sum is at most (4/e)d. This completes the proof of Lemma 3.5.9.

We are finally ready to give the proof of Theorem 9.

*Proof of Theorem 9.* We have three cases.

- If  $0 and <math>p_1 = \cdots = p_m = 2$ , then the algorithm and guarantee on the weights follow from Algorithm 7 and Lemma 3.5.4.
- If  $p \ge 2$  and  $p_1 = \cdots = p_m = 2$ , then the algorithm and guarantee on the weights follow from Algorithm 8 and Lemma 3.5.6.
- If p = 2 and  $p_1, \ldots, p_m \ge 2$ , then the algorithm and guarantee on the weights follow from Algorithm 9 and Lemma 3.5.9.

We plug these guarantees into Theorem 11 and conclude the proof of Theorem 9.

#### 3.5.3. Minimizing sums of Euclidean norms (Proof of Theorem 10)

Recall the minimizing sums of Euclidean norms (MSN) problem (3.1.3). Given  $\mathbf{A} \in \mathbb{R}^{n \times d}$ , a partition  $S_1, \ldots, S_m$  of [n], and  $\mathbf{b}_1 \in \mathbb{R}^{|S_1|}, \ldots, \mathbf{b}_m \in \mathbb{R}^{|S_m|}$ , we would like to find  $\hat{\mathbf{x}}$  such that

$$\sum_{i=1}^{m} \|\mathbf{A}_{S_i} \widehat{\mathbf{x}} - \mathbf{b}_i\|_2 \le (1+\varepsilon) \min_{\mathbf{x} \in \mathbb{R}^d} \sum_{i=1}^{m} \|\mathbf{A}_{S_i} \mathbf{x} - \mathbf{b}_i\|_2$$

Xue and Ye [XY97] give an algorithm with iteration complexity  $O(\sqrt{m} \log (1/\epsilon))$  for the above problem (though for an additive approximation guarantee instead of a multiplicative one), where each iteration reduces to solving linear systems in matrices  $\mathbf{A}^{\mathsf{T}}\mathbf{D}\mathbf{A}$  for block-diagonal matrices, where each block has size  $(|S_i| + 1) \times (|S_i| + 1)$ . Their algorithm is based on the primal-dual interior point method framework.

Each system solve takes the following form. Let **A** be a block matrix with  $-\mathbf{I}_m$  in one block, and **A** in another. The goal is to find y in the system

$$\widetilde{\mathbf{A}}^{\top}\mathbf{D}\widetilde{\mathbf{A}}y = z$$

where **D** is a block matrix, with one  $(1 + |S_i|) \times (1 + |S_i|)$  sized block for each group (see [XY97, equation (4.13)]).

Our main result of this section is Theorem 10, which gives an improved iteration complexity for (3.1.3) when  $m \gg n$ .

**Theorem 10** (Minimizing sums of Euclidean norms). Let  $\mathbf{A} \in \mathbb{R}^{n \times d}$  and  $\mathbf{b} \in \mathbb{R}^{n}$ , and  $S_{1}, \ldots, S_{m}$  be a partition of k. There exists an algorithm that, with probability  $\geq 1 - \delta$ , returns  $\hat{\mathbf{x}}$  such that

$$\sum_{i=1}^{m} \|\mathbf{A}_{S_i} \widehat{\mathbf{x}} - \mathbf{b}_{S_i}\|_2 \leq (1+\varepsilon) \min_{\mathbf{x} \in \mathbb{R}^d} \sum_{i=1}^{m} \|\mathbf{A}_{S_i} \mathbf{x} - \mathbf{b}_{S_i}\|_2.$$

The algorithm runs in  $\widetilde{O}\left(\sqrt{d}/\varepsilon \cdot \sqrt{\log(1/\delta)}\right)$  calls to a linear system solver in matrices of the form  $\mathbf{A}^{\top}\mathbf{D}\mathbf{A}$  for block-diagonal matrices  $\mathbf{D}$ , where each block has size  $(|S_i| + 1) \times (|S_i| + 1)$ .

We prove Theorem 10 by sparsifying the objective (3.1.3) using Theorem 9 and then applying the primal-dual interior point method from [XY97]. We state the guarantee of this algorithm in Lemma 3.5.13.

**Lemma 3.5.13.** Let  $\mathbf{A} \in \mathbb{R}^{n \times d}$  and  $\mathbf{b} \in \mathbb{R}^n$ , and  $S_1, \ldots, S_m$  be a partition of k. There exists an algorithm that returns  $\hat{\mathbf{x}}$  such that

$$\sum_{i=1}^{m} \left\| \mathbf{A}_{S_{i}} \widehat{\boldsymbol{x}} - \boldsymbol{b}_{S_{i}} \right\|_{2} \leq (1 + \varepsilon) \min_{\boldsymbol{x} \in \mathbb{R}^{d}} \sum_{i=1}^{m} \left\| \mathbf{A}_{S_{i}} \boldsymbol{x} - \boldsymbol{b}_{S_{i}} \right\|_{2}.$$

The algorithm runs in  $\widetilde{O}(\sqrt{m}\log(1/\varepsilon))$  calls to a linear system solver in matrices of the form  $\mathbf{A}^{\mathsf{T}}\mathbf{D}\mathbf{A}$  for block-diagonal matrices  $\mathbf{D}$ , where each block has size  $(|S_i| + 1) \times (|S_i| + 1)$ .

*Proof of Lemma 3.5.13.* The guarantee we will reduce to is [XY97, Theorem 5.2]. However, the guarantee there is stated for an additive approximation, and we desire a multiplicative approximation. We therefore apply a few transformations to our problem so that we can apply this guarantee.

Let

$$\mathbf{x}_0 \coloneqq \operatorname*{argmin}_{\mathbf{x} \in \mathbb{R}^d} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2.$$

This can be found in one linear system solve. Let  $V \coloneqq ||\mathbf{A}\mathbf{x}_0 - \mathbf{b}||_2$  and consider the following modified optimization problem:

$$\min_{\boldsymbol{x}-\boldsymbol{x}_{0}\in\mathbb{R}^{d}}\frac{1}{V}\sum_{i=1}^{m}\left\|\mathbf{A}_{S_{i}}(\boldsymbol{x}-\boldsymbol{x}_{0})-(\boldsymbol{b}_{S_{i}}-\mathbf{A}_{S_{i}}\boldsymbol{x}_{0})\right\|_{2}.$$
(3.5.4)

We will invoke [XY97, Theorem 5.2] on the above problem (3.5.4), folding the  $\frac{1}{V}$  factor into **A** and **b**. Clearly, this problem is equivalent to the problem we started with. Next, let  $x^*$  be given by

$$\mathbf{x}^{\star} \coloneqq \operatorname*{argmin}_{\mathbf{x} \in \mathbb{R}^d} \sum_{i=1}^m \|\mathbf{A}_{S_i}\mathbf{x} - \mathbf{b}_{S_i}\|_2,$$

where we use  $\boldsymbol{b} \in \mathbb{R}^n$  for the vector formed by stacking  $\boldsymbol{b}_{S_1}, \ldots, \boldsymbol{b}_{S_m}$ . Let OPT :=  $1/v \cdot \sum_{i=1}^m \|\mathbf{A}_{S_i} \boldsymbol{x}^* - \boldsymbol{b}_{S_i}\|_2$ . Because  $\|\cdot\|_2 \leq \|\cdot\|_1$ , we have

$$1 = \frac{\|\mathbf{A}\mathbf{x}_{0} - \mathbf{b}\|_{2}}{V} \le \frac{\|\mathbf{A}\mathbf{x}^{\star} - \mathbf{b}\|_{2}}{V} \le \frac{1}{V} \sum_{i=1}^{m} \|\mathbf{A}_{S_{i}}\mathbf{x}^{\star} - \mathbf{b}_{S_{i}}\|_{2} = \mathsf{OPT}.$$

Furthermore, we have by  $\|\cdot\|_1 \le \sqrt{m} \|\cdot\|_2$  (applied by considering the summation over *i* as an  $\ell_1$  norm) that

$$\max_{1 \le i \le m} \frac{1}{V} \| \boldsymbol{b}_{S_i} - \mathbf{A}_{S_i} \boldsymbol{x}_0 \|_2 \le \frac{1}{V} \sum_{i=1}^m \| \boldsymbol{b}_{S_i} - \mathbf{A}_{S_i} \boldsymbol{x}_0 \|_2 \le \sqrt{m} \cdot \frac{\| \mathbf{A} \boldsymbol{x}_0 - \boldsymbol{b} \|_2}{V} = \sqrt{m}.$$

This implies that all the offset vectors of our transformed problem (3.5.4) have polynomially bounded norm. This suffices for our iteration complexity bound, because the iteration complexity of the algorithm in [XY97] depends logarithmically on the maximum norm of these vectors. Now, using their method we can solve (3.5.4) up to  $\varepsilon$  additive error. This means we find  $\hat{x}$  such that

$$\sum_{i=1}^{m} \left\| \mathbf{A}_{S_{i}}(\widehat{\mathbf{x}} - \mathbf{x}_{0}) - (\mathbf{b}_{S_{i}} - \mathbf{A}_{S_{i}}\mathbf{x}_{0}) \right\|_{2} \leq \sum_{i=1}^{m} \left\| \mathbf{A}_{S_{i}}(\mathbf{x}^{\star} - \mathbf{x}_{0}) - (\mathbf{b}_{S_{i}} - \mathbf{A}_{S_{i}}\mathbf{x}_{0}) \right\|_{2} + V\varepsilon$$
$$= V \cdot \mathsf{OPT}\left( 1 + \frac{\varepsilon}{\mathsf{OPT}} \right) \leq V \cdot \mathsf{OPT}\left( 1 + \varepsilon \right).$$

where in the last inequality, we use that  $OPT \ge 1$ . Since  $V \cdot OPT$  is the original optimal objective value, this completes the proof of Lemma 3.5.13.

We remark that instead of the  $\sqrt{m}$  above, one can get a  $\sqrt{d}$ -factor relationship between the optimal objective for a least squares relaxation of our problem by using the block Lewis weights with  $p_1 = \cdots = p_m = 2$  and p = 1. However, this will only impact lower order terms.

We are now ready to prove Theorem 10.

*Proof of Theorem 10.* We apply Theorem 9 for  $p_1 = \cdots = p_m = 2$  and p = 1 with approximation  $\varepsilon/3$  to the group matrices  $[\mathbf{A}_{S_i}|\mathbf{b}_{S_i}]$  to find a sparsified objective with  $\widetilde{m} = O(\varepsilon^{-2} \cdot d(\log d)^2 \log(d/\varepsilon))$  terms. This requires  $\widetilde{O}(1)$  linear system solves. This implies a  $(1 + \varepsilon)$  approximation to an un-sparsified objective over any vector in  $\mathbb{R}^{d+1}$ , and the approximation we need comes by only considering vectors in  $\mathbb{R}^{d+1}$  whose last entry is -1. We plug this into the guarantee of Lemma 3.5.13 with approximation  $\varepsilon/3$ . Since  $(1 + \varepsilon/3)^2 \le 1 + \varepsilon$  and  $(1 + \varepsilon/3)/(1 - \varepsilon/3) \le 1 + \varepsilon$ , this returns a  $(1 + \varepsilon)$ -approximate minimizer to (3.1.3), completing the proof of Theorem 10.

# 4. Block Lewis weights for distributionally robust linear regression

In this chapter, we continue studying applications of block Lewis weights. This time, we use them for minimizing a multidistributional linear regression loss. The material in this chapter is based on a joint work with Kumar Kshitij Patel [MP24].

# 4.1. Introduction

Machine learning algorithms and their training datasets have grown tremendously in the past decade, both in size and complexity. This increased model complexity has made it challenging to interpret and predict their behavior in unobserved scenarios. Hence, many applications that involve societal decisions still rely on simple, interpretable models like linear regression, often after feature engineering. Examples of such applications are predicting housing prices across cities, estimating wages across industries, forecasting loan amounts across banks, predicting life insurance premiums for different groups, and projecting energy consumption in various communities.

A shared safety and sometimes legal concern across the above applications is the potential for wildly different model qualities for different distributions, i.e., outputting a notably worse model for some source data distributions [Dat14; BS16; HPS16; VVB18; SBFVV19; BHJKR21; CGNSG23; Cho16; KLMR18; ADW19; CGKMN24; SVWZ24]. Specifically, consider fitting a linear model  $x \in \mathbb{R}^d$  to make real predictions on some task over *m* groups where group *i*'s dataset consists of  $n_i$  entries and is denoted by  $S_i = \{(a_i^j, b_i^j)\}_{j \in [n_i]}$ . The *utilitarian* or the total-cost-minimizing objective minimizes the average squared prediction error across groups, i.e.,

$$\min_{\boldsymbol{x}\in\mathbb{R}^d} \frac{1}{m} \sum_{i\in[m]} \frac{1}{n_i} \|\mathbf{A}_{S_i}\boldsymbol{x} - \boldsymbol{b}_{S_i}\|_2^2 \quad , \tag{4.1.1}$$

where  $\mathbf{A}_{S_i} \coloneqq [a_i^1 \dots a_i^{n_i}]^\top \in \mathbb{R}^{n_i \times d}$  is the feature matrix and  $\mathbf{b}_{S_i} \coloneqq [b_i^1 \dots b_i^{n_i}]^\top \in \mathbb{R}^{n_i}$  is the label vector for group  $i \in [m]$ .

Because of the inherent heterogeneity of the datasets, the model derived from optimizing objective (4.1.1) may be particularly bad for some groups, in that the prediction error could be disproportionately higher for these groups. To overcome these limitations, the following *egalitarian* or group Distributionally Robust Optimization (DRO) objective has been considered in several recent works [BDDMR13; DGN16; SKHL20; LCDS20; SGJ22; AAKMRZ22; SVWZ24],

$$\min_{\boldsymbol{x} \in \mathbb{R}^d} \max_{i \in [m]} \frac{1}{n_i} \| \mathbf{A}_{S_i} \boldsymbol{x} - \boldsymbol{b}_{S_i} \|_2^2 \quad .$$
(4.1.2)

Objective (4.1.2) is the "fairest" objective among all objectives that balance utility and distributional robustness by ensuring that no one group has a loss that is too high [KLMR18; CR18; ANSS22; CGSB22; RVFRWYT19; GNPS24] Since (4.1.2) is a convex problem, it is natural to apply standard black-box optimization techniques to solve them. However, we identify several challenges in applying existing methods:

Efficient first-order algorithms have geometry-dependent rates. To our knowledge, using an efficient first-order method (such as sub-gradient descent) will incur a geometry-dependent runtime. In particular, if the matrices  $\mathbf{A}_{S_i}$  or if the stacked matrix  $\mathbf{A} \coloneqq [\mathbf{A}_{S_1}^{\mathsf{T}} \dots \mathbf{A}_{S_M}^{\mathsf{T}}]^{\mathsf{T}}$  are poorly conditioned, then this will be reflected accordingly in the convergence rates. This is a drawback of the existing results by Abernethy, Awasthi, Kleindessner, Morgenstern, Russell, and Zhang [AAKMRZ22] and Song, Vakilian, Woodruff, and Zhou [SVWZ24].

**Objective** (4.1.2) **is not smooth.** Since the objective is the pointwise maximum of several continuous functions, the derivative is not well-defined at the points at which the maximizing function changes. Thus, applying subgradient descent to this objective without a customized analysis will result in a rather unimpressive  $1/\varepsilon^2$  dependence in the iteration complexity.

**Min-max optimization approaches have a**  $1/\varepsilon^2$  **dependence on iteration complexity.** Since problem 4.1.2 is a min-max optimization objective, it is also natural to try to use game theory-inspired approaches that use some oracle (such as gradients) for each group as a black box. Perhaps the most basic such algorithm is casting objective (4.1.2) as a repeated game between a min player (equipped with a no-regret algorithm) and a max player (equipped with the best response oracle). The main shortcoming of this approach is that even though the function for each group is smooth, the iteration complexity (to get  $\varepsilon$  average regret) for smooth online convex optimization still has an unimpressive  $1/\varepsilon^2$  dependence (as opposed to  $1/\varepsilon$  for smooth convex optimization) [SGJ22; ZZZYZ24]. Thus, this approach is no better than directly applying sub-gradient descent to objective (4.1.2).

Interior point methods have a poor iteration complexity for large *m*. Another natural approach (that can partially address the previous two issues), following the discussion by Boyd and Vandenberghe [BV04, Section 6.4], is to rewrite the problem (4.1.2) in its epigraph form and use an interior point method (IPM) to solve the resulting problem (which, in this case, is a quadratically constrained linear program). Unfortunately, this will give an algorithm whose analysis is only known to yield an iteration complexity of  $O(\sqrt{m})$ , where each iteration solves a linear system in matrices of the form  $\mathbf{A}^{\top}\mathbf{B}\mathbf{A}$  for a block-diagonal **B** (see Remark 4.1.1). A naïve implementation of this algorithm will thus have a superlinear runtime in the number of groups, which is undesirable when the number of groups is large. Furthermore, note that (4.1.2) is not a linear program when at least one group *i* is such that  $n_i > 1$ . So, we cannot immediately apply recent advances in linear programming that get iteration complexities independent of the number of constraints [LS19].

Hence, designing an algorithm without these shortcomings requires novel ideas.

# 4.1.1. Our results

In this chapter, we give a new algorithm (Algorithm 12) to approximately optimize (4.1.2) that overcomes the above difficulties. We state our algorithm's iteration complexity in the following theorem.

**Theorem 12** (Robust regression). Let  $\mathbf{A}_{S_i} \in \mathbb{R}^{n_i \times d}$  and  $\mathbf{b}_{S_i} \in \mathbb{R}^{n_i}$  for all  $i \in [m]$ . Denote their concatenations by  $\mathbf{A} := [\mathbf{A}_{S_1}^\top \dots \mathbf{A}_{S_M}^\top]^\top \in \mathbb{R}^{n \times d}$  and  $\mathbf{b} := [\mathbf{b}_{S_1}^\top \dots \mathbf{b}_{S_M}^\top]^\top \in \mathbb{R}^n$  where  $n := \sum_{i \in [m]} n_i$ . Let  $\varepsilon > 0$ . Then Algorithm 12 returns  $\widehat{\mathbf{x}}$  such that,

$$\max_{i \in [m]} \frac{1}{\sqrt{n_i}} \left\| \mathbf{A}_{S_i} \widehat{\mathbf{x}} - \mathbf{b}_{S_i} \right\|_2 \le (1 + \varepsilon) \cdot \min_{\mathbf{x} \in \mathbb{R}^d} \max_{i \in [m]} \frac{1}{\sqrt{n_i}} \left\| \mathbf{A}_{S_i} \mathbf{x} - \mathbf{b}_{S_i} \right\|_2 , \qquad (4.1.3)$$

and it runs in

$$O\left(\frac{\min\left\{\mathsf{rank}(\mathbf{A}),m\right\}^{1/3}\left(\log\left(\frac{n\log m}{\varepsilon}\right)^{14/3}+\log\left(m\right)\right)}{\varepsilon^{2/3}}\right)$$

*linear system solves in matrices of the form*  $\mathbf{A}^{\top}\mathbf{B}\mathbf{A}$ *, where*  $\mathbf{B}$  *is a block-diagonal matrix for which block i has size*  $n_i \times n_i$ *.* 

We prove Theorem 12 in Section 4.5.

We compare the guarantee of Theorem 12 against the other baselines in Table 4.1. Unlike the aforementioned first-order methods, our algorithm has no geometry-dependent terms. Additionally, our algorithm improves over the standard log-barrier IPM when the desired accuracy  $\varepsilon \ge m^{-1/4}$  — this improvement is more pronounced when  $m \gg \operatorname{rank}(\mathbf{A})$ , i.e. when the number of data sources is much larger than the dimension of the parameter vector  $\mathbf{x}$ . Additionally, for  $\varepsilon \ge \operatorname{rank}(\mathbf{A})^{-1/4}$ , our guarantee matches the best known guarantee for  $\ell_{\infty}$ regression [LS19; JLS22].

**Remark 4.1.1** (Why use linear-system-solve complexity?). We benchmark our algorithms using the number of linear-system-solves for a few reasons. First, this is typically how second-order algorithms are compared, such as interior point methods for linear programming [LS19]. Second, the particular structure of the linear system solves presents the possibility for a faster amortized runtime for the systems over the whole algorithm. This observation, combined with an understanding of how the linear systems changed between iterations, was used recently to get fast runtimes for linear programming [LS19] and  $\ell_{\infty}$  regression [AJK24].

**Interpolating between robust and nonrobust optimization.** We also study the following family of objectives that interpolate between (4.1.1) and (4.1.2) for different values of  $p \ge 2$ ,

$$\min_{\boldsymbol{x}\in\mathbb{R}^d} \frac{1}{m} \sum_{i\in[m]} \left( \frac{1}{n_i} \|\mathbf{A}_{S_i} \boldsymbol{x} - \boldsymbol{b}_{S_i}\|_2^2 \right)^{p/2} \quad .$$
(4.1.4)

In particular, note that choosing p = 2 in the above objective gives us the average least-squares problem in objective (4.1.1), while  $p \rightarrow \infty$  recovers objective (4.1.2). Varying p from 2 to  $\infty$  and minimizing gives solutions that interpolate between utilitarian and egalitarian approaches, allowing for a smooth trade-off between utility and robustness. To this end, we give an algorithm (Algorithm 14) to approximately optimize (4.1.4) and prove the following guarantee about its iteration complexity.

**Theorem 13** (Trading off utility and robustness). Let  $\mathbf{A}_{S_i} \in \mathbb{R}^{n_i \times d}$  and  $\mathbf{b}_{S_i} \in \mathbb{R}^{n_i}$  for all  $i \in [m]$ . Denote their concatenations by  $\mathbf{A} := [\mathbf{A}_{S_1}^\top \dots \mathbf{A}_{S_M}^\top]^\top \in \mathbb{R}^{n \times d}$  and  $\mathbf{b} := [\mathbf{b}_{S_1}^\top \dots \mathbf{b}_{S_M}^\top]^\top \in \mathbb{R}^n$  where  $n \coloneqq \sum_{i \in [m]} n_i$ . Let  $p \ge 2$  and  $\varepsilon > 0$ . Then Algorithm 14 returns  $\widehat{x}$  such that,

$$\left(\sum_{i=1}^{m} \left(\frac{1}{\sqrt{n_i}} \left\|\mathbf{A}_{S_i} \widehat{\mathbf{x}} - \mathbf{b}_{S_i}\right\|_2\right)^p\right)^{1/p} \le (1+\varepsilon) \cdot \min_{\mathbf{x} \in \mathbb{R}^d} \left(\sum_{i=1}^{m} \left(\frac{1}{\sqrt{n_i}} \left\|\mathbf{A}_{S_i} \mathbf{x} - \mathbf{b}_{S_i}\right\|_2\right)^p\right)^{1/p}$$
(4.1.5)

and runs in

$$O\left(p^{O(1)}\min\left\{\mathsf{rank}\left(\mathbf{A}\right),m
ight\}^{rac{p-2}{3p-2}}\log\left(rac{pd}{\varepsilon}
ight)^{3}
ight)$$

*linear system solves in matrices of the form*  $\mathbf{A}^{\top}\mathbf{B}\mathbf{A}$ *, where*  $\mathbf{B}$  *is a block-diagonal matrix for which block i has size*  $n_i \times n_i$ .

We prove Theorem 13 in Section 4.6.

In the special case where  $n_i = 1$  for all *i* (and therefore the problem is  $\ell_p$  regression for  $p \ge 2$ ), the complexity promised by Theorem 13 is comparable to that promised by Jambulapati, Liu, and Sidford [JLS22] for  $\ell_p$  regression. The main difference is that our iteration complexity is unconditionally polynomial in *p*. In contrast, the comparable result from [JLS22] seems to require mild assumptions on the problem parameters (see the "Discussion on numerical stability" in [JLS22, Section 4]).

**Remark 4.1.2** (Large values of *p*). Note that for values of *p* larger than  $\log(m)$ , solving (4.1.2) is morally equivalent to solving (4.1.4). To intuitively see this, first recall that for any vector  $\mathbf{x} \in \mathbb{R}^d$  and  $p = \log_2(m)$  we have,  $\|\mathbf{x}\|_{\infty} \leq \|\mathbf{x}\|_p \leq 2 \cdot \|\mathbf{x}\|_{\infty}$ . This implies that for all  $i \in [m]$  we have the following for objective (4.1.4) (for  $p = \log_2(m)$ ) for any  $\mathbf{x} \in \mathbb{R}^d$ ,

$$\max_{i \in [m]} \|\mathbf{A}_{S_i} \mathbf{x} - \mathbf{b}_{S_i}\|_2 \le \left(\sum_{i \in [m]} \|\mathbf{A}_{S_i} \mathbf{x} - \mathbf{b}_{S_i}\|_2^p\right)^{1/p} \le 2 \cdot \max_{i \in [m]} \|\mathbf{A}_{S_i} \mathbf{x} - \mathbf{b}_{S_i}\|_2 .$$

In particular, this means that minimizing the interpolating objective (4.1.4) also minimizes the robust objective (4.1.2) (up to numerical constants) and vice versa. Thus, for  $p = \Omega(\log_2(m))$ , for our intended applications, it makes sense to minimize the robust objective instead. This is why, in Theorem 13, we do not care too much about the exponent on p in the iteration complexity. Our main goal is to show that we can get a  $O(\text{poly}(p, \log(\frac{1}{\epsilon})) \min{\{\text{rank}(\mathbf{A}), m\}}^{1/3})$  iteration complexity.

#### 4.1.2. Prior results, connections, and open problems

Here, we discuss prior work that conceptually and technically relates to ours. We then suggest natural directions for future work.

**Multi-distribution learning.** Many learning problems involve multiple data sources, for instance, when multiple agents generate their data independently. One can formulate these multi-distribution problems as standard learning or optimization problems by considering a mixture of their distributions, as in objective (4.1.1). However, this approach often biases solutions toward dominant data sources, leading to poor performance on outliers—an issue stemming from statistical heterogeneity. This limitation motivates the study of multi-objective optimization problems [Mie99; Ehr05], where each agent *m* has a distribution  $\mathcal{D}_m$  that defines its objective as  $\mathbb{E}_{z\sim\mathcal{D}_m}[f(\mathbf{x}_m; z)]$ , and where models  $\mathbf{x}_m$  can vary across agents—a framework known as personalization.

One of the earliest algorithms for such problems was introduced by Blum, Haghtalab, Procaccia, and Qiao [BHPQ17], where each agent's objective must be minimized to a pre-specified threshold  $\epsilon$  with high probability, framed within a PAC learning framework [Val84; Vap13]. Subsequent research has refined these algorithms, achieving optimal sample complexity guarantees for learning from multiple distributions [CZZ18; NZ18; HK19; HJZ22; ZZCDL24]. Our objectives (4.1.2) and (4.1.4) offer different approaches to multi-distribution learning, where data distributions correspond to empirical agent distributions. In particular, Mohri, Sivek, and Suresh [MSS19] analyzed objective (4.1.2) to establish generalization bounds for unknown mixtures of agents' distributions.

Beyond sample efficiency, researchers have also examined other challenges, such as communication costs in large-scale distributed optimization [MMRHyA17]. A particularly relevant study is that of Bullins, Patel, Shamir, Srebro, and Woodworth [BPSSW21], which employs an efficient distributed quadratic sub-solver [WPSDBMSS20; PGZWSCJS24] to implement an inexact Newton method for optimizing quasi-self-concordant functions (see Definition 4.2.3).

**Group fairness.** Recently, interest in algorithmic fairness has intensified [BS16; ABKLRR20; KA21] with researchers exploring fairness across various domains, including supervised learning [CKP09; DHPRZ12; HPS16; KLRS17; GYF18; ULP19], resource allocation [BFT11; BFT12; HW12; DK20; MNR21], scheduling [MR21], online matching [CKLV19; MXX23], assortment planning [SJ18; BGW18; SJ19; CGSB22], and facility location [GMS22]. The extensive literature on algorithmic fairness falls into three main categories: (1) individual fairness, which ensures that similar individuals receive comparable predictions [DHPRZ12; LHH19; CGSB22], (2) group fairness, which aims for equal treatment of different demographic groups, often in terms of resource allocation or performance parity [SJ18; BLM21], and (3) subgroup fairness, which blends aspects of both individual and group fairness [KNRW18; KNRW19].

This chapter focuses on a well-studied group fairness notion in machine learning literature: the group DRO problem [BDDMR13; DGN16; SKHL20]. The idea of interpolating between robust-ness and utility is also common [GNPS24] and closely related to multi-objective optimization, where scalarization [Mie99; Ehr05] helps recover desired solutions along the Pareto frontier.

**Linear programming and**  $\ell_p$  **regression.** In the last several years, there has been a surge of work in obtaining second-order, condition-free algorithms for linear programming and  $\ell_p$  regression [BCLL18; LS19; AKPS19; JLS22]. Observe that  $\ell_p$  regression is a special case of the problem we study in objective (4.1.4), which is recovered when all  $n_i = 1$ , and  $\ell_{\infty}$  regression is captured by linear programming. Note that neither of these problem families is expressive enough to capture the objectives we study. In general, to get iteration complexities in the smaller of the two dimensions for these problems, it seems that a geometric understanding of the solution space is required – these ideas were central to the improvements obtained by [LS19; JLS22] and in our work.

**Open problems.** Our work raises several open questions. One limitation of Theorem 12 is that its iteration complexity is not high-accuracy, meaning its dependence on  $\varepsilon$  is not  $polylog(1/\varepsilon)$ .

Designing a high-accuracy solver under the same conditions as Theorem 13 with iteration complexity  $\widetilde{O}(\operatorname{poly}(\min \{\operatorname{rank}(\mathbf{A}), m\}, \log (\frac{1}{\varepsilon})))$  remains an open problem.

A more ambitious general goal is to design algorithms for convex quadratic programs with the aforementioned iteration complexity. This would generalize analogous results for linear programming [LS19]. We view the current work as a first step towards this goal, as the objective (4.1.2) is a structured convex quadratic program for which we get an iteration complexity independent of *m*. It would also be interesting to consider other complexity measures beyond rank (**A**), for instance, assumptions about the ground-truth labeling vector  $x_i^*$  for each group's data  $S_i$ .

Finally, our results suggest that optimizing for " $\ell_p$ -interpolants" between non-robust and robust objectives may be computationally easier than optimizing for the robust objective alone. A more precise statistical characterization of how robustness and utility trade-off as p varies in collaborative, fair, or multi-distributional learning settings would be valuable. Additionally, exploring interpolations or solution concepts along the Pareto frontier of the *m*-dimensional multi-objective optimization problem could yield further insights.

# 4.1.3. Chapter outline

In the rest of this chapter we will outline the key details of our approach as well as give a proof outline for our theoretical results. In Section 4.2, we give proof sketches of our main results. In Section 4.3, we give an analysis of mirror descent under inexact subproblem solves – we will need this in the proof of Theorem 13. In Section 4.4, we modify an acceleration scheme due to [CHJJS22], which we will use to iterate calls to the proximal subproblem solver (4.2.2) for the proof of Theorem 13. In Section 4.5, we prove Theorem 12. In Section 4.6, we prove Theorem 13.

# 4.2. Technical overview

In this section, we sketch our proofs for Theorem 12 and Theorem 13.

**Notation.** Here and in the rest of the chapter, we ignore the dataset size normalization factors  $1/\sqrt{n_i}$  as we can fold this into  $\mathbf{A}_{S_i}$  and  $\mathbf{b}_{S_i}$ . Additionally, let  $f(\mathbf{x}) \coloneqq \sum_{i=1}^m \|\mathbf{A}_{S_i}\mathbf{x} - \mathbf{b}_{S_i}\|_2^p$  if  $2 \le p < \infty$  and let  $f(\mathbf{x}) \coloneqq \max_{1 \le i \le m} \|\mathbf{A}_{S_i}\mathbf{x} - \mathbf{b}_{S_i}\|_2$  if  $p = \infty$ . Note that in the  $2 \le p < \infty$  case, we let  $f(\mathbf{x})$  be the *p*th power of the objective written in Theorem 13; this is to make future calculations easier and makes a difference of only polynomial factors in *p* in the iteration complexity. Without loss of generality (by rescaling), let  $\mathsf{OPT} = 1$ , where  $\mathsf{OPT} \coloneqq f(\mathbf{x}^*)$ . So, it is enough to get an  $\varepsilon$ -additive optimal solution  $\hat{\mathbf{x}}$ . Also without loss of generality, let **A** be such that rank (**A**) = *d*. For a positive semidefinite  $\mathbf{M} \in \mathbb{R}^{d \times d}$ , denote  $\|\mathbf{x}\|_{\mathbf{M}} \coloneqq \sqrt{\mathbf{x}^\top \mathbf{M} \mathbf{x}}$ . As shorthand, for  $\mathbf{y} \in \mathbb{R}^n$ , we will often refer to the norm  $\|\mathbf{y}\|_{\mathcal{G}_p} \coloneqq \left(\sum_{i=1}^m \|\mathbf{y}_{S_i}\|_2^p\right)^{1/p}$  for  $p \ge 1$ , where with a slight abuse of notation  $\mathbf{y}_{S_i}$  denotes the coordinates of  $\mathbf{y}$  indexed by  $S_i$ . Finally, in an abuse of notation, for symmetric matrices **M**, let  $\mathbf{M}^{-1}$  denote the pseudoinverse of **M**.

With this notation in mind, we note that many iterative methods for convex optimization can be seen as decomposing a complex problem into a series of simpler subproblems. Our algorithms

Algorithm	Iteration Complexity	Each Iteration	
Subgradient descent	$\frac{\left\ \boldsymbol{x}^{\star}\right\ _{2} \max_{1 \leq i \leq m} \frac{1}{\sqrt{n_{i}}} \left\ \boldsymbol{A}_{\mathcal{S}_{i}}\right\ _{\mathrm{op}}}{\varepsilon^{2}}$	Evaluate $\nabla f(\mathbf{x})$	
Nesterov acceleration on smoothened objective	$\frac{\left\ \boldsymbol{x}^{\star}\right\ _{2} \left(\max_{1 \leq i \leq m} \frac{1}{\sqrt{n_{i}}} \left\ \boldsymbol{A}_{S_{i}}\right\ _{\mathrm{op}}\right)^{1/2}}{\varepsilon}$	Evaluate $\nabla \widetilde{f}_{\beta,\delta}(\mathbf{x})$	
[AAKMRZ22]	$\frac{\left\ \boldsymbol{x}^{\star}\right\ _{2}\max_{1\leq i\leq m}\frac{1}{\sqrt{n_{i}}}\left\ \mathbf{A}_{\boldsymbol{S}_{i}}\right\ _{\mathrm{op}}}{\varepsilon}$	Evaluate $\nabla \widetilde{f}_{\beta,\delta}(\boldsymbol{x})$	
Interior point with log barrier [BV04]	$m^{1/2}\log\left(rac{1}{arepsilon} ight)$	Linear system solve in $\mathbf{A}^{T}\mathbf{B}\mathbf{A}$	
<b>This chapter</b> (naïve geometry)	$\frac{m^{1/3}}{\varepsilon^{2/3}}$	Linear system solve in $\mathbf{A}^{T}\mathbf{B}\mathbf{A}$	
$\ell_{\infty}$ regression with Lewis weights [JLS22]	$\frac{rank(\mathbf{A})^{1/3}}{\varepsilon^{2/3}}$	Linear system solve in $\mathbf{A}^{T}\mathbf{D}\mathbf{A}$	
$\ell_{\infty}$ regression with IPM [LS19]	$rank\left(\mathbf{A} ight)^{1/2}\log\left(rac{1}{arepsilon} ight)$	Linear system solve in <b>A</b> <sup>T</sup> <b>DA</b>	
This chapter (Theorem 12)	$\frac{\min\{rank(\mathbf{A}),m\}^{1/3}}{\varepsilon^{2/3}}$	Linear system solve in $\mathbf{A}^{T}\mathbf{B}\mathbf{A}$	

Table 4.1.: Here, we list the complexities of algorithms for optimizing (4.1.2) or for the special case of  $\ell_{\infty}$  regression, assuming OPT = 1 (the first three guarantees are additive approximations) and ignoring polylog(n, m) terms. We write **D** to be a diagonal matrix and **B** to be a block-diagonal matrix where each block has size  $(n_i + o(1)) \times (n_i + o(1))$ . We remark that in the special case where  $n_i = 1$ , our algorithm exactly recovers that of [JLS22].

for distributionally robust linear regression follow this pattern, where the simple subproblem resembles

$$O(q) \coloneqq \min_{\|\mathbf{x}-q\|_{\mathbf{M}} \le r_q} \quad f(\mathbf{x}) \quad , \tag{4.2.1}$$

for some positive semidefinite M and for some ball radius  $r_q$  which may depend on the query q. Sub-routines like (4.2.1) are central to many trust-region methods [CGT00], and, importantly when f is the sum of a linear function and a self-concordant barrier, interior point methods derived from the self-concordant barrier framework <sup>1</sup> [NN94].

With such a subproblem structure in hand, three questions arise. (1) How do we choose the "local geometry" M? (2) How do we solve the subproblems efficiently? (3) How do we combine our subproblem solutions to arrive at our final answer? We address these concerns in order in the following discussion.

#### 4.2.1. The geometry of the proximal subproblems

Observe that when we solve (4.2.1), we are solving an optimization problem over the sublevel sets  $\{x : ||x||_M \le r_q\}$  – these are ellipsoids. Now, consider choosing the  $\ell_2$  geometry that best

<sup>&</sup>lt;sup>1</sup>In this case, the matrix  $\mathbf{M}$  is given by the Hessian of the barrier function evaluated at the subproblem's solution.

approximates our loss function. Specifically, ignoring the offset *b* for now, for a norm loss function  $\|\cdot\|$  and for some *distortion*  $\Delta \ge 1$  that is as close to 1 as possible, we want

for all 
$$x \in \mathbb{R}^d$$
:  $||x||_{\mathbf{M}} \le ||\mathbf{A}x|| \le \triangle ||x||_{\mathbf{M}}$ 

Observe that this captures the families of losses we study – in particular, we can check that for  $y \in \mathbb{R}^n$ , the functions  $\|y\|_{\mathcal{G}_p} = \left(\sum_{i=1}^m \|y_{S_i}\|_2^p\right)^{1/p}$  for  $1 \le p \le \infty$  are norms. To see what kinds of distortion guarantees we can hope for, recall that we can use John's theorem (Theorem 1.1.1) as a benchmark. For convenience, we restate it below.

**Theorem 4.2.1** (John's theorem, [Joh48]). For any symmetric convex body  $K \subset \mathbb{R}^d$ , let  $\mathcal{E}(K)$  denote the ellipsoid of maximum volume contained within K. Then, we have

$$\mathcal{E}(K) \subseteq K \subseteq \sqrt{d} \cdot \mathcal{E}(K) \ .$$

*Moreover, the*  $\sqrt{d}$  *is worst-case optimal (e.g. let K be the unit*  $\ell_{\infty}$  *ball).* 

It is easy to see that sublevel sets of norms, i.e., sets of the form  $\{x \in \mathbb{R}^d : ||x|| \le 1\}$ , are symmetric convex bodies. Hence, using John's theorem, we see that for our normed losses, there exists **M** that achieves distortion  $\triangle \le \sqrt{d}$ . However, as written, this is only an existence result. To make this useful for us and actually find **M**, we need an algorithm to calculate John's ellipsoid for the level sets of our losses (or some other ellipsoid that gets an even better approximation factor). To this end, we repurpose and renotate an earlier result from Chapter 3 (also found in [MO25]). It gives us an efficient algorithm to find this  $\ell_2$  geometry for the loss families we consider.

**Theorem 4.2.2** (Combining Lemmas 5.6, 5.8, Equation (1.8) from [MO25]). Let  $p \ge 2$ . There exists an algorithm that finds a positive diagonal matrix  $\mathbf{W} \in \mathbb{R}^{n \times n}$  such that for all  $\mathbf{x} \in \mathbb{R}^d$  and all  $c \in \mathbb{R}$ , we have

$$\frac{\left\|\mathbf{W}^{\frac{1}{2}-\frac{1}{p}}\left(\mathbf{A}\boldsymbol{x}-c\boldsymbol{b}\right)\right\|_{2}}{\left(2(\operatorname{\mathsf{rank}}\left(\mathbf{A}\right)+1)\right)^{\frac{1}{2}-\frac{1}{p}}} \le \left(\sum_{i=1}^{m} \left\|\mathbf{A}_{S_{i}}\boldsymbol{x}-c\boldsymbol{b}_{S_{i}}\right\|_{2}^{p}\right)^{\frac{1}{p}} \le \left\|\mathbf{W}^{\frac{1}{2}-\frac{1}{p}}\left(\mathbf{A}\boldsymbol{x}-c\boldsymbol{b}\right)\right\|_{2}$$

*The algorithm runs in*  $O(\log m)$  *linear system solves in matrices of the form*  $\mathbf{A}^{\top}\mathbf{D}\mathbf{A}$  *for positive diagonal matrices*  $\mathbf{D}$ *.* 

The diagonal entries of matrix **W** are rescaled versions *block Lewis weights* that we discussed in the previous chapter. Recall that this is a generalization of Lewis weights, and both objects have been used previously for various matrix approximation problems [BLM89; MMWY22; JLS23; JLLS23; MO25]. Furthermore, Lewis weights are central to improvements in the iteration complexities for linear programming and vanilla  $\ell_p$  regression [LS19; JLS22].

Additionally, notice that the distortion of  $O(\operatorname{rank}(\mathbf{A})^{1/2-1/p})$  guaranteed by Theorem 4.2.2 is optimal. To see this, let  $\mathbf{A} \in \mathbb{R}^{n \times d}$  be such that for  $i \in [d]$ , row  $a_i = e_i$ , where  $e_i$  is the *i*th standard basis vector. Then, for all  $d + 1 \le i \le n$ , let  $a_i = 0$ . In words,  $\mathbf{A}$  is the *d*-dimensional identity matrix atop a large matrix of all 0s. It is easy to see that for any *p*, we have  $\|\mathbf{A}\mathbf{x}\|_p = \|\mathbf{x}\|_p$ , and the best distortion we can get for relating  $\|\mathbf{x}\|_p$  to any *d*-dimensional  $\ell_2$  norm is  $d^{[1/2-1/p]}$ .

With Theorem 4.2.2 and its near optimality in hand, it is natural to choose  $\mathbf{M} = \mathbf{A}^{\top} \mathbf{W}^{1-\frac{2}{p}} \mathbf{A}$  if rank ( $\mathbf{A}$ )  $\leq m$  and  $\mathbf{M} = \mathbf{A}^{\top} \mathbf{A}$  if rank ( $\mathbf{A}$ )  $\geq m$  (in the latter case, we get a  $\sqrt{m}$  distortion for free

from relating  $\ell_2^m$  to  $\ell_{\infty}^m$ ). This gives us an  $\ell_2$  geometry that nearly optimally approximates our losses. In the following sections, we will see how this helps us both implement our proximal subproblem solvers and combine these to solve the whole original problem.

# 4.2.2. Solving proximal subproblems

In this subsection, let **M** be any positive semidefinite matrix, as the arguments here apply for any **M**. In particular, these arguments are required to analyze even the naïve geometry obtained by choosing  $\mathbf{M} = \mathbf{I}$  or for any other ellipsoidal approximation given by any choice of **M**. For our best results, we will finally choose  $\mathbf{M} = \mathbf{I}$  or  $\mathbf{M} = \mathbf{A}^{\top} \mathbf{W}^{1-\frac{2}{p}} \mathbf{A}$  depending on which ellipsoidal approximation is better for our input.

Here, we discuss how to solve problems of the form (4.2.1) for a fixed query q. Our strategy follows two general steps. First, we establish some form of local stability for  $\nabla^2 f(x)$  within the ball we are solving in, i.e., we want  $\nabla^2 f(x)$  to not change too much inside the ball  $\{x \in \mathbb{R}^d : ||x - q||_M \le r_q\}$ . Second, we use this to show that an appropriate second-order algorithm enjoys a good convergence rate to an approximate solution for our subproblem. We handle the  $p = \infty$  and  $2 \le p < \infty$  cases separately below.

#### The robust case ( $p = \infty$ ).

Unfortunately, since f is not even differentiable (it is the pointwise maximum of Euclidean norms, each of which is also not differentiable), we cannot directly argue about the stability of  $\nabla^2 f(\mathbf{x})$ . We therefore first need to find some surrogate objective  $\tilde{f}$  so that:

- 1. The approximation error  $\left\| \widetilde{f} f \right\|_{\infty}$  is small;
- 2. The surrogate objective  $\tilde{f}$  is smooth in  $\|\cdot\|_{\mathbf{M}}$  in such a way that we can solve the proximal subproblems fast.

To smoothen f(x), we use the family of objectives parameterized by  $\beta$ ,  $\delta$ 

$$\widetilde{f}_{\beta,\delta}(\boldsymbol{x}) \coloneqq \beta \log \left( \sum_{i=1}^{m} \exp \left( \frac{\sqrt{\delta^2 + \|\mathbf{A}_{S_i} \boldsymbol{x} - \boldsymbol{b}_{S_i}\|_2^2} - \delta}{\beta} \right) \right) \ .$$

This can be seen as composing the softmax function with temperature  $\beta$  with "inner functions"  $\sqrt{\delta^2 + \|\mathbf{A}_{S_i} \mathbf{x} - \mathbf{b}_{S_i}\|_2^2} - \delta$ . It is straightforward to show that for all  $\mathbf{x} \in \mathbb{R}^d$ ,  $\left| \widetilde{f}_{\beta,\delta}(\mathbf{x}) - f(\mathbf{x}) \right| \leq \beta \log m + \delta$ . So, setting  $\beta = \varepsilon/4 \log m$  and  $\delta = \varepsilon/4$ , it is sufficient to optimize  $\widetilde{f}_{\beta,\delta}$  up to  $\varepsilon/2$  additive error to get an  $\varepsilon$ -additive suboptimal solution to our original objective. Furthermore, we prove that  $\widetilde{f}_{\beta,\delta}$  is  $O(1/\beta + 1/\delta)$ -smooth in the norm  $\|\mathbf{A}\mathbf{x}\|_{\mathcal{G}_{\infty}} \coloneqq \max_{1 \leq i \leq m} \|\mathbf{A}\mathbf{x}\|_2$ . From Theorem 4.2.2, this means that  $\widetilde{f}_{\beta,\delta}$  is also  $O(1/\beta + 1/\delta)$ -smooth in the norm  $\|\mathbf{x}\|_{\mathbf{M}}$  where **M** is chosen according to the previous subsection (notice that the only fact we need about **M** here is that  $\max_{1 \leq i \leq m} \|\mathbf{A}\mathbf{x}\|_2 \leq \|\mathbf{x}\|_{\mathbf{M}}$ ).

Next, Carmon, Jambulapati, Jiang, Jin, Lee, Sidford, and Tian [CJJJLST20] show that if  $f_{\beta,\delta}$  satisfies a higher-order smoothness condition called *quasi-self-concordance* with respect to the
norm  $\|\cdot\|_{\mathbf{M}}$ , then we can get the required Hessian stability for a *fixed*  $r_q = \Theta(1/\varepsilon)$  (in particular,  $r_q$  does not depend on q here). To be more clear, we define quasi-self-concordance below.

**Definition 4.2.3** (Quasi-self-concordance, adapted from [KSJ18, Appendix A]). Let  $f : \mathbb{R}^k \to \mathbb{R}$ . We say that f is v-quasi-self-concordant in the norm  $\|\cdot\|$  if for all vectors  $y \in \mathbb{R}^k$ , directions  $d \in \mathbb{R}^k$ , and  $t \in \mathbb{R}$ , we have

$$\left| \left( \frac{d}{dt} \right)^3 f(\boldsymbol{y} + t \, \boldsymbol{d}) \right| \le \nu \, \|\boldsymbol{d}\| \left( \frac{d}{dt} \right)^2 f(\boldsymbol{y} + t \, \boldsymbol{d}).$$

Then, [CJJJLST20] shows how to leverage this Hessian stability to implement (4.2.1) with low linear-system-solve iteration complexity. However, previously, it was only shown that the composition of the softmax function with linear functions is quasi-self-concordant. So, it was unknown whether composing softmax with other functions could also be quasi-self-concordant.

To resolve this, we prove a much more general composition result, which may be of independent interest. It essentially states that if we compose the softmax function with any combination of "inner" functions that are quasi-self-concordant, the resulting function is also quasi-self-concordant. For a more formal statement, see Lemma 4.5.3.

Hence, to show the requisite Hessian stability, we use the following steps. We show that the "inner" functions  $\sqrt{\delta^2 + \|\mathbf{A}_{S_i} \mathbf{x} - \mathbf{b}_{S_i}\|_2^2} - \delta$  are each  $O(1/\delta)$ -quasi-self-concordant in the norm  $\|\mathbf{A}_{S_i} \mathbf{x}\|_2$ . So, we can apply our composition result Lemma 4.5.3 to prove that  $\tilde{f}_{\beta,\delta}$  is  $O(1/\beta + 1/\delta)$ -quasi-self-concordant in the norm  $\max_{i \in [m]} \|\mathbf{A}_{S_i} \mathbf{x}\|_2$ . Following from Theorem 4.2.2,  $\tilde{f}$  is  $O(1/\beta + 1/\delta)$ -quasi-self concordant in  $\|\mathbf{x}\|_M$  as well (and in particular when  $\mathbf{M} = \mathbf{A}^\top \mathbf{W}^{1-\frac{2}{p}} \mathbf{A}$  and  $\mathbf{M} = \mathbf{A}^\top \mathbf{A}$ ). We then apply the recipe given in [CJJJLST20] and get our subproblem solver for the  $p = \infty$  case.

The interpolating case ( $2 \le p < \infty$ ).

Instead of explicitly constraining  $r_q$  like in the  $p = \infty$  case, we regularize our movement from q in the norm  $\|\cdot\|_M$ . Specifically, the subproblem we solve for any query q is

$$\underset{\boldsymbol{x} \in \mathbb{R}^d}{\operatorname{argmin}} f(\boldsymbol{x}) + e p^p \|\boldsymbol{x} - \boldsymbol{q}\|_{\mathbf{M}}^p \quad . \tag{4.2.2}$$

This is the natural generalization of the proximal problem that [JLS22] use to get their results for  $\ell_p$  regression, and the outline of our solver for these subproblems is similar to what [JLS22] use for this special case (see their Section 4).

However, we go a step further and show how to obtain approximate stationary points to (4.2.2) instead of just getting a small objective value. This is because the acceleration scheme we use to iterate subproblem solutions to get our final answer  $\hat{x}$  requires us to obtain an approximate stationary point for (4.2.2). The main new technical tool we develop for this purpose is a form of strong convexity for functions of the form  $\|y\|_2^p$  for  $y \in \mathbb{R}^k$  for any  $k \ge 1$ . See Lemma 4.6.3.

**Lemma 4.6.3** (Strong convexity of  $||\mathbf{y}||_2^p$ ). Let  $\mathbf{v} \in \mathbb{R}^k$  for  $k \ge 1$ . For any  $\Delta \in \mathbb{R}^k$ , we have

$$\|v + \Delta\|_{2}^{p} \ge \|v\|_{2}^{p} + p \|v\|_{2}^{p-2} \langle v, \Delta \rangle + \frac{4}{2^{p}} \|\Delta\|_{2}^{p}$$

With Lemma 4.6.3, we can argue about the strong convexity of  $||x - q||_{\mathbf{M}'}^p$ , which means that we can convert an approximately optimal solution to (4.2.2) in function value to one that is approximately optimal in parameter space as well. We combine this with a local gradient Lipschitzness property of the objective (4.2.2) to get our approximate stationary point, which is enough for our purposes. The local gradient Lipschitzness property itself follows from a form of Hessian stability that we show for the objective (4.2.2). See Lemma 4.6.9.

Finally, to obtain an approximately optimal solution to (4.2.2) in function value, we again apply the Hessian stability property to conclude that (4.2.2) is relatively smooth and relatively strongly convex in a simpler reference function. We show how to solve optimization problems in this reference function up to an approximate optimality that is sufficient for the rest of our applications – this requires a mild modification of the standard mirror descent analysis, and we do this in Section 4.3. Combining all of these building blocks gives us our subproblem solver for the  $2 \le p < \infty$  case.

## 4.2.3. Iterating proximal calls

We now discuss the last item. Recall that we think of O(q) as answering a proximal problem for the query q. It is not hard to show that under reasonable conditions on f and on the structure of the subproblems, we can iterate calls to O(q) to optimize f (see, e.g., [CJJJLST20, Appendix A]). This conceptually simple approach will already give us condition-free, group-independent rates for the problems we study.

But, we can do better. An acceleration framework originally due to Monteiro and Svaiter [MS13] and generalized/refined in subsequent works [BJLLS19; CJJJLST20; CHJJS22] gives a recipe to iterate calls of O(q) to optimize the original function f. From these, the iteration complexity we need for an  $\varepsilon$ -additive solution with an initialization  $x_0$  and optimum  $x^*$  is roughly  $(||x_0 - x^*||_M / \varepsilon)^{2/3}$  (see Theorem 4.4.3 for a more formal statement). Combining this with Theorem 4.2.2, which implies that we can find  $x_0$  such that  $||x_0 - x^*||_M \le \sqrt{d}$ , we see that we should expect rates of roughly  $\widetilde{O}(d^{1/3}\varepsilon^{-2/3})$  for our problems. Indeed, Theorem 12 and Theorem 13 attain rates that are at least this good up to logarithmic factors. In this step, we again use the strong convexity that we prove for our objective (Lemma 4.6.3) to show that after enough steps of this algorithm, the problem diameter will have noticeably shrunk. Iterating this gives the high-accuracy result of Theorem 13.

Interestingly, our algorithm for the  $2 \le p < \infty$  case uses a form of this accelerated scheme developed in [CHJJS22] that does not require solving an implicit equation for the query point, improving over the results from [JLS22] for  $\ell_p$  regression. It would be nice to obtain this for the  $p = \infty$  case (in Section 4.4, we discuss a technical challenge in obtaining this).

## 4.3. Mirror descent with inexact updates

**Notation warning.** This section is meant to be a self-contained, standalone analysis of mirror descent under inexact updates. The notation is chosen to be consistent with most material we could find on mirror descent and therefore conflicts with the notation used in the rest of the chapter.

In this section, we give an analysis of unconstrained mirror descent when each Bregman

proximal problem is solved only approximately (Algorithm 10). Although we expect that this is a standard fact about mirror descent, we could not find an appropriate reference. Hence, we produce it here.

Algorithm 10 ApproximateMirrorDescent: Implements mirror descent to optimize convex and differentiable f given *L*-relative smoothness and  $\mu$ -relative strong convexity in the reference h when we may not be able to solve each proximal problem exactly.

**Require:** Initial point 
$$x_0$$
, iteration count  $T$ .  
1: Define  
 $D_h(x, y) \coloneqq h(x) - h(y) - \langle \nabla h(y), x - y \rangle$   
 $x^* \coloneqq \operatorname{argmin} f(x)$   
2: for  $i = 1, ..., T$  do  
3:  $\begin{aligned} x_i^* = \operatorname{argmin} f(x_{i-1}) + \langle \nabla f(x_{i-1}), \tilde{x} - x_{i-1} \rangle + LD_h(\tilde{x}, x_{i-1}) \triangleright We \text{ may only be able to approx-} \\ \tilde{x} \in \mathbb{R}^d \\ \text{imate } x_i^* - \text{see the next line.} \end{aligned}$   
4: Let  $x_i$  be an approximate stationary point for the above objective.  
return  $\operatorname{argmin} f(x_i)$   
 $0 \le i \le T$ 

In Algorithm 10, we assume that the function f is  $\mu$ -relatively strongly convex and L-smooth in a *reference function* h. This means that for all  $x, y \in \mathbb{R}^d$ , we have

$$\mu D_h(\boldsymbol{x}, \boldsymbol{y}) \le f(\boldsymbol{x}) - f(\boldsymbol{y}) - \langle \nabla f(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y} \rangle \le L D_h(\boldsymbol{x}, \boldsymbol{y})$$

Using [LFN18, Proposition 1.1], when *f* is twice-differentiable, this condition is equivalent to asking for all  $x \in \mathbb{R}^d$ ,

$$\mu \nabla^2 h(\mathbf{x}) \le \nabla^2 f(\mathbf{x}) \le L \nabla^2 h(\mathbf{x}).$$

We are now ready to state the performance guarantee of Algorithm 10. See Theorem 4.3.1.

**Theorem 4.3.1.** Let index *j* be the index output by Algorithm 10. Let  $\triangle_i$  be defined such that

$$\Delta_i \coloneqq \nabla f(\mathbf{x}_{i-1}) + L \left( \nabla h(\mathbf{x}_i) - \nabla h(\mathbf{x}_{i-1}) \right).$$

Then, we have

$$f(\mathbf{x}_j) - f(\mathbf{x}^{\star}) \leq L \left(1 - \frac{\mu}{L}\right)^T D_h(\mathbf{x}^{\star}, \mathbf{x}_0) + \max_{1 \leq i \leq n} \left\langle \Delta_i, \mathbf{x}_i - \mathbf{x}^{\star} \right\rangle.$$

To prove Theorem 4.3.1, we begin with a few standard facts about the mirror descent iterations.

**Lemma 4.3.2.** Let  $y \in \mathbb{R}^d$  be arbitrary. We have

$$\langle \nabla f(\boldsymbol{x}_{i-1}), \boldsymbol{x}_i - \boldsymbol{y} \rangle = L\left(D_h(\boldsymbol{y}, \boldsymbol{x}_{i-1}) - D_h(\boldsymbol{y}, \boldsymbol{x}_i) - D_h(\boldsymbol{x}_i, \boldsymbol{x}_{i-1})\right) + \langle \Delta_i, \boldsymbol{x}_i - \boldsymbol{y} \rangle.$$

Proof of Lemma 4.3.2. By the three point identity (see, e.g., [SYLS16, Equation (A.9)]), we have

$$D_h(\boldsymbol{y}, \boldsymbol{x}_{i-1}) - D_h(\boldsymbol{y}, \boldsymbol{x}_i) - D_h(\boldsymbol{x}_i, \boldsymbol{x}_{i-1}) = -\langle \nabla h(\boldsymbol{x}_i) - \nabla h(\boldsymbol{x}_{i-1}), \boldsymbol{x}_i - \boldsymbol{y} \rangle$$
$$= \frac{1}{L} \langle \nabla f(\boldsymbol{x}_{i-1}) - \Delta_i, \boldsymbol{x}_i - \boldsymbol{y} \rangle,$$

completing the proof of Lemma 4.3.2.

**Lemma 4.3.3** (Mirror descent lemma under approximate stationary point updates). Let  $y \in \mathbb{R}^d$  be arbitrary. For every iteration *i*, we have

$$f(\boldsymbol{x}_i) - f(\boldsymbol{y}) \leq (L - \mu)D_h(\boldsymbol{y}, \boldsymbol{x}_{i-1}) - LD_h(\boldsymbol{y}, \boldsymbol{x}_i) + \langle \Delta_i, \boldsymbol{x}_i - \boldsymbol{y} \rangle.$$

*Proof of Lemma 4.3.3.* The definition of  $\mu$ -relative strong convexity tells us that

$$f(\boldsymbol{x}_{i-1}) - f(\boldsymbol{y}) \leq \langle \nabla f(\boldsymbol{x}_{i-1}), \boldsymbol{x}_{i-1} - \boldsymbol{y} \rangle - \mu D_h(\boldsymbol{y}, \boldsymbol{x}_{i-1}).$$

We now write

$$f(\mathbf{x}_{i}) - f(\mathbf{y}) \leq f(\mathbf{x}_{i-1}) - f(\mathbf{y}) + \langle \nabla f(\mathbf{x}_{i-1}), \mathbf{x}_{i} - \mathbf{x}_{i-1} \rangle + LD_{h}(\mathbf{x}_{i}, \mathbf{x}_{i-1})$$
(L-RS)  
$$\leq \langle \nabla f(\mathbf{x}_{i-1}), \mathbf{x}_{i} - \mathbf{y} \rangle - \mu D_{h}(\mathbf{y}, \mathbf{x}_{i-1}) + LD_{h}(\mathbf{x}_{i}, \mathbf{x}_{i-1})$$
( $\mu$ -RSC)  
$$\leq (L - \mu)D_{h}(\mathbf{y}, \mathbf{x}_{i-1}) - LD_{h}(\mathbf{y}, \mathbf{x}_{i}) + \langle \Delta_{i}, \mathbf{x}_{i} - \mathbf{y} \rangle,$$
(Lemma 4.3.2)

completing the proof of Lemma 4.3.3.

We now have the tools to complete the proof of Theorem 4.3.1.

*Proof of Theorem 4.3.1.* Let  $E_i := f(x_i) - f(x^*) - \langle \triangle_i, x_i - x^* \rangle$ . Substituting  $y = x^*$  and rearranging the conclusion of Lemma 4.3.3 gives

$$E_i \leq (L-\mu)D_h(\boldsymbol{x^{\star}}, \boldsymbol{x}_{i-1}) - LD_h(\boldsymbol{x^{\star}}, \boldsymbol{x}_i).$$

We multiply both sides by  $\left(\frac{L}{L-\mu}\right)^i$  and write

$$\left(\frac{L}{L-\mu}\right)^{i} E_{i} \leq \frac{L^{i}}{(L-\mu)^{i-1}} D_{h}(\boldsymbol{x}^{\star}, \boldsymbol{x}_{i-1}) - \frac{L^{i+1}}{(L-\mu)^{i}} D_{h}(\boldsymbol{x}^{\star}, \boldsymbol{x}_{i}).$$

Adding over all *T* iterations yields

$$\sum_{i=1}^{T} \left(\frac{L}{L-\mu}\right)^{i} E_{i} \leq LD_{h}(\boldsymbol{x}^{\star},\boldsymbol{x}_{0}) - \left(\frac{L}{L-\mu}\right)^{T} LD_{h}(\boldsymbol{x}^{\star},\boldsymbol{x}_{T}) \leq LD_{h}(\boldsymbol{x}^{\star},\boldsymbol{x}_{0}).$$

Expanding out the definition of  $E_i$  and rearranging gives

$$\sum_{i=1}^{T} \left( \frac{L}{L-\mu} \right)^{i} \left( f(\boldsymbol{x}_{i}) - f(\boldsymbol{x}^{\star}) \right) \leq LD_{h}(\boldsymbol{x}^{\star}, \boldsymbol{x}_{0}) + \sum_{i=1}^{T} \left( \frac{L}{L-\mu} \right)^{i} \left\langle \Delta_{i}, \boldsymbol{x}_{i} - \boldsymbol{x}^{\star} \right\rangle.$$

By the geometric series summation formula, we define and have

$$C_T := \sum_{i=1}^T \left(\frac{L}{L-\mu}\right)^i = \frac{L}{\mu} \left( \left(1 + \frac{\mu}{L-\mu}\right)^T - 1 \right).$$

Let *j* be the index that Algorithm 10 returns. It is easy to check that

$$\sum_{i=1}^{T} \left( \frac{L}{L-\mu} \right)^{i} \left( f(\boldsymbol{x}_{i}) - f(\boldsymbol{x}^{\star}) \right) \geq C_{T} \left( f(\boldsymbol{x}_{j}) - f(\boldsymbol{x}^{\star}) \right)$$

and

$$\sum_{i=1}^{T} \left( \frac{L}{L-\mu} \right)^{i} \left\langle \Delta_{i}, \boldsymbol{x}_{i} - \boldsymbol{x}^{\star} \right\rangle \leq C_{T} \max_{1 \leq i \leq n} \left\langle \Delta_{i}, \boldsymbol{x}_{i} - \boldsymbol{x}^{\star} \right\rangle.$$

This gives us

$$f(\boldsymbol{x}_j) - f(\boldsymbol{x}^{\star}) \leq \frac{L}{C_T} D_h(\boldsymbol{x}^{\star}, \boldsymbol{x}_0) + \max_{1 \leq i \leq n} \left\langle \Delta_i, \boldsymbol{x}_i - \boldsymbol{x}^{\star} \right\rangle.$$

Finally, notice that

$$\frac{L}{C_T} = \frac{\mu}{\left(1 + \frac{\mu}{L - \mu}\right)^T - 1} \le L \left(1 - \frac{\mu}{L}\right)^T.$$

Combining everything completes the proof of Theorem 4.3.1.

Finally, we add another useful lemma that quantifies the descent, if any, in the objective value between iterations.

Lemma 4.3.4. For every iteration *i*, we have

$$f(\mathbf{x}_i) - f(\mathbf{x}_{i-1}) \leq -LD_h(\mathbf{x}_{i-1}, \mathbf{x}_i) + \langle \Delta_i, \mathbf{x}_i - \mathbf{x}_{i-1} \rangle.$$

In particular, if  $\langle \triangle_i, x_i - x_{i-1} \rangle \leq LD_h(x_{i-1}, x_i)$ , then iteration *i* is a descent step.

*Proof of Lemma 4.3.4.* We substitute  $y = x_{i-1}$  in the conclusion of Lemma 4.3.3. This gives

$$f(\mathbf{x}_i) - f(\mathbf{x}_{i-1}) \leq -LD_h(\mathbf{x}_{i-1}, \mathbf{x}_i) + \langle \Delta_i, \mathbf{x}_i - \mathbf{x}_{i-1} \rangle,$$

completing the proof of Lemma 4.3.4.

# 4.4. Optimal MS acceleration under custom Euclidean geometry

In this section, we adapt the bisection-free Monteiro-Svaiter acceleration framework developed in [CHJJS22] to handle custom Euclidean geometries. The object of interest here is Algorithm 11, which we will call with different choices of the oracle  $O_{MS}$  for our algorithms.

### Algorithm 11 OptimalMSAcceleration: optimizes function f given MS oracle $O_{MS}$ .

**Require:** Initial  $x_0$ , function f, oracle  $O_{MS}$ , initial  $\lambda'_0$ , multiplicative adjustment factor  $\alpha > 1$ , iteration count T

1: Set 
$$v_0 = x_0, A_0 = 0, A'_0 = 0.$$
  
2: Set  $\tilde{x}_1, \lambda_1 = O(x_0; \lambda'_0)$  and  $\lambda'_1 = \lambda_1.$   
3: for  $t = 0, ..., T$  do  
4:  $a'_{t+1} = \frac{1}{2\lambda'_{t+1}} \left(1 + \sqrt{1 + 4\lambda'_{t+1}A_t}\right)$   
5:  $A'_{t+1} = A_t + a'_{t+1}$   
6:  $q_t = \frac{A_t}{A'_{t+1}} x_t + \frac{a'_{t+1}}{A'_{t+1}} v_t$   
7: if  $t > 0$  then  $\tilde{x}_{t+1}, \lambda_{t+1} = O_{MS}(q_t; \lambda'_{t+1})$   
8:  $\gamma_{t+1} = \min\left\{1, \frac{\lambda'_{t+1}}{\lambda_{t+1}}\right\}$   
9:  $a_{t+1} = \gamma_{t+1}a'_{t+1}$  and  $A_{t+1} = A_t + a_{t+1}$   
10:  $x_{t+1} = \frac{(1 - \gamma_{t+1})A_t}{A_{t+1}} x_t + \frac{\gamma_{t+1}A'_{t+1}}{A_{t+1}} \tilde{x}_{t+1}$   
11: if  $\gamma_{t+1} = 1$  then  
12:  $|\lambda'_{t+2} = \frac{1}{\alpha}\lambda'_{t+1}$   
13: else  
14:  $|\lambda'_{t+1} = v_t - a_{t+1}M^{-1}\nabla f(\tilde{x}_{t+1})$ 

In order to state the performance guarantee of Algorithm 11, we require the notions of an *MS oracle* and a *movement bound*. See Definition 4.4.1 and Definition 4.4.2.

**Definition 4.4.1** (MS oracle, generalization of [CHJJS22, Definition 1]). Let  $\mathbf{M} \in \mathbb{R}^{d \times d}$  be a positive semidefinite matrix. An oracle  $O: \mathbb{R}^d \times \mathbb{R}_{\geq 0} \to \mathbb{R}^d \times \mathbb{R}_{\geq 0}$  is a  $\sigma$ -MS oracle for function  $f: \mathbb{R}^d \to \mathbb{R}$  if for every  $q \in \mathbb{R}^d$  and  $\lambda' > 0$ , the points  $(\mathbf{x}, \lambda) = O(q; \lambda')$  satisfy

$$\left\| \boldsymbol{x} - \boldsymbol{q} + \frac{1}{\lambda} \mathbf{M}^{-1} \nabla f(\boldsymbol{x}) \right\|_{\mathbf{M}} \le \sigma \left\| \boldsymbol{x} - \boldsymbol{q} \right\|_{\mathbf{M}}$$

**Definition 4.4.2** (Movement bound [CHJJS22, Definition 2]). For a norm  $\|\cdot\|_{\mathbf{M}}$  induced by positive semidefinite  $\mathbf{M} \in \mathbb{R}^{d \times d}$ , numbers  $s \ge 1, c, \lambda > 0$ , and  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ , we say that  $(\mathbf{x}, \mathbf{y}, \lambda)$  satisfies a (s, c)-movement bound if

$$\|\boldsymbol{x} - \boldsymbol{y}\|_{\mathbf{M}} \geq \begin{cases} \left(\frac{\lambda}{c^s}\right)^{\frac{1}{s-1}} & \text{if } s < \infty \\ \frac{1}{c} & \text{if } s = \infty \end{cases}$$

With these in hand, we are ready to state the convergence guarantee we get with Algorithm 11. See Theorem 4.4.3.

**Theorem 4.4.3** (Modification of [CHJJS22, Theorem 1]). Let  $f : \mathbb{R}^d \to \mathbb{R}$  be convex and differentiable. Consider running Algorithm 11 with parameters  $\alpha = \exp\left(3 - \frac{2}{s+1}\right)$  and a  $\sigma$ -MS oracle with  $0 \le \sigma < 0.99$  (Definition 4.4.1). Let  $s \ge 1$  and c > 0 and suppose that for all t such that  $\lambda_t > \lambda'_t$  or t = 1, the iterates  $(\tilde{\mathbf{x}}_t, \mathbf{q}_{t-1}, \lambda_t)$  satisfy an (s, c)-movement bound (Definition 4.4.2). Let C be a universal constant. For any iteration count T satisfying

$$T \ge C \begin{cases} s \left( \frac{c^s \| \boldsymbol{x}_0 - \boldsymbol{x}^{\star} \|_{\mathbf{M}}^{s+1}}{\varepsilon} \right)^{\frac{2}{3s+1}} & \text{if } s < \infty \\ (c \| \boldsymbol{x}_0 - \boldsymbol{x}^{\star} \|_{\mathbf{M}})^{2/3} \log \left( \frac{\lambda_1 \| \boldsymbol{x}_0 - \boldsymbol{x}^{\star} \|_{\mathbf{M}}^2}{\varepsilon} \right) & \text{if } s = \infty \end{cases},$$

we have

$$f(\boldsymbol{x}_T) - f(\boldsymbol{x}^{\star}) \leq \varepsilon.$$

The proof of Theorem 4.4.3 follows the same recipe as the proof of [CHJJS22, Theorem 1]. The only modification needed is that stated in Lemma 4.4.4.

**Lemma 4.4.4** (Replaces [CHJJS22, Proposition 1]). In the context of Theorem 4.4.3, let  $E_t := f(\mathbf{x}_t) - f(\mathbf{x}^*)$ ,  $D_t := \frac{1}{2} \|\mathbf{v}_t - \mathbf{x}^*\|_{\mathbf{M}}^2$ ,  $N_{t+1} := \frac{1}{2} \|\widetilde{\mathbf{x}}_{t+1} - \mathbf{q}_t\|_{\mathbf{M}}^2$ . Then, for all  $t \ge 0$ , we have

$$A_{t+1}E_{t+1} + D_{t+1} + (1 - \sigma^2)A'_{t+1}\min\left\{\lambda_{t+1}, \lambda'_{t+1}\right\}N_{t+1} \le A_tE_t + D_t$$

Consequently, for all  $T \ge 1$ ,  $\sqrt{A_T} \ge \frac{1}{2} \sum_{t \in S_T^{\leq}} \frac{1}{\sqrt{\lambda'_t}}$ ,

$$E_T \leq \frac{D_0}{A_T}$$
 and  $(1 - \sigma^2) \sum_{t \in \mathcal{S}_T^{\geq}} A_t \lambda_t' N_t \leq D_0 - A_T E_T.$ 

*Proof of Lemma 4.4.4*. This proof is a straightforward modification of [CHJJS22, Proposition 1]. We have

$$D_{t+1} = \frac{1}{2} \left\| \boldsymbol{v}_{t+1} - \boldsymbol{x}^{\star} \right\|_{\mathbf{M}}^{2} = \frac{1}{2} \left\| \boldsymbol{v}_{t} - \boldsymbol{a}_{t+1} \mathbf{M}^{-1} \nabla f(\widetilde{\boldsymbol{x}}_{t+1}) - \boldsymbol{x}^{\star} \right\|_{\mathbf{M}}^{2}$$
  
=  $D_{t} + \boldsymbol{a}_{t+1} \left\langle \mathbf{M}^{-1} \nabla f(\widetilde{\boldsymbol{x}}_{t+1}), \boldsymbol{x}^{\star} - \boldsymbol{v}_{t} \right\rangle_{\mathbf{M}} + \frac{a_{t+1}^{2}}{2} \left\| \mathbf{M}^{-1} \nabla f(\widetilde{\boldsymbol{x}}_{t+1}) \right\|_{\mathbf{M}}^{2}$ 

By definition of  $q_t$  and  $A'_{t+1} = A_t + a'_{t+1}$ , we have

$$a'_{t+1}v_t = A'_{t+1}q_t - A_tx_t = a'_{t+1}\tilde{x}_{t+1} + A'_{t+1}(q_t - \tilde{x}_{t+1}) - A_t(x_t - \tilde{x}_{t+1}).$$

Subtracting  $a'_{t+1}x^*$  and taking the inner product with  $\mathbf{M}^{-1}\nabla f(\widetilde{\mathbf{x}}_{t+1})$  gives

$$\begin{aligned} & a'_{t+1} \left\langle \mathbf{M}^{-1} \nabla f(\widetilde{\mathbf{x}}_{t+1}), \mathbf{x}^{\star} - \mathbf{v}_{t} \right\rangle_{\mathbf{M}} \\ &= \left\langle \mathbf{M}^{-1} \nabla f(\widetilde{\mathbf{x}}_{t+1}), a'_{t+1}(\mathbf{x}^{\star} - \widetilde{\mathbf{x}}_{t+1}) + A'_{t+1} \left( \widetilde{\mathbf{x}}_{t+1} - \mathbf{q}_{t} \right) + A_{t} \left( \mathbf{x}_{t} - \widetilde{\mathbf{x}}_{t+1} \right) \right\rangle_{\mathbf{M}} \\ &\leq a'_{t+1} \left( f(\mathbf{x}^{\star}) - f(\widetilde{\mathbf{x}}_{t+1}) \right) + A'_{t+1} \left\langle \mathbf{M}^{-1} \nabla f(\widetilde{\mathbf{x}}_{t+1}), \widetilde{\mathbf{x}}_{t+1} - \mathbf{q}_{t} \right\rangle_{\mathbf{M}} + A_{t} \left( f(\mathbf{x}_{t}) - f(\widetilde{\mathbf{x}}_{t+1}) \right) \\ &\leq A_{t} E_{t} - A'_{t+1} \left( f(\widetilde{\mathbf{x}}_{t+1}) - f(\mathbf{x}^{\star}) \right) + A'_{t+1} \left\langle \mathbf{M}^{-1} \nabla f(\widetilde{\mathbf{x}}_{t+1}), \widetilde{\mathbf{x}}_{t+1} - \mathbf{q}_{t} \right\rangle_{\mathbf{M}}. \end{aligned}$$

Rearranging gives

$$\begin{aligned} A_{t+1}'\left(f(\widetilde{\boldsymbol{x}}_{t+1}) - f(\boldsymbol{x}^{\star})\right) &\leq A_t E_t + a_{t+1}' \left\langle \mathbf{M}^{-1} \nabla f(\widetilde{\boldsymbol{x}}_{t+1}), \boldsymbol{v}_t - \boldsymbol{x}^{\star} \right\rangle_{\mathbf{M}} \\ &+ A_{t+1}' \left\langle \mathbf{M}^{-1} \nabla f(\widetilde{\boldsymbol{x}}_{t+1}), \widetilde{\boldsymbol{x}}_{t+1} - \boldsymbol{q}_t \right\rangle_{\mathbf{M}}. \end{aligned}$$

Next, recall that by Definition 4.4.1, we have

$$\left\|\mathbf{M}^{-1}\nabla f(\widetilde{\mathbf{x}}_{t+1}) + \lambda_{t+1}\left(\widetilde{\mathbf{x}}_{t+1} - \mathbf{q}_{t}\right)\right\|_{\mathbf{M}}^{2} \leq \lambda_{t+1}^{2}\sigma^{2}\left\|\widetilde{\mathbf{x}}_{t+1} - \mathbf{q}_{t}\right\|_{\mathbf{M}}^{2}.$$

We use this to write

$$\lambda_{t+1} \left\langle \mathbf{M}^{-1} \nabla f(\widetilde{\mathbf{x}}_{t+1}), \widetilde{\mathbf{x}}_{t+1} - \mathbf{q}_t \right\rangle_{\mathbf{M}}$$
  
=  $\frac{1}{2} \left\| \mathbf{M}^{-1} \nabla f(\widetilde{\mathbf{x}}_{t+1}) + \lambda_{t+1} (\widetilde{\mathbf{x}}_{t+1} - \mathbf{q}_t) \right\|_{\mathbf{M}}^2 - \frac{1}{2} \left\| \mathbf{M}^{-1} \nabla f(\widetilde{\mathbf{x}}_{t+1}) \right\|_{\mathbf{M}}^2 - \frac{\lambda_{t+1}^2}{2} \left\| \widetilde{\mathbf{x}}_{t+1} - \mathbf{q}_t \right\|_{\mathbf{M}}^2$ 

$$\leq -\lambda_{t+1}^2 (1-\sigma^2) N_{t+1} - \frac{1}{2} \left\| \mathbf{M}^{-1} \nabla f(\widetilde{\mathbf{x}}_{t+1}) \right\|_{\mathbf{M}}^2,$$

from which we conclude

$$\left\langle \mathbf{M}^{-1}\nabla f(\widetilde{\mathbf{x}}_{t+1}), \widetilde{\mathbf{x}}_{t+1} - \mathbf{q}_t \right\rangle_{\mathbf{M}} \le -\lambda_{t+1}(1 - \sigma^2)N_{t+1} - \frac{1}{2\lambda_{t+1}} \left\| \mathbf{M}^{-1}\nabla f(\widetilde{\mathbf{x}}_{t+1}) \right\|_{\mathbf{M}}^2$$

Substituting back gives

$$\begin{aligned} A'_{t+1}\left(f(\widetilde{\boldsymbol{x}}_{t+1}) - f(\boldsymbol{x}^{\star})\right) &\leq A_{t}E_{t} + a'_{t+1}\left\langle \mathbf{M}^{-1}\nabla f(\widetilde{\boldsymbol{x}}_{t+1}), \boldsymbol{v}_{t} - \boldsymbol{x}^{\star}\right\rangle_{\mathbf{M}} \\ &+ A'_{t+1}\left\langle \mathbf{M}^{-1}\nabla f(\widetilde{\boldsymbol{x}}_{t+1}), \widetilde{\boldsymbol{x}}_{t+1} - \boldsymbol{q}_{t}\right\rangle_{\mathbf{M}} \\ &\leq A_{t}E_{t} + a'_{t+1}\left\langle \mathbf{M}^{-1}\nabla f(\widetilde{\boldsymbol{x}}_{t+1}), \boldsymbol{v}_{t} - \boldsymbol{x}^{\star}\right\rangle_{\mathbf{M}} \\ &- A'_{t+1}\lambda_{t+1}(1 - \sigma^{2})N_{t+1} - \frac{A'_{t+1}}{2\lambda_{t+1}}\left\|\mathbf{M}^{-1}\nabla f(\widetilde{\boldsymbol{x}}_{t+1})\right\|_{\mathbf{M}}^{2}.\end{aligned}$$

Next, recall that  $\gamma_{t+1}a'_{t+1} = a_{t+1}$  and  $\gamma_{t+1}\lambda_{t+1} = \min\{\lambda_{t+1}, \lambda'_{t+1}\}$ , by construction. Let  $\widehat{\lambda}_{t+1} := \min\{\lambda_{t+1}, \lambda'_{t+1}\}$  We multiply both sides by  $\gamma_{t+1}$  and conclude

$$\begin{aligned} \gamma_{t+1}A'_{t+1}\left(f(\widetilde{\boldsymbol{x}}_{t+1}) - f(\boldsymbol{x}^{\star})\right) &\leq \gamma_{t+1}A_{t}E_{t} + a_{t+1}\left\langle \mathbf{M}^{-1}\nabla f(\widetilde{\boldsymbol{x}}_{t+1}), \boldsymbol{v}_{t} - \boldsymbol{x}^{\star}\right\rangle_{\mathbf{M}} \\ &- A'_{t+1}\widehat{\lambda}_{t+1}(1 - \sigma^{2})N_{t+1} - \frac{\gamma_{t+1}A'_{t+1}}{2\lambda_{t+1}}\left\|\mathbf{M}^{-1}\nabla f(\widetilde{\boldsymbol{x}}_{t+1})\right\|_{\mathbf{M}}^{2}.\end{aligned}$$

Now, by convexity of *f* and from the definition of  $x_{t+1}$ , we have

$$f(\mathbf{x}_{t+1}) - f(\mathbf{x}^{\star}) \le \frac{(1 - \gamma_{t+1})A_t}{A_{t+1}} \left( f(\mathbf{x}_t) - f(\mathbf{x}^{\star}) \right) + \frac{\gamma_{t+1}A'_{t+1}}{A_{t+1}} \left( f(\widetilde{\mathbf{x}}_{t+1}) - f(\mathbf{x}^{\star}) \right)$$

Recall the definition of  $E_t$ , multiply both sides by  $A_{t+1}$ , apply our bound on  $\gamma_{t+1}A'_{t+1}(f(\tilde{x}_{t+1}) - f(x^*))$ , and we get

$$\begin{aligned} A_{t+1}E_{t+1} &\leq (1 - \gamma_{t+1})A_tE_t + \gamma_{t+1}A'_{t+1}\left(f(\widetilde{\mathbf{x}}_{t+1}) - f(\mathbf{x}^{\star})\right) \\ &\leq A_tE_t + a_{t+1}\left\langle \mathbf{M}^{-1}\nabla f(\widetilde{\mathbf{x}}_{t+1}), \mathbf{v}_t - \mathbf{x}^{\star}\right\rangle_{\mathbf{M}} \\ &- A'_{t+1}\widehat{\lambda}_{t+1}(1 - \sigma^2)N_{t+1} - \frac{\gamma_{t+1}A'_{t+1}}{2\lambda_{t+1}}\left\|\mathbf{M}^{-1}\nabla f(\widetilde{\mathbf{x}}_{t+1})\right\|_{\mathbf{M}}^2 \end{aligned}$$

After shifting terms around, we see that it remains to show

$$a_{t+1} \left\langle \mathbf{M}^{-1} \nabla f(\widetilde{\mathbf{x}}_{t+1}), \mathbf{v}_t - \mathbf{x}^{\star} \right\rangle_{\mathbf{M}} - \frac{\gamma_{t+1} A'_{t+1}}{2\lambda_{t+1}} \left\| \mathbf{M}^{-1} \nabla f(\widetilde{\mathbf{x}}_{t+1}) \right\|_{\mathbf{M}}^2 \stackrel{?}{\leq} D_t - D_{t+1}.$$

In fact, by the choice of  $a'_{t+1}$  and the definition of  $A'_{t+1}$ , we have

$$\lambda'_{t+1}(a'_{t+1})^2 = a'_{t+1} + A_t = A'_{t+1}.$$

Multiply both sides by  $\gamma_{t+1}^2/(2\lambda_{t+1}')$  and we get

$$\frac{a_{t+1}^2}{2} = \frac{\gamma_{t+1}^2 A_{t+1}'}{2\lambda_{t+1}'} = \frac{\min\left\{1, \frac{\lambda_{t+1}'}{\lambda_{t+1}}\right\} \gamma_{t+1} A_{t+1}'}{2\lambda_{t+1}'} \le \frac{\gamma_{t+1} A_{t+1}'}{2\lambda_{t+1}}.$$

We recycle an earlier computation and know that

$$D_t - D_{t+1} = a_{t+1} \left\langle \mathbf{M}^{-1} \nabla f(\widetilde{\mathbf{x}}_{t+1}), \mathbf{v}_t - \mathbf{x}^{\star} \right\rangle_{\mathbf{M}} - \frac{a_{t+1}^2}{2} \left\| \mathbf{M}^{-1} \nabla f(\widetilde{\mathbf{x}}_{t+1}) \right\|_{\mathbf{M}}^2$$

$$\geq a_{t+1} \left\langle \mathbf{M}^{-1} \nabla f(\widetilde{\mathbf{x}}_{t+1}), \mathbf{v}_t - \mathbf{x}^{\star} \right\rangle_{\mathbf{M}} - \frac{\gamma_{t+1} A'_{t+1}}{2\lambda_{t+1}} \left\| \mathbf{M}^{-1} \nabla f(\widetilde{\mathbf{x}}_{t+1}) \right\|_{\mathbf{M}}^2$$

which completes the proof of the potential decrease.

The remaining statements follow as written in [CHJJS22, Proof of Proposition 1], and we conclude the proof of Lemma 4.4.4. □

Now that we have shown Lemma 4.4.4, we refer the reader to [CHJJS22, Appendix A] for the proof of Theorem 4.4.3, as it now follows exactly as written there.

We also give additional bounds on the movement of the iterates in  $\|\cdot\|_{M}$ , which is a straightforward adaptation of [CJJJLST20, Lemma 31] to the improved framework from [CHJJS22].

**Lemma 4.4.5.** For all  $t \ge 1$ , we have both

$$\begin{aligned} \left\| \boldsymbol{v}_{t} - \boldsymbol{x}^{\star} \right\|_{\mathbf{M}} &\leq \sqrt{2} \left\| \boldsymbol{x}_{0} - \boldsymbol{x}^{\star} \right\|_{\mathbf{M}} \\ \left\| \boldsymbol{x}_{t} - \boldsymbol{x}^{\star} \right\|_{\mathbf{M}} &\leq \left( \sqrt{2} + \max_{1 \leq i \leq t} \frac{\lambda_{i}'}{\lambda_{i}} \cdot \sqrt{\frac{2}{1 - \sigma^{2}}} \right) \left\| \boldsymbol{x}_{0} - \boldsymbol{x}^{\star} \right\|_{\mathbf{M}} \end{aligned}$$

In the statement of Lemma 4.4.5, the cost of overshooting the guess  $\lambda'_i$  becomes evident – without an additional strong convexity guarantee, it is challenging to ensure that each iterate remains in a small ball around  $x^*$ . This is the main reason we are unable to apply the framework of [CHJJS22] to the  $p = \infty$  case.

Proof of Lemma 4.4.5. Using the same notation as in Lemma 4.4.4 and in that proof, we define

$$P_t := A_t E_t + D_t$$
$$\widehat{\lambda}_t := \min \left\{ \lambda_t, \lambda'_t \right\}$$

By induction on the conclusion of Lemma 4.4.4, for  $t \ge 1$  we have

$$\frac{1}{2} \| \boldsymbol{v}_t - \boldsymbol{x}^{\star} \|_{\mathbf{M}}^2 = D_t \le P_t + (1 - \sigma^2) \sum_{k=1}^t A'_k \widehat{\lambda}_k N_k \le P_0 = \| \boldsymbol{x}_0 - \boldsymbol{x}^{\star} \|_{\mathbf{M}}^2.$$

Thus,

$$\left\|\boldsymbol{v}_{t}-\boldsymbol{x}^{\star}\right\|_{\mathbf{M}}\leq\sqrt{2}\left\|\boldsymbol{x}_{0}-\boldsymbol{x}^{\star}\right\|_{\mathbf{M}}.$$

For the second conclusion, we introduce the following notation.

$$\begin{split} \alpha_{t+1} &\coloneqq \frac{(1-\gamma_{t+1})A_t}{A_{t+1}} \\ \beta_{t+1} &\coloneqq \frac{A_t}{A'_{t+1}} \\ \delta_{t+1} &\coloneqq 1 - (1-\alpha_{t+1})(1-\beta_{t+1}) = 1 - \frac{\gamma_{t+1}A'_{t+1}}{A_{t+1}} \cdot \frac{a'_{t+1}}{A'_{t+1}} = \frac{A_t}{A_{t+1}} \end{split}$$

We also establish for any *i*,

$$\frac{\gamma_i A_i'}{\lambda_i a_i^2} = \frac{A_i'}{\lambda_i \gamma_i (a_i')^2} = \frac{1}{\gamma_i} \cdot \frac{\lambda_i'}{\lambda_i} = \max\left\{\frac{\lambda_i'}{\lambda_i}, 1\right\},$$

which implies

$$\frac{\gamma_i A_i'}{\lambda_i} = a_i^2 \max\left\{\frac{\lambda_i'}{\lambda_i}, 1\right\}.$$

Notice that

$$\begin{split} \|\mathbf{x}_{i+1} - \mathbf{x}^{\star}\|_{\mathbf{M}} &\leq \alpha_{i+1} \|\mathbf{x}_{i} - \mathbf{x}^{\star}\|_{\mathbf{M}} + (1 - \alpha_{i+1}) \left\| \|\mathbf{x}_{i+1} - \mathbf{x}^{\star} \|_{\mathbf{M}} + (1 - \alpha_{i+1}) \left( \|\mathbf{q}_{i} - \mathbf{x}^{\star} \|_{\mathbf{M}} + \|\mathbf{x}_{i+1} - q_{i} \|_{\mathbf{M}} \right) \\ &\leq \alpha_{i+1} \|\mathbf{x}_{i} - \mathbf{x}^{\star} \|_{\mathbf{M}} \\ &+ (1 - \alpha_{i+1}) \left( \beta_{i+1} \|\mathbf{x}_{i} - \mathbf{x}^{\star} \|_{\mathbf{M}} + (1 - \beta_{i+1}) \|\mathbf{v}_{i} - \mathbf{x}^{\star} \|_{\mathbf{M}} \\ &+ (1 - \alpha_{i+1}) \left( 1 - \beta_{i+1} \right) \|\mathbf{v}_{i} - \mathbf{x}^{\star} \|_{\mathbf{M}} + (1 - \alpha_{i+1}) \left\| \mathbf{x}_{i+1} - q_{i} \right\|_{\mathbf{M}} \\ &= \delta_{i+1} \|\mathbf{x}_{i} - \mathbf{x}^{\star} \|_{\mathbf{M}} + (1 - \delta_{i+1}) \|\mathbf{v}_{i} - \mathbf{x}^{\star} \|_{\mathbf{M}} + (1 - \alpha_{i+1}) \left\| \mathbf{x}_{i+1} - q_{i} \right\|_{\mathbf{M}} \\ &\leq \delta_{i+1} \|\mathbf{x}_{i} - \mathbf{x}^{\star} \|_{\mathbf{M}} + (1 - \delta_{i+1}) \|\mathbf{v}_{i} - \mathbf{x}^{\star} \|_{\mathbf{M}} + (1 - \alpha_{i+1}) \left\| \mathbf{x}_{i+1} - q_{i} \right\|_{\mathbf{M}} \\ &\leq \delta_{i+1} \|\mathbf{x}_{i} - \mathbf{x}^{\star} \|_{\mathbf{M}} + \left( 1 - \int_{i=0}^{i} \delta_{i+1} \right) \|\mathbf{v}_{i} - \mathbf{x}^{\star} \|_{\mathbf{M}} \\ &= \delta_{i+1} \|\mathbf{x}_{i} - \mathbf{x}^{\star} \|_{\mathbf{M}} + \left( 1 - \int_{i=0}^{i} \delta_{i+1} \right) \|\mathbf{v}_{i} - \mathbf{x}^{\star} \|_{\mathbf{M}} \\ &= \delta_{i+1} \|\mathbf{x}_{i} - \mathbf{x}^{\star} \|_{\mathbf{M}} + \left( 1 - \int_{i=0}^{i} \delta_{i+1} \right) \|\mathbf{v}_{i} - \mathbf{x}^{\star} \|_{\mathbf{M}} \\ &= \sqrt{2} \|\mathbf{x}_{0} - \mathbf{x}^{\star} \|_{\mathbf{M}} + \sum_{i=1}^{i+1} \int_{i=1}^{i+1} \delta_{i}(1 - \alpha_{i}) \|\mathbf{x}_{i} - q_{i-1} \|_{\mathbf{M}} \\ &= \sqrt{2} \|\mathbf{x}_{0} - \mathbf{x}^{\star} \|_{\mathbf{M}} + \sum_{i=1}^{i+1} \frac{A_{i}}{A_{i+1}} \cdot \frac{\gamma_{i}A_{i}'}{A_{i}} \|\mathbf{x}_{i} - q_{i-1} \|_{\mathbf{M}} \\ &= \sqrt{2} \|\mathbf{x}_{0} - \mathbf{x}^{\star} \|_{\mathbf{M}} + \frac{1}{\sum_{i=1}^{i+1} \frac{A_{i}}{A_{i+1}} \cdot \frac{\gamma_{i}A_{i}'}{A_{i}} \|\mathbf{x}_{i} - q_{i-1} \|_{\mathbf{M}} \\ &= \sqrt{2} \|\mathbf{x}_{0} - \mathbf{x}^{\star} \|_{\mathbf{M}} + \frac{1}{\sum_{i=1}^{i+1} \frac{A_{i}}{A_{i+1}} \cdot \frac{\gamma_{i}A_{i}'}{A_{i}} \|\mathbf{x}_{i} - q_{i-1} \|_{\mathbf{M}} \\ &\leq \sqrt{2} \|\mathbf{x}_{0} - \mathbf{x}^{\star} \|_{\mathbf{M}} + \frac{\sum_{i=1}^{i+1} \frac{A_{i}}{A_{i+1}} \cdot \frac{\gamma_{i}A_{i}'}{A_{i}} \|\mathbf{x}_{i} - q_{i-1} \|_{\mathbf{M}} \\ &\leq \sqrt{2} \|\mathbf{x}_{0} - \mathbf{x}^{\star} \|_{\mathbf{M}} + \frac{\sum_{i=1}^{i+1} \frac{A_{i}}{A_{i+1}} \cdot \frac{\gamma_{i}A_{i}'}{A_{i}} \cdot \sqrt{A_{i}\gamma_{i}A_{i}'} \|\mathbf{x}_{i} - q_{i-1} \|_{\mathbf{M}} \\ &\leq \sqrt{2} \|\mathbf{x}_{0} - \mathbf{x}^{\star} \|_{\mathbf{M}} + \frac{\sum_{i=1}^{i+1} \frac{A_{i}}{A_{i+1}} \cdot \frac{\gamma_{i}A_{i}'}{A_{i}} \cdot \sqrt{\frac{1}{2} - \sigma^{2}} \|\mathbf{x}_{0} - \mathbf{x}^{\star} \|_{\mathbf{M}} \\ &\leq \sqrt{2} \|\mathbf{x}_{0} - \mathbf{x}^{\star} \|_{\mathbf{M}} + \frac{\sum_$$

completing the proof of Lemma 4.4.5.

# 4.5. Minimizing the distributionally robust loss

The goal of this section is to prove Theorem 12. We break up the proof into parts as described in Section 4.2. We structure the section as follows. In the rest of this subsection, we present Algorithm 12, our algorithm that minimizes the distributionally robust loss. In Section 4.5.1, we introduce our smooth approximation for the objective (4.1.2) and show that it is a good additive approximation (this is a standard argument, but we include it as it provides important intuition).

As the main of the difficulty of the proof in Theorem 12 is to establish a Hessian stability for our surrogate loss, we devote the bulk of this section to proving this. Recall that in Section 4.2.2, we claimed that a higher-order smoothness condition called *quasi-self-concordance* gives us the needed Hessian stability – in fact, this follows from [CJJJLST20, Lemma 11]. In light of this, it is enough to prove that our surrogate loss is quasi-self-concordant.

In Section 4.5.2, we work out some calculus facts related to the softmax function. In particular, it is in Section 4.5.2 that we prove the general composition result Lemma 4.5.3 that states that if we take the softmax of several quasi-self-concordant functions, then the resulting function is also quasi-self-concordant. In Section 4.5.3, we apply this composition fact to prove that our surrogate objective is quasi-self-concordant. Finally, in Section 4.5.4, we combine these building blocks with the acceleration framework in [CJJJLST20] and complete the proof of Theorem 12.

Algorithm 12 MinMaxRegression: optimizes (4.1.2) to  $(1 + \varepsilon)$ -multiplicative error

**Require:** Regression problems  $(\mathbf{A}_{S_1}, \mathbf{b}_{S_1}), \dots, (\mathbf{A}_{S_m}, \mathbf{b}_{S_m})$ , accuracy  $\varepsilon > 0$ 

1: Using [MO25, Algorithm 2] with input [A|b], find nonnegative diagonal W and weights  $w_1, \ldots, w_m$  such that for all  $j \in S_i$ ,  $W[j][j] = w_i$  and for all  $x \in \mathbb{R}^d$  and  $c \in \mathbb{R}$ ,

$$\|\mathbf{A}\boldsymbol{x} - c\boldsymbol{b}\|_{\mathcal{G}_{\infty}} \leq \left\|\mathbf{W}^{1/2}\mathbf{A}\boldsymbol{x} - c\mathbf{W}^{1/2}\boldsymbol{b}\right\|_{2} \leq \sqrt{2(\operatorname{rank}(\mathbf{A}) + 1)} \|\mathbf{A}\boldsymbol{x} - c\boldsymbol{b}\|_{\mathcal{G}_{\infty}}.$$

2: if 
$$\sum_{i=1}^{m} w_i \ge m$$
 then  
3:  $\lfloor$  Reset  $\mathbf{W} = \mathbf{I}_n$ .  
4: Let  $x_0 = (\mathbf{A}^\top \mathbf{W} \mathbf{A})^{-1} \mathbf{A}^\top \mathbf{W} \mathbf{b}$ .  
5: Let  

$$\widetilde{f}_{\beta,\delta}(\mathbf{x}) \coloneqq \beta \log \left( \sum_{i=1}^{m} \exp\left(\frac{\sqrt{\delta^2 + \|\mathbf{A}_{S_i} \mathbf{x} - \mathbf{b}_{S_i}\|_2^2} - \delta}{\beta}\right) \right)$$
where  $\beta = \frac{\varepsilon}{4 \log m}$  and  $\delta = \frac{\varepsilon}{4}$ .  
6: Let  $\widehat{f}(\mathbf{x}) \coloneqq \widetilde{f}_{\varepsilon/4 \log m, \varepsilon/4}(\mathbf{x}) + \frac{\varepsilon}{1000 \min\{\operatorname{rank}(\mathbf{A}), m\}} \|\mathbf{W}^{1/2} \mathbf{A}(\mathbf{x} - \mathbf{x}_0)\|_2^2$ .  
7: Using [CJJJLST20, Algorithm 3], implement a  $\left(\frac{C}{\min\{\operatorname{rank}(\mathbf{A}), m\}}, \frac{C}{\varepsilon}\right)$ -ball optimization oracle

- for *f*, where *C* is a universal constant.  $\triangleright$  Iteration complexity guaranteed by Lemma 4.5.5 8: Using [CJJJLST20, Algorithm 2], implement a  $\frac{1}{2}$ -MS oracle for  $\hat{f}$ .
- 9: Run [CJJJLST20, Algorithm 1] for  $\widetilde{O}\left(\frac{\min\{\operatorname{rank}(\mathbf{A}),m\}^{1/3}\log(\frac{d}{\varepsilon})}{\varepsilon^{2/3}}\right)$  iterations using the MS oracle from the previous line and with initial point  $\mathbf{x}_0$  and final point  $\widehat{\mathbf{x}}$ .

10: return  $\widehat{x}$ 

### 4.5.1. Smoothly approximating the objective

Recall that for  $y \in \mathbb{R}^n$ , let  $\|y\|_{\mathcal{G}_{\infty}} := \max_{1 \le i \le m} \|y_{S_i}\|_2$ , where for  $y \in \mathbb{R}^n$  we let  $y_{S_i}$  refer to the vector in  $\mathbb{R}^{n_i}$  indexed by the indices in  $S_i$ . Also, for  $y \in \mathbb{R}^m$ , let  $\mathsf{lse}_\beta(y)$  refer to the function

$$\mathsf{lse}_{\beta}(\boldsymbol{y}) \coloneqq \beta \log \left( \sum_{i=1}^{m} \exp \left( \frac{y_i}{\beta} \right) \right).$$

At a high level, our algorithm will minimize the function

$$\widetilde{f}_{\beta,\delta}(\boldsymbol{x}) \coloneqq \beta \log \left( \sum_{i=1}^{m} \exp \left( \frac{\sqrt{\delta^2 + \|\mathbf{A}_{S_i} \boldsymbol{x} - \boldsymbol{b}_{S_i}\|_2^2} - \delta}{\beta} \right) \right)$$

for appropriate choices of the parameters  $\beta$  and  $\delta$ . This choice of smoothening is natural because of the following approximation statement – see Lemma 4.5.1.

**Lemma 4.5.1.** *For all*  $x \in \mathbb{R}^d$ *, we have* 

$$\left|\widetilde{f}_{\beta,\delta}(\boldsymbol{x}) - \|\mathbf{A}\boldsymbol{x} - \boldsymbol{b}\|_{\mathcal{G}_{\infty}}\right| \leq \beta \log m + \delta.$$

*Proof of Lemma 4.5.1.* These guarantees are well-known, but we prove them anyway for the sake of self-containment. We first prove that for any  $v \in \mathbb{R}^m$ , we have

$$\max_{1 \le i \le m} v_i \le \mathsf{lse}_{\beta}(v) \le \max_{1 \le i \le m} v_i + \beta \log m.$$

In one direction, we have

$$\mathsf{lse}_{\beta}(v) \leq \beta \log \left( \sum_{i=1}^{m} \exp \left( \frac{\max_{1 \leq i \leq m} v_i}{\beta} \right) \right) = \beta \log m + \max_{1 \leq i \leq m} v_i,$$

and in the other, we have

$$\operatorname{lse}_{\beta}(v) \ge \beta \log \left( \exp \left( \frac{\max_{1 \le i \le m} v_i}{\beta} \right) \right) = \max_{1 \le i \le m} v_i.$$

Next, for  $v \in \mathbb{R}^m$ , we will show that

$$\|v\|_{2} - \delta \leq \sqrt{\delta^{2} + \|v\|_{2}^{2}} - \delta \leq \|v\|_{2}.$$

Indeed, we have

$$\sqrt{\delta^2 + \|\boldsymbol{v}\|_2^2} - \delta \le \sqrt{\delta^2} + \sqrt{\|\boldsymbol{v}\|_2^2} - \delta = \|\boldsymbol{v}\|_2,$$

and

$$\sqrt{\delta^2 + \|\boldsymbol{v}\|_2^2} - \delta \ge \sqrt{\|\boldsymbol{v}\|_2^2} - \delta = \|\boldsymbol{v}\|_2 - \delta.$$

From this, we get

$$\widetilde{f}_{\beta,\delta}(\boldsymbol{x}) \leq \max_{1 \leq i \leq m} \left( \sqrt{\delta^2 + \|\mathbf{A}_{S_i}\boldsymbol{x} - \boldsymbol{b}_{S_i}\|_2^2} - \delta \right) + \beta \log m \leq \|\mathbf{A}\boldsymbol{x} - \boldsymbol{b}\|_{\mathcal{G}_{\infty}} + \beta \log m$$

and

$$\widetilde{f}_{\beta,\delta}(\boldsymbol{x}) \geq \beta \log \left( \sum_{i=1}^{m} \exp \left( \frac{\|\mathbf{A}_{S_i} \boldsymbol{x} - \boldsymbol{b}_{S_i}\|_2 - \delta}{\beta} \right) \right) \geq \|\mathbf{A} \boldsymbol{x} - \boldsymbol{b}\|_{\mathcal{G}_{\infty}} - \delta.$$

Putting these together gives

$$\left|\widetilde{f}_{\beta,\delta}(\boldsymbol{x}) - \|\mathbf{A}\boldsymbol{x} - \boldsymbol{b}\|_{\mathcal{G}_{\infty}}\right| \leq \max\left(\beta \log m, \delta\right) \leq \beta \log m + \delta,$$

completing the proof of Lemma 4.5.1.

Eventually, we will choose  $\beta = \varepsilon/(4 \log m)$  and  $\delta = \varepsilon/4$  and then minimize  $\tilde{f}_{\beta,\delta}$  to  $\varepsilon/2$  additive error. In light of Lemma 4.5.1, this will be enough to get an  $\varepsilon$ -additive approximation to the optimum for  $\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_{\mathcal{G}_{\infty}}$ .

### 4.5.2. Calculus for LogSumExp

We investigate certain properties of  $\text{lse}_{\beta}(y)$  when each entry  $[y]_i$  is a function  $h_i(t)$  for  $t \in \mathbb{R}$  for all  $i \in [m]$ . Let  $h(t) \in \mathbb{R}^m$  denote the vector where its *i*th entry is given by  $h_i(t)$ . We treat each  $h_i$  as a one-dimensional restriction of a function  $g_i \colon \mathbb{R}^m \to \mathbb{R}$ , so  $h_i(t) = g_i(y + td)$  for center yand direction d (we omit the parameters y, d in the notation  $h_i$  as it will be clear from context). Finally, recall the definition of quasi-self-concordance (Definition 4.2.3).

We begin with calculating the first two derivatives of  $lse_{\beta}(h(t))$  with respect to *t* in Lemma 4.5.2.

**Lemma 4.5.2.** Let  $\lambda_i(t) := \exp(h_i(t)/\beta)$ . Then, we have

$$\begin{pmatrix} \frac{d}{dt} \end{pmatrix} \operatorname{lse}_{\beta}(h(t)) = \frac{\sum_{i=1}^{m} \left(\lambda_{i}(t) \cdot h_{i}'(t)\right)}{\sum_{i=1}^{m} \lambda_{i}(t)} \\ \left(\frac{d}{dt}\right)^{2} \operatorname{lse}_{\beta}(h(t)) = \frac{1}{\beta} \left(\frac{\sum_{i=1}^{m} \lambda_{i}(t)h_{i}'(t)^{2}}{\sum_{i=1}^{m} \lambda_{i}(t)} - \left(\frac{\sum_{i=1}^{m} \lambda_{i}(t)h_{i}'(t)}{\sum_{i=1}^{m} \lambda_{i}(t)}\right)^{2}\right) + \frac{\sum_{i=1}^{m} \lambda_{i}(t)h_{i}''(t)}{\sum_{i=1}^{m} \lambda_{i}(t)}.$$

Proof of Lemma 4.5.2. The first derivative follows from the chain rule. Indeed, we have

$$\mathsf{lse}_{\beta}'(h(t)) = \beta \cdot \frac{\sum_{i=1}^{m} \lambda_i'(t)}{\sum_{i=1}^{m} \lambda_i(t)} = \beta \cdot \frac{\sum_{i=1}^{m} \left(\lambda_i(t) \cdot \frac{h_i'(t)}{\beta}\right)}{\sum_{i=1}^{m} \lambda_i(t)} = \frac{\sum_{i=1}^{m} \left(\lambda_i(t) \cdot h_i'(t)\right)}{\sum_{i=1}^{m} \lambda_i(t)} \le \max_i h_i'(t).$$

For the second derivative, we use the differentiation rule for multiplication and division and the chain rule, giving

$$\mathsf{lse}_{\beta}^{\prime\prime}(h(t)) = \frac{\left[\left(\sum_{i=1}^{m} \lambda_{i}^{\prime}(t)h_{i}^{\prime}(t) + \lambda_{i}(t)h_{i}^{\prime\prime}(t)\right)\left(\sum_{i=1}^{m} \lambda_{i}(t)\right)\right] - \frac{1}{\beta}\left(\sum_{i=1}^{m} \lambda_{i}(t)h_{i}^{\prime}(t)\right)^{2}}{\left(\sum_{i=1}^{m} \lambda_{i}(t)\right)^{2}}$$
$$= \frac{\left[\frac{1}{\beta}\left(\sum_{i=1}^{m} \lambda_{i}(t)h_{i}^{\prime}(t)^{2} + \beta\lambda_{i}(t)h_{i}^{\prime\prime}(t)\right)\left(\sum_{i=1}^{m} \lambda_{i}(t)\right)\right] - \frac{1}{\beta}\left(\sum_{i=1}^{m} \lambda_{i}(t)h_{i}^{\prime}(t)\right)^{2}}{\left(\sum_{i=1}^{m} \lambda_{i}(t)\right)^{2}}$$

$$=\frac{1}{\beta}\left(\frac{\sum_{i=1}^{m}\lambda_{i}(t)h_{i}'(t)^{2}}{\sum_{i=1}^{m}\lambda_{i}(t)}-\frac{\left(\sum_{i=1}^{m}\lambda_{i}(t)h_{i}'(t)\right)^{2}}{\left(\sum_{i=1}^{m}\lambda_{i}(t)\right)^{2}}\right)+\frac{\sum_{i=1}^{m}\lambda_{i}(t)h_{i}''(t)}{\sum_{i=1}^{m}\lambda_{i}(t)}$$

This completes the proof of Lemma 4.5.2.

Next, we prove a general fact regarding composing lse with a vector formed by functions that are themselves quasi self concordant. See Lemma 4.5.3.

**Lemma 4.5.3** (Composing softmax with quasi-self-concordant functions). Let  $\|\cdot\|$  be an arbitrary norm and  $h_1, \ldots, h_m$  be such that for all  $1 \le i \le m$  and for all  $y, d \in \mathbb{R}^m$  and  $t \in \mathbb{R}$ ,

$$\left(\frac{d}{dt}\right)h_i(t) \le \|d\|$$
 (Lipschitzness)  
$$\left|\left(\frac{d}{dt}\right)^3 h_i(t)\right| \le \nu \|d\| \left(\frac{d}{dt}\right)^2 h_i(t)$$
 (quasi-self-concordance).

*Then, for all*  $y, d \in \mathbb{R}^m$  *and all*  $t \in \mathbb{R}$ *, we have* 

$$\left| \left( \frac{d}{dt} \right)^3 \operatorname{lse}_{\beta}(h(t)) \right| \le \left( \frac{16}{\beta} + \nu \right) \|d\| \left( \frac{d}{dt} \right)^2 \operatorname{lse}_{\beta}(h(t)).$$

As far as we are aware, this type of composition result was not previously known and may be of independent interest.

To prove Lemma 4.5.3, we need Lemma 4.5.4.

**Lemma 4.5.4.** *For any two random variables X*, *Y*, *we have* 

$$\operatorname{Var}[XY] \le 2 \|Y\|_{\infty}^{2} \operatorname{Var}[X] + 2 \|X\|_{\infty}^{2} \operatorname{Var}[Y].$$

*Proof of Lemma* 4.5.4. The proof follows that of [Gir14], but we reproduce it here for completeness. First, notice that for random variables *U*, *V*, we have

$$2\mathsf{Var}\left[U\right] + 2\mathsf{Var}\left[V\right] - \mathsf{Var}\left[U+V\right] = \mathsf{Var}\left[U\right] + \mathsf{Var}\left[V\right] - 2\mathsf{Cov}\left[U,V\right] = \mathsf{Var}\left[U-V\right] \ge 0$$

Let  $U = (X - \mathbb{E}[X])Y$  and  $V = \mathbb{E}[X]Y$ . Then, U + V = XY, and we have

$$\operatorname{Var}[XY] \le 2\operatorname{Var}[(X - \mathbb{E}[X])Y] + 2\operatorname{Var}[\mathbb{E}[X]Y] = 2\operatorname{Var}[(X - \mathbb{E}[X])Y] + 2\mathbb{E}[X]^2\operatorname{Var}[Y].$$

It remains to bound  $Var[(X - \mathbb{E}[X])Y]$ . By Hölder's inequality, we have

$$\operatorname{Var}\left[(X - \mathbb{E}\left[X\right])Y\right] \le \mathbb{E}\left[((X - \mathbb{E}\left[X\right])Y)^2\right] \le \mathbb{E}\left[(X - \mathbb{E}\left[X\right])^2\right] \|Y\|_{\infty}^2 = \operatorname{Var}\left[X\right] \|Y\|_{\infty}^2.$$

Combining everything gives us the conclusion of Lemma 4.5.4.

We are now ready to prove Lemma 4.5.3.

*Proof of Lemma* 4.5.3. Let  $\lambda_i(t) := \exp(h_i(t)/\beta)$ .

In this proof, we will encounter many weighted averages of vectors  $z \in \mathbb{R}^m$  of the form

$$\frac{\sum_{i=1}^m \lambda_i(t) z_i}{\sum_{i=1}^m \lambda_i(t)}.$$

Let  $\mathcal{D}$  be the distribution over [m] whose entries are given by  $\mathcal{D}_j = \lambda_j(t) / \sum_{i=1}^m \lambda_i(t)$ . In the rest of this proof, all expected values, variances, and covariances will be taken with respect to this distribution. In an abuse of notation, let h(t) denote the "random" variable that is  $h_i(t)$  with probability  $\mathcal{D}_i$ . Define h'(t), h''(t), h'''(t) analogously.

To find the third derivative of  $lse_{\beta}(h(t))$ , we start with its second derivative. By Lemma 4.5.2, it is given by

$$\operatorname{lse}_{\beta}^{\prime\prime}(h(t)) = \underbrace{\frac{1}{\beta} \left( \underbrace{\frac{\sum_{i=1}^{m} \lambda_{i}(t) h_{i}^{\prime}(t)^{2}}{\sum_{i=1}^{m} \lambda_{i}(t)} - \left(\frac{\sum_{i=1}^{m} \lambda_{i}(t) h_{i}^{\prime}(t)}{\sum_{i=1}^{m} \lambda_{i}(t)}\right)^{2} \right)}_{T_{1}} + \underbrace{\frac{\sum_{i=1}^{m} \lambda_{i}(t) h_{i}^{\prime\prime}(t)}{\sum_{i=1}^{m} \lambda_{i}(t)}}_{T_{2}} + \underbrace{\frac{\sum_{i=1}^{m} \lambda_{i}(t) h_{i}^{\prime\prime}(t)}{\sum_{i=1}^{m} \lambda_{i}(t)}}_{T_{2}}$$

We now differentiate the above term by term. First, we have

$$\begin{split} T_{2}'(t) &= \frac{\sum_{i=1}^{m} \lambda_{i}(t) \left( \left( \frac{h_{i}'(t)h_{i}''(t)}{\beta} \right) + h_{i}'''(t) \right)}{\sum_{i=1}^{m} \lambda_{i}(t)} - \frac{1}{\beta} \cdot \frac{\left( \sum_{i=1}^{m} \lambda_{i}(t)h_{i}'(t) \right) \left( \sum_{i=1}^{m} \lambda_{i}(t)h_{i}'(t) \right)}{\left( \sum_{i=1}^{m} \lambda_{i}(t) \right)^{2}} \\ &= \frac{1}{\beta} \left( \frac{\sum_{i=1}^{m} \lambda_{i}(t)h_{i}'(t)h_{i}''(t)}{\sum_{i=1}^{m} \lambda_{i}(t)} - \frac{\left( \sum_{i=1}^{m} \lambda_{i}(t)h_{i}'(t) \right) \left( \sum_{i=1}^{m} \lambda_{i}(t)h_{i}''(t) \right)}{\left( \sum_{i=1}^{m} \lambda_{i}(t) \right)^{2}} \right) + \frac{\sum_{i=1}^{m} \lambda_{i}(t)h_{i}''(t)}{\sum_{i=1}^{m} \lambda_{i}(t)} \\ &= \frac{1}{\beta} \mathsf{Cov} \left[ h'(t), h''(t) \right] + \mathbb{E} \left[ h'''(t) \right]. \end{split}$$

Next, we have

$$\frac{d}{dt}\mathbb{E}\left[h'(t)\right]^2 = 2\mathbb{E}\left[h'(t)\right] \cdot \frac{d}{dt}\mathbb{E}\left[h'(t)\right] = 2\mathbb{E}\left[h'(t)\right] \left(\frac{1}{\beta}\mathsf{Var}\left[h'(t)\right] + \mathbb{E}\left[h''(t)\right]\right)$$

and

$$\begin{split} &\frac{d}{dt} \mathbb{E} \left[ h'(t)^2 \right] \\ &= \frac{\left( \sum_{i=1}^m \lambda'_i(t) h'_i(t)^2 + 2h'_i(t) h''_i(t) \lambda_i(t) \right) \left( \sum_{i=1}^m \lambda_i(t) \right) - \frac{1}{\beta} \left( \sum_{i=1}^m \lambda_i(t) h'_i(t) \right) \left( \sum_{i=1}^m \lambda_i(t) h'_i(t)^2 \right) \right)}{\left( \sum_{i=1}^m \lambda_i(t) \right)^2} \\ &= \frac{\left( \sum_{i=1}^m \lambda'_i(t) h'_i(t)^2 + 2h'_i(t) h''_i(t) \lambda_i(t) \right)}{\sum_{i=1}^m \lambda_i(t)} - \frac{1}{\beta} \cdot \frac{\left( \sum_{i=1}^m \lambda_i(t) h'_i(t) \right) \left( \sum_{i=1}^m \lambda_i(t) h'_i(t)^2 \right)}{\left( \sum_{i=1}^m \lambda_i(t) \right)^2} \\ &= \frac{\sum_{i=1}^m \lambda_i(t) \left( \frac{h'_i(t)^3}{\beta} + 2h'_i(t) h''_i(t) \right)}{\sum_{i=1}^m \lambda_i(t)} - \frac{1}{\beta} \cdot \frac{\left( \sum_{i=1}^m \lambda_i(t) h'_i(t) \right) \left( \sum_{i=1}^m \lambda_i(t) h'_i(t)^2 \right)}{\left( \sum_{i=1}^m \lambda_i(t) \right)^2} \\ &= \frac{1}{\beta} \mathsf{Cov} \left[ h'(t), h'(t)^2 \right] + 2\mathbb{E} \left[ h'(t) h''(t) \right]. \end{split}$$

Combining everything gives us

$$\begin{split} & \operatorname{lse}_{\beta}^{\prime\prime\prime}(h(t)) \\ &= \frac{1}{\beta} \left( \frac{1}{\beta} \operatorname{Cov} \left[ h'(t), h'(t)^2 \right] + 2\mathbb{E} \left[ h'(t) h''(t) \right] - 2\mathbb{E} \left[ h'(t) \right] \left( \frac{1}{\beta} \operatorname{Var} \left[ h'(t) \right] + \mathbb{E} \left[ h''(t) \right] \right) \right) \\ &+ \frac{1}{\beta} \operatorname{Cov} \left[ h'(t), h''(t) \right] + \mathbb{E} \left[ h'''(t) \right] \\ &= \frac{1}{\beta^2} \operatorname{Cov} \left[ h'(t), h'(t)^2 \right] - \frac{2}{\beta^2} \mathbb{E} \left[ h'(t) \right] \operatorname{Var} \left[ h'(t) \right] + \frac{3}{\beta} \operatorname{Cov} \left[ h'(t), h''(t) \right] + \mathbb{E} \left[ h'''(t) \right] . \end{split}$$

We first analyze the terms that only depend on h'(t). To do so, we use Lemma 4.5.4 to write

$$\left|\operatorname{Cov}\left[h'(t),h'(t)^{2}\right]\right| \leq \sqrt{\operatorname{Var}\left[h'(t)\right]}\sqrt{\operatorname{Var}\left[h'(t)^{2}\right]} \leq 2 \|d\|\operatorname{Var}\left[h'(t)\right].$$

Now, we have

$$\begin{aligned} &\frac{1}{\beta^2} \left| \mathsf{Cov} \left[ h'(t), h'(t)^2 \right] - 2\mathbb{E} \left[ h'(t) \right] \mathsf{Var} \left[ h'(t) \right] \right| \\ &\leq \frac{1}{\beta^2} \left| \mathsf{Cov} \left[ h'(t), h'(t)^2 \right] \right| + \frac{2}{\beta^2} \left| \mathbb{E} \left[ h'(t) \right] \mathsf{Var} \left[ h'(t) \right] \right| \\ &\leq \frac{4}{\beta^2} \left\| d \right\| \mathsf{Var} \left[ h'(t) \right] \leq \frac{4}{\beta} \left\| d \right\| \mathsf{Ise}_{\beta}''(h(t)). \end{aligned}$$

Next, we take care of the remaining terms. We have

$$\begin{split} \frac{3}{\beta} \left| \mathsf{Cov}\left[h'(t), h''(t)\right] \right| + \left| \mathbb{E}\left[h'''(t)\right] \right| &\leq \frac{6}{\beta} \left( \max_{i} h'_{i}(t) \right) \mathbb{E}\left[ \left|h''(t) - \mathbb{E}\left[h''(t)\right] \right| \right] + \left| \mathbb{E}\left[h'''(t)\right] \right| \\ &\leq \frac{12}{\beta} \left\| d \right\| \left| \mathsf{lse}_{\beta}''(h(t)) + \mathbb{E}\left[ \left|h'''(t)\right| \right] \\ &\leq \frac{12}{\beta} \left\| d \right\| \left| \mathsf{lse}_{\beta}''(h(t)) + \nu \left\| d \right\| \mathbb{E}\left[h''(t)\right] \\ &\leq \left( \frac{12}{\beta} + \nu \right) \left\| d \right\| \left| \mathsf{lse}_{\beta}''(h(t)), \end{split}$$

where the penultimate line follows from Lemma 4.5.7. Combining these conclusions yields

$$\left|\mathsf{lse}_{\beta}^{\prime\prime\prime}(h(t))\right| \leq \left(\frac{16}{\beta} + \nu\right) \|d\| \,\mathsf{lse}_{\beta}^{\prime\prime}(h(t)),$$

completing the proof of Lemma 4.5.3.

### 4.5.3. Smoothness and quasi-self-concordance of the modified objective

The main result of this subsection is Lemma 4.5.5.

**Lemma 4.5.5.** Let **W** be such that for all  $z \in \mathbb{R}^d$ , we have  $\|\mathbf{A}z\|_{\mathcal{G}_{\infty}} \leq \|\mathbf{W}^{1/2}\mathbf{A}z\|_2$ . For all  $x, z \in \mathbb{R}^d$  and  $t \in \mathbb{R}$ , we have

$$\left(\frac{d}{dt}\right)^2 \widetilde{f}_{\beta,\delta}(\mathbf{x}+t\mathbf{z}) \le \left(\frac{1}{\delta}+\frac{1}{\beta}\right) \left\|\mathbf{W}^{1/2}\mathbf{A}\mathbf{z}\right\|_2^2$$
 (smoothness)  
$$\left|\left(\frac{d}{dt}\right)^3 \widetilde{f}_{\beta,\delta}(\mathbf{x}+t\mathbf{z})\right| \le \left(\frac{16}{\delta}+\frac{3}{\beta}\right) \left\|\mathbf{W}^{1/2}\mathbf{A}\mathbf{z}\right\|_2 \left(\frac{d}{dt}\right)^2 \widetilde{f}_{\beta,\delta}(\mathbf{x}+t\mathbf{z})$$
 (quasi-self-concordance).

Our goal in the rest of this section is to prove Lemma 4.5.5.

We begin with defining  $h_i(t)$  as (absorb the  $\delta$ , y, d parameters into the definition of  $h_i$ )

$$h_i(t) \coloneqq \sqrt{\delta^2 + \left\| \boldsymbol{y}_{S_i} + t \, \boldsymbol{d}_{S_i} \right\|_2^2}$$

Let h(t) denote the vector whose *i*th entry is  $h_i(t)$ . Then, observe that

$$\mathsf{lse}_{\beta}(h(t)) = \beta \log\left(\sum_{i=1}^{m} \exp\left(\frac{h_i(t)}{\beta}\right)\right) = \beta \log\left(\sum_{i=1}^{m} \exp\left(\frac{\sqrt{\delta^2 + \|\boldsymbol{y}_{S_i} + t\,\boldsymbol{d}_{S_i}\|_2^2}}{\beta}\right)\right).$$

It is easy to see that every one-dimensional restriction of  $\tilde{f}_{\beta,\delta}$  can be obtained by an affine transformation of  $|se_{\beta}(h(t))|$  after appropriate choices of  $y, d \in \mathbb{R}^m$ . Hence, we first analyze  $|se_{\beta}(h(t))|$  for all  $y, d \in \mathbb{R}^m$ .

We begin with proving the smoothness of  $|se_{\beta}(h(t))|$  with respect to  $\|\cdot\|_{\mathcal{G}_{\infty}}$ .

**Lemma 4.5.6.** *For all*  $y, d \in \mathbb{R}^m$  *and all*  $t \in \mathbb{R}$ *, we have* 

$$\left(\frac{d}{dt}\right)^{2} |\mathsf{se}_{\beta}(h(t))| \leq \left(\frac{1}{\delta} + \frac{1}{\beta}\right) ||d||_{\mathcal{G}_{\infty}}^{2}.$$

Proof of Lemma 4.5.6. By direct calculation, it is easy to see that

$$h_{i}'(t) = \frac{\langle y_{S_{i}} + t d_{S_{i}}, d_{S_{i}} \rangle}{h_{i}(t)}$$

$$h_{i}''(t) = \frac{\|d_{S_{i}}\|_{2}^{2} h_{i}(t) - h_{i}'(t)^{2} h_{i}(t)}{h_{i}(t)^{2}} = \frac{\|d_{S_{i}}\|_{2}^{2} - h_{i}'(t)^{2}}{h_{i}(t)}.$$
(4.5.1)

We plug this into the result of Lemma 4.5.2 and get

$$\begin{aligned} |\mathsf{se}_{\beta}''(h(t)) &\leq \frac{1}{\beta} \max_{i} h_{i}'(t)^{2} + \max_{i} h_{i}''(t) \\ &= \frac{1}{\beta} \max_{i} \left( \frac{\langle y_{S_{i}} + t d_{S_{i}}, d_{S_{i}} \rangle}{\sqrt{\delta^{2} + \|y_{S_{i}} + t d_{S_{i}}\|_{2}^{2}}} \right)^{2} + \max_{i} \frac{\|d_{S_{i}}\|_{2}^{2} - h_{i}'(t)^{2}}{\sqrt{\delta^{2} + \|y_{S_{i}} + t d_{S_{i}}\|_{2}^{2}}} \\ &\leq \frac{1}{\beta} \max_{i} \|d_{S_{i}}\|_{2}^{2} + \frac{1}{\delta} \max_{i} \|d_{S_{i}}\|_{2}^{2} = \left(\frac{1}{\beta} + \frac{1}{\delta}\right) \|d\|_{\mathcal{G}_{\infty}}^{2}, \end{aligned}$$

completing the proof of Lemma 4.5.6.

Our next task is to show that  $|se_{\beta}(h(t))| = O(1/\beta + 1/\delta)$ -quasi-self-concordant in  $|| \cdot ||_{\mathcal{G}_{\infty}}$ . To do so, we will appeal to Lemma 4.5.3. To be able to do this, we first have to prove the quasi-self-concordance of each component function in  $|se_{\beta}(h(t))|$ .

**Lemma 4.5.7.** For all  $y, d \in \mathbb{R}^m$  and all  $t \in \mathbb{R}$ , we have

$$\left| \left( \frac{d}{dt} \right)^3 \sqrt{\delta^2 + \left\| \boldsymbol{y}_{S_i} + t \, \boldsymbol{d}_{S_i} \right\|_2^2} \right| \le \frac{3}{\delta} \left\| \boldsymbol{d}_{S_i} \right\|_2 \left( \left( \frac{d}{dt} \right)^2 \sqrt{\delta^2 + \left\| \boldsymbol{y}_{S_i} + t \, \boldsymbol{d}_{S_i} \right\|_2^2} \right)$$

*Proof of Lemma 4.5.7.* Although a similar fact appears in [OB20, Section 2.1.2], it is not in the exact form we need. So, we prove the required statement here.

Recycling the computation from (4.5.1), recall

$$h_i''(t) = \frac{\|d_{S_i}\|_2^2 - h_i'(t)^2}{h_i(t)}$$

which gives

$$h_i'''(t) = \frac{-2h_i'(t)h_i''(t)h_i(t) - h_i'(t)(h_i(t)h_i''(t))}{h_i(t)^2} = -\frac{3h_i'(t)h_i''(t)}{h_i(t)}.$$

Finally, again recalling (4.5.1), notice that

$$\left|\frac{h_{i}'(t)}{h_{i}(t)}\right| = \left|\frac{\langle \boldsymbol{y}_{S_{i}} + t\,\boldsymbol{d}_{S_{i}},\boldsymbol{d}_{S_{i}}\rangle}{h_{i}(t)^{2}}\right| = \left|\left\langle\frac{\boldsymbol{y}_{S_{i}} + t\,\boldsymbol{d}_{S_{i}}}{\sqrt{\delta^{2} + \left\|\boldsymbol{y}_{S_{i}} + t\,\boldsymbol{d}_{S_{i}}\right\|_{2}^{2}}, \frac{\boldsymbol{d}_{S_{i}}}{\sqrt{\delta^{2} + \left\|\boldsymbol{y}_{S_{i}} + t\,\boldsymbol{d}_{S_{i}}\right\|_{2}^{2}}}\right\rangle\right| \le \frac{\left\|\boldsymbol{d}_{S_{i}}\right\|_{2}}{\delta}.$$

ī

Combining everything completes the proof of Lemma 4.5.7.

We are now ready to prove the quasi-self-concordance of  $|se_{\beta}(h(t))$  in  $\|\cdot\|_{\mathcal{G}_{\infty}}$ .

**Lemma 4.5.8.** For all  $y, d \in \mathbb{R}^m$  and  $t \in \mathbb{R}$ , we have

$$\left| \left( \frac{d}{dt} \right)^3 \mathsf{lse}_\beta(h(t)) \right| \le \left( \frac{16}{\beta} + \frac{3}{\delta} \right) \| d \|_{\mathcal{G}_\infty} \left( \frac{d}{dt} \right)^2 \mathsf{lse}_\beta(h(t)).$$

*Proof of Lemma* 4.5.8. In the statement of Lemma 4.5.3, let  $\|\cdot\| = \|\cdot\|_{\mathcal{G}_{\infty}}$ . By the definition of  $\|\cdot\|_{\mathcal{G}_{\infty}}$  and  $h_i$ , we have for all i and t that  $h'_i(t) \leq \|d\|_{\mathcal{G}_{\infty}}$ . Additionally, from Lemma 4.5.7, we have that the  $h_i(t)$  are  $3/\delta$ -quasi-self-concordant in the norm  $\|d\|_{\mathcal{G}_{\infty}}$  for all i. Lemma 4.5.8 now follows immediately from Lemma 4.5.3.

Finally, we can prove Lemma 4.5.5.

*Proof of Lemma 4.5.5.* By the conclusion of Lemma 4.5.6, we know that for all  $y, d \in \mathbb{R}^m$  and  $t \in \mathbb{R}$  that

$$\left(\frac{d}{dt}\right)^{2} \operatorname{lse}_{\beta}(h(t)) \leq \left(\frac{1}{\delta} + \frac{1}{\beta}\right) \|\boldsymbol{z}\|_{\mathcal{G}_{\infty}}^{2}.$$

Let y = Ax - b for some x and d = Az for some z. Let

$$g(\boldsymbol{y}) \coloneqq \beta \log \left( \sum_{i=1}^{m} \exp \left( \frac{\sqrt{\delta^2 + \|\boldsymbol{y}_{S_i}\|_2^2 - \delta}}{\beta} \right) \right).$$

Then,

$$\left(\frac{d}{dt}\right)^2 \widetilde{f}_{\beta,\delta}(\boldsymbol{x}+t\boldsymbol{z}) = \left(\frac{d}{dt}\right)^2 g(\mathbf{A}\boldsymbol{x}-\boldsymbol{b}+t\mathbf{A}\boldsymbol{z}) \leq \left(\frac{1}{\delta}+\frac{1}{\beta}\right) \|\mathbf{A}\boldsymbol{z}\|_{\mathcal{G}_{\infty}}^2.$$

With the exact same reasoning applied to the conclusion of Lemma 4.5.8, we also see that

$$\left| \left( \frac{d}{dt} \right)^3 \widetilde{f}_{\beta,\delta}(\mathbf{x} + t\mathbf{z}) \right| \leq \left( \frac{16}{\delta} + \frac{3}{\beta} \right) \|\mathbf{A}\mathbf{z}\|_{\mathcal{G}_{\infty}} \left( \frac{d}{dt} \right)^2 \widetilde{f}_{\beta,\delta}(\mathbf{x} + t\mathbf{z}).$$

The conclusion of Lemma 4.5.5 then follows from remembering that we have **W** such that for all  $z \in \mathbb{R}^d$ ,  $\|\mathbf{A}z\|_{\mathcal{G}_{\infty}} \leq \|\mathbf{W}^{1/2}\mathbf{A}z\|_2$  (following from Theorem 4.2.2).

#### 4.5.4. Analysis of Algorithm 12

In this subsection, we use the calculus facts from the previous two subsections to analyze Algorithm 12. The outline of this proof follows that of [JLS22, Theorem 2], which in turn builds up to using the proof used in [CJJJLST20, Corollary 12]. The main idea is to define the algorithm based on the norm given by a good choice of positive semidefinite **M**, given by Theorem 4.2.2.

In the rest of this section, let **W** be factor-2 block Lewis weight overestimates for  $[\mathbf{A}|\mathbf{b}]$ . As in Line 1 of Algorithm 12 and from the corresponding guarantee given in [MO25, Lemmas 5.6, 5.8], this means that within  $2 \log m$  linear system solves in  $\mathbf{A}^{\top} \mathbf{D} \mathbf{A}$  for diagonal **D**, we can find **W** such that for all  $\mathbf{x} \in \mathbb{R}^d$  and  $c \in \mathbb{R}$  we have

$$\|\mathbf{A}\boldsymbol{x} - c\boldsymbol{b}\|_{\mathcal{G}_{\infty}} \leq \left\|\mathbf{W}^{1/2}\mathbf{A}\boldsymbol{x} - c\mathbf{W}^{1/2}\boldsymbol{b}\right\|_{2} \leq \sqrt{2(\operatorname{rank}(\mathbf{A}) + 1)} \|\mathbf{A}\boldsymbol{x} - c\boldsymbol{b}\|_{\mathcal{G}_{\infty}}$$

Note that choosing c = 1 yields our original objective on either side of the above inequality. Motivated by the above, it is natural to use the norm given by  $\mathbf{M} := \mathbf{A}^{\top} \mathbf{W} \mathbf{A}$  to give the geometry for the ball optimization oracle and for the analysis. Additionally, without loss of generality and for the sake of the analysis, let us rescale the problem so that

$$1 = \mathsf{OPT} := \left\| \mathbf{A} \mathbf{x}^{\star} - \mathbf{b} \right\|_{\mathcal{G}_{\infty}}.$$

Also, as mentioned earlier, assume without loss of generality that  $rank(\mathbf{A}) = d$ .

We begin with Lemma 4.5.9, which bounds our initial suboptimality in  $\tilde{f}$  and in  $\|\cdot\|_{\mathbf{M}}$ .

**Lemma 4.5.9.** Let  $\widetilde{x}_{\beta,\delta} \coloneqq \underset{x \in \mathbb{R}^d}{\operatorname{argmin}} \widetilde{f}_{\beta,\delta}(x)$ . Then,

$$\begin{aligned} \left\| \widetilde{\boldsymbol{x}}_{\boldsymbol{\beta},\boldsymbol{\delta}} - \boldsymbol{x}_0 \right\|_{\mathbf{M}} &\leq (2 + 2(\beta \log m + \delta))\sqrt{2(d+1)} \\ \widetilde{f}_{\boldsymbol{\beta},\boldsymbol{\delta}}(\boldsymbol{x}_0) - \widetilde{f}_{\boldsymbol{\beta},\boldsymbol{\delta}}(\widetilde{\boldsymbol{x}}_{\boldsymbol{\beta},\boldsymbol{\delta}}) &\leq \sqrt{2(d+1)} - 1 + 2(\beta \log m + \delta) \end{aligned}$$

*Proof of Lemma* 4.5.9. It is easy to check that

$$\boldsymbol{x}_0 \coloneqq \left( \mathbf{A}^\top \mathbf{W} \mathbf{A} \right)^{-1} \mathbf{A}^\top \mathbf{W} \boldsymbol{b} = \operatorname*{argmin}_{\boldsymbol{x} \in \mathbb{R}^d} \left\| \mathbf{W}^{1/2} \mathbf{A} \boldsymbol{x} - \mathbf{W}^{1/2} \boldsymbol{b} \right\|_2.$$

By Lemma 4.5.1, for all  $x \in \mathbb{R}^d$ ,

$$\left|\widetilde{f}_{\beta,\delta}(\boldsymbol{x}) - \|\mathbf{A}\boldsymbol{x} - \boldsymbol{b}\|_{\mathcal{G}_{\infty}}\right| \leq \beta \log m + \delta,$$

implying

$$\left\|\left\|\mathbf{A}\boldsymbol{x}^{\star}-\boldsymbol{b}\right\|_{\boldsymbol{\mathcal{G}}_{\infty}}-\widetilde{f}_{\boldsymbol{\beta},\boldsymbol{\delta}}(\widetilde{\boldsymbol{x}}_{\boldsymbol{\beta},\boldsymbol{\delta}})\right\|\leq\beta\log m+\boldsymbol{\delta}.$$

Combining this with Theorem 4.2.2, we get

$$1 \le \left\| \mathbf{A} \mathbf{x}^{\star} - \mathbf{b} \right\|_{\mathcal{G}_{\infty}} \le \left\| \mathbf{A} \mathbf{x}_0 - \mathbf{b} \right\|_{\mathcal{G}_{\infty}} \le \left\| \mathbf{W}^{1/2} \mathbf{A} \mathbf{x}_0 - \mathbf{W}^{1/2} \mathbf{b} \right\|_2$$

and

$$\frac{\left\|\mathbf{W}^{1/2}\mathbf{A}\boldsymbol{x}_{0}-\mathbf{W}^{1/2}\boldsymbol{b}\right\|_{2}}{\sqrt{2(d+1)}} \leq \frac{\left\|\mathbf{W}^{1/2}\mathbf{A}\boldsymbol{x}^{\star}-\mathbf{W}^{1/2}\boldsymbol{b}\right\|_{2}}{\sqrt{2(d+1)}} \leq \left\|\mathbf{A}\boldsymbol{x}^{\star}-\boldsymbol{b}\right\|_{\mathcal{G}_{\infty}} = 1.$$

Combining these gives

$$1 \le \left\| \mathbf{W}^{1/2} \mathbf{A} \mathbf{x}_0 - \mathbf{W}^{1/2} \mathbf{b} \right\|_2 \le \sqrt{2(d+1)}.$$

Additionally,

$$\begin{split} \left\| \mathbf{W}^{1/2} \mathbf{A} \widetilde{\mathbf{x}}_{\beta,\delta} - \mathbf{W}^{1/2} \mathbf{b} \right\|_{2} &\leq \sqrt{2(d+1)} \left\| \mathbf{A} \widetilde{\mathbf{x}}_{\beta,\delta} - \mathbf{b} \right\|_{\mathcal{G}_{\infty}} \\ &\leq \sqrt{2(d+1)} \left( \widetilde{f}_{\beta,\delta} (\widetilde{\mathbf{x}}_{\beta,\delta}) + \beta \log m + \delta \right) \\ &\leq \sqrt{2(d+1)} \left( \left\| \mathbf{A} \mathbf{x}^{\star} - \mathbf{b} \right\|_{\mathcal{G}_{\infty}} + 2(\beta \log m + \delta) \right) \\ &= \sqrt{2(d+1)} (1 + 2(\beta \log m + \delta)). \end{split}$$

Then,

$$\begin{split} \|\widetilde{\boldsymbol{x}} - \boldsymbol{x}_0\|_{\mathbf{M}} &= \left\| \left( \mathbf{W}^{1/2} \mathbf{A} \widetilde{\boldsymbol{x}}_{\beta,\delta} - \mathbf{W}^{1/2} \boldsymbol{b} \right) - \left( \mathbf{W}^{1/2} \mathbf{A} \boldsymbol{x}_0 - \mathbf{W}^{1/2} \boldsymbol{b} \right) \right\|_2 \\ &\leq \left\| \mathbf{W}^{1/2} \mathbf{A} \widetilde{\boldsymbol{x}}_{\beta,\delta} - \mathbf{W}^{1/2} \boldsymbol{b} \right\|_2 + \left\| \mathbf{W}^{1/2} \mathbf{A} \boldsymbol{x}_0 - \mathbf{W}^{1/2} \boldsymbol{b} \right\|_2 \\ &\leq (2 + 2(\beta \log m + \delta)) \sqrt{2(d+1)}, \end{split}$$

and

$$\begin{split} \widetilde{f}_{\beta,\delta}(\boldsymbol{x}_0) &- \widetilde{f}_{\beta,\delta}(\widetilde{\boldsymbol{x}}_{\beta,\delta}) \leq \|\mathbf{A}\boldsymbol{x}_0 - \boldsymbol{b}\|_{\mathcal{G}_{\infty}} - \|\mathbf{A}\boldsymbol{x}^{\star} - \boldsymbol{b}\|_{\mathcal{G}_{\infty}} + 2(\beta \log m + \delta) \\ &\leq \left\|\mathbf{W}^{1/2}\mathbf{A}\boldsymbol{x}_0 - \mathbf{W}^{1/2}\boldsymbol{b}\right\|_2 - \mathsf{OPT} + 2(\beta \log m + \delta) \\ &\leq \sqrt{2(d+1)} - 1 + 2(\beta \log m + \delta). \end{split}$$

This completes the proof of Lemma 4.5.9.

We are now ready to prove Theorem 12.

*Proof of Theorem 12.* Algorithm 12 optimizes the regularization of  $\tilde{f}$  given by

$$\widehat{f}(\boldsymbol{x}) \coloneqq \widetilde{f}_{\beta,\delta}(\boldsymbol{x}) + \frac{\varepsilon}{110R^2} \left\| \mathbf{W}^{1/2} \mathbf{A}(\boldsymbol{x} - \boldsymbol{x}_0) \right\|_2^2,$$

where *R* is such that  $\|x_0 - \tilde{x}_{\beta,\delta}\|_{\mathbf{M}} \leq R$ . Let  $\hat{x} \coloneqq \operatorname{argmin}_{x \in \mathbb{R}^d} \widehat{f}(x)$ . Using [CJJJLST20, Proof of Corollary 12], we know that for every iterate *x* of Algorithm 12,

$$\left|\widehat{f}(\boldsymbol{x}) - \widetilde{f}_{\beta,\delta}(\boldsymbol{x})\right| \leq \frac{\varepsilon}{4}.$$

We now choose  $\beta = \varepsilon/(4 \log m)$  and  $\delta = \varepsilon/4$ , so that  $\tilde{f}_{\beta,\delta}$  approximates f up to error  $\varepsilon/2$  on every point. Using Lemma 4.5.9, this gives  $R = (2 + \varepsilon)\sqrt{2(d + 1)}$ . It is therefore sufficient to optimize  $\hat{f}$  up to  $\varepsilon/4$  additive error.

Next, using Lemma 4.5.5 and [CJJJLST20, Lemmas 11, 43], we have that  $\hat{f}$  is  $(1/\nu, e)$ -Hessian stable in  $\|\cdot\|_{M}$  for  $\nu = \Omega(1/(\varepsilon \log m))$ . We now invoke [CJJJLST20, Theorem 9], which tells us that we can implement a  $(C/\sqrt{d}, C/\varepsilon)$ -ball optimization oracle for f with  $O\left(\log\left(\frac{d}{\varepsilon}\right)^{2}\right)$  linear system solves.

The next step is to turn the ball optimization oracle into a  $\frac{1}{2}$ -MS oracle (Definition 4.4.1). Using [CJJJLST20, Proposition 5], we get a ball oracle complexity of  $O\left(\log\left(\frac{d}{\varepsilon}\right)\right)$  to implement the MS oracle. In total, our linear system solve complexity for implementing the MS oracle for iteration t is  $O\left(\log\left(\frac{d}{\varepsilon}\right)^3\right)$ .

Finally, using [CJJJLST20, Theorem 6], we get that Algorithm 12 has a Newton iteration complexity of

$$O\left(\left(\frac{(1+\varepsilon)\sqrt{d}\log m}{\varepsilon}\right)^{2/3}\log\left(\frac{\sqrt{d}+\varepsilon}{\varepsilon}\right)\left(\log\left(\frac{(\log m/\varepsilon)d(1+(1+\varepsilon)\sqrt{d}\log m/\varepsilon)}{\varepsilon}\right)\right)^3\right)$$
$$=O\left(\frac{d^{1/3}}{\varepsilon^{2/3}}\log\left(\frac{d\log m}{\varepsilon}\right)^{14/3}\right),$$

as promised.

Next, we analyze what happens if we fall in the case where  $\mathbf{W} = \mathbf{I}_m$ . Here, by using the  $\sqrt{m}$  distortion from approximating  $\ell_{\infty}^m$  with  $\ell_2^m$ , we have for all  $\mathbf{x} \in \mathbb{R}^d$ ,

$$\frac{\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2}{\sqrt{m}} \leq \|\mathbf{A}\mathbf{x}-\mathbf{b}\|_{\mathcal{G}_{\infty}} \leq \|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2.$$

Using this and repeating the previous analysis with this choice of M gives us a rate of

$$O\left(\frac{m^{1/3}}{\varepsilon^{2/3}}\log\left(\frac{m\log m}{\varepsilon}\right)^{14/3}\right),\,$$

as required.

It remains to determine the form of the Newton steps. For this, it is sufficient to understand the Hessian of  $\hat{f}$ . A straightforward calculation shows that it is of the form  $\mathbf{A}^{\top}\mathbf{B}\mathbf{A}$  where **B** is a block-diagonal matrix where each block has size  $|S_i| \times |S_i|$ . Thus, each Newton step solves a linear system of the form  $\mathbf{A}^{\top}\mathbf{B}\mathbf{A}\mathbf{z} = \mathbf{v}$ .

Combining this with the iteration complexity guarantee to find W (see Theorem 4.2.2) completes the proof of Theorem 12.

## 4.6. Interpolating between average and robust losses

In this section, we prove Theorem 13. As before, our proof follows the outline in Section 4.2. The main technical challenges are to establish a form of strong convexity for our objective f and then to build a solver for the proximal problem (4.2.2).

The rest of this section is organized as follows. In Section 4.6.1, we derive calculus facts about our objective f, including bounds on its Hessian and the promised strong convexity (particularly Lemma 4.6.2 and the more general result it builds on, Lemma 4.6.3). In Section 4.6.2, we prove some facts about the iterates of Algorithm 11 when applied to our setting. In Section 4.6.3, we more precisely define and analyze our solver for proximal sub-problems. This section is fairly technical and we give a more detailed outline there. Finally, in Section 4.6.4, we assemble all these components and analyze Algorithm 14, thereby proving Theorem 13.

Throughout this analysis, we rescale the problem so that  $f(x^*) = 1$ . It is now sufficient to solve for an  $\varepsilon$ -additive error solution.

### 4.6.1. Calculus for the objective

In this section, we work out some calculus facts related to our objective  $\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_{\mathcal{G}_p}^p$ . Throughout this discussion, let  $f(\mathbf{x}) \coloneqq \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_{\mathcal{G}_p}^p$ .

**Lemma 4.6.1.** *For any*  $z \in \mathbb{R}^d$ *, we have* 

$$p\sum_{i=1}^{m} \|\mathbf{A}_{S_{i}}\boldsymbol{x} - \boldsymbol{b}_{S_{i}}\|_{2}^{p-2} \|\mathbf{A}_{S_{i}}\boldsymbol{z}\|_{2}^{2} \leq \boldsymbol{z}^{\top} \left(\nabla^{2} f(\boldsymbol{x})\right) \boldsymbol{z} \leq p(p-1)\sum_{i=1}^{m} \|\mathbf{A}_{S_{i}}\boldsymbol{x} - \boldsymbol{b}_{S_{i}}\|_{2}^{p-2} \|\mathbf{A}_{S_{i}}\boldsymbol{z}\|_{2}^{2}.$$

*Proof of Lemma 4.6.1.* Let us first calculate the derivative and hessian for  $f(\cdot)$  using the chain rule and usual matrix differentiation rules:

$$f(\mathbf{x}) = \sum_{i=1}^{m} \|\mathbf{A}_{S_{i}}\mathbf{x} - \mathbf{b}_{S_{i}}\|_{2}^{p} ,$$

$$\nabla f(\mathbf{x}) = p \sum_{i=1}^{m} \|\mathbf{A}_{S_{i}}\mathbf{x} - \mathbf{b}_{S_{i}}\|_{2}^{p-2} \mathbf{A}_{S_{i}}^{\top} (\mathbf{A}_{S_{i}}\mathbf{x} - \mathbf{b}_{S_{i}}) ,$$

$$\nabla^{2} f(\mathbf{x}) = p \sum_{i=1}^{m} \|\mathbf{A}_{S_{i}}\mathbf{x} - \mathbf{b}_{S_{i}}\|_{2}^{p-2} \mathbf{A}_{S_{i}}^{\top} \mathbf{A}_{S_{i}}$$

$$+ p(p-2) \sum_{i=1}^{m} \|\mathbf{A}_{S_{i}}\mathbf{x} - \mathbf{b}_{S_{i}}\|_{2}^{p-4} \left(\mathbf{A}_{S_{i}}^{\top} (\mathbf{A}_{S_{i}}\mathbf{x} - \mathbf{b}_{S_{i}}) (\mathbf{A}_{S_{i}}\mathbf{x} - \mathbf{b}_{S_{i}})^{\top} \mathbf{A}_{S_{i}}\right) .$$
(4.6.2)

Using this formula, we take the quadratic form with respect to a vector *z*. By Cauchy-Schwarz, notice that

$$z^{\top} \|\mathbf{A}_{S_{i}} \boldsymbol{x} - \boldsymbol{b}_{S_{i}}\|_{2}^{p-4} \left(\mathbf{A}_{S_{i}}^{\top} (\mathbf{A}_{S_{i}} \boldsymbol{x} - \boldsymbol{b}_{S_{i}}) (\mathbf{A}_{S_{i}} \boldsymbol{x} - \boldsymbol{b}_{S_{i}})^{\top} \mathbf{A}_{S_{i}} \right) z$$
  
=  $\|\mathbf{A}_{S_{i}} \boldsymbol{x} - \boldsymbol{b}_{S_{i}}\|_{2}^{p-4} \langle \mathbf{A}_{S_{i}} \boldsymbol{z}, \mathbf{A}_{S_{i}} \boldsymbol{x} - \boldsymbol{b}_{S_{i}} \rangle^{2} \leq \|\mathbf{A}_{S_{i}} \boldsymbol{x} - \boldsymbol{b}_{S_{i}}\|_{2}^{p-2} \|\mathbf{A}_{S_{i}} \boldsymbol{z}\|_{2}^{2}$ 

With that, we have

$$\boldsymbol{z}^{\top} \left( \nabla^{2} f(\boldsymbol{x}) \right) \boldsymbol{z} \leq p \sum_{i=1}^{m} \| \mathbf{A}_{S_{i}} \boldsymbol{x} - \boldsymbol{b}_{S_{i}} \|^{p-2} \| \mathbf{A}_{S_{i}} \boldsymbol{z} \|_{2}^{2} + (p-2) \| \mathbf{A}_{S_{i}} \boldsymbol{x} - \boldsymbol{b}_{S_{i}} \|^{p-2} \| \mathbf{A}_{S_{i}} \boldsymbol{z} \|_{2}^{2} ,$$
  
$$= p(p-1) \sum_{i=1}^{m} \| \mathbf{A}_{S_{i}} \boldsymbol{x} - \boldsymbol{b}_{S_{i}} \|_{2}^{p-2} \| \mathbf{A}_{S_{i}} \boldsymbol{z} \|_{2}^{2} .$$
(4.6.3)

For the lower bound, we use our calculation for  $\nabla^2 f(x)$  to write

$$\boldsymbol{z}^{\top} \left( \nabla^2 f(\boldsymbol{x}) \right) \boldsymbol{z} \geq p \sum_{i=1}^m \| \mathbf{A}_{S_i} \boldsymbol{x} - \boldsymbol{b}_{S_i} \|_2^{p-2} \| \mathbf{A}_{S_i} \boldsymbol{z} \|_2^2,$$

completing the proof of Lemma 4.6.1.

### Strong convexity of the objective

The main pair of results of this section are Lemma 4.6.2 and Lemma 4.6.3. We can think of Lemma 4.6.2 as a form of strong convexity for our objective.

**Lemma 4.6.2** (Strong convexity of *f*). Let  $f(\mathbf{x}) \coloneqq \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_{\mathcal{G}_p}^p$ . For all  $\mathbf{d} \in \mathbb{R}^d$ , we have

$$f(\boldsymbol{x} + \boldsymbol{d}) \ge f(\boldsymbol{x}) + \langle \nabla f(\boldsymbol{x}), \boldsymbol{d} \rangle + \frac{4}{2^p} \|\mathbf{A}\boldsymbol{d}\|_{\mathcal{G}_p}^p$$

and therefore

$$\|\mathbf{x} - \mathbf{x}^{\star}\|_{\mathbf{M}} \le 2^{3/2 - 3/p} d^{1/2 - 1/p} (f(\mathbf{x}) - f(\mathbf{x}^{\star}))^{1/p}$$

**Lemma 4.6.3** (Strong convexity of  $\|\boldsymbol{y}\|_2^p$ ). Let  $\boldsymbol{v} \in \mathbb{R}^k$  for  $k \ge 1$ . For any  $\Delta \in \mathbb{R}^k$ , we have

$$\|\boldsymbol{v} + \boldsymbol{\Delta}\|_{2}^{p} \ge \|\boldsymbol{v}\|_{2}^{p} + p \|\boldsymbol{v}\|_{2}^{p-2} \langle \boldsymbol{v}, \boldsymbol{\Delta} \rangle + \frac{4}{2^{p}} \|\boldsymbol{\Delta}\|_{2}^{p}$$

To motivate Lemma 4.6.3, let us see how Lemma 4.6.3 implies Lemma 4.6.2.

Proof of Lemma 4.6.2. Note that

$$\nabla f(\boldsymbol{x}) = \sum_{i=1}^{m} p \| \mathbf{A}_{S_i} \boldsymbol{x} - \boldsymbol{b}_{S_i} \|_2^{p-2} \mathbf{A}_{S_i}^{\mathsf{T}} (\mathbf{A}_{S_i} \boldsymbol{x} - \boldsymbol{b}_{S_i}) \|_2^{p-2}$$

This implies

$$\sum_{i=1}^{m} p \left\| \mathbf{A}_{S_{i}} \mathbf{x} - \mathbf{b}_{S_{i}} \right\|_{2}^{p-2} \left\langle \mathbf{A}_{S_{i}} \mathbf{x} - \mathbf{b}_{S_{i}}, \mathbf{A}_{S_{i}} \mathbf{d} \right\rangle = \left\langle \nabla f(\mathbf{x}), \mathbf{d} \right\rangle \quad .$$

Combining this and applying Lemma 4.6.3 (which is a strong convexity lemma for  $\|\cdot\|_2^p$  that we prove subsequently in this section), we get

$$f(\mathbf{x} + \mathbf{d}) = \|\mathbf{A}(\mathbf{x} + \mathbf{d}) - \mathbf{b}\|_{\mathcal{G}_p}^p = \|\mathbf{A}\mathbf{d} + (\mathbf{A}\mathbf{x} - \mathbf{b})\|_{\mathcal{G}_p}^p ,$$

$$= \sum_{i=1}^{m} \|\mathbf{A}_{S_{i}}d + (\mathbf{A}_{S_{i}}x - b_{S_{i}})\|_{2}^{p} ,$$

$$\geq^{(\text{Lemma 4.6.3)}} \sum_{i=1}^{m} \|\mathbf{A}_{S_{i}}x - b_{S_{i}}\|_{2}^{p} + p\|\mathbf{A}_{S_{i}}x - b_{S_{i}}\|_{2}^{p-2} \langle (\mathbf{A}_{S_{i}}x - b_{S_{i}}), \mathbf{A}_{S_{i}}d \rangle + \frac{4}{2^{p}} \|\mathbf{A}_{S_{i}}d\|_{2}^{p} ,$$

$$= \sum_{i=1}^{m} \|\mathbf{A}_{S_{i}}x - b_{S_{i}}\|_{2}^{p} + \langle p\|\mathbf{A}_{S_{i}}x - b_{S_{i}}\|_{2}^{p-2} \mathbf{A}_{S_{i}}^{\top} (\mathbf{A}_{S_{i}}x - b_{S_{i}}), d \rangle + \frac{4}{2^{p}} \|\mathbf{A}_{S_{i}}d\|_{2}^{p} ,$$

$$=^{(4.6.1)} \|\mathbf{A}x - b\|_{\mathcal{G}_{p}}^{p} + \langle \nabla f(x), d \rangle + \frac{4}{2^{p}} \|\mathbf{A}d\|_{\mathcal{G}_{p}}^{p} = f(x) + \langle \nabla f(x), d \rangle + \frac{4}{2^{p}} \|\mathbf{A}d\|_{\mathcal{G}_{p}}^{p} .$$

We now take care of the second statement. Observe that at optimality, we have  $\nabla f(\mathbf{x}^*) = 0$ . Plugging this in (replace  $\mathbf{x}$  by  $\mathbf{x}^*$  and d by  $\mathbf{x} - \mathbf{x}^*$  above), rearranging, and taking pth roots gives

$$\left\|\mathbf{A}(\mathbf{x}-\mathbf{x}^{\star})\right\|_{\mathcal{G}_p} \le \left(\frac{4}{2^p}\right)^{-1/p} (f(\mathbf{x})-f(\mathbf{x}^{\star}))^{1/p} = \frac{2}{4^{1/p}} (f(\mathbf{x})-f(\mathbf{x}^{\star}))^{1/p} .$$

Next, recall that by Theorem 4.2.2,

$$\left\|\boldsymbol{x} - \boldsymbol{x}^{\star}\right\|_{\mathbf{M}} = \left\|\mathbf{W}^{1/2-1/p}\mathbf{A}(\boldsymbol{x} - \boldsymbol{x}^{\star})\right\|_{2} \le (2d)^{1/2-1/p}\left\|\mathbf{A}(\boldsymbol{x} - \boldsymbol{x}^{\star})\right\|_{\mathcal{G}_{p}} \quad .$$

,

Stitching the inequalities together completes the proof of Lemma 4.6.2.

In the rest of this subsection, we prove Lemma 4.6.3. We begin with a few numerical inequalities.

**Lemma 4.6.4.** For  $\alpha \leq -1/2$  and  $p \geq 2$ ,  $g(\alpha) := \frac{1+p\alpha}{(-(2\alpha+1))^{p/2}}$  is nonincreasing in  $\alpha$ .

*Proof of Lemma 4.6.4.* We first take the derivative of g with respect to  $\alpha$ ,

$$g'(\alpha) = \frac{p(-(2\alpha+1))^{p/2} - \left((-2)\frac{p}{2}\left(-(2\alpha+1)\right)^{p/2-1}\right)(1+p\alpha)}{(-(2\alpha+1))^p}$$
$$= \frac{p(-(2\alpha+1)^{p/2}) + p\left(-(2\alpha+1)\right)^{p/2-1}(1+p\alpha)}{(-(2\alpha+1))^p} ,$$
$$= p \cdot \frac{(-(2\alpha+1)) + (1+p\alpha)}{(-(2\alpha+1))^{p/2+1}} ,$$
$$= p \cdot \frac{(p-2)\alpha}{(-(2\alpha+1))^{p/2+1}} \le 0 ,$$

where in the final inequality we used that  $p \ge 2$  and  $\alpha \le -1/2$ . This completes the proof of the lemma.

We also need the following lemma, which is similar to a result due to Adil, Kyng, Peng, and Sachdeva [AKPS19, Lemma 4.5]. It amounts to proving Lemma 4.6.3 when the dimension k = 1.

**Lemma 4.6.5** (Case A. of Lemma 4.6.6). *For any*  $\alpha \in \mathbb{R}$  *and*  $p \ge 2$ *,* 

$$|1 + \alpha|^p \ge 1 + p\alpha + \frac{4}{2^p} |\alpha|^p$$
.

*Proof of Lemma 4.6.5.* Note that the inequality is true when p = 2 and becomes an equality. We consider the case when p > 2 and use  $h(\alpha)$  to denote the error function,

$$h(\alpha) \coloneqq |1 + \alpha|^p - \left(1 + p\alpha + \frac{4}{2^p} |\alpha|^p\right)$$

We aim to show  $h(\alpha) \ge 0$  for all  $\alpha \in \mathbb{R}$ . Let us first write the derivatives of *h*.

$$\begin{split} h'(\alpha) &= p\left(\left|1+\alpha\right|^{p-2}(1+\alpha) - \left(1+\frac{4}{2^{p}}\left|\alpha\right|^{p-2}\alpha\right)\right) \ ,\\ h''(\alpha) &= p(p-1)\left(\left|1+\alpha\right|^{p-2} - \frac{4}{2^{p}}\left|\alpha\right|^{p-2}\right) = p(p-1)\left(\left|1+\alpha\right|^{p-2} - \left|\frac{\alpha}{2}\right|^{p-2}\right) \end{split}$$

It is now easy to verify the following statements about h,

- I. h'(-2) = h''(-2) = 0 and  $h''(\alpha) > 0$  for  $\alpha < -2$ ,  $\Rightarrow$  within the range  $(-\infty, -2]$  the function h is minimized at -2;
- II. h'(-2) = 0 and  $h''(\alpha) \le 0$  for  $\alpha \in (-2, -2/3] \Rightarrow h'(\alpha) < 0$  in the range (-2, -2/3], i.e., in that range the function *h* is minimized at -2/3;
- III. h'(-2/3) < 0 = h'(0) and  $h''(\alpha) > 0$  for  $\alpha > -2/3 \Rightarrow$  the function *h* is decreasing in (-2/3, 0) and increasing in  $[0, \infty)$ , i.e., within the range  $(-2/3, \infty)$  the function *h* is minimized at 0.

As a result of the above observations, it is enough to check the inequality at the inputs  $\alpha \in \{-2, -2/3, 0\}$ . We have for p > 2,

$$\begin{split} h(-2) &= 1 - (1 - 2p + 4) = 2p - 4 > 0 \ , \\ h\left(-\frac{2}{3}\right) &= \frac{1}{3^p} - \left(1 - \frac{2p}{3} + \frac{4}{2^p} \left|\frac{2}{3}\right|^p\right) = \frac{1}{3^p} - 1 + \frac{2p}{3} - \frac{4}{3^p} = -1 + \frac{2p}{3} - \frac{3}{3^p} > 0 \\ h(0) &= 1 - 1 = 0 \ . \end{split}$$

This implies that  $h(\alpha) \ge 0$  for all values of  $\alpha$ , concluding the proof of Lemma 4.6.5.

Next, we prove a special case of Lemma 4.6.3.

**Lemma 4.6.6.** For any  $\alpha \in \mathbb{R}$ ,  $\beta \ge 0$ , and  $p \ge 2$ , we have

$$((1+\alpha)^2 + \beta^2)^{p/2} \ge 1 + p\alpha + \frac{4}{2^p} (\alpha^2 + \beta^2)^{p/2}$$

*Proof of Lemma 4.6.6.* Let us study the difference of both sides of the inequality using the following function,

$$h(\alpha,\beta) \coloneqq \left( (1+\alpha)^2 + \beta^2 \right)^{p/2} - \left( 1 + p\alpha + \frac{4}{2^p} \left( \alpha^2 + \beta^2 \right)^{p/2} \right)$$

We want to show that for  $\alpha \in \mathbb{R}$ ,  $\beta \ge 0$ , and  $p \ge 2$ ,  $h(\alpha, \beta) \ge 0$ . We will break this proof into three cases: **A**.  $\alpha \in \mathbb{R}$  and  $\beta = 0$ ; **B**.  $\alpha \in (-\infty, -2] \cup [-2/3, \infty)$  and  $\beta > 0$ ; and **C**.  $\alpha \in (-2, -2/3)$  and  $\beta > 0$ . These cases together cover of the entire range of  $\alpha \in \mathbb{R}$  and  $\beta \ge 0$ .

**<u>Case A.</u>** When  $\beta = 0$ , the proof simply follows from the statement of Lemma 4.6.5 by noting  $|\alpha|^p = (\sqrt{\alpha^2})^p = (\alpha^2)^{p/2}$ .

In the remaining two cases we will show that for any  $\alpha \in \mathbb{R}$ , increasing the value of  $\beta$  still maintains  $h(\alpha, \beta) \ge 0$ . To see this, we first note that the derivative of  $h(\alpha, \beta)$  w.r.t.  $\beta$  is given by,

$$\nabla_{\beta} h(\alpha, \beta) = p\beta \left( \left( (1+\alpha)^2 + \beta^2 \right)^{p/2-1} - \frac{4}{2^p} \left( \alpha^2 + \beta^2 \right)^{p/2-1} \right)$$

For  $\beta > 0$ , ensuring this derivative is positive is equivalent to the following,

$$\begin{aligned} \nabla_{\beta} h(\alpha, \beta) > 0 &\equiv p\beta \left( (1+\alpha)^2 + \beta^2 \right)^{p/2-1} > p\beta \cdot \frac{4}{2^p} \left( \alpha^2 + \beta^2 \right)^{p/2-1} ,\\ &\equiv {}^{(p\beta>0)} \left( 1+\alpha \right)^2 + \beta^2 > \left( \frac{1}{2^{p-2}} \right)^{2/(p-2)} \cdot \left( \alpha^2 + \beta^2 \right) ,\\ &\equiv (1+\alpha)^2 + \beta^2 > \frac{1}{4} \cdot \left( \alpha^2 + \beta^2 \right) ,\\ &\equiv (3\alpha^2 + 8\alpha + 4) + 3\beta^2 > 0 ,\\ &\equiv \beta^2 > - \left( \alpha^2 + \frac{8}{3}\alpha + \frac{4}{3} \right) . \end{aligned}$$
(4.6.4)

<u>**Case B.</u>** Note that the roots of the quadratic function  $3\alpha^2 + 8\alpha + 4$  are given by  $\alpha_1 = -2$  and  $\alpha_2 = -2/3$ . This means that for  $\alpha \in (-\infty, -2] \cup [-2/3, \infty)$  we have  $3\alpha^2 + 8\alpha + 4 \ge 0$  which is **sufficient** to ensure using (4.6.4) that  $\nabla_{\beta}h(\alpha, \beta) > 0$ , and hence  $h(\alpha, \beta) > 0$ . This takes care of Case B.</u>

<u>**Case C.**</u> Now we only need to consider the range  $\alpha \in (-2, -2/3)$  with  $\beta > 0$ . In this range, the recall the equivalence (4.6.4),

$$\nabla_{\beta}h(\alpha,\beta) > 0 \equiv \beta > \sqrt{-\left(\alpha^2 + \frac{8}{3}\alpha + \frac{4}{3}\right)} =: \beta_0(\alpha)$$

Thus for all  $\beta > \beta_0(\alpha)$  we know that  $h(\alpha, \beta)$  is increasing in  $\beta$  and vice-versa. This allows us for any given  $\alpha \in (-2, -2/3)$  to further break Case C into two sub-cases:

**<u>Case C.I</u>** For  $\beta \in [0, \beta_0)$ , since  $h(\alpha, \beta)$  is decreasing in  $\beta$  its lowest value is attained at  $\beta = 0$  and we only need to verify that  $h(\alpha, 0) \ge 0$ . We get this directly from Lemma 4.6.5.

**<u>Case C.II</u>** For  $\beta \in [\beta_0, \infty)$ , since  $h(\alpha, \beta)$  is increasing in  $\beta$  its lowest value is attained at  $\beta = \beta_0$  and we only need to verify that  $h(\alpha, \beta_0(\alpha)) \ge 0$ . We first simplify the expression for  $h(\alpha, \beta_0(\alpha))$ ,

$$\begin{split} h(\alpha,\beta_0(\alpha)) &= \left((1+\alpha)^2 + \beta_0^2\right)^{p/2} - \left(1 + p\alpha + K_p \left(\alpha^2 + \beta_0^2\right)^{p/2}\right) ,\\ &= \left(-\frac{1}{3} - \frac{2}{3}\alpha\right)^{p/2} - \left(1 + p\alpha + \frac{4}{2^p} \left(-\frac{8}{3}\alpha - \frac{4}{3}\right)^{p/2}\right) ,\\ &= \left(-\frac{1}{3} - \frac{2}{3}\alpha\right)^{p/2} - \left(1 + p\alpha + 4 \left(-\frac{2}{3}\alpha - \frac{1}{3}\right)^{p/2}\right) ,\\ &= -1 - p\alpha - 3 \left(-\frac{2}{3}\alpha - \frac{1}{3}\right)^{p/2} ,\end{split}$$

$$= -1 - p\alpha - \frac{1}{3^{p/2-1}} (-2\alpha - 1)^{p/2} ,$$
  
$$= -(-2\alpha - 1)^{p/2} \left( \frac{1 + p\alpha}{(-2\alpha - 1)^{p/2}} + \frac{1}{3^{p/2-1}} \right)$$

Now since  $\alpha \in (-2, -2/3) < -1/2$  we can use Lemma 4.6.4 to note that the first term is nondecreasing in  $\alpha$  which means that its lowest value in this range can be lower bounded by its value at  $\alpha = -2$ , i.e., for  $\alpha \in (-2, -2/3)$ ,

$$\begin{split} h(\alpha,\beta_0(\alpha)) &\geq h(-2,\beta_0(-2)) \ , \\ &= -3^{p/2} \left( \frac{1-2p}{3^{p/2}} + \frac{1}{3^{p/2-1}} \right) \ , \\ &= 2p-1-3 = 2(p-2) > 0 \ , \end{split}$$

which finishes the proof of Case C.II and also Case C. Together Cases A, B and C complete the proof of Lemma 4.6.6.

We are now ready to prove Lemma 4.6.3.

*Proof of Lemma 4.6.3.* First, assume that  $||v||_2 = 1$ . We will later extend the result to all v.

Since  $||v||_2 = 1$ , we can write  $\Delta = \alpha v + \beta w$  where  $\langle v, w \rangle = 0$  and  $||w||_2 = 1$ , so that we have  $||\Delta||_2^2 = \alpha^2 + \beta^2$ . Without loss of generality, we have  $\beta \ge 0$ . Fixing w and  $\alpha$  for now, it is enough to show that for all  $\beta \ge 0$ , we have

$$\|(1+\alpha)v + \beta w\|_{2}^{p} = \left((1+\alpha)^{2} + \beta^{2}\right)^{p/2} \stackrel{?}{\geq} 1 + p\alpha + \frac{4}{2^{p}} \|\Delta\|_{2}^{p} = 1 + p\alpha + \frac{4}{2^{p}} \left(\alpha^{2} + \beta^{2}\right)^{p/2}.$$

This follows immediately by Lemma 4.6.6.

We now extend the result for all v. Let  $\bar{v} \coloneqq v/||v||_2$  and note that

$$\begin{aligned} \|\boldsymbol{v} + \boldsymbol{\Delta}\|_{2}^{p} &= \|\boldsymbol{v}\|_{2}^{p} \left\| \bar{\boldsymbol{v}} + \frac{\boldsymbol{\Delta}}{\|\boldsymbol{v}\|_{2}} \right\|_{2}^{p} \geq \|\boldsymbol{v}\|_{2}^{p} \left( 1 + \left\langle \bar{\boldsymbol{v}}, \frac{\boldsymbol{\Delta}}{\|\boldsymbol{v}\|_{2}} \right\rangle + \frac{4}{2^{p}} \left\| \frac{\boldsymbol{\Delta}}{\|\boldsymbol{v}\|_{2}} \right\|_{2}^{p} \right) \\ &= \|\boldsymbol{v}\|_{2}^{p} + p \|\boldsymbol{v}\|_{2}^{p-2} \left\langle \boldsymbol{v}, \boldsymbol{\Delta} \right\rangle + \frac{4}{2^{p}} \|\boldsymbol{\Delta}\|_{2}^{p}, \end{aligned}$$

completing the proof of Lemma 4.6.3.

### Smoothness of the objective

The main result of this subsection is Lemma 4.6.7.

**Lemma 4.6.7.** *For all*  $x \in \mathbb{R}^d$ *, we have* 

$$f(\mathbf{x}) - f(\mathbf{x}^{\star}) \leq \frac{p(p-1)}{2} f(\mathbf{x})^{1-\frac{2}{p}} \left\| \mathbf{A}(\mathbf{x} - \mathbf{x}^{\star}) \right\|_{\mathcal{G}_{p}}^{2}.$$

*Proof of Lemma 4.6.7.* By Taylor's/mean-value theorem, we can write for some y on the line connecting  $x^*$  and x,

$$f(\mathbf{x}) = f(\mathbf{x}^{\star}) + \left\langle \nabla f(\mathbf{x}^{\star}), \mathbf{x} - \mathbf{x}^{\star} \right\rangle + \frac{1}{2} (\mathbf{x} - \mathbf{x}^{\star})^{\top} \nabla^2 f(\mathbf{y}) (\mathbf{x} - \mathbf{x}^{\star})$$

 1	-		1

$$\leq^{(4.6.3)} f(\mathbf{x}^{\star}) + \frac{p(p-1)}{2} \sum_{i=1}^{m} \|\mathbf{A}_{S_{i}}\mathbf{y} - \mathbf{b}_{S_{i}}\|_{2}^{p-2} \|\mathbf{A}_{S_{i}}(\mathbf{x} - \mathbf{x}^{\star})\|_{2}^{2}$$

$$\leq f(\mathbf{x}^{\star}) + \frac{p(p-1)}{2} \left(\sum_{i=1}^{m} \|\mathbf{A}_{S_{i}}\mathbf{y} - \mathbf{b}_{S_{i}}\|_{2}^{p}\right)^{\frac{p-2}{p}} \left(\sum_{i=1}^{m} \|\mathbf{A}_{S_{i}}(\mathbf{x} - \mathbf{x}^{\star})\|_{2}^{p}\right)^{\frac{2}{p}}$$

$$\leq f(\mathbf{x}^{\star}) + \frac{p(p-1)}{2} f(\mathbf{x})^{1-\frac{2}{p}} \|\mathbf{A}(\mathbf{x} - \mathbf{x}^{\star})\|_{\mathcal{G}_{p}}^{2},$$

completing the proof of Lemma 4.6.7.

### 4.6.2. Facts about the iterates

The main result of this section is Lemma 4.6.8. In words, Lemma 4.6.8 tells us that each proximal query we make in Algorithm 11 (see Line 7 of Algorithm 11) has bounded objective value. We will need this later when we argue about the convergence rates for the algorithms used to solve the proximal subproblems.

**Lemma 4.6.8.** For all queries  $q_t$ , we have

$$f(\boldsymbol{q}_t) \le f(\boldsymbol{x}_t) + (9p(p-1))^{\frac{p}{2}} d^{\frac{p}{2}-1}.$$

*Proof of Lemma 4.6.8.* We establish the following upper bound on  $f(v_t) - f(x^*)$  using the ingredients developed so far:

$$\begin{split} f(\boldsymbol{v}_{t}) - f(\boldsymbol{x}^{\star}) &\leq \frac{p(p-1)}{2} f(\boldsymbol{v}_{t})^{1-\frac{2}{p}} \left\| \mathbf{A}(\boldsymbol{v}_{t} - \boldsymbol{x}^{\star}) \right\|_{\mathcal{G}_{p}}^{2} \qquad \text{(Lemma 4.6.7)} \\ &\leq \frac{p(p-1)}{2} f(\boldsymbol{v}_{t})^{1-\frac{2}{p}} \left\| \boldsymbol{v}_{t} - \boldsymbol{x}^{\star} \right\|_{\mathbf{M}}^{2} \qquad \text{(Theorem 4.2.2)} \\ &\leq p(p-1) f(\boldsymbol{v}_{t})^{1-\frac{2}{p}} \left\| \boldsymbol{x}_{0} - \boldsymbol{x}^{\star} \right\|_{\mathbf{M}}^{2} \qquad \text{(Lemma 4.4.5)} \\ &\leq p(p-1) f(\boldsymbol{v}_{t})^{1-\frac{2}{p}} 2^{2} (2d)^{1-\frac{2}{p}} \qquad \text{(Theorem 4.2.2)} \\ &\leq 8d^{1-\frac{2}{p}} p(p-1) f(\boldsymbol{v}_{t})^{1-\frac{2}{p}} \ . \end{split}$$

Now, recall that we assume by rescaling that  $f(x^*) = 1$ . From this, it trivially follows that  $1 \le d^{1-\frac{2}{p}}p(p-1)f(v_t)^{1-\frac{2}{p}}$ . Combining these and re-arranging the above inequality leads to the following polynomial inequality in  $f(v_t)$ ,

$$0 \ge f(\boldsymbol{v}_{t}) - 8d^{1-\frac{2}{p}}p(p-1)f(\boldsymbol{v}_{t})^{1-\frac{2}{p}} - 1 ,$$
  

$$= f(\boldsymbol{v}_{t}) - 9d^{1-\frac{2}{p}}p(p-1)f(\boldsymbol{v}_{t})^{1-\frac{2}{p}} + d^{1-\frac{2}{p}}p(p-1)f(\boldsymbol{v}_{t})^{1-\frac{2}{p}} - 1 ,$$
  

$$\ge f(\boldsymbol{v}_{t}) - 9d^{1-\frac{2}{p}}p(p-1)f(\boldsymbol{v}_{t})^{1-\frac{2}{p}} , \qquad (4.6.5)$$

where in the last inequality we used the fact that the optimal value  $f(x^*) = 1$  (due to our rescaling), which implies that for  $p \ge 2$ ,

$$1 \le f(\boldsymbol{v}_t) \le d^{1-\frac{2}{p}} p(p-1) f(\boldsymbol{v}_t)^{1-\frac{2}{p}}$$

Solving for  $f(v_t)$  in (4.6.5), we get

$$f(\boldsymbol{v}_t) \le (9p(p-1))^{\frac{p}{2}} d^{\frac{p}{2}-1}$$

Using the definition of  $q_t$  from Algorithm 11 (Line 6) along with the convexity of f (Jensen's inequality), and using our bound on  $f(v_t)$  we note that,

$$f(\boldsymbol{q}_t) \le f(\boldsymbol{x}_t) + f(\boldsymbol{v}_t) ,$$
  
$$\le f(\boldsymbol{x}_t) + (9p(p-1))^{\frac{p}{2}} d^{\frac{p}{2}-1}$$

1

which completes the proof of Lemma 4.6.8.

#### 4.6.3. Proximal subproblems – calculus, algorithms, proofs

Let

$$f_{\boldsymbol{q}_t}(\widetilde{\boldsymbol{x}}) \coloneqq f(\widetilde{\boldsymbol{x}}) + ep^p \|\widetilde{\boldsymbol{x}} - \boldsymbol{q}_t\|_{\mathbf{M}}^p$$

In this subsection, we design and analyze an algorithm (Algorithm 13) that approximately solves the subproblem

$$\underset{\widetilde{\mathbf{x}}\in\mathbb{R}^d}{\operatorname{argmin}}f_{\boldsymbol{q}_t}(\widetilde{\mathbf{x}}).$$

Specifically, we will output  $(\tilde{x}_{t+1}, \lambda_{t+1})$  that satisfy the  $\frac{1}{2}$ -MS oracle condition (Definition 4.4.1) and an appropriate movement bound (Definition 4.4.2).

This subproblem is the workhorse of Algorithm 14, and once we implement and analyze the solver, it is very straightforward to plug this into Algorithm 11 and Theorem 4.4.3 to get our final iteration complexity.

**Algorithm 13** GpRegressionProxOracle: Implements  $\frac{1}{2}$ -MS oracle for  $\|\cdot\|_{\mathcal{G}_p}$  regression (see Lemma 4.6.20 and Algorithm 10.

**Require:** Query  $q_t$ , previous iterate  $x_t$ , intended parameter distance  $\gamma$ . 1: Define

$$\begin{aligned} f_{q_t}(\widetilde{\mathbf{x}}) &\coloneqq f(\widetilde{\mathbf{x}}) + ep^p \left\| \widetilde{\mathbf{x}} - q_t \right\|_{\mathbf{M}}^p \\ h_{q_t}(\widetilde{\mathbf{x}}) &\coloneqq \left\| \widetilde{\mathbf{x}} - q_t \right\|_{\nabla^2 f(q_t)}^2 + ep^p \left\| \widetilde{\mathbf{x}} - q_t \right\|_{\mathbf{M}}^p \\ D_{h_{q_t}}(\mathbf{x}, \mathbf{y}) &\coloneqq h_{q_t}(\mathbf{x}) - h_{q_t}(\mathbf{y}) - \left\langle \nabla h_{q_t}(\mathbf{y}), \mathbf{x} - \mathbf{y} \right\rangle^{\cdot} \\ \widetilde{\mathbf{x}}_{q_t} &\coloneqq \operatorname*{argmin}_{\widetilde{\mathbf{x}} \in \mathbb{R}^d} f_{q_t}(\widetilde{\mathbf{x}}) \end{aligned}$$

2: Let  $T \ge Cp^{O(1)}e \log \left(dpeh_{q_t}(\widetilde{x}_{q_t})\left(\frac{4}{p\gamma}\right)^r\right)$ .

3: Run Algorithm 10 with input iteration count *T*, base function  $f_{q_t}$ , reference function  $h_{q_t}$ , and initialization  $q_t$ .

The goal of the rest of this section is to analyze Algorithm 13. The analysis follows several steps:

1. We find a reference function  $h_{q_t}$  that depends on the query point  $q_t$  for which the proximal objective  $f_{q_t}$  is relatively smooth and relatively strongly convex with  $O(p^{O(1)})$  condition number (see Section 4.3 for a sense of why this is useful). The main result here is Lemma 4.6.9.

- 2. We show that  $f_{q_t}$  is strongly convex, following from Lemma 4.6.3. This will help us understand the argument suboptimality for any point that approximately optimizes  $f_{q_t}$  in function value. We also show that the reference function  $h_{q_t}$  is strongly convex, using the same tools, for the same reason.
- 3. We show a form of smoothness for  $f_{q_t}$ . This helps us bound the gradient of any point that approximately optimizes  $f_{q_t}$ . Combining these later will tell us that an approximate solution to  $f_{q_t}$  in argument value is also an approximate stationary point, i.e., it satisfies the  $\frac{1}{2}$ -MS condition (Definition 4.4.1).
- 4. We solve the proximal subproblems. This solution itself follows a few steps:
  - a) We apply Theorem 4.3.1. This tells us that as long as we can approximately solve the Bregman proximal problems (approximately implementing Line 3 in Algorithm 10), we will be in good shape.
  - b) This means we have to figure out how to approximately solve problems of the form  $\underset{x \in \mathbb{R}^d}{\operatorname{argmin}} \langle g, x \rangle + Lh_{q_t}(x)$ , where *L* is the smoothness constant derived for  $f_{q_t}$  with  $\underset{x \in \mathbb{R}^d}{\operatorname{respect}}$  to  $h_{q_t}$ . We do this up to an accuracy that approximate mirror descent can handle (see Theorem 4.3.1 for details on what we want this approximation to look like). For the approximation to work, we need to approximately solve this problem up to both argument accuracy and approximate stationarity. The main technical result of interest here is Lemma 4.6.18.
- 5. We use the smoothness and strong convexity guarantees to show that our solution from the previous step satisfies the  $\frac{1}{2}$ -MS oracle (Definition 4.4.1), which means we can plugand-play into Theorem 4.4.3.

### Hessian stability

Throughout this section, we adopt the following notation:

$$C_p \coloneqq ep^p$$

$$f(\mathbf{x}) \coloneqq \sum_{i=1}^m \|\mathbf{A}_{S_i}\mathbf{x} - \mathbf{b}_{S_i}\|_2^p$$

$$f_q(\mathbf{x}) \coloneqq f(\mathbf{x}) + C_p \|\mathbf{x} - \mathbf{q}\|_{\mathbf{M}}^p$$

$$h_q(\mathbf{x}) \coloneqq \|\mathbf{x} - \mathbf{q}\|_{\nabla^2 f(\mathbf{q})}^2 + C_p \|\mathbf{x} - \mathbf{q}\|_{\mathbf{M}}^p$$

We begin with proving our Hessian stability fact, which should also be equivalently viewed as showing that  $f_{q_t}$  is relatively smooth and relatively strongly convex in  $h_{q_t}$  with  $O(p^{O(1)})$  condition number. Our main result is Lemma 4.6.9 which relies on analytical results Lemma 4.6.10 and Lemma 4.6.11 that we prove later.

**Lemma 4.6.9.** For all  $x \in \mathbb{R}^d$  and  $p \ge 2$ , we have

$$\frac{1}{2p \cdot e} \nabla^2 h_q(x) \leq \nabla^2 f_q(x) \leq p \cdot e \nabla^2 h_q(x) \ .$$

*Proof of Lemma 4.6.9.* Using an arbitrary  $z \in \mathbb{R}^d$  we can write the following quadratic form of

the hessian of f,

$$\begin{aligned} z^{\top} \nabla^{2} f(\mathbf{x}) z &\leq^{(a)} p \cdot (p-1) \sum_{i=1}^{m} \|\mathbf{A}_{S_{i}} \mathbf{x} - \mathbf{b}_{S_{i}}\|_{2}^{p-2} \|\mathbf{A}_{S_{i}} z\|_{2}^{2} , \\ &= p \cdot (p-1) \sum_{i=1}^{m} \|\mathbf{A}_{S_{i}} (\mathbf{x} - q) + \mathbf{A}_{S_{i}} q - \mathbf{b}_{S_{i}}\|_{2}^{p-2} \|\mathbf{A}_{S_{i}} z\|_{2}^{2} , \\ &\leq^{(b)} p \cdot (p-1) \sum_{i=1}^{m} \left( \alpha_{p}^{p-2} \|\mathbf{A}_{S_{i}} (\mathbf{x} - q)\|_{2}^{p-2} \|\mathbf{A}_{S_{i}} z\|_{2}^{2} + \beta_{p}^{p-2} \|\mathbf{A}_{S_{i}} q - \mathbf{b}_{S_{i}}\|_{2}^{p-2} \|\mathbf{A}_{S_{i}} z\|_{2}^{2} \right) , \\ &\leq^{(c)} p \cdot (p-1) \cdot \alpha_{p}^{p-2} \sum_{i=1}^{m} \|\mathbf{A}_{S_{i}} (\mathbf{x} - q)\|_{2}^{p-2} \|\mathbf{A}_{S_{i}} z\|_{2}^{2} + (p-1) \cdot \beta_{p}^{p-2} z^{\top} \nabla^{2} f(q) z , \\ &\leq^{(d)} p \cdot (p-1) \cdot \alpha_{p}^{p-2} \left( \|\mathbf{x} - q\|_{\mathbf{M}}^{p} \right)^{(p-2)/p} \left( \|z\|_{\mathbf{M}}^{p} \right)^{2/p} + (p-1) \cdot \beta_{p}^{p-2} z^{\top} \nabla^{2} f(q) z , \\ &= p \cdot (p-1) \cdot \alpha_{p}^{p-2} \|\mathbf{x} - q\|_{\mathbf{M}}^{p-2} \|z\|_{\mathbf{M}}^{2} + (p-1) \cdot \beta_{p}^{p-2} z^{\top} \nabla^{2} f(q) z , \\ &\leq^{(e)} \frac{(p-1) \cdot \alpha_{p}^{p-2}}{C_{p}} z^{\top} \nabla^{2} g_{q} (\mathbf{x}) z + (p-1) \cdot \beta_{p}^{p-2} z^{\top} \nabla^{2} f(q) z , \end{aligned}$$

$$(4.6.6)$$

where in (a) we apply the upper bound from Lemma 4.6.1, in (b) we pick  $\alpha_p$ ,  $\beta_p \ge 1$  such that  $1/\alpha_p + 1/\beta_p = 1$  (we will choose them later), in (c) we apply the lower bound from Lemma 4.6.1, in (d) we use the choice of our weights in designing **M** and Theorem 4.2.2 and finally in (e) we use the following calculations for the regularizer term for some  $z \in \mathbb{R}^d$ ,

$$\begin{split} g_{q}(x) &\coloneqq C_{p} \left\| x - q \right\|_{\mathbf{M}}^{p} ,\\ \nabla g_{q}(x) &= pC_{p} \left\| x - q \right\|_{\mathbf{M}}^{p-2} \mathbf{M}(x - q) ,\\ \nabla^{2} g_{q}(x) &= pC_{p} \left\| x - q \right\|_{\mathbf{M}}^{p-2} \mathbf{M} + p(p - 2)C_{p} \left\| x - q \right\|_{\mathbf{M}}^{p-4} \mathbf{M}(x - q)(x - q)^{\mathsf{T}} \mathbf{M} ,\\ z^{\mathsf{T}} \nabla^{2} g_{q}(x) &z = pC_{p} \left\| x - q \right\|_{\mathbf{M}}^{p-2} \left\| z \right\|_{\mathbf{M}}^{2} + p(p - 2)C_{p} \left\| x - q \right\|_{\mathbf{M}}^{p-4} \left( (x - q)^{\mathsf{T}} \mathbf{M} z \right)^{2} \geq^{(p \ge 2)} 0 . \end{split}$$

Combining (4.6.6) with the definition of  $f_q$  gives us,

$$\begin{aligned} \boldsymbol{z}^{\top} \nabla^2 f_{\boldsymbol{q}}(\boldsymbol{x}) \boldsymbol{z} &= \boldsymbol{z}^{\top} \nabla^2 f(\boldsymbol{x}) \boldsymbol{z} + \boldsymbol{z}^{\top} \nabla^2 g_{\boldsymbol{q}}(\boldsymbol{x}) \boldsymbol{z} \ ,\\ &\leq^{\text{using (4.6.6)}} (p-1) \cdot \beta_p^{p-2} \boldsymbol{z}^{\top} \nabla^2 f(\boldsymbol{q}) \boldsymbol{z} + \left(1 + \frac{(p-1) \cdot \alpha_p^{p-2}}{C_p}\right) \boldsymbol{z}^{\top} \nabla^2 g_{\boldsymbol{q}}(\boldsymbol{x}) \boldsymbol{z} \ . \end{aligned}$$

Thus, in order to finish the proof for the upper bound we need to pick  $\alpha_p$ ,  $\beta_p$ . We split the analysis here into two cases: **A**. p > 2 and **B**. p = 2.

**<u>Case A.</u>** (p > 2) For simplicity we will just pick  $\alpha_p = p - 1$  and  $\beta_p = \frac{p-1}{p-2}$  which implies,

$$\begin{split} z^{\top} \nabla^{2} f_{q}(\mathbf{x}) z &\leq (p-1) \cdot \left(1 + \frac{1}{p-2}\right)^{p-2} z^{\top} \nabla^{2} f(q) z + \left(1 + \frac{(p-1) \cdot (p-1)^{p-2}}{C_{p}}\right) z^{\top} \nabla^{2} g_{q}(\mathbf{x}) z \ , \\ &\leq (p-1) \cdot e z^{\top} \nabla^{2} f(q) z + \left(1 + \frac{(p-1)^{p-1}}{C_{p}}\right) z^{\top} \nabla^{2} g_{q}(\mathbf{x}) z \ , \\ &= \frac{(p-1) \cdot e}{2} z^{\top} \left(\nabla^{2} h_{q}(\mathbf{x}) - \nabla^{2} g_{q}(\mathbf{x})\right) z + \left(1 + \frac{(p-1)^{p-1}}{C_{p}}\right) z^{\top} \nabla^{2} g_{q}(\mathbf{x}) z \ , \\ &\leq ^{(p\geq 2)} p \cdot e z^{\top} \nabla^{2} h_{q}(\mathbf{x}) z + \left(1 + \frac{(p-1)^{p-1}}{C_{p}} - \frac{(p-1) \cdot e}{2}\right) z^{\top} \nabla^{2} g_{q}(\mathbf{x}) z \ , \end{split}$$

$$= p \cdot e \boldsymbol{z}^{\top} \nabla^2 h_{\boldsymbol{q}}(\boldsymbol{x}) \boldsymbol{z} + \left(1 + \frac{(p-1)^{p-1}}{e p^p} - \frac{(p-1) \cdot e}{2}\right) \boldsymbol{z}^{\top} \nabla^2 g_{\boldsymbol{q}}(\boldsymbol{x}) \boldsymbol{z} ,$$
  
$$\leq^{(\text{Lemma 4.6.10)}} p \cdot e \boldsymbol{z}^{\top} \nabla^2 h_{\boldsymbol{q}}(\boldsymbol{x}) \boldsymbol{z} ,$$

where in the final inequality we use Lemma 4.6.10 which tell us that for  $p \ge 2$  the constant in front of  $z^{\top}\nabla^2 g_q(x)z$  is negative along with the fact that  $z^{\top}\nabla^2 g_q(x)z$  is non-negative. To get the lower bound we first exchange x, q in (4.6.6) (and use the values of  $\alpha_p$  and  $\beta_p$ ) to get,

$$\begin{split} z^{\top} \nabla^{2} f(q) z &\leq \frac{(p-1) \cdot (p-1)p - 2}{ep^{p}} z^{\top} \nabla^{2} g_{x}(q) z + (p-1) \left( 1 + \frac{1}{p-2} \right)^{p-2} z^{\top} \nabla^{2} f(x) z \ , \\ \Rightarrow z^{\top} \nabla^{2} f(q) z &\leq \frac{(p-1)^{p-1}}{ep^{p}} z^{\top} \nabla^{2} g_{x}(q) z + (p-1)e z^{\top} \nabla^{2} f(x) z \ , \\ \Rightarrow \frac{1}{(p-1)e} z^{\top} \nabla^{2} f(q) z - \frac{(p-1)^{p-2}}{e^{2} p^{p}} z^{\top} \nabla^{2} g_{x}(q) z \leq z^{\top} \nabla^{2} f(x) z \ . \end{split}$$

We can finally lower bound,

$$\begin{split} z^{\top} \nabla^{2} f_{q}(x) z &= z^{\top} \nabla^{2} f(x) z + z^{\top} \nabla^{2} g_{q}(x) z \ , \\ &\geq \frac{1}{(p-1)e} z^{\top} \nabla^{2} f(q) z - \frac{(p-1)^{p-2}}{e^{2} p^{p}} z^{\top} \nabla^{2} g_{x}(q) z + z^{\top} \nabla^{2} g_{q}(x) z \ , \\ &= \frac{1}{2(p-1)e} z^{\top} \left( \nabla^{2} h_{q}(x) - \nabla^{2} g_{q}(x) \right) z - \frac{(p-1)^{p-2}}{e^{2} p^{p}} z^{\top} \nabla^{2} g_{x}(q) z + z^{\top} \nabla^{2} g_{q}(x) z \ , \\ &\geq ^{(g_{q}(x)=g_{x}(q))} \frac{1}{2pe} z^{\top} \nabla^{2} h_{q}(x) z + \left( 1 - \frac{1}{2(p-1)e} - \frac{(p-1)^{p-2}}{e^{2} p^{p}} \right) z^{\top} \nabla^{2} g_{q}(x) z \ , \\ &\geq ^{(\text{Lemma 4.6.11)}} \frac{1}{2pe} z^{\top} \nabla^{2} h_{q}(x) z \ , \end{split}$$

where in the final inequality we use Lemma 4.6.11 and the fact that  $z^{\top}\nabla^2 g_q(x)z$  is non-negative. This finishes the proof for Case A.

We finally consider the corner case with p = 2.

<u>**Case B.**</u> (p = 2) In this case the proof is trivial, and follows from simply writing the quadratic forms for  $f_q$  and  $h_q$ . We do so below,

$$\begin{aligned} \boldsymbol{z}^{\mathsf{T}} \nabla^2 f_{\boldsymbol{q}}(\boldsymbol{x}) \boldsymbol{z} &= \boldsymbol{z}^{\mathsf{T}} \nabla^2 f(\boldsymbol{x}) \boldsymbol{z} + \boldsymbol{z}^{\mathsf{T}} \nabla^2 g_{\boldsymbol{q}}(\boldsymbol{x}) \boldsymbol{z} \ , \\ &= \boldsymbol{z}^{\mathsf{T}} \nabla^2 f(\boldsymbol{x}) \boldsymbol{z} + 2\boldsymbol{C}_2 \, \|\boldsymbol{z}\|_{\mathbf{M}}^2 \ , \\ &\leq 2\boldsymbol{z}^{\mathsf{T}} \nabla^2 f(\boldsymbol{x}) \boldsymbol{z} + 2\boldsymbol{C}_2 \, \|\boldsymbol{z}\|_{\mathbf{M}}^2 = \boldsymbol{z}^{\mathsf{T}} \nabla^2 h_{\boldsymbol{q}}(\boldsymbol{x}) \boldsymbol{z} \ , \end{aligned}$$

which shows the relative smoothness with a constant of 1 which is smaller (and hence better) than the claimed constant (for p = 2) of 2e in the lemma. Now for the relative strong convexity we do the same,

$$\begin{aligned} \boldsymbol{z}^{\top} \nabla^2 f_{\boldsymbol{q}}(\boldsymbol{x}) \boldsymbol{z} &= \boldsymbol{z}^{\top} \nabla^2 f(\boldsymbol{x}) \boldsymbol{z} + 2C_2 \|\boldsymbol{z}\|_{\mathbf{M}}^2 \quad, \\ &\geq \frac{1}{2} \cdot \left( 2\boldsymbol{z}^{\top} \nabla^2 f(\boldsymbol{x}) \boldsymbol{z} + 2C_2 \|\boldsymbol{z}\|_{\mathbf{M}}^2 \right) \quad, \\ &= \frac{1}{2} \boldsymbol{z}^{\top} \nabla^2 h_{\boldsymbol{q}}(\boldsymbol{x}) \boldsymbol{z} \quad, \end{aligned}$$

which shows relative strong-convexity with a constant of  $\frac{1}{2}$  which is larger (and hence better) than the claimed constant (for p = 2) of  $\frac{1}{4e}$  in the lemma. This finishes the proof for Case B.

This completes the proof of Lemma 4.6.9.

We prove two small technical lemmas that we used in the above proof now.

**Lemma 4.6.10.** For all  $p \ge 2$ ,  $g(p) = 1 + \frac{(p-1)^{p-1}}{ep^p} - \frac{(p-1) \cdot e}{2} \le 0$ .

*Proof.* First note that at p = 2 the function takes a strictly negative value,

$$g(2) = 1 + \frac{(1)}{e^{2^2}} - \frac{e}{2} = \frac{4e + 1 - 2e^2}{4e} < 0$$
.

We will now show that the function is increasing in *p* for  $p \ge 2$ ,

$$g'(p) = -\frac{(p-1)^{p-1}p^p(\ln(p)+1)}{p^2p} + \frac{(p-1)^{p-1}(\ln(p-1)+1)}{p^p} - \frac{e}{2} ,$$
$$= -\frac{(p-1)^{p-1}\ln(p/(p-1))}{p^p} - \frac{e}{2} < 0 .$$

Thus, the function attains its maximum value at p = 2 in the range  $p \ge 2$ , implying it is strictly negative in that range.

**Lemma 4.6.11.** For all  $p \ge 2$ ,  $g(p) = 1 - \frac{1}{2(p-1)e} - \frac{(p-1)^{p-2}}{e^2 p^p} \ge 0$ .

*Proof.* First note that at p = 2 the function takes a strictly positive value,

$$g(2) = 1 - \frac{1}{2e} - \frac{1^0}{e^2 2^2} = 1 - \frac{1}{2e} - \frac{1}{4e^2} = \frac{4e^2 - 2e - 1}{4e^2} > 0 .$$

We will now show that the function is increasing in *p* for  $p \ge 2$ ,

$$\begin{split} g'(p) &= \frac{1}{2(p-1)^2 e} + \frac{(p-1)^{p-2} p^p (\ln(p)+1)}{e^2 p^{2p}} - \frac{(p-1)^{p-2} (\ln(p-1)+(p-2)/(p-1))}{e^2 p^p} \\ &= \frac{1}{2(p-1)^2 e} + \frac{(p-1)^{p-2} (\ln(p)+1)}{e^2 p^p} - \frac{(p-1)^{p-2} (\ln(p-1)+1-1/(p-1))}{e^2 p^p} \ , \\ &= \frac{1}{2(p-1)^2 e} + \frac{(p-1)^{p-2} (\ln(p/(p-1))+1/(p-1))}{e^2 p^p} > 0 \ . \end{split}$$

Thus, the function *g* attains its minimum value at p = 2 in the range  $p \ge 2$ , implying that it is strictly positive in that range.

### Strong convexity of the proximal objective and friends

We begin with showing that the proximal objective enjoys a form of strong convexity.

**Lemma 4.6.12.** For all  $x, d \in \mathbb{R}^d$ , we have

$$f_{q}(\boldsymbol{x}+\boldsymbol{d}) \geq f_{q}(\boldsymbol{x}) + \left\langle \nabla f_{q}(\boldsymbol{x}), \boldsymbol{d} \right\rangle + \frac{4}{2^{p}} \left( \left\| \mathbf{A} \boldsymbol{d} \right\|_{\mathcal{G}_{p}}^{p} + C_{p} \left\| \boldsymbol{d} \right\|_{\mathbf{M}}^{p} \right).$$

*Proof of Lemma 4.6.12.* Let  $K_p := \frac{4}{2^p}$ .

The plan is to apply Lemma 4.6.3 to  $f_q(x + d)$ . We start with the regularizer. Notice that

$$\|\mathbf{x} + \mathbf{d} - \mathbf{q}\|_{\mathbf{M}}^{p} = \|\mathbf{M}^{1/2}(\mathbf{x} + \mathbf{d} - \mathbf{q})\|_{2}^{p} = \|\mathbf{M}^{1/2}(\mathbf{x} - \mathbf{q}) + \mathbf{M}^{1/2}\mathbf{d}\|_{2}^{p} ,$$

$$\geq^{(\text{Lemma 4.6.3)}} \|\mathbf{M}^{1/2}(\mathbf{x} - \mathbf{q})\|_{2}^{p} \qquad (4.6.7)$$

$$+ \left\langle p \|\mathbf{M}^{1/2}(\mathbf{x} - \mathbf{q})\|_{2}^{p-2} \mathbf{M}^{1/2}(\mathbf{x} - \mathbf{q}), \mathbf{M}^{1/2}\mathbf{d} \right\rangle + K_{p} \|\mathbf{M}^{1/2}\mathbf{d}\|_{2}^{p} ,$$

$$= \|\mathbf{x} - \mathbf{q}\|_{\mathbf{M}}^{p} + \left\langle p \|\mathbf{x} - \mathbf{q}\|_{\mathbf{M}}^{p-2} \mathbf{M}(\mathbf{x} - \mathbf{q}), \mathbf{d} \right\rangle + K_{p} \|\mathbf{d}\|_{\mathbf{M}}^{p} ,$$

$$= \|\mathbf{x} - \mathbf{q}\|_{\mathbf{M}}^{p} + \left\langle \nabla_{\mathbf{x}} \left(\|\mathbf{x} - \mathbf{q}\|_{\mathbf{M}}^{p}\right), \mathbf{d} \right\rangle + K_{p} \|\mathbf{d}\|_{\mathbf{M}}^{p} . \qquad (4.6.8)$$

We combine this with the conclusion of Lemma 4.6.2, giving

$$f_{q}(\mathbf{x} + d) = f(\mathbf{x} + d) + C_{p} \|\mathbf{x} + d - q\|_{\mathbf{M}}^{p} ,$$

$$\geq^{(\text{Lemma 4.6.2)}} f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), d \rangle + K_{p} \|\mathbf{A}d\|_{\mathcal{G}_{p}}^{p} + C_{p} \|\mathbf{x} + d - q\|_{\mathbf{M}}^{p} ,$$

$$\geq^{(4.6.8)} f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), d \rangle + K_{p} \|\mathbf{A}d\|_{\mathcal{G}_{p}}^{p} + C_{p} \|\mathbf{x} - q\|_{\mathbf{M}}^{p} ,$$

$$+ C_{p} \left\langle \nabla_{\mathbf{x}} \left( \|\mathbf{x} - q\|_{\mathbf{M}}^{p} \right), d \right\rangle + K_{p} C_{p} \|d\|_{\mathbf{M}}^{p} ,$$

$$= f(\mathbf{x}) + C_{p} \|\mathbf{x} - q\|_{\mathbf{M}}^{p} + \left\langle \nabla_{\mathbf{x}} \left( f(\mathbf{x}) + C_{p} \|\mathbf{x} - q\|_{\mathbf{M}}^{p} \right), d \right\rangle + K_{p} \|\mathbf{A}d\|_{\mathcal{G}_{p}}^{p} + K_{p} C_{p} \|d\|_{\mathbf{M}}^{p} ,$$

$$= f_{q}(\mathbf{x}) + \left\langle \nabla f_{q}(\mathbf{x}), d \right\rangle + K_{p} \left( \|\mathbf{A}d\|_{\mathcal{G}_{p}}^{p} + C_{p} \|d\|_{\mathbf{M}}^{p} \right) .$$

completing the proof of Lemma 4.6.12.

We also show that the subproblems we solve in Line 3 of Algorithm 10 are strongly convex.

**Lemma 4.6.13.** *Fix* z, q,  $d \in \mathbb{R}^d$  and let L > 0. Consider the function

$$g(\mathbf{x}) \coloneqq \langle \mathbf{z}, \mathbf{x} \rangle + L\left( \|\mathbf{x} - \mathbf{q}\|_{\nabla^2 f(\mathbf{q})}^2 + C_p \|\mathbf{x} - \mathbf{q}\|_{\mathbf{M}}^p \right) .$$

Then,

$$g(\boldsymbol{x} + \boldsymbol{d}) \ge g(\boldsymbol{x}) + \langle \nabla g(\boldsymbol{x}), \boldsymbol{d} \rangle + L \left( \|\boldsymbol{d}\|_{\nabla^2 f(\boldsymbol{q})}^2 + \frac{4C_p}{2^p} \|\boldsymbol{d}\|_{\mathbf{M}}^p \right)$$

In particular, if z is the minimizer for g, then for any  $d \in \mathbb{R}^d$ , we have

$$\|d\|_{\mathbf{M}} \leq \frac{2}{p \cdot (4e)^{1/p}} \left(\frac{g(z+d) - g(z)}{L}\right)^{1/p}$$
.

*Proof of Lemma 4.6.13.* This is pretty much the same proof as Lemma 4.6.12. It is easy to check that

$$\|(\mathbf{x}+d)-\mathbf{q}\|_{\nabla^{2}f(q)}^{2} = \|\mathbf{x}-\mathbf{q}\|_{\nabla^{2}f(q)}^{2} + \left\langle 2\nabla^{2}f(\mathbf{q})(\mathbf{x}-\mathbf{q}), d\right\rangle + \|d\|_{\nabla^{2}f(q)}^{2} , \qquad (4.6.9)$$

and using Lemma 4.6.3 in the same way as in the proof of Lemma 4.6.12, we have

$$\|(x+d) - q\|_{\mathbf{M}}^{p} \ge^{(4.6.8)} \|x - q\|_{\mathbf{M}}^{p} + \left\langle p \|x - q\|_{\mathbf{M}}^{p-2} \mathbf{M}(x-q), d \right\rangle + \frac{4}{2^{p}} \|d\|_{\mathbf{M}}^{p}$$

Combining this with the definition of g gives the following,

$$\begin{split} g(x+d) &= \langle z, x+d \rangle + L \left( \|x+d-q\|_{\nabla^2 f(q)}^2 + C_p \|x+d-q\|_{\mathbf{M}}^p \right) ,\\ &\geq^{(4.6.9), (4.6.8)} \langle z, x \rangle + \langle z, d \rangle + L \|x-q\|_{\nabla^2 f(q)}^2 + L \left\langle 2\nabla^2 f(q)(x-q), d \right\rangle \\ &+ L \|d\|_{\nabla^2 f(q)}^2 + L C_p \left( \|x-q\|_{\mathbf{M}}^p + \left\langle p \|x-q\|_{\mathbf{M}}^{p-2} \mathbf{M}(x-q), d \right\rangle + \frac{4}{2^p} \|d\|_{\mathbf{M}}^p \right) ,\\ &= g(x) + \left\langle z + 2L \nabla^2 f(q)(x-q) + L C_p p \|x-q\|_{\mathbf{M}}^{p-2} \mathbf{M}(x-q), d \right\rangle \\ &+ L \left( \|d\|_{\nabla^2 f(q)}^2 + \frac{4C_p}{2^p} \|d\|_{\mathbf{M}}^p \right) ,\\ &= g(x) + \left\langle \nabla g(x), d \right\rangle + L \left( \|d\|_{\nabla^2 f(q)}^2 + \frac{4C_p}{2^p} \|d\|_{\mathbf{M}}^p \right) , \end{split}$$

which proves the first result of the lemma.

To get the second result, we observe that  $\nabla g(z) = 0$  by the optimality of z. Ignoring the  $\|d\|_{\nabla^2 f(q)}$  terms and rearranging gives the conclusion of Lemma 4.6.13.

### Smoothness of the proximal objective

We first bound the operator norm of a matrix related to the Hessian of the proximal objective.

**Lemma 4.6.14.** For all  $q, y \in \mathbb{R}^d$ , we have

$$\left\| \mathbf{M}^{-1/2} \left( \nabla^2 f_{\boldsymbol{q}}(\boldsymbol{y}) \right) \mathbf{M}^{-1/2} \right\|_{\text{op}} \le e p^2 (p-1) \left( 2f(\boldsymbol{q})^{1-\frac{2}{p}} + C_p \left\| \boldsymbol{y} - \boldsymbol{q} \right\|_{\mathbf{M}}^{p-2} \right) .$$

*Proof of Lemma* 4.6.14. Recall from the proof of Lemma 4.6.9 the definition of the regularization term  $g_q(y) \coloneqq C_p ||y - q||_M^p$  for  $C_p = ep^p$  as well as the following calculations,

$$g_{\boldsymbol{q}}(\boldsymbol{y}) \coloneqq C_{p} \|\boldsymbol{y} - \boldsymbol{q}\|_{\mathbf{M}}^{p},$$
  

$$\nabla g_{\boldsymbol{q}}(\boldsymbol{y}) = pC_{p} \|\boldsymbol{y} - \boldsymbol{q}\|_{\mathbf{M}}^{p-2} \mathbf{M}(\boldsymbol{y} - \boldsymbol{q}),$$
  

$$\nabla^{2} g_{\boldsymbol{q}}(\boldsymbol{y}) = pC_{p} \|\boldsymbol{y} - \boldsymbol{q}\|_{\mathbf{M}}^{p-2} \mathbf{M} + p(p-2)C_{p} \|\boldsymbol{y} - \boldsymbol{q}\|_{\mathbf{M}}^{p-4} \mathbf{M}(\boldsymbol{y} - \boldsymbol{q})(\boldsymbol{y} - \boldsymbol{q})^{\mathsf{T}} \mathbf{M}.$$

By Lemma 4.6.9, we know that

$$\nabla^2 f_{\boldsymbol{q}}(\boldsymbol{y}) \leq ep\left(2\nabla^2 f(\boldsymbol{q}) + \nabla^2 g_{\boldsymbol{q}}(\boldsymbol{y})\right).$$

Observe that

$$\begin{split} \mathbf{M}^{-1/2} \left( \nabla^2 g_{q}(\boldsymbol{y}) \right) \mathbf{M}^{-1/2} &= p C_{p} \left( \left\| \boldsymbol{y} - \boldsymbol{q} \right\|_{\mathbf{M}}^{p-2} + (p-2) \left\| \boldsymbol{y} - \boldsymbol{q} \right\|_{\mathbf{M}}^{p-4} \mathbf{M}^{1/2} (\boldsymbol{y} - \boldsymbol{q}) (\boldsymbol{y} - \boldsymbol{q})^{\mathsf{T}} \mathbf{M}^{1/2} \right) &, \\ &\leq p C_{p} \left\| \boldsymbol{y} - \boldsymbol{q} \right\|_{\mathbf{M}}^{p-2} \mathbf{I} + (p-2) \left\| \boldsymbol{y} - \boldsymbol{q} \right\|_{\mathbf{M}}^{p-4} \left\| \mathbf{M}^{1/2} (\boldsymbol{y} - \boldsymbol{q}) (\boldsymbol{y} - \boldsymbol{q})^{\mathsf{T}} \mathbf{M}^{1/2} \right\|_{\mathrm{op}} \mathbf{I} &, \\ &\leq p C_{p} \left\| \boldsymbol{y} - \boldsymbol{q} \right\|_{\mathbf{M}}^{p-2} \mathbf{I} + (p-2) \left\| \boldsymbol{y} - \boldsymbol{q} \right\|_{\mathbf{M}}^{p-4} \left\| \mathbf{M}^{1/2} (\boldsymbol{y} - \boldsymbol{q}) \right\|_{2}^{2} \mathbf{I} &, \end{split}$$

$$\leq p(p-1)C_p \|\boldsymbol{y} - \boldsymbol{q}\|_{\mathbf{M}}^{p-2} \mathbf{I}$$
,

and, applying Lemma 4.6.1 (with  $\mathbf{M}^{-1/2} \mathbf{z}$  as the vectors in the quadratic form) and Hölder inequality with norms  $\|\cdot\|_{p/(p-2)}$ ,  $\|\cdot\|_{p/2}$ , for  $\mathbf{z} \in \mathbb{R}^d$  we have

$$\begin{split} \boldsymbol{z}^{\top} \mathbf{M}^{-1/2} \left( \nabla^2 f(\boldsymbol{q}) \right) \mathbf{M}^{-1/2} \boldsymbol{z} &\leq p(p-1) \sum_{i=1}^m \left\| \mathbf{A}_{S_i} \boldsymbol{q} - \boldsymbol{b}_{S_i} \right\|_2^{p-2} \left\| \left\| \mathbf{A}_{S_i} \mathbf{M}^{-1/2} \boldsymbol{z} \right\|_2^2 \\ &\leq p(p-1) \left( \sum_{i=1}^m \left\| \mathbf{A}_{S_i} \boldsymbol{q} - \boldsymbol{b}_{S_i} \right\|_2^p \right)^{\frac{p-2}{p}} \left( \sum_{i=1}^m \left\| \mathbf{A}_{S_i} \mathbf{M}^{-1/2} \boldsymbol{z} \right\|_2^p \right)^{\frac{2}{p}} \\ &\leq p(p-1) f(\boldsymbol{q})^{1-\frac{2}{p}} \left\| \mathbf{M}^{-1/2} \boldsymbol{z} \right\|_{\mathbf{M}}^2 = p(p-1) f(\boldsymbol{q})^{1-\frac{2}{p}} \left\| \boldsymbol{z} \right\|_2^2. \end{split}$$

Combining gives

$$\begin{split} \mathbf{M}^{-1/2} \left( \nabla^2 f_{\boldsymbol{q}}(\boldsymbol{y}) \right) \mathbf{M}^{-1/2} &\leq e p \mathbf{M}^{-1/2} \left( 2 \nabla^2 f(\boldsymbol{q}) + \nabla^2 g_{\boldsymbol{q}(\boldsymbol{y})} \right) \mathbf{M}^{-1/2} ,\\ &\leq 2 e p^2 (p-1) f(\boldsymbol{q})^{1-\frac{2}{p}} + e p^2 (p-1) C_p \left\| \boldsymbol{y} - \boldsymbol{q} \right\|_{\mathbf{M}}^{p-2} ,\\ &\leq e p^2 (p-1) \left( 2 f(\boldsymbol{q})^{1-\frac{2}{p}} + C_p \left\| \boldsymbol{y} - \boldsymbol{q} \right\|_{\mathbf{M}}^{p-2} \right), \end{split}$$

completing the proof of Lemma 4.6.14.

Next, we show a bound on the norm of the gradient of any solution x that is approximately optimal for  $f_q$ .

**Lemma 4.6.15.** For all  $q, x \in \mathbb{R}^d$ , we have

$$\left\|\mathbf{M}^{-1}\nabla f_{q}(\boldsymbol{x})\right\|_{\mathbf{M}} \leq ep^{2}(p-1)\left(f(q)^{1-\frac{2}{p}} + C_{p}\max\left\{\left\|\boldsymbol{x} - \boldsymbol{q}\right\|_{\mathbf{M}}, \left\|\boldsymbol{x}_{q} - \boldsymbol{q}\right\|_{\mathbf{M}}\right\}^{p-2}\right)\left\|\boldsymbol{x} - \boldsymbol{x}_{q}\right\|_{\mathbf{M}}$$

*Proof of Lemma 4.6.15.* We use a continuity argument. By Taylor's theorem, we know for some y along the line connecting x and  $x_q$  (minimizer of  $f_q$ ) that

$$\nabla f_q(\mathbf{x}) = \nabla f_q(\mathbf{x}_q) + \nabla^2 f_q(\mathbf{y})(\mathbf{x} - \mathbf{x}_q) = \nabla^2 f_q(\mathbf{y})(\mathbf{x} - \mathbf{x}_q) \ .$$

Taking  $M^{-1}$ -norm of both sides gives,

$$\begin{split} \left\| \nabla f_{q}(\boldsymbol{x}) \right\|_{\mathbf{M}^{-1}} &= \left\| \mathbf{M}^{-1/2} \nabla f_{q}(\boldsymbol{x}) \right\|_{2} , \\ &= \left\| \mathbf{M}^{-1/2} \nabla^{2} f_{q}(\boldsymbol{y}) (\boldsymbol{x} - \boldsymbol{x}_{q}) \right\|_{2} , \\ &= \left\| \mathbf{M}^{-1/2} \nabla^{2} f_{q}(\boldsymbol{y}) \mathbf{M}^{-1/2} \mathbf{M}^{1/2} (\boldsymbol{x} - \boldsymbol{x}_{q}) \right\|_{2} , \\ &\leq \left\| \mathbf{M}^{-1/2} \left( \nabla^{2} f_{q}(\boldsymbol{y}) \right) \mathbf{M}^{-1/2} \right\|_{\mathrm{op}} \cdot \left\| \boldsymbol{x} - \boldsymbol{x}_{q} \right\|_{\mathbf{M}} \end{split}$$

The rest of the proof involves bounding the operator norm term. This follows directly from Lemma 4.6.14, from which we get (using convexity of  $\|\cdot\|_{M}$ ),

$$\begin{split} \left\| \mathbf{M}^{-1/2} \nabla^2 f_{\boldsymbol{q}}(\boldsymbol{y}) \mathbf{M}^{-1/2} \right\|_{\text{op}} &\leq e p^2 (p-1) \left( 2 f(\boldsymbol{q})^{1-\frac{2}{p}} + C_p \|\boldsymbol{y} - \boldsymbol{q}\|_{\mathbf{M}}^{p-2} \right) \\ &\leq e p^2 (p-1) \left( 2 f(\boldsymbol{q})^{1-\frac{2}{p}} + C_p \max \left\{ \|\boldsymbol{x} - \boldsymbol{q}\|_{\mathbf{M}}, \|\boldsymbol{x}_{\boldsymbol{q}} - \boldsymbol{q}\|_{\mathbf{M}} \right\}^{p-2} \right). \end{split}$$

1	-	-	
Putting everything together, we get

$$\begin{split} \left\| \mathbf{M}^{-1} \nabla f_{q}(\mathbf{x}) \right\|_{\mathbf{M}} &= \left\| \nabla f_{q}(\mathbf{x}) \right\|_{\mathbf{M}^{-1}} ,\\ &\leq e p^{2} (p-1) \left( 2 f(q)^{1-\frac{2}{p}} + C_{p} \max \left\{ \left\| \mathbf{x} - q \right\|_{\mathbf{M}}, \left\| \mathbf{x}_{q} - q \right\|_{\mathbf{M}} \right\}^{p-2} \right) \left\| \mathbf{x} - \mathbf{x}_{q} \right\|_{\mathbf{M}} , \end{split}$$

completing the proof of Lemma 4.6.15.

#### Solving the proximal subproblems

We begin by showing that the optimal solution to the proximal problem  $x_{q_t} := \underset{x \in \mathbb{R}^d}{\operatorname{argmin}} f_{q_t}(x)$  is not too far from  $x^*$ .

**Lemma 4.6.16.** For all proximal queries  $q_t$ , we have

$$\left\|\boldsymbol{x}_{\boldsymbol{q}_{t}}-\boldsymbol{x}^{\star}\right\|_{\mathbf{M}}\leq d^{\frac{1}{2}-\frac{1}{p}}\left(2^{\frac{3}{2}}f(\boldsymbol{x}_{t})+4\right).$$

*Proof.* In the rest of this proof, we omit the subscript *t* wherever it is clear which iterates we are working with.

We first show that

$$\left\|x_{q}-q\right\|_{\mathbf{M}}\leq\left\|x^{\star}-q\right\|_{\mathbf{M}}$$

To see this, suppose this is not the case. Then, we have

$$f(\mathbf{x}^{\star}) + C_p \left\| \mathbf{x}^{\star} - \mathbf{q} \right\|_{\mathbf{M}}^p < f(\mathbf{x}_q) + C_p \left\| \mathbf{x}_q - \mathbf{q} \right\|_{\mathbf{M}}^p$$

which contradicts the optimality of  $x_q$  for  $f_q$ .

We now write

$$\begin{aligned} \left\| \boldsymbol{x}_{\boldsymbol{q}_{t}} - \boldsymbol{x}^{\star} \right\|_{\mathbf{M}} &\leq \left\| \boldsymbol{x}_{\boldsymbol{q}_{t}} - \boldsymbol{q}_{t} \right\|_{\mathbf{M}} + \left\| \boldsymbol{x}^{\star} - \boldsymbol{q}_{t} \right\|_{\mathbf{M}} ,\\ &\leq 2 \left\| \boldsymbol{x}^{\star} - \boldsymbol{q}_{t} \right\|_{\mathbf{M}} ,\\ &\leq 2 \left( \left\| \boldsymbol{x}_{t} - \boldsymbol{x}^{\star} \right\|_{\mathbf{M}} + \left\| \boldsymbol{v}_{t} - \boldsymbol{x}^{\star} \right\|_{\mathbf{M}} \right) \end{aligned}$$

where in the last inequality, we used the definition of  $q_t$  from Line 6 in Algorithm 11 and the convexity of  $\|\cdot\|_{\mathrm{M}}$ . The required control on  $\|v_t - x^*\|_{\mathrm{M}}$  comes from Lemma 4.4.5 and Theorem 4.2.2 (along with re-scaling assumption to make the optimal value 1) – we have

$$\|v_t - x^{\star}\|_{\mathbf{M}} \le \sqrt{2} \|x_0 - x^{\star}\|_{\mathbf{M}} \le 4d^{\frac{1}{2} - \frac{1}{p}}$$

For the other term, we apply Lemma 4.6.2 and get

$$\left\|\boldsymbol{x}_{t}-\boldsymbol{x}^{\star}\right\|_{\mathbf{M}} \leq 2^{\frac{3}{2}} d^{\frac{1}{2}-\frac{1}{p}} \left(f(\boldsymbol{x}_{t})-f(\boldsymbol{x}^{\star})\right)^{\frac{1}{p}} < 2^{\frac{3}{2}} d^{\frac{1}{2}-\frac{1}{p}} f(\boldsymbol{x}_{t})^{\frac{1}{p}}.$$

Adding gives us the conclusion of Lemma 4.6.16.

The next few lemmas are targeted at solving the proximal subproblems. We begin with a calculation that we will use in showing that the initial Bregman divergence between our initialization and the optimum is small.

**Lemma 4.6.17.** In the same setting as Lemma 4.6.9, for all  $x, y \in \mathbb{R}^d$ , we have

$$h_q(x_q) \le p(p-1)f(q)^{1-\frac{2}{p}} \|x_q - q\|_{\mathbf{M}}^2 + C_p \|x_q - q\|_{\mathbf{M}}^p < f(q) + C_p \|x_q - q\|_{\mathbf{M}}^p \le 2f(q).$$

*Proof of Lemma* 4.6.17. By optimality of  $x_q$  for the subproblem, we have

$$f(\boldsymbol{x}_{\boldsymbol{q}}) + C_p \left\| \boldsymbol{x}_{\boldsymbol{q}} - \boldsymbol{q} \right\|_{\mathbf{M}}^p \leq f(\boldsymbol{q}) + C_p \left\| \boldsymbol{q} - \boldsymbol{q} \right\|_{\mathbf{M}}^p = f(\boldsymbol{q}).$$

Rearranging gives,

$$\|\mathbf{x}_{q} - \mathbf{q}\|_{\mathbf{M}}^{p} \le \frac{f(\mathbf{q}) - f(\mathbf{x}_{q})}{C_{p}} \le \frac{f(\mathbf{q})}{C_{p}}$$
 (4.6.10)

We now use the definition of  $h_q$  and Lemma 4.6.1 to write

$$\begin{split} h_{q}(x_{q}) &= \left\| x_{q} - q \right\|_{\nabla^{2} f(q)}^{2} + C_{p} \left\| x_{q} - q \right\|_{\mathbf{M}}^{p} ,\\ &\leq^{\text{Lemma 4.6.1}} p(p-1) \sum_{i=1}^{m} \left\| \mathbf{A}_{S_{i}} q - \mathbf{b}_{S_{i}} \right\|_{2}^{p-2} \left\| \mathbf{A}_{S_{i}}(x_{q} - q) \right\|_{2}^{2} + C_{p} \left\| x_{q} - q \right\|_{\mathbf{M}}^{p} ,\\ &\leq^{(a)} p(p-1) \left( \sum_{i=1}^{m} \left\| \mathbf{A}_{S_{i}} q - \mathbf{b}_{S_{i}} \right\|_{2}^{p} \right)^{1-\frac{2}{p}} \left( \sum_{i=1}^{m} \left\| \mathbf{A}_{S_{i}}(x_{q} - q) \right\|_{2}^{p} \right)^{\frac{2}{p}} + C_{p} \left\| x_{q} - q \right\|_{\mathbf{M}}^{p} ,\\ &\leq^{(b)} p(p-1) f(q)^{1-\frac{2}{p}} \left\| x_{q} - q \right\|_{\mathbf{M}}^{2} + C_{p} \left\| x_{q} - q \right\|_{\mathbf{M}}^{p} ,\\ &\leq^{(4.6.10)} p(p-1) f(q)^{1-\frac{2}{p}} \left( \frac{f(q)}{C_{p}} \right)^{\frac{2}{p}} + C_{p} \left\| x_{q} - q \right\|_{\mathbf{M}}^{p} ,\\ &=^{(C_{p} = ep^{p})} \frac{(p-1)}{ep} f(q) + C_{p} \left\| x_{q} - q \right\|_{\mathbf{M}}^{p} ,\\ &$$

where in (a) we used Hölder inequality with norms  $\|\cdot\|_{p/(p-2)}$ ,  $\|\cdot\|_{p/2}$  and in (b) we used Theorem 4.2.2.

This completes the proof for the series of inequalities in Lemma 4.6.17.

We now have the tools to show how to approximately solve problems in Line 3 of Algorithm 10 when applied in our setting. Although this and future complexity bounds depend on  $f(x_t)$ , we will later be able to use Theorem 4.4.3 to "bootstrap" and get an unconditional upper bound below.

**Lemma 4.6.18.** Let  $\alpha \le 1/2$ . In the context of Algorithm 14, there exists an algorithm that approximately solves subproblems of the form (for  $p \ge 2$  and L = pe),

$$\boldsymbol{z} \coloneqq \operatorname*{argmin}_{\boldsymbol{x} \in \mathbb{R}^d} \left\langle \boldsymbol{g}, \boldsymbol{x} \right\rangle + L \left( \|\boldsymbol{x} - \boldsymbol{q}\|_{\nabla^2 f(\boldsymbol{q})}^2 + C_p \|\boldsymbol{x} - \boldsymbol{q}\|_{\mathbf{M}}^p \right) \ ,$$

in the sense that we output x for which,

$$\max\left\{ \|\boldsymbol{x} - \boldsymbol{z}\|_{\mathbf{M}}, \left\| \mathbf{M}^{-1}\boldsymbol{g} + 2L\left(\mathbf{M}^{-1}\nabla^{2}f(\boldsymbol{q})(\boldsymbol{x} - \boldsymbol{q}) + C_{p} \|\boldsymbol{x} - \boldsymbol{q}\|_{\mathbf{M}}^{p-2}(\boldsymbol{x} - \boldsymbol{q})\right) \right\|_{\mathbf{M}} \right\} \leq \alpha \ .$$

The algorithm takes  $p^{O(1)}\log\left(\frac{pd \cdot f(q)}{\alpha}\right)$  linear system solves in matrices of the form  $\mathbf{A}^{\top}\mathbf{B}\mathbf{A}$  for blockdiagonal **B**, where each block in **B** has size  $|S_i| \times |S_i|$ . *Proof of Lemma 4.6.18.* This proof is long, and splitting it into lemmas would break up the intended reading flow. So we break it up into several key components here.

**Motivation for the lemma.** First, let us see why this lemma is even useful. In each iteration of Algorithm 13, which in turn calls Algorithm 10, the main primitive is computing

$$\begin{split} \widetilde{x}_{i} &= \underset{\widetilde{x} \in \mathbb{R}^{d}}{\operatorname{argmin}} f_{q_{i}}(\widetilde{x}_{i-1}) + \left\langle \nabla f_{q_{t}}(\widetilde{x}_{i-1}), \widetilde{x} - \widetilde{x}_{i-1} \right\rangle + peD_{h_{q_{i}}}(\widetilde{x}, \widetilde{x}_{i-1}) \ , \\ &= \underset{\widetilde{x} \in \mathbb{R}^{d}}{\operatorname{argmin}} f_{q_{i}}(\widetilde{x}_{i-1}) + \left\langle \nabla f_{q_{t}}(\widetilde{x}_{i-1}), \widetilde{x} - \widetilde{x}_{i-1} \right\rangle + pe\left(h_{q_{t}}(\widetilde{x}) - h_{q_{t}}(\widetilde{x}_{i-1}) - \left\langle \nabla h_{q_{t}}(\widetilde{x}_{i-1}), \widetilde{x} - \widetilde{x}_{i-1} \right\rangle \right) \ , \\ &= \underset{\widetilde{x} \in \mathbb{R}^{d}}{\operatorname{argmin}} f_{q_{i}}(\widetilde{x}_{i-1}) - peh_{q_{i}}(\widetilde{x}_{i-1}) + \left\langle \nabla f_{q_{t}}(\widetilde{x}_{i-1}) - pe\nabla h_{q_{t}}(\widetilde{x}_{i-1}), \widetilde{x} - \widetilde{x}_{i-1} \right\rangle + peh_{q_{t}}(\widetilde{x}) \ , \\ &= \underset{\widetilde{x} \in \mathbb{R}^{d}}{\operatorname{argmin}} \left\langle \nabla f_{q_{t}}(\widetilde{x}_{i-1}) - pe\nabla h_{q_{t}}(\widetilde{x}_{i-1}), \widetilde{x} \right\rangle + peh_{q_{t}}(\widetilde{x}) \ . \end{split}$$

Observe that the subproblem is of the form

$$z = \underset{x \in \mathbb{R}^d}{\operatorname{argmin}} \langle g, x \rangle + peh_q(x) ,$$
  
= 
$$\underset{x \in \mathbb{R}^d}{\operatorname{argmin}} \langle g, x \rangle + pe\left( \|x - q\|_{\nabla^2 f(q)}^2 + C_p \|x - q\|_{\mathbf{M}}^p \right) , \qquad (4.6.11)$$

and so our goal is to show how to solve these types of problems.

**The general algorithm.** Consider solving the related subproblem (instead of (4.6.11)),

$$\operatorname{argmin}_{\boldsymbol{x} \in \mathbb{R}^d} \langle \boldsymbol{g}, \boldsymbol{x} \rangle + L \left( \|\boldsymbol{x} - \boldsymbol{q}\|_{\nabla^2 f(\boldsymbol{q})}^2 + C_p \tau \|\boldsymbol{x} - \boldsymbol{q}\|_{\mathbf{M}}^2 \right)$$

for some fixed  $\tau \ge 0$ . This is a quadratic problem, and we can therefore solve it in 1 linear system solve. It is easy to check that at optimality, we have

$$g + 2pe\left(\nabla^2 f(q)(x-q) + C_p \tau \mathbf{M}(x-q)\right) = 0 ,$$

which rearranges to<sup>2</sup>

$$\boldsymbol{x} - \boldsymbol{q} = -\frac{1}{2pe} \left( \nabla^2 f(\boldsymbol{q}) + C_p \tau \mathbf{M} \right)^{-1} \boldsymbol{g}$$

Note that at optimality for our original subproblem (4.6.11), we have  $\tau^* := ||z - q||_{\mathbf{M}}^{p-2}$  where z is the solution of subproblem (4.6.11). Also note that  $||x - q||_{\mathbf{M}}$  is a decreasing function in  $\tau$  because,

$$\|\boldsymbol{x} - \boldsymbol{q}\|_{\mathbf{M}}^{2} = \frac{1}{4p^{2}e^{2}} \|\boldsymbol{g}\|_{\left(\nabla^{2}f(\boldsymbol{q}) + C_{p}\tau\mathbf{M}\right)^{-1}\mathbf{M}\left(\nabla^{2}f(\boldsymbol{q}) + C_{p}\tau\mathbf{M}\right)^{-1}}^{2},$$

and for  $\tau_1 \leq \tau_2$ ,

$$\left(\nabla^2 f(\boldsymbol{q}) + C_p \tau_1 \mathbf{M}\right)^{-1} \mathbf{M} \left(\nabla^2 f(\boldsymbol{q}) + C_p \tau_1 \mathbf{M}\right)^{-1} \ge \left(\nabla^2 f(\boldsymbol{q}) + C_p \tau_2 \mathbf{M}\right)^{-1} \mathbf{M} \left(\nabla^2 f(\boldsymbol{q}) + C_p \tau_2 \mathbf{M}\right)^{-1}$$

We therefore see that if  $\tau > ||x - q||_{\mathbf{M}}^{p-2}$  — where *x* is the optimal solution for a fixed  $\tau$  — then we are over-regularizing and need to decrease  $\tau$  and vice-versa. This means we can binary

<sup>&</sup>lt;sup>2</sup>Recall that  $\nabla^2 f(q) = \mathbf{A}^\top \mathbf{B}_1 \mathbf{A}$  for block-diagonal  $\mathbf{B}_1$  and by construction,  $\mathbf{M} = \mathbf{A}^\top \mathbf{W}^{1-\frac{2}{p}} \mathbf{A}$  where  $\mathbf{W}$  consists of the block Lewis weights on the diagonal. Thus,  $\nabla^2 f(q) + C_p \tau \mathbf{M} = \mathbf{A}^\top \mathbf{B}_2 \mathbf{A}$  for block-diagonal  $\mathbf{B}_2$ .

search for the appropriate value of  $\tau$ . To execute this, we first need to establish the accuracy up to which we have to identify  $\tau$ .

**Convergence in Argument.** By Lemma 4.6.13 (setting d = x - z), recall that it is enough to solve sub-problem (4.6.11) up to additive accuracy  $(p/2)^p L\alpha^p$  to get  $||x - z||_M \le \alpha$ . Suppose we find  $\tau$  for which  $\tau^* \le \tau \le \tau^* + \delta$ . By writing the objectives and comparing, we see that the x we find from using  $\tau$  gives us at most a  $\delta \cdot d$ -suboptimal solution compared to z. Plugging this into the bound from Lemma 4.6.13 tells us that we should choose  $\delta = (p/2)^p L\alpha^p/d$ , and plugging this into the binary search over  $\tau \in [0, d^p(1 + f(q))]$  gives us  $p^{O(1)} \log \left(\frac{pd \cdot f(q)}{\alpha}\right)$  steps, as needed.

First-order stationary point. We first claim that it is enough to get

$$\left\|\mathbf{M}^{-1}\nabla h_{\boldsymbol{q}}(\boldsymbol{x})-\mathbf{M}^{-1}\nabla h_{\boldsymbol{q}}(\boldsymbol{z})\right\|_{\mathbf{M}}\leq\frac{\alpha}{L}.$$

Indeed, let z be the optimal solution for the subproblem. This means that it must satisfy the first order stationary condition, namely,

$$g + L\nabla h_q(z) = 0.$$

Multiplying both sides by  $M^{-1}$ , subtracting, and dividing both sides by *L* gives us the expression we are interested in.

Writing first order stationary conditions gives both

$$g + 2L\left(\nabla^2 f(q)(x-q) + C_p \tau \mathbf{M}(x-q)\right) = 0$$
  
$$g + 2L\left(\nabla^2 f(q)(z-q) + C_p \tau^* \mathbf{M}(z-q)\right) = 0$$

Multiplying both sides of both equalities by  $M^{-1}$  and subtracting these gives

$$2L\left(\mathbf{M}^{-1}\nabla^2 f(\boldsymbol{q})(\boldsymbol{x}-\boldsymbol{z}) + C_p\left(\tau(\boldsymbol{x}-\boldsymbol{q}) - \tau^{\star}(\boldsymbol{z}-\boldsymbol{q})\right)\right) = 0.$$

Expanding out  $L(\mathbf{M}^{-1}\nabla h_q(\mathbf{x}) - \mathbf{M}^{-1}h_q(\mathbf{z}))$  and subtracting the above gives the desired condition

$$2L\left|\tau-\|\boldsymbol{x}-\boldsymbol{q}\|_{\mathbf{M}}^{p-2}\right|\cdot\|\boldsymbol{x}-\boldsymbol{q}\|_{\mathbf{M}}\overset{?}{\leq}\alpha.$$

Next, let us run the binary search from above so that we get argument convergence, i.e.  $||x - z||_{M} \le \alpha^{C} \ll 0.1\alpha$  for some constant *C*. Using the fact that the approximate mirror descent step using *z* decreases the objective value (Lemma 4.3.4), observe that

$$\|x - q\|_{\mathbf{M}} \le \|z - q\|_{\mathbf{M}} + \|x - z\|_{\mathbf{M}} \le \|q - z\|_{\mathbf{M}} + 0.1\alpha \le \sqrt{d(1 + f(q))}.$$

It then follows that binary searching  $\tau$  to additive accuracy  $\alpha(\sqrt{d}(1 + f(q)))^{-1}/L$  is sufficient. By the same argument as above, this takes  $p^{O(1)}\log\left(\frac{pd \cdot f(q_i)}{\alpha}\right)$  steps, completing the proof of Lemma 4.6.18.

We now combine Lemma 4.6.18 with Theorem 4.3.1 and Algorithm 10 to obtain approximate argument optimality for each proximal subproblem.

**Lemma 4.6.19.** Let  $\gamma > 0$  and  $x_q := \underset{x \in \mathbb{R}^d}{\operatorname{argmin}} f_q(x)$ . There exists an algorithm that returns x for which

$$\left\|x-x_q\right\|_{\mathbf{M}}\leq \gamma.$$

The algorithm takes at most  $O\left(p^{O(1)}\log\left(ph_q(x_q)\left(\frac{4}{p\gamma}\right)^p\right)\right)$  iterations of solving subproblems of the form  $\underset{x \in \mathbb{R}^d}{\operatorname{argmin}} \langle g, x \rangle + eph_q(x)$  for fixed vectors g and q.

*Proof of Lemma 4.6.19.* This proof resembles [JLS22, Lemma 4.5], which uses an exact version of mirror descent arising from Lu, Freund, and Nesterov [LFN18]. The main difference between our argument and that of [JLS22, Lemma 4.5] is that we rigorously identify a concrete upper bound on the complexity needed to satisfy the MS condition and argue that the mirror descent algorithm can handle the inexact Bregman proximal problem solves.

First, we use Lemma 4.6.12 on the approximate solution x and true solution  $x_q$  and get,

$$f_{q}(\mathbf{x}) \geq f_{q}(\mathbf{x}_{q}) + \frac{4}{2^{p}} \left( \left\| \mathbf{A}(\mathbf{x} - \mathbf{x}_{q}) \right\|_{\mathcal{G}_{p}}^{p} + C_{p} \left\| \mathbf{x}_{q} - \mathbf{x} \right\|_{\mathbf{M}}^{p} \right) ,$$
  
 
$$\geq f_{q}(\mathbf{x}) + \frac{4C_{p}}{2^{p}} \left\| \mathbf{x}_{q} - \mathbf{x} \right\|_{\mathbf{M}}^{p} .$$

Rearranging, we get

$$\begin{split} \left\| \mathbf{x}_{q} - \mathbf{x} \right\|_{\mathbf{M}} &\leq \left( \frac{2^{p}}{4C_{p}} \right)^{1/p} \left( f_{q}(\mathbf{x}) - f_{q}(\mathbf{x}_{q}) \right)^{1/p} , \\ &= \left( \frac{2^{p}}{4ep^{p}} \right)^{1/p} \left( f_{q}(\mathbf{x}) - f_{q}(\mathbf{x}_{q}) \right)^{1/p} , \\ &< \frac{2}{p} \left( f_{q}(\mathbf{x}) - f_{q}(\mathbf{x}_{q}) \right)^{1/p} . \end{split}$$

Using the notation from [LFN18], for convex  $h : \mathbb{R}^d \to \mathbb{R}$ , let

$$D_h(\mathbf{x}, \mathbf{y}) \coloneqq h(\mathbf{x}) - h(\mathbf{y}) - \langle \nabla h(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle.$$

Recall the conclusion of Lemma 4.6.9 – we have for  $\mu = 1/(2pe)$  and L = pe that

$$\mu \nabla^2 h_q(\mathbf{x}) \leq \nabla^2 f_q(\mathbf{x}) \leq L \nabla^2 h_q(\mathbf{x}).$$

By Theorem 4.3.1 and Lemma 4.6.9, using the same notation from Lemma 4.6.9, we have for all iterations *t* of Algorithm 10 (with  $f = f_q$  and  $h = h_q$ ) that,

$$f_{q}(\boldsymbol{x}_{t}) - f_{q}(\boldsymbol{x}_{q}) \leq L \left(1 - \frac{\mu}{L}\right)^{t} D_{h_{q}}(\boldsymbol{x}_{q}, \boldsymbol{q}) + \max_{1 \leq i \leq t} \left\langle \Delta_{i}, \boldsymbol{x}_{t} - \boldsymbol{x}_{q} \right\rangle ,$$
$$= 2L \left(1 - \frac{\mu}{L}\right)^{t} h_{q}(\boldsymbol{x}_{q}) + \max_{1 \leq i \leq t} \left\langle \Delta_{i}, \boldsymbol{x}_{t} - \boldsymbol{x}_{q} \right\rangle .$$

Hence, for  $t \ge \frac{L}{\mu} \log \left( Lh_q(x_q) \left( \frac{4}{p\gamma} \right)^p \right)$ , it is easy to check that for  $p \ge 2$ ,

$$\begin{split} f_{q}(\boldsymbol{x}_{t}) - f_{q}(\boldsymbol{x}_{q}) &\leq 2L \left(\frac{1}{e}\right)^{\log\left(Lh_{q}(\boldsymbol{x}_{q})\left(\frac{4}{p\gamma}\right)^{p}\right)} h_{q}(\boldsymbol{x}_{q}) + \max_{1 \leq i \leq t} \left\langle \Delta_{i}, \boldsymbol{x}_{t} - \boldsymbol{x}_{q} \right\rangle \ , \\ &= 2 \left(\frac{p\gamma}{4}\right)^{p} + \max_{1 \leq i \leq t} \left\langle \Delta_{i}, \boldsymbol{x}_{t} - \boldsymbol{x}_{q} \right\rangle \ , \\ &\leq \left(\frac{p\gamma}{2}\right)^{p} + \max_{1 \leq i \leq t} \left\langle \Delta_{i}, \boldsymbol{x}_{t} - \boldsymbol{x}_{q} \right\rangle \ , \end{split}$$

and combining this with Lemma 4.6.18 to make the error term on the order of our accuracy, we get  $\|x_q - x\|_M \leq \gamma$ . We thus conclude the proof of Lemma 4.6.19.

The last step is to use our proximal problem solver to build a valid MS oracle.

**Lemma 4.6.20.** *In the context of Algorithm* 11*, there exists an algorithm*  $(\tilde{x}_{t+1}, \lambda_{t+1}) = O_{\text{prox}}(q_t)$  that approximately solves

$$\underset{\widetilde{\mathbf{x}} \in \mathbb{R}^d}{\operatorname{argmin}} f(\widetilde{\mathbf{x}}) + ep^p \left\| \widetilde{\mathbf{x}} - \boldsymbol{q}_t \right\|_{\mathbf{M}}^p$$

using  $O\left(p^{O(1)}\log\left(\frac{pd\cdot f(\mathbf{x}_t)}{\varepsilon}\right)\right)$  linear system solves in  $\mathbf{A}^{\top}\mathbf{B}\mathbf{A}$ , in the sense that

$$\left\|\frac{1}{ep^{p+1}\left\|\widetilde{\boldsymbol{x}}_{t+1}-\boldsymbol{q}_{t}\right\|_{\mathbf{M}}^{p-2}}\mathbf{M}^{-1}\nabla f(\widetilde{\boldsymbol{x}}_{t+1})+(\widetilde{\boldsymbol{x}}_{t+1}-\boldsymbol{q}_{t})\right\|_{\mathbf{M}} \leq \frac{1}{2}\left\|\widetilde{\boldsymbol{x}}_{t+1}-\boldsymbol{q}_{t}\right\|_{\mathbf{M}}.$$

*Proof of Lemma 4.6.20.* The point of this proof is to give an analysis of Algorithm 13.

For notational simplicity, let  $x = \tilde{x}_{t+1}$  and  $\lambda = \lambda_{t+1}$ . We will reintroduce the indices when it is essential to clarify the iterations we are discussing.

First, it is helpful to see why the stated notion of approximation is useful. Let  $C_p \coloneqq ep^p$ . Observe that at exact optimality, we have

$$\nabla f(\boldsymbol{x}_{\boldsymbol{q}}) + \underbrace{ep^{p+1} \| \boldsymbol{x}_{\boldsymbol{q}} - \boldsymbol{q} \|_{\mathbf{M}}^{p-2}}_{\lambda^{\star}} \mathbf{M}(\boldsymbol{x} - \boldsymbol{q}) = 0 \quad .$$
(4.6.12)

This motivates the approximation in our lemma statement, with us asking for a  $\frac{1}{2}$ -approximate MS oracle (Definition 4.4.1) for *f*. This also tells us that at optimality in (4.6.12), we have,

$$\begin{aligned} \nabla f(\boldsymbol{x}_{q}) + ep^{p+1} \left\| \boldsymbol{x}_{q} - \boldsymbol{q} \right\|_{\mathbf{M}}^{p-2} \mathbf{M}(\boldsymbol{x} - \boldsymbol{q}) &= 0 , \\ \Leftrightarrow \mathbf{M}^{-1/2} f(\boldsymbol{x}_{q}) &= -pC_{p} \left\| \boldsymbol{x}_{q} - \boldsymbol{q} \right\|_{\mathbf{M}}^{p-2} \mathbf{M}^{1/2}(\boldsymbol{x} - \boldsymbol{q}) , \\ \Rightarrow \left\| \mathbf{M}^{-1/2} f(\boldsymbol{x}_{q}) \right\|_{2} &= pC_{p} \left\| \boldsymbol{x}_{q} - \boldsymbol{q} \right\|_{\mathbf{M}}^{p-2} \left\| \mathbf{M}^{1/2}(\boldsymbol{x} - \boldsymbol{q}) \right\|_{2} , \\ \Leftrightarrow \left\| \boldsymbol{x}_{q} - \boldsymbol{q} \right\|_{\mathbf{M}} &= \left( \frac{\left\| \mathbf{M}^{-1} \nabla f(\boldsymbol{x}_{q}) \right\|_{\mathbf{M}}}{pC_{p}} \right)^{\frac{1}{p-1}} . \end{aligned}$$

We now break up our analysis into two cases. In the first, suppose that  $\|\mathbf{M}^{-1}\nabla f(\mathbf{x}_q)\|_{\mathbf{M}} \leq \varepsilon/\|\mathbf{x}_q - \mathbf{x}^{\star}\|_{\mathbf{M}}$ . Then, by convexity, we have

$$f(\boldsymbol{x}_q) - f(\boldsymbol{x}^{\star}) \leq \left\langle \nabla f(\boldsymbol{x}_q), \boldsymbol{x}_q - \boldsymbol{x}^{\star} \right\rangle \leq \left\| \mathbf{M}^{-1} \nabla f(\boldsymbol{x}_q) \right\|_{\mathbf{M}} \left\| \boldsymbol{x}_q - \boldsymbol{x}^{\star} \right\|_{\mathbf{M}} \leq \varepsilon.$$

Hence, for the rest of the proof, assume that  $\|\mathbf{M}^{-1}\nabla f(x_q)\| \ge \varepsilon/\|x_q - x^*\|_{\mathbf{M}}$  (because if this is not the case, in the algorithm we can simply check whether the MS condition is satisfied – if not, then we know this assumption was violated and we are done anyway). We run the algorithm implied by Lemma 4.6.19 and obtain an approximate solution x for which

$$\|\mathbf{x} - \mathbf{x}_{q}\|_{\mathbf{M}} \le \alpha \|\mathbf{x}_{q} - q\|_{\mathbf{M}} \text{ for } \alpha = \frac{1}{5} \min \left\{ \frac{C_{p}}{ep(p-1)} \left( \frac{\|\mathbf{x}_{q} - q\|_{\mathbf{M}}}{f(q)^{\frac{1}{p}}} \right)^{p-2}, 1 \right\}$$
 (4.6.13)

Since  $\alpha < 1$  the guarantee in (4.6.13) gives us,

$$\left\| \boldsymbol{x} - \boldsymbol{x}_{\boldsymbol{q}} \right\|_{\mathbf{M}} \le \alpha \left\| \boldsymbol{x} - \boldsymbol{q} \right\|_{\mathbf{M}} \le \frac{\alpha}{1 - \alpha} \left\| \boldsymbol{x} - \boldsymbol{q} \right\|_{\mathbf{M}} \quad , \tag{4.6.14}$$

and further applying triangle inequality gives us

$$\begin{aligned} \left\| x_{q} - q \right\|_{\mathbf{M}} &\leq \left\| x - q \right\|_{\mathbf{M}} + \left\| x_{q} - x \right\|_{\mathbf{M}}, \\ &\leq \frac{1 - \alpha}{1 - \alpha} \left\| x - q \right\|_{\mathbf{M}} + \frac{\alpha}{1 - \alpha} \left\| x - q \right\|_{\mathbf{M}}, \\ &\leq \frac{1}{1 - \alpha} \left\| x - q \right\|_{\mathbf{M}}. \end{aligned}$$

$$(4.6.15)$$

Hence, we get

$$\frac{ep(p-1)f(q)^{1-\frac{2}{p}}}{C_{p} \|\mathbf{x}-q\|_{\mathbf{M}}^{p-2}} \cdot \|\mathbf{x}-\mathbf{x}_{q}\|_{\mathbf{M}} = \frac{ep(p-1)}{C_{p}} \cdot \left(\frac{f(q)^{\frac{1}{p}}}{\|\mathbf{x}-q\|_{\mathbf{M}}}\right)^{p-2} \cdot \|\mathbf{x}-\mathbf{x}_{q}\|_{\mathbf{M}} ,$$

$$\leq^{(4.6.13)} \frac{1}{5} \|\mathbf{x}_{q}-q\|_{\mathbf{M}} ,$$

$$\leq^{(4.6.15)} \frac{1}{5} \cdot \frac{1}{1-\alpha} \|\mathbf{x}-q\|_{\mathbf{M}} ,$$

$$\leq \frac{1}{4} \|\mathbf{x}-q\|_{\mathbf{M}} , \qquad (4.6.16)$$

where in the last inequality, we used that  $\alpha \leq \frac{1}{5}$  due to our choice in (4.6.13). We now call Lemma 4.6.15, divide both sides by  $\lambda$ , and get

$$\begin{split} & \left\| \frac{1}{ep^{p+1} \|\mathbf{x} - q\|_{\mathbf{M}}^{p-2}} \mathbf{M}^{-1} \nabla f(\mathbf{x}) + (\mathbf{x} - q) \right\|_{\mathbf{M}} \\ \leq ^{(\text{Lemma 4.6.15)}} ep(p-1) \left( \frac{f(q)^{1-\frac{2}{p}}}{C_{p} \|\mathbf{x} - q\|_{\mathbf{M}}^{p-2}} + \max \left\{ 1, \left( \frac{\|\mathbf{x}_{q} - q\|_{\mathbf{M}}}{\|\mathbf{x} - q\|_{\mathbf{M}}} \right)^{p-2} \right\} \right) \|\mathbf{x} - \mathbf{x}_{q}\|_{\mathbf{M}} , \\ \leq ^{(4.6.15)} ep(p-1) \left( \frac{f(q)^{1-\frac{2}{p}}}{C_{p} \|\mathbf{x} - q\|_{\mathbf{M}}^{p-2}} + \frac{1}{(1-\alpha)^{p-2}} \right) \|\mathbf{x} - \mathbf{x}_{q}\|_{\mathbf{M}} , \\ \leq ^{(4.6.14)} \frac{ep(p-1)f(q)^{1-\frac{2}{p}}}{C_{p} \|\mathbf{x} - q\|_{\mathbf{M}}^{p-2}} \cdot \|\mathbf{x} - \mathbf{x}_{q}\|_{\mathbf{M}} + \frac{ep(p-1)\alpha}{(1-\alpha)^{p-1}} \|\mathbf{x} - q\|_{\mathbf{M}} , \\ \leq ^{(4.6.15), (4.6.13)} \frac{1}{4} \|\mathbf{x} - q\|_{\mathbf{M}} + \frac{ep(p-1)5^{p-2}}{4^{p-1}} \|\mathbf{x} - q\|_{\mathbf{M}} , \\ \leq \frac{1}{2} \|\mathbf{x} - q\|_{\mathbf{M}} , \end{split}$$

giving us the approximation guarantee.

It remains to understand the complexity of solving the proximal subproblem to the accuracy required in (4.6.13). Plugging in  $\gamma = \alpha ||x_q - q||_{\text{M}}$  into Lemma 4.6.19 and using our bound on  $h_q(x_q)$  from Lemma 4.6.17 gives an iteration complexity of (ignoring the constant in front of the big-O)

$$p^{O(1)} \log \left( ph_{q}(\boldsymbol{x}_{q}) \left( \frac{2}{p\alpha \|\boldsymbol{x}_{q} - \boldsymbol{q}\|_{\mathbf{M}}} \right)^{p} \right)$$

$$\leq p^{O(1)} \log \left( p \left( p(p-1)f(\boldsymbol{q})^{1-\frac{2}{p}} \|\boldsymbol{x}_{q} - \boldsymbol{q}\|_{\mathbf{M}}^{2} + C_{p} \|\boldsymbol{x}_{q} - \boldsymbol{q}\|_{\mathbf{M}}^{p} \right) \left( \frac{2}{p\alpha \|\boldsymbol{x}_{q} - \boldsymbol{q}\|_{\mathbf{M}}} \right)^{p} \right)$$

$$= p^{O(1)} \log \left( \left( \frac{2}{p} \right)^{p} p \left( \frac{p(p-1)f(\boldsymbol{q})^{1-\frac{2}{p}} \|\boldsymbol{x}_{q} - \boldsymbol{q}\|_{\mathbf{M}}^{2} + C_{p} \|\boldsymbol{x}_{q} - \boldsymbol{q}\|_{\mathbf{M}}^{p}}{\alpha^{p} \|\boldsymbol{x}_{q} - \boldsymbol{q}\|_{\mathbf{M}}^{p}} \right) \right)$$

$$= p^{O(1)} \log \left( \left(\frac{2}{p}\right)^p p \left( \frac{p(p-1)f(\boldsymbol{q})^{1-\frac{2}{p}}}{\alpha^p \left\| \boldsymbol{x}_{\boldsymbol{q}} - \boldsymbol{q} \right\|_{\mathbf{M}}^{p-2}} + \frac{C_p}{\alpha^p} \right) \right)$$

We have two cases to analyze for the value of  $\alpha$ . In the first, suppose we get  $\alpha = \frac{1}{5}$ . By the definition of  $\alpha$ , this means we have

$$\frac{C_p}{ep(p-1)} \left( \frac{\left\| \boldsymbol{x}_{\boldsymbol{q}} - \boldsymbol{q} \right\|_{\mathbf{M}}}{f(\boldsymbol{q})^{\frac{1}{p}}} \right)^{p-2} \ge 1,$$

which means the complexity we get is  $p^{O(1)} \log p$ . We now handle the other case, i.e.,  $\alpha = \frac{C_p}{5ep(p-1)} \left(\frac{\|x_q - q\|_M}{f(q)^{\frac{1}{p}}}\right)^{p-2}$ . Here, it will be useful to keep track of the timestep *t* that we are working with. Recall that

$$\left\|\boldsymbol{x}_{\boldsymbol{q}_{t}}-\boldsymbol{q}_{t}\right\|_{\mathbf{M}}^{p} = \left(\frac{\left\|\mathbf{M}^{-1}\nabla f(\boldsymbol{x}_{\boldsymbol{q}_{t}})\right\|_{\mathbf{M}}}{pC_{p}}\right)^{\frac{p}{p-1}} \ge \left(\frac{\varepsilon}{pC_{p}\left\|\boldsymbol{x}_{\boldsymbol{q}_{t}}-\boldsymbol{x}^{\star}\right\|_{\mathbf{M}}}\right)^{\frac{p}{p-1}}, \quad (4.6.17)$$

so the complexity we want to control is given by

$$\begin{split} p^{O(1)} \log \left( \left(\frac{2}{p}\right)^p p\left(\frac{2f(q_t)}{a^p \, \|\mathbf{x}_{q_t} - q_t\|_{\mathbf{M}}^p}\right) \right) \\ & \lesssim^{(4.6.13)} p^{O(1)} \log \left( \left(\frac{2}{p}\right)^p p\left(\frac{2\left(5ep(p-1)\right)^p f(q_t)^{p-1}}{C_p^p \, \|\mathbf{x}_{q_t} - q_t\|_{\mathbf{M}}^{p(p-2)} \, \|\mathbf{x}_{q_t} - q_t\|_{\mathbf{M}}^p} \right) \right) , \\ & \lesssim p^{O(1)} \log \left( p\left(\frac{2\left(10(p-1)\right)^p f(q_t)^{p-1}}{p^{p^2} \, \|\mathbf{x}_{q_t} - q_t\|_{\mathbf{M}}^{p(p-1)}} \right) \right) , \\ & \lesssim^{(4.6.17)} p^{O(1)} \log \left( p\left(\frac{2\left(10e(p-1)\right)^p p^{p(p+1)} f(q_t)^{p-1}}{p^{p^2} \epsilon^p} \right) \, \|\mathbf{x}_{q_t} - \mathbf{x}^\star\|_{\mathbf{M}}^p \right) , \\ & \lesssim^{(4.6.17)} p^{O(1)} \log \left( \left(\frac{2\left(10e(p-1)\right)^p p^{p+1} f(q_t)^{p-1}}{\epsilon^p} \right) \, \|\mathbf{x}_{q_t} - \mathbf{x}^\star\|_{\mathbf{M}}^p \right) , \\ & \lesssim p^{O(1)} \log \left( \frac{pf(q_t) \, \|\mathbf{x}_{q_t} - \mathbf{x}^\star\|_{\mathbf{M}}}{\epsilon} \right) , \\ & \lesssim p^{O(1)} \log \left( \frac{pf(q_t) \, \|\mathbf{x}_{q_t} - \mathbf{x}^\star\|_{\mathbf{M}}}{\epsilon} \right) , \\ & \lesssim^{(\text{Lemma 4.6.16)} p^{O(1)} \log \left( \frac{pf(q_t) df(\mathbf{x}_t)}{\epsilon} \right) , \\ & \lesssim^{(\text{Lemma 4.6.8)} p^{O(1)} \log \left( \frac{pf(\mathbf{x}_t)}{\epsilon} \right) , \end{split}$$

completing the proof of Lemma 4.6.20.

#### 4.6.4. The algorithm

We are now ready to combine the results from the previous two subsections to build our algorithm for  $\mathcal{G}_p$ -regression and prove Theorem 13. The main algorithmic object here is Algorithm 14.

#### **Algorithm 14** GpRegression: Optimizes (4.1.4) up to $(1 + \varepsilon)$ -multiplicative error

**Require:** Regression problems  $(\mathbf{A}_{S_1}, \boldsymbol{b}_{S_1}), \ldots, (\mathbf{A}_{S_m}, \boldsymbol{b}_{S_m})$ , accuracy  $\varepsilon > 0$ 

1: Using [MO25, Algorithm 2] with input [A|b], find nonnegative diagonal W such that for all  $x \in \mathbb{R}^d$  and  $c \in \mathbb{R}$ ,

$$\|\mathbf{A}\mathbf{x} - c\mathbf{b}\|_{\mathcal{G}_{\infty}} \leq \left\|\mathbf{W}^{\frac{1}{2} - \frac{1}{p}}\mathbf{A}\mathbf{x} - c\mathbf{W}^{1/2}\mathbf{b}\right\|_{2} \leq (2(d+1))^{\frac{1}{2} - \frac{1}{p}} \|\mathbf{A}\mathbf{x} - c\mathbf{b}\|_{\mathcal{G}_{\infty}}.$$
  
2: Let  $\mathbf{x}_{0} = \left(\mathbf{A}^{\mathsf{T}}\mathbf{W}^{1 - \frac{2}{p}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathsf{T}}\mathbf{W}^{1 - \frac{2}{p}}\mathbf{b}.$   $\triangleright \mathbf{x}_{0} \coloneqq \operatorname*{argmin}_{\mathbf{x} \in \mathbb{R}^{d}} \left\|\mathbf{W}^{\frac{1}{2} - \frac{1}{p}}\mathbf{A}\mathbf{x} - \mathbf{W}^{\frac{1}{2} - \frac{1}{p}}\mathbf{b}\right\|_{2}$ 

- 3: Using Algorithm 13 and Lemma 4.6.20, implement a  $\frac{1}{2}$ -MS oracle for f (Definition 4.4.1)
- 4: Run Algorithm 11 with the oracle from the previous line and with  $x_0$  as the initialization for  $O\left(\mathsf{poly}(p)\min\{\mathsf{rank}(\mathbf{A}), m\}^{\frac{p-2}{3p-2}}\log\left(\frac{d}{\varepsilon}\right)^3\right)$  iterations.
- 5: return  $\hat{x}$  the output of the previous step.

*Proof of Theorem 13.* By writing the stationary condition of the proximal problem, it makes sense to choose  $\lambda_{t+1} = ep^{p+1} \| \widetilde{x}_{t+1} - q_t \|_{\mathbf{M}}^{p-2}$ .

It is easy to check that

$$\left\| \widetilde{\mathbf{x}}_{t+1} - \mathbf{q}_t \right\|_{\mathbf{M}} = \left( \frac{ep^{p+1} \left\| \widetilde{\mathbf{x}}_{t+1} - \mathbf{q}_t \right\|_{\mathbf{M}}^{p-2}}{\left( (ep^{p+1})^{\frac{1}{p-1}} \right)^{p-1}} \right)^{(p-1)},$$

and therefore the triple  $(\tilde{x}_{t+1}, q_t, ep^{p+1} \| \tilde{x}_{t+1} - q_t \|_{\mathbf{M}}^{p-2})$  always satisfies a  $(p-1, (ep^{p+1})^{1/(p-1)})$ -movement bound (Definition 4.4.2).

Next, we calculate the iteration complexity we need to reduce the error to half of what we started with. For an arbitrary initial iterate x, let  $\delta = 0.5(f(x) - f(x^*))$ . By Lemma 4.6.2, we have

$$\|\mathbf{x} - \mathbf{x}^{\star}\|_{\mathbf{M}}^{s+1} = \|\mathbf{x} - \mathbf{x}^{\star}\|_{\mathbf{M}}^{p} \le 2^{3p/2} d^{p/2-1} (f(\mathbf{x}) - f(\mathbf{x}^{\star})),$$

so combining this along with the fact that  $c^s = ep^{p+1}$  and applying Theorem 4.4.3 with our proximal solver Lemma 4.6.20 yields

$$T_{\min} = \frac{p-1}{3} \left( pC_p \cdot 2^{3p/2+1} d^{p/2-1} \right)^{\frac{2}{3p-2}} \lesssim p^{5/3} d^{\frac{p-2}{3p-2}}$$

Next, we initialize  $\mathbf{x}_0 \coloneqq \left(\mathbf{A}^\top \mathbf{W}^{1-2/p} \mathbf{A}\right)^{-1} \mathbf{A}^\top \mathbf{W}^{1-2/p} \mathbf{b}$ . Using Theorem 4.2.2, we have

$$f(\boldsymbol{x}_0) \le (2d)^{p/2-1} f(\boldsymbol{x}^{\star}),$$

so reaching an iterate x for which  $f(x) - f(x^*) \leq \varepsilon f(x^*)$  takes  $T_{\min} \cdot \log \left( \frac{d^{p/2-1}}{\varepsilon} \right) = p^{8/3} d^{\frac{p-2}{3p-2}} \log \left( \frac{d}{\varepsilon} \right)$  calls to  $O_{\text{prox}}$ .

We now resolve the full iteration complexity, including the bootstrapping step to show that  $f(x_t)$  is reasonably bounded so that we get an unconditional upper bound from Lemma 4.6.20. At the end of iteration t, from (loosely) inverting the bound in Theorem 4.4.3, we know that

$$f(\mathbf{x}_t) - f(\mathbf{x}^{\star}) \le \frac{(Cp^3)^{\frac{3p-2}{2}}(2d)^{\frac{p}{2}-1}}{t^{\frac{3p-2}{2}}}.$$

Since  $\tilde{x}_{t+1}$  only depends on  $q_t$ , which in turn only depends on  $x_t$  and  $v_t$ , it suffices to use the above bound for  $f(x_t)$ , which gives us an iteration complexity of  $p^{O(1)} \log \left(\frac{pd}{\varepsilon}\right)$  to compute  $\tilde{x}_{t+1}$  (which we get from plugging into Lemma 4.6.20).

Combining this with the iteration complexity of  $O_{\text{prox}}$  gives us the result of Theorem 13.

# 5. Dueling optimization with a monotone adversary

In this chapter, we give randomized algorithms for the problem of *dueling optimization with a monotone adversary*. The content here is based on joint work with Avrim Blum, Meghal Gupta, Gene Li, Aadirupa Saha, and Chloe Yang [BGLMSY24].

# 5.1. Introduction

A growing body of literature studies learning with preference-based feedback [BV06; SJ11], with tremendous empirical success in recommendation systems, search engine optimization, information retrieval, and robotics. More recently, preference-based feedback has received a lot of attention as a mechanism to train large language models [OWJ<sup>+</sup>22]. Moreover, in recommender systems [BOHG13], a natural approach is to learn from users' preferences relations on a set of recommended items and update the system's belief for better future recommendations [JRTZ16] (e.g., given these items, which one do you prefer the most?).

Such preference-based feedback is not readily addressed by classical formulations for online decision making, such as bandits and reinforcement learning. In particular, algorithms for these problems rely on ordinal feedback per item (e.g., on a scale of 1 to 10, how much did the user like a particular item?). To address this, a long line of work studies the *dueling bandit framework* for online decision making under pairwise/preference-based feedback. There exist efficient algorithms with provable guarantees for the standard multi-armed bandit setup [YBKJ12; AKJ14; KHKN15], contextual bandits [DHSSZ15; SK22], as well as dueling convex optimization [JNR12; SKM21; SKM22], to name a few. The dueling bandit framework is especially applicable in settings where real-valued feedback is scarce or impossible to obtain, but preference-based feedback is readily available.

However, a key limitation of the dueling bandit framework is that the feedback that the learner receives is essentially "in-list". That is, the users are restricted to selecting items exclusively from the list of recommended items. This feedback model fails to capture the real-world scenarios where the users might select an out-of-list item they prefer. To illustrate, music streaming services like Spotify create personalized playlists for users. Concretely, each song can be encoded as a feature vector  $x \in \mathbb{R}^d$ , and the goal is to recommend the songs with the highest utility for a hidden, well-structured utility function of x. However, the users can also search for and play the songs they have a stronger preference (i.e., higher utility) than all recommendations.

This out-of-list feedback model falls into a monotone adversarial framework (see the chapter by Feige [Fei21]). In such models, an adversary is only allowed to make "helpful" changes. For example, in a graph clustering problem, the adversary is only allowed to add edges within communities and delete edges that cross communities (see, e.g., the chapter by Moitra [Moi21b]).

In our setting, the adversary is only allowed to respond with an item that is at least as good as any recommended item. A clear adaptation of the dueling bandit framework to this new feedback type is not evident.

#### 5.1.1. Problem statement

As our main conceptual contribution, we introduce a theoretical formulation for this setting that we call *dueling optimization with a monotone adversary*. As we will see, our formulation supports "out-of-list" feedback.

**Problem 5.1** (Dueling optimization with a monotone adversary). Let  $X \subseteq \mathbb{R}^d$  be a decision space, and let  $f : X \to \mathbb{R}$  be a cost function with an unknown global minimum  $x^*$ . A learner interacts with an adversary over rounds t = 1, 2, ..., where each round is of the following form.

- 1. The learner proposes m points  $x_t^{(1)}, \ldots, x_t^{(m)} \in \mathcal{X}$ .
- 2. The adversary responds with a point  $\mathbf{x}_t^{\star}$  that satisfies  $f(\mathbf{x}_t^{\star}) \leq \min_{1 \leq j \leq m} \left\{ f\left(\mathbf{x}_t^{(j)}\right) \right\}$ .

The goal is to design algorithms that:

- 1. for some prespecified  $\varepsilon > 0$ , minimize the number of iterations to find a point x for which  $f(x) f(x^*) \le \varepsilon$ ;
- 2. minimize the total cost  $\sum_{t=1}^{\infty} \left( \max_{1 \le j \le m} \left\{ f\left( \boldsymbol{x}_{t}^{(j)} \right) \right\} f(\boldsymbol{x}^{\star}) \right)$ .

Note that in Problem 5.1, we are interested in both the iteration complexity and the total cost. The first objective is a standard metric for measuring the performance of an iterative optimization algorithm. The second objective is motivated by online settings in which a practitioner may wish to minimize the total regret (cost) of its recommendations over an indefinitely long interaction with a user. In fact, the algorithms we propose in this chapter simultaneously achieve both small iteration complexity (for any choice of  $\varepsilon$ ) as well as total cost — see our technical overview in Section 5.1.3 for more details.

Problem 5.1 is a natural extension of (noiseless) dueling optimization [JNR12; SKM21; SKM22] to handle "out-of-list" responses, as in the Spotify recommendation example. The vanilla (noiseless) dueling optimization setup corresponds to the requirement that the user's response satisfies  $x_t^{\star} \in \{x_t^{(1)}, x_t^{(2)}\}$ . We allow the user to be potentially adversarial by allowing it to respond with any improvement to the learner's suggestions (in the sequel, we exclusively refer to the user as the adversary).

Even though the monotone adversary is only improving upon the learner's suggestions, existing algorithms for dueling optimization cannot be freely extended to handle the monotone feedback. At a high level, the difficulty arises from the fact that existing algorithms carefully select the queries  $x_t^{(1)}$ ,  $x_t^{(2)}$  so that learning whether  $f(x_t^{(1)}) > f(x_t^{(2)})$  reveals information about the underlying f. However, a monotone adversary can return a point  $x_t^*$  that reveals no information about the relationship between  $x_t^{(1)}$  and  $x_t^{(2)}$ .

To illustrate this point, consider a natural coordinate-wise binary search algorithm for the dueling optimization problem when  $f(\mathbf{x}) = \|\mathbf{x} - \mathbf{x}^{\star}\|_{2}^{2}$  for some  $\mathbf{x}^{\star} \in \mathcal{B}_{2}^{d} \coloneqq \{\mathbf{x} : \|\mathbf{x}\|_{2} \le 1\}$ .

For coordinates  $i = 1, \dots, d$ , query points of the form  $x_t^{(1)} = c_1 \cdot e_i, x_t^{(2)} = c_2 \cdot e_i$  and progressively refine the values  $c_1, c_2 \in \mathbb{R}$  to search for the value of  $x^*[i]$  (i.e., the *i*-th entry of  $x^*$ ). It is easy to show that this approach has a query complexity of  $O(d \log (1/\varepsilon))$  in the vanilla dueling optimization setting. However, a monotone adversary can return orthogonal responses of the form  $x_t^* = Ce_j$  (where  $j \neq i$  and *C* is a constant) that do not allow the learner to search along the intended coordinate *i*. Furthermore, Jamieson, Nowak, and Recht [JNR12] and Saha, Koren, and Mansour [SKM21] give more sophisticated algorithms for the dueling optimization problem that inherently depend upon the "in-list" feedback, which clearly cannot apply to our setting. We therefore need novel insights to solve Problem 5.1.

# 5.1.2. Our results

We study Problem 5.1 for various natural classes of functions f and provide tight upper and lower bounds on the number of queries required to find an  $\varepsilon$ -optimal point.

**Upper bound for linear functions.** First, we study dueling optimization with a monotone adversary when the function *f* is linear. This is a natural class to consider. In particular, an algorithm that solves Problem 5.1 can be adapted to achieve constant regret for (noiseless) linear contextual bandits [CLRS11], where the reward function is  $r(x) \coloneqq \langle x, x^* \rangle$ . Note that the key difference in the setup is that the learner does not get to observe the actual linear costs but instead only an improvement to the actions (points) that the learner selects.

**Theorem 14.** Let m = 2, let  $X = \mathbb{S}_2^d$ , let  $x^*$  be such that  $||x^*||_2 = 1$ , and let  $f: X \to \mathbb{R}$  be  $f(x) = -\langle x, x^* \rangle$ . Fix any  $\varepsilon > 0$ . There exists an algorithm that, in the setting of Problem 5.1, with probability at least  $1 - \exp(-O(d))$ :

- outputs a point x satisfying  $\langle x^{\star} x, x^{\star} \rangle \leq \varepsilon$  within  $O(d \log (1/\varepsilon)^2)$  iterations;
- *incurs total cost O(d).*

Each pair of guesses at time t can be computed in O(d) time.

We prove Theorem 14 in Section 5.2.2, and the cost is near-optimal with respect to *d*.

Gollapudi, Guruganesh, Kollias, Manurangsi, Leme, and Schneider [GGKMLS21] study a closely related setup that they call *local contextual recommendation*. Their result (see their Theorem 6.4) can be interpreted as showing that if the action set X is a discrete set (namely a packing over the unit sphere), there exists a  $2^{\Omega(d)}$  lower bound on the iteration complexity to find a point with constant suboptimality. In contrast, our Theorem 14 shows a much smaller upper bound when the domain is the entire unit sphere.

**Upper bound for smooth and PŁ functions.** Next, we study whether we can show guarantees for a large class of functions. We show a positive result for functions that are both  $\beta$ -smooth and  $\alpha$ -Polyak-Łojasiewicz (abbreviated as PŁ). These assumptions are standard in optimization.

**Definition 5.1.1** ( $\beta$ -smooth function [Bub15, Lemma 3.4]). We say f is  $\beta$ -smooth if it satisfies

(5.1.1).

For all 
$$\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$$
:  $|f(\mathbf{x}) - f(\mathbf{y}) - \langle \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle| \le \frac{\beta}{2} \cdot ||\mathbf{x} - \mathbf{y}||_2^2$  (5.1.1)

**Definition 5.1.2** ( $\alpha$ -PL function). We say f is  $\alpha$ -PL if it satisfies (5.1.2).

For all 
$$\mathbf{x} \in \mathbb{R}^d$$
 and minimizers  $\mathbf{x}^{\star}$ :  $f(\mathbf{x}) - f(\mathbf{x}^{\star}) \le \frac{1}{2\alpha} \|\nabla f(\mathbf{x})\|_2^2$  (5.1.2)

Our main result for this setting is Theorem 15.

**Theorem 15.** Let  $X = \mathbb{R}^d$ , and suppose f is  $\beta$ -smooth (Definition 5.1.1) and  $\alpha$ -PL (Definition 5.1.2). Fix any  $\varepsilon > 0$ , as well as a known point  $\mathbf{x}_1$  and a value B satisfying  $B \ge f(\mathbf{x}_1) - f(\mathbf{x}^*)$ . For all d larger than a universal constant, there exists an algorithm that, in the setting of Problem 5.1 with  $m \ge 2$ , with probability at least  $1 - \exp(-O(d))$ :

- outputs a point x satisfying  $f(x) f(x^*) \le \varepsilon$  within  $O\left(\beta/\alpha \cdot d/\log m \cdot \log (B/\varepsilon)^2\right)$  iterations;
- *incurs total cost*  $O(\beta/\alpha \cdot B \cdot d/\log m)$ .

The list of m guesses at time t can be computed in O(md) time.

We prove Theorem 15 in Section 5.2.3. Importantly, observe that the results above generalize those of Saha, Feldman, Mansour, and Koren [SFMK24], as our methods also work under a monotone adversary. Additionally, these achieve the dependence on the list size *m* that we observe in our lower bounds (to be presented momentarily), which therefore makes our results tight. Finally, although Theorem 15 is only stated for smooth and PL functions, it is straightforward to adapt this to a result for convex and smooth functions (we describe this adaptation in a moment).

As an application, we show a positive result when the loss function is the Euclidean distance, and the decision space  $X = \mathcal{B}_2^d$  is a unit ball:

**Theorem 16.** Let m = 2, let  $X = \mathcal{B}_2^d$ , let  $x^*$  be such that  $||x^*||_2 \leq 1$ , and let  $f: X \to \mathbb{R}$  be  $f(x) = ||x - x^*||_2$ . Fix any  $\varepsilon > 0$ . There exists an algorithm that, in the setting of Problem 5.1, with probability at least  $1 - \exp(-O(d))$ :

• outputs a point x satisfying  $||x - x^{\star}||_2 \le \varepsilon$  within  $O\left(d \cdot \log\left(\frac{B}{\varepsilon}\right)^2\right)$  iterations;

• *incurs total cost O (d).* 

Each pair of guesses at time t can be computed in O(d) time.

We prove Theorem 16 in Section 5.2.4.

Note that unlike in Theorem 15, Theorem 16 applies to the setting where the algorithm must guess points belonging to a given constraint set X. Hence, in the proof of Theorem 16, we have to be careful to ensure that the convergence argument still holds when we apply the algorithm for Theorem 15 along with a projection step. It is not clear that this argument holds by default

for all *f* satisfying the conditions requested by Theorem 15. Furthermore, as will become evident, we really only require that X be any convex body (though we state the result with  $X = \mathcal{B}_2^d$  to emphasize the consistency with our following lower bounds).

Finally, as another corollary to Theorem 15, we prove a low-accuracy result for optimizing functions that are  $\beta$ -smooth and convex.

**Theorem 17.** Let  $X = \mathbb{R}^d$ , and suppose f is convex and  $\beta$ -smooth (Definition 5.1.1). Fix any  $\varepsilon > 0$ , as well as a known point  $\mathbf{x}_1$  and a value B satisfying  $B \ge f(\mathbf{x}_1) - f(\mathbf{x}^*)$  and  $\|\mathbf{x}_1 - \mathbf{x}^*\|_2 \le \sqrt{D}$ . For all d larger than a universal constant, there exists an algorithm that, in the setting of Problem 5.1 with  $m \ge 2$ , with probability at least  $1 - \exp(-O(d))$ , outputs a point  $\mathbf{x}$  satisfying  $f(\mathbf{x}) - f(\mathbf{x}^*) \le \varepsilon$  within  $O\left(\frac{\beta D}{\varepsilon} \cdot \frac{d}{\log m} \cdot \log\left(\frac{B}{\varepsilon}\right)^2\right)$  iterations. The list of m guesses at time t can be computed in O(md) time.

We prove Theorem 17 in Section 5.2.5.

**Lower bounds.** We also prove that the dependence on *d* in our results is tight. In particular, when *f* is either a linear function or the distance to the target (as in Theorem 16), then  $\Omega(d)$  queries are necessary to identify  $x^*$ . This will translate to a  $\Omega(d)$  cost over an infinite number of rounds. In fact, our lower bound is valid when the adversary must return one of the two queried points, as in vanilla dueling optimization framework.

Our lower bound also covers a more general setting than that stated in Problem 5.1. Thus far, we have only discussed the setting where the algorithm can query only two points and is told the better of the two. In many practical instances, the algorithm can query m points and learn the point with the best objective value (we call this *m*-ary dueling optimization). In our construction, we prove that unless m is polynomial in d, we cannot decrease the total cost substantially below  $\Omega(d)$ . Thus, Theorem 15 is tight.

See Theorem 18 for a formal statement of our lower bound.

**Theorem 18** (Lower bound,  $\ell_2$  distance). Let  $X = \mathcal{B}_2^d$ . For any randomized algorithm for *m*-ary dueling optimization, there exists a choice of minimizer  $x^* \in \mathcal{B}_2^d$  and function  $f(x) := ||x - x^*||_2$  such that the algorithm must:

- perform  $\Omega(d/\log m)$  iterations in expectation to find a point x for which  $f(x) f(x^*) \le \varepsilon$ .
- incur cost  $\Omega(d/\log m)$  in expectation.

*Here,*  $\varepsilon > 0$  *is an absolute numerical constant.* 

We prove Theorem 18 in Section 5.3. Using the same construction, we can also demonstrate that Theorem 14 is tight when X is the unit sphere.

**Corollary 5.1.3** (Lower bound, linear *f*). Let  $X = \mathbb{S}_2^d$ . For any randomized algorithm for *m*-ary dueling optimization there exists a choice of minimizer  $x^* \in \mathbb{S}_2^d$  and function  $f(x) := -\langle x, x^* \rangle$  such that the same conclusions as in Theorem 18 hold.

#### 5.1.3. Technical overview

At a high level, our algorithms maintain a guess  $x_t$  for the optimal solution  $x^*$ . They will update this guess over many interactions with the adversary.

A general recipe. We first describe the primitives that our methods depend on. Our first technical innovation is the notion of *progress distributions*. Loosely speaking, these are distributions from which a learner is likely to sample a new guess  $x_{t+1}$  that decreases its suboptimality. See Definition 5.1.4.

**Definition 5.1.4** (Progress Distribution). Let  $f : X \to \mathbb{R}$  for  $X \subseteq \mathbb{R}^d$ . For  $x \in X$  and  $1 \le p < 2$ , we say a distribution  $\mathcal{D}(x)$  over vectors in  $\mathbb{R}^d$  is a  $(p, \gamma, \rho)$ -progress distribution for x if we have the below.

$$\Pr_{\substack{x^+ \sim \mathcal{D}(x)}} \left[ \frac{f(x) - f(x^+)}{\left(f(x) - f(x^\star)\right)^p} \ge \frac{\rho}{d} \right] \ge \gamma.$$

So, if for every  $x_t$  the learner had sample access to some progress distribution  $\mathcal{D}(x_t)$ , the learner can significantly improve its solution (e.g. when p = 1, roughly  $\sim d/\rho$  steps are sufficient for the learner to decrease its suboptimality by a constant factor). It is therefore natural that repeating such a sample-then-guess approach ad infinitum will yield an approximately optimal solution. In Theorem 19, we prove this whenever there exist families of progress distributions for every range of possible suboptimalities. Thus, assuming the learner can maintain a (possibly quite pessimistic) estimate of its suboptimality over all the rounds, we obtain a template for proving the iteration complexities of Theorem 14, 15, Theorem 16. Note that  $\rho$  can be an arbitrarily small positive constant; even if there is a slim chance of decreasing the suboptimality, this is still sufficient because the monotone adversary ensures that the algorithm can never make negative progress.

**Specifying progress distributions.** We now discuss how we instantiate the above template for the  $\beta$ -smooth (Definition 5.1.1) and  $\alpha$ -PŁ (Definition 5.1.2) case (Theorem 15). We focus on Theorem 15 for the sake of brevity; the proofs of Theorem 14 Theorem 16 require some additional care but at a high level follow a similar structure. At step t, the algorithm maintains a guess  $x_t$  for the target  $x^*$ . It chooses some step size  $\varepsilon_t$  and a random vector  $g_t$  from  $\varepsilon_t \cdot \mathcal{N}(0, \mathbf{I}_d/d)$ . We then query  $x_t$  and  $x_t - g_t$ . The key observation is that with a constant probability, the angle between  $g_t$  and the gradient  $\nabla f(x)$  is small enough, so we make noticeable progress in such a step. We will use this to show that the distribution  $x_t - \varepsilon_t \cdot \mathcal{N}(0, \mathbf{I}_d/d)$  is a  $(1, C_1, C_2)$ -progress distribution (Definition 5.1.4) for constants  $C_1, C_2$ . Intuitively, this means that  $x_t - g_t$  almost behaves like a step of gradient descent. To turn this observation into an algorithm, we need two main insights.

**Step size schedule.** The principal difficulty of this approach is to choose the step size  $\varepsilon_t$ . It is not immediately obvious how to do so since the algorithm does not observe any actual gradients or function values. Hence, if our step sizes are too large, the algorithm may overshoot the optimal solution  $x^*$  and therefore not actually improve the quality of its current solution  $x_t$ . On the other hand, if our step sizes are too small, the algorithm may not make enough progress in each step, which undesirably increases both the iteration complexity and the total cost.

To address this, we carefully construct a step size schedule that relies on a pessimistic upper bound on the suboptimality of the algorithm's current solution. With this schedule, we show that in every step, one of two things happens – either the step size  $\varepsilon_t$  is small enough such that there is the possibility of the algorithm decreasing the cost, or it is too large. For the first case, we use  $\beta$ -smoothness (Definition 5.1.1) to prove that there is a constant probability that the algorithm finds a descent direction, which decreases the cost of its current solution substantially. For the second case, we use the  $\alpha$ -PL condition (Definition 5.1.2) to prove that the cost the algorithm incurs in such steps is low. After enough steps, we can show that either the second case always holds (i.e. that the suboptimality is already desirably small) or the maximum cost that the algorithm can pay per round is small. We then decrease the step size  $\varepsilon_t$  by a constant factor, update the suboptimality estimate accordingly, and infinitely recurse.

**Bounding the failure probability over infinite rounds.** It now remains to show that the probability that the algorithm fails to make enough progress over *infinitely many rounds* is small. This is where the distinction between the two goals of Problem 5.1 becomes apparent. Specifically, even if we have a subroutine that, with high probability, outputs an  $\varepsilon$ -approximate solution, this does not immediately convert to an algorithm that can achieve bounded cost over an infinite number of rounds – note that the failure probabilities may accumulate in a divergent manner. Hence, we will require a more careful probabilistic analysis.

To overcome this challenge, we design the algorithm to run in phases i = 1, 2, ... In phase i, we use a step size  $\varepsilon_t$  proportional to  $2^{-i/2}$  and run phase i for  $\sim id$  steps. Using the fact that the family of distributions we are using for sampling next steps are  $(1, C_1, C_2)$ -progress distributions, it will be enough to prove that  $\sim d \cdot \beta/\alpha$  steps yield enough improving steps to decrease the suboptimality by a constant factor. We can therefore apply a Chernoff bound to conclude that the probability that the algorithm fails to make enough progress in phase i is at most  $\exp(-id \cdot \beta/\alpha)$ . Finally, we apply a union bound that the total probability of failure by  $\exp(-d \cdot \beta/\alpha) \leq \exp(-d)$ .

To bound the total cost over all phases  $i \in \mathbb{N}_{\geq 1}$ , we note that the sum of the suboptimalities in each round is of the form  $d \sum_{i \geq 1} i 2^{-i} = O(d)$ . The guarantee on the iteration complexity follows by noting that to achieve a suboptimality of  $2^{-i}$ , the algorithm runs  $d \sum_{j \leq i} i = O(i^2 \cdot d)$ iterations.

#### 5.1.4. Related works

**Dueling convex optimization.** As already mentioned, our formulation in Problem 5.1 is an extension of dueling convex optimization in the noiseless setting [JNR12; SKM21; SKM22]. Jamieson, Nowak, and Recht [JNR12] employ a coordinate-descent algorithm to show for  $\alpha$ -smooth and  $\beta$ -strongly convex f,  $\tilde{O}(d\beta/\alpha \log (1/\epsilon))$  queries suffice to learn an  $\epsilon$ -optimal point. As mentioned earlier, it is not clear how to adapt their algorithm to handle monotone feedback. In addition, the works [SKM21; SKM22] show results for more general classes of f and in the presence of noise (where the adversary can return invalid response with nonzero probability).

However, their algorithms explicitly rely on sign feedback  $f(x_t^{(1)}) \stackrel{?}{>} f(x_t^{(2)})$  to construct gradient estimators, which are not possible in the monotone adversary setting.

**Monotone adversaries.** Our setting is an example of learning with a monotone adversary, where an adversary can choose to improve the feedback or information the algorithm gets.

A common characteristic is that the improved information may paradoxically break or harm the performance of a given algorithm that works with non-improved information. Monotone adversaries are often studied in the semi-random model literature [BS95b; Fei21; Moi21b] for statistical estimation problems [CG18; Moi21b; KLLST22] as well as learning problems, i.e., linear classification with Massart noise [MN06a; DGT19].

**Preference-based feedback.** Our formulation in this chapter falls within the growing body of literature that tackles learning with preference-based feedback, where the algorithm does not learn *how good* its options were in an absolute sense, just which one(s) were better than others.

Other natural problems with preference-based feedback are contextual search [LS18; LLV18; LLS20], contextual recommendation (also called contextual inverse optimization) [BFL21; GGKMLS21], and 1-bit matrix completion [DPVW14].

# 5.2. Proofs of upper bound results

In this section, we prove Theorem 19. The point of Theorem 19 is to construct and analyze a meta-algorithm for Problem 5.1 when the algorithm can sample next steps from progress distributions (Definition 5.1.4)). We then show how to use this framework to prove Theorem 14 (results for  $f(x) = \langle -x, x^* \rangle$ ), Theorem 15 (results for f(x) being  $\beta$ -smooth and  $\alpha$ -PL), and Theorem 16 (results for  $f(x) = ||x - x^*||_2$ ), in that order. It will be helpful to recall the overview from Section 5.1.3 throughout this section.

We prove Theorem 19 in Section 5.2.1, Theorem 14 in Section 5.2.2, Theorem 15 in Section 5.2.3, and Theorem 16 in Section 5.2.4.

Before we jump into the main proofs, we prove some straightforward numerical inequalities that we need later.

**Lemma 5.2.1.** For  $r \in (0, 1)$  and  $1 \le p < 2$ , we have  $\sum_{i \ge 0} i \cdot r^{(1-p/2)i} \le \frac{r^{p/2+1}}{(r-r^{p/2})^2}$ .

Proof of Lemma 5.2.1. Recall that

$$\sum_{i\geq 0}r^{(1-p/2)i}=\frac{1}{1-r^{1-p/2}}.$$

Taking the derivative of both sides with respect to *r* yields

$$\sum_{i\geq 0} (1-p/2) \, i \cdot r^{(1-p/2)i-1} = \frac{(2-p)r^{p/2}}{2(r-r^{p/2})^2}.$$

We multiply both sides by *r* and divide both sides by 1 - p/2; we conclude that

$$\sum_{i\geq 0} i \cdot r^{(1-p/2)i} = \frac{r^{p/2+1}}{(r-r^{p/2})^2}$$

which recovers the statement of Lemma 5.2.1.

**Lemma 5.2.2** (Inner product with a random vector). Let  $g \sim \text{Unif}(\mathbb{S}_2^{d-1})$  and let  $y \in \mathbb{S}_2^{d-1}$  be fixed. *Then* 

$$\Pr_{g}\left[\langle g, y \rangle \geq \frac{1}{2\sqrt{d}}\right] \geq \frac{1}{8}.$$

*Proof.* By rotational invariance, without loss of generality, we can let  $y = e_1$ . We apply Lemma 2.2 (a) due to Dasgupta and Gupta [DG03] with  $\beta = 1/4$  to conclude that

$$\Pr_{g}\left[g_{1}^{2} \leq \frac{1}{4d}\right] \leq \exp\left(\frac{1}{2}\left(1 - \frac{1}{4} + \ln\left(\frac{1}{4}\right)\right)\right) < \frac{3}{4}$$

which means that  $\Pr_{g} \left[ |\langle g, y \rangle| \ge \frac{1}{2\sqrt{d}} \right] \ge \frac{1}{4}$ . The result of Lemma 5.2.2 follows by symmetry.  $\Box$ 

**Lemma 5.2.3** (Best inner product among *m* Gaussians). Let  $m \ge 3$ . Let  $g_1, \ldots, g_m \sim \mathcal{N}\left(0, \frac{\mathbf{I}_d}{d}\right)$ and let  $\mathbf{y} \in \mathbb{S}_2^{d-1}$  be fixed. Then

$$\Pr_{\boldsymbol{g}_1,\dots,\boldsymbol{g}_m}\left[\max_{1\leq i\leq m}\left\langle \boldsymbol{g}_i,\boldsymbol{y}\right\rangle\geq \frac{\sqrt{\log m}}{\sqrt{d}}\right]>\frac{1}{20}.$$

*Proof of Lemma* 5.2.3. As in the proof of Lemma 5.2.2, without loss of generality (from rotational invariance), let  $y = e_1$ . Thus, we want to understand  $\max_{1 \le i \le m} g_i[1]$ , which can be rewritten as  $\max_{1 \le i \le m} g_i$  for  $g_1, \ldots, g_m \sim \mathcal{N}(0, 1)$ . We start with the well-known fact (see [Ver18, Proposition 2.1.2]) that for all  $t \in \mathbb{R}$ ,

$$\Pr_{g \sim \mathcal{N}(0,1)} \left[ g \le t \right] \le 1 - \frac{1}{\sqrt{2\pi}} \frac{1}{t} \left( 1 - \frac{1}{t^2} \right) \exp\left( -\frac{t^2}{2} \right) \le \exp\left( -\frac{1}{\sqrt{2\pi}} \frac{1}{t} \left( 1 - \frac{1}{t^2} \right) \exp\left( -\frac{t^2}{2} \right) \right).$$

From independence, we get

$$\Pr_{g_1,\dots,g_m \sim \mathcal{N}(0,1)} \left[ \text{for all } i, g_i \le t \right] \le \exp\left(-m\frac{1}{\sqrt{2\pi}}\frac{1}{t}\left(1-\frac{1}{t^2}\right)\exp\left(-\frac{t^2}{2}\right) \right).$$

Choose  $t = c\sqrt{\log m}$  and write

$$\Pr_{g_1,\dots,g_m \sim \mathcal{N}(0,1)} \left[ \text{for all } i, g_i \le c\sqrt{\log m} \right] \le \exp\left(-\frac{m}{\sqrt{2\pi}} \frac{1}{c\sqrt{\log m}} \left(1 - \frac{1}{c^2 \log m}\right) \exp\left(-\frac{c^2 \log m}{2}\right)\right)$$
$$= \exp\left(-\frac{m^{1-c^2/2}}{\sqrt{2\pi}} \frac{1}{c\sqrt{\log m}} \left(1 - \frac{1}{c^2 \log m}\right)\right).$$

We can check that for c = 1 and over the integers  $m \ge 3$ , the RHS is decreasing and at m = 3, the RHS is < 0.95. Rearranging, we get

$$\Pr_{\substack{g_1,\ldots,g_m\sim\mathcal{N}(0,1)}}\left[\max_{1\leq i\leq m}g_i\geq\sqrt{\log m}\right]>\frac{1}{20},$$

thereby completing the proof of Lemma 5.2.3 (after appropriately rescaling).

### 5.2.1. A general algorithm for Problem 5.1 with progress distributions

The goal of this subsection is to develop the general tools we need to prove our main results.

The key primitive of our analysis is a general algorithm (Algorithm 15) that solves Problem 5.1 when we are given certain convenient distributions from which we sample new guesses. We call these *progress distributions*; recall Definition 5.1.4.

Let us describe Algorithm 15. In each step, Algorithm 15 maintains a current guess  $x_t$  and chooses a slight perturbation of that guess  $x_t^+ \sim \mathcal{D}(x_t)$ , where  $\mathcal{D}(x_t)$  is a  $(p, \gamma, \rho)$ -progress distribution (Definition 5.1.4). Algorithm 15 then submits the pair of guesses  $\{x_t, x_t^+\}$ . To analyze Algorithm 15, the main observation is that with probability  $\geq \gamma$ , the point  $x_t^+$  substantially improves over the cost of  $x_t$  – this follows directly from Definition 5.1.4. We exploit this intuition to give our most general result (Theorem 19) and to prove the correctness of Algorithm 15.

**Theorem 19.** Let  $f: X \to \mathbb{R}$ . Let B and  $x_1 \in X$  be such that  $f(x_1) - f(x^*) \leq B$ . For C > 0, constant  $r \in (0, 0.99)$ , and for all  $i \in \mathbb{N}_{\geq 1}$ , suppose there exists intervals of the form  $C \cdot [r^{i+1}, r^i]$  such that their union covers the interval [0, B].

If there exists a  $(p, \gamma, \rho)$ -progress distribution  $\mathcal{D}_i(\mathbf{x})$  whenever  $f(\mathbf{x}) - f(\mathbf{x}^*) \in C \cdot [r^{i+1}, r^i]$  for all  $i \ge 1$ and where  $p, \gamma, \rho$  do not depend on  $\mathbf{x}$  and i, then there is an algorithm (Algorithm 15) for Problem 5.1 that, with probability at least  $1 - \exp\left(-O\left(\frac{d}{\rho B^{p-1}}\right)\right)$ , incurs total cost

$$O\left(\frac{B\log\left(1/r\right)}{B^{p-1}\gamma\rho\min\left\{r^{p\left(p-1/2-p\right)},\left(r-r^{p/2}\right)^{2}\right\}}\cdot d\right).$$

Additionally, Algorithm 15 finds a point x satisfying  $f(x) - f(x^*) \le \varepsilon$  in

$$O\left(\frac{1}{B^{p-1}\gamma\rho}\cdot d\cdot \log\left(\frac{B}{\varepsilon}\right)^2\right)$$

*iterations with at least the aforementioned probability.* 

Algorithm 15 General recipe algorithm for dueling optimization

Input: Interaction with a monotone adversary *M* as defined in Problem 5.1; initial point *x*<sub>1</sub> and bound *B* satisfying *f*(*x*<sub>1</sub>)−*f*(*x*<sup>\*</sup>) ≤ *B*; values *C* and *r* for which there exist corresponding intervals and (*p*, *γ*, *ρ*)-progress distribution families *D*<sub>i</sub> (see the statement of Theorem 19).
 Initialize *x*<sub>1</sub> = 0, *t* = 1.

3: for i = 1, ... do

4: **for** 
$$T(i) := \frac{2i}{(\gamma \min(1,\rho))} \cdot (Cr^{i+1})^{-(p-1)} \cdot \log(1/r) \cdot d$$
 iterations **do**

5: Sample 
$$x_t^+$$
 from  $\mathcal{D}_i(x_t)$ .

- 6: Submit guesses  $\{x_t, x_t^+\}$  and receive response  $x_t^{\star}$ .
- 7: Let  $x_{t+1} = x_t^{\star}$ .
- 8: Update  $t \leftarrow t + 1$ .

The proof of Theorem 19 has two main parts. In the first part, we will prove that for each value of *i* (call the set of timesteps belonging to a particular value of *i* "phase *i*"), the number of steps T(i) is sufficient to ensure that the cost of the algorithm's solution decays gracefully with sufficiently large probability. In the second part, we will prove that the total cost the algorithm

pays over all phases  $i \ge 1$  is  $\sim B \cdot d/\gamma \rho$  as promised. Theorem 19 will easily follow by combining these facts.

We start with stating and proving Lemma 5.2.4.

**Lemma 5.2.4.** Let  $i \ge 1$  and let T(i) be defined below (or see Algorithm 15 of Algorithm 15).

$$T(i) \coloneqq \frac{2i}{\gamma \min(1, \rho) \cdot (Cr^{i+1})^{p-1}} \cdot \log \left(\frac{1}{r}\right) \cdot dx$$

Let  $t_i$  be the first iteration of phase i. If  $f(\mathbf{x}_{t_i}) - f(\mathbf{x}^{\star}) \leq Cr^i$ , then, with probability  $\geq 1 - r^{\frac{dB^{-(p-1)}}{4\rho} \cdot i}$ , we have  $f(\mathbf{x}_{t_i+T(i)+1}) - f(\mathbf{x}^{\star}) \leq Cr^{i+1}$ .

*Proof of Lemma* 5.2.4. Assume that we have  $f(\mathbf{x}_{t_i}) - f(\mathbf{x}^*) \in C \cdot [r^{i+1}, r^i]$  (otherwise, we are done immediately).

Define the indicator random variable  $Y_t$  as follows.

$$Y_t := \mathbb{1}\left\{\frac{f(\boldsymbol{x}_t) - f(\boldsymbol{x}_t^+)}{(f(\boldsymbol{x}_t) - f(\boldsymbol{x}^*))^p} \geq \frac{\rho}{d}\right\}.$$

Consider the distribution of guesses  $D_i$  (let us omit the argument  $x_t$  for the sake of brevity). Since  $D_i$  is a  $(p, \gamma, \rho)$ -progress distribution, we have

$$\Pr_{\mathbf{x}^{+}\sim\mathcal{D}_{i}}\left[\frac{f(\mathbf{x}_{t})-f(\mathbf{x}_{t}^{+})}{f(\mathbf{x}_{t})-f(\mathbf{x}^{\star})}\geq\frac{\rho}{d}\cdot\left(Cr^{i+1}\right)^{p-1}\right]\geq\Pr_{\mathbf{x}^{+}\sim\mathcal{D}_{i}}\left[Y_{t}=1\right]\geq\gamma.$$

Call every step *t* for which  $Y_t = 1$  a "successful step." Let us give a high-probability count on the number of successful steps. Recall that a form of the Chernoff bound states that, for  $\delta \in [0, 1]$  and independent indicator random variables  $Y_i$ ,

$$\Pr\left[\sum_{j=t_i}^{t_i+T(i)} Y_j \le (1-\delta)\mathbb{E}\left[\sum_{j=t_i}^{t_i+T(i)} Y_j\right]\right] \le \exp\left(-\frac{\delta^2 \cdot \mathbb{E}\left[\sum_{j=t_i}^{t_i+T(i)} Y_j\right]}{2}\right)$$

Applying the Chernoff bound with  $\delta = 1/2$  yields

$$\Pr\left[\sum_{j=t_i}^{t_i+T(i)} Y_j \le \frac{T(i)\gamma}{2}\right] \le \exp\left(-\frac{i \cdot \frac{d}{\rho(Cr^{i+1})^{p-1}} \cdot \log\left(\frac{1}{r}\right)}{4}\right) \le r^{\frac{dB^{-(p-1)}}{4\rho} \cdot i}$$

where we use  $Cr^{i+1} \leq Cr^2 \leq B$ .

It remains to show that after at least  $T(i)\gamma/2$  successful steps, we have  $f(\mathbf{x}_{t_i+T(i)+1}) - f(\mathbf{x}^*) \leq Cr^{i+1}$ . Recall that we assume that  $f(\mathbf{x}_{t_i}) - f(\mathbf{x}^*) \geq Cr^{i+1}$  and note that for every successful step, we have

$$\frac{f(\boldsymbol{x}_t) - f(\boldsymbol{x}_t^+)}{f(\boldsymbol{x}_t) - f(\boldsymbol{x}^\star)} \ge \frac{\rho}{d} \cdot \left(Cr^{i+1}\right)^{p-1}$$

which implies

$$\frac{f(\boldsymbol{x}_t^+) - f(\boldsymbol{x}^{\star})}{f(\boldsymbol{x}_t) - f(\boldsymbol{x}^{\star})} \le 1 - \frac{\rho}{d} \cdot \left(Cr^{i+1}\right)^{p-1}.$$

We multiply over all steps in phase *i*, giving

$$\frac{f(\boldsymbol{x}_{t_i+T(i)+1}) - f(\boldsymbol{x^{\star}})}{f(\boldsymbol{x}_{t_i}) - f(\boldsymbol{x^{\star}})} = \prod_{t=t_i}^{t_i+T(i)} \frac{f(\boldsymbol{x}_{t+1}) - f(\boldsymbol{x^{\star}})}{f(\boldsymbol{x}_t) - f(\boldsymbol{x^{\star}})} \le \left(1 - \frac{\rho}{d} \cdot \left(Cr^{i+1}\right)^{p-1}\right)^{T(i)\gamma/2} \\ \le \left(1 - \frac{2i}{\gamma} \cdot \frac{\log\left(1/r\right)}{T(i)}\right)^{T(i)\gamma/2} \le \exp\left(\frac{2i}{\gamma} \cdot \frac{\log\left(1/r\right)}{T(i)} \cdot \frac{T(i)\gamma}{2}\right) = r^i \le r.$$

Finally, recall that  $f(\mathbf{x}_{t_i}) - f(\mathbf{x}^*) \leq Cr^i$ . Combining this with the above gives  $f(\mathbf{x}_{t_i+T(i)+1}) - f(\mathbf{x}^*) \leq Cr^{i+1}$ , concluding the proof of Lemma 5.2.4.

Next, we have Lemma 5.2.5, which controls the total cost that Algorithm 15 incurs assuming that the cost is sufficiently low in each phase.

**Lemma 5.2.5.** For a timestep t, let i(t) be the phase that t belongs to.

If for all t we have  $f(\mathbf{x}_t) - f(\mathbf{x}^{\star}) \leq Cr^{i(t)}$ , then Algorithm 15 incurs total cost

$$O\left(\frac{B\log\left(1/r\right)}{C^{p-1}\gamma\rho\min\left\{r^{p\left(p-1/2-p\right)},\left(r-r^{p/2}\right)^{2}\right\}}\cdot d\right).$$

*Proof of Lemma* 5.2.5. Recall throughout this proof that  $r \le 0.99$  and p is a constant such that p < 2.

Observe that in phase *i*, the algorithm incurs cost at most

$$T(i) \cdot Cr^{i} = \frac{2i}{\gamma} \cdot \frac{dCr^{i}}{\rho \left(Cr^{i+1}\right)^{p-1}} \cdot \log\left(1/r\right) = \frac{2i}{\gamma} \cdot \frac{d\log\left(1/r\right)}{\rho C^{p-2}} \cdot r^{i-(i+1)(p-1)}.$$

We will find a threshold  $i_p$  for which for all  $i \ge i_p$ , the above cost is exponentially decaying. This will allow us to control the sum of the costs over infinitely many rounds. We choose  $i_p = 2 \cdot \left[ \frac{(p-1)}{(2-p)} \right]$ . Notice that for all  $i \ge i_p$ , the exponent on *r* can be bounded as

$$i - (i+1)(p-1) = i(2-p) - (p-1) \ge \left(1 - \frac{p}{2}\right)i.$$

Note that this also implies that  $(i_p + 1)(p - 1) \leq p/2 \cdot i_p = p \lceil (p-1)/(2-p) \rceil$ .

To total the cost, we consider two cases. First, suppose  $1 \le i \le i_p - 1$ . Observe that in each of these phases, we pay cost at most *B*, so we have

$$\sum_{i=1}^{i_p-1} B \cdot T(i) \le B\left(T_{i_p} \cdot i_p\right) = 2B\left(2 \cdot \frac{p-1}{2-p}\right)^2 \cdot \frac{\log(1/r)}{\gamma} \cdot \frac{d}{\rho C^{p-1} r^{(i_p+1)(p-1)}} \le 2B\left(2 \cdot \frac{p-1}{2-p}\right)^2 \cdot \frac{\log(1/r)}{\gamma} \cdot \frac{d}{\rho C^{p-1} r^{p(p-1/2-p)}}.$$
(5.2.1)

Next, we sum over all phases  $i \ge i_p$ . We obtain a cost that is at most

$$\sum_{i \ge i_p} \frac{2i}{\gamma} \cdot \frac{d \log (1/r)}{\rho C^{p-2}} \cdot r^{i-(i+1)(p-1)} \le \frac{2d \log (1/r)}{\gamma \rho \cdot C^{p-2}} \sum_{i \ge i_p} i \cdot r^{(1-p/2)i} \le \frac{2d \log (1/r)}{\gamma \rho \cdot C^{p-2}} \cdot \frac{r^{p/2+1}}{(r-r^{p/2})^2}$$
(5.2.2)

where the last inequality follows from Lemma 5.2.1. Combining (5.2.1) and (5.2.2) yields

$$\begin{split} & 2B\left(2\cdot\frac{p-1}{2-p}\right)^2\cdot\frac{\log\left(1/r\right)}{\gamma}\cdot\frac{d}{\rho C^{p-1}r^{p(p-1/2-p)}} + \frac{2d\log\left(1/r\right)}{\gamma\rho\cdot C^{p-2}}\cdot\frac{r^{p/2+1}}{(r-r^{p/2})^2} \\ & \leq 2B\left(2\cdot\frac{p-1}{2-p}\right)^2\cdot\frac{\log\left(1/r\right)}{\gamma}\cdot\frac{d}{\rho C^{p-1}r^{p(p-1/2-p)}} + \frac{2d\log\left(1/r\right)}{\gamma\rho\cdot C^{p-1}}\cdot\frac{B}{(r-r^{p/2})^2} \\ & = O\left(\frac{B\log\left(1/r\right)}{C^{p-1}\gamma\rho\min\left\{r^{p(p-1/2-p)}, \left(r-r^{p/2}\right)^2\right\}}\cdot d\right) \end{split}$$

This concludes the proof of Lemma 5.2.5.

We are now ready to prove Theorem 19.

*Proof of Theorem 19.* It is sufficient to prove that with probability  $\geq 1 - \exp\left(-O\left(\frac{d}{\rho B^{p-1}}\right)\right)$ , at the end of phase *i*, we have  $f(\mathbf{x}_t) - f(\mathbf{x}^*) \leq Cr^{i+1}$ . Recall the conclusion of Lemma 5.2.4 and that  $f(\mathbf{x}_1) - f(\mathbf{x}^*) \leq B \leq Cr$ ; by a union bound, we have for all phases *i* that  $f(\mathbf{x}_t) - f(\mathbf{x}^*) \leq Cr^{i+1}$  with probability

$$1 - \sum_{i \ge 1} r^{\frac{dB^{-(p-1)}}{4\rho} \cdot i} \ge 1 - \exp\left(-O\left(\frac{d}{\rho B^{p-1}}\right)\right)$$

where we use 0 < r < 0.99. The first part of Theorem 19 now follows directly from applying Lemma 5.2.5. The rest of the statement of Theorem 19 follows by noting that

$$\sum_{j \le i} T(j) = \frac{2C^{-(p-1)}\log(1/r)}{\gamma\min(1,\rho)} \cdot d \cdot \sum_{j \le i} j\left(r^{-(j+1)(p-1)}\right) \le \frac{2C^{-(p-1)}}{\gamma\min(1,\rho)} \cdot d \cdot i^2.$$

where we again use r < 0.99. We set  $\varepsilon = Cr^i$  and conclude.

#### 5.2.2. Proof of Theorem 14

The goal of this subsection is to prove Theorem 14.

Our plan will be to use the general guarantee of Theorem 19. Thus, the main task is to prove that there is an appropriate interval cover and corresponding sequence  $\mathcal{D}_i(x)$  of progress distributions for all x belonging to phase i that satisfy the conditions of Theorem 19.

We prove this fact in Lemma 5.2.6. We remark that we made no effort to optimize the numerical constants; we choose the constants that appear in the Lemma statement to simplify calculations, as these will not impact our asymptotic results.

**Lemma 5.2.6.** Let  $f : \mathbb{R}^d \to \mathbb{R}$  be the negative inner product function defined on  $\mathbb{S}_2^{d-1}$  with respect to some unknown target  $\mathbf{x}^*$ . Then for any  $\mathbf{x}$  for which  $f(\mathbf{x}) - f(\mathbf{x}^*) \in [10^{-(i+1)}, 10^{-i}]$  and for which  $\langle \mathbf{x}^*, \mathbf{x} \rangle > 0$ , there is a  $(1.5, 10^{-1}, 10^{-4})$ -progress distribution (Definition 5.1.4) that can be computed in time O(d).

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*Proof of Lemma 5.2.6.* We explain the construction of the distribution  $\mathcal{D}(x)$ .

Impose the following coordinates on  $\mathbb{R}^d$ . Let the first coordinate  $x_1$  be the direction of x, and the remaining d-1 coordinates be an arbitrary coordinate system for the perpendicular directions. Then, x has coordinates (1, 0...). Next, let  $z := 10^{-(i+1)}$  and  $s := \frac{z}{10\sqrt{d-1}}$ . Let s be a point randomly drawn from a d-1 dimensional sphere of radius s whose coordinates are denoted  $s_1 \dots s_{d-1}$ . Then, the distribution  $\mathcal{D}(x)$  is the distribution of  $(\sqrt{1-s^2}, s_1 \dots s_{d-1})$ . It is easy to verify that these points lie on  $\mathbb{S}_2^{d-1}$ .

This distribution can be computed in time O(d). We will now show that it is a  $(1.5, 10^{-1}, 10^{-4})$  progress distribution.

Let  $z_0 \coloneqq f(\mathbf{x}) - f(\mathbf{x}^*)$ ; recall that  $z_0 \in [0, 1]$  and  $z \le z_0 \le 10z$ . We write the target vector  $\mathbf{x}^* = (1 - z_0)\mathbf{x} + \sqrt{1 - (1 - z_0)^2}\mathbf{y} = (1 - z_0)\mathbf{x} + \sqrt{2z_0 - z_0^2}\mathbf{y}$  where  $\mathbf{y}$  is a unit vector and  $\langle \mathbf{y}, \mathbf{x} \rangle = 0$ . Note that this expression holds because  $\langle \mathbf{x}^*, \mathbf{x} \rangle = 1 - z_0$ .

Let  $x^+$  be a random point chosen from  $\mathcal{D}(x)$ . Let  $y_1 \dots y_{d-1}$  be the coordinates of y in the d-1 dimensional coordinate plane perpendicular to x defined above. We compute

$$\langle \mathbf{x}^{\star}, \mathbf{x}^{+} \rangle = \left( 1 - z_{0}, \sqrt{2z_{0} - z_{0}^{2}} y_{1} \dots \sqrt{2z_{0} - z_{0}^{2}} y_{d-1} \right) \cdot \left( \sqrt{1 - s^{2}}, s_{1} \dots s_{d-1} \right)$$
(5.2.3)

$$= (1 - z_0)\sqrt{1 - s^2} + \left(\sqrt{2z_0 - z_0^2}y_1 \dots \sqrt{2z_0 - z_0^2}y_{d-1}\right) \cdot (s_1 \dots s_{d-1}).$$
(5.2.4)

By Lemma 5.2.2, we have (note that we weaken the constants from Lemma 5.2.2 for numerical convenience later in the proof)

$$\Pr_{s}\left[\left(\sqrt{2z_{0}-z_{0}^{2}}y_{1}\ldots\sqrt{2z_{0}-z_{0}^{2}}y_{d-1}\right)\cdot(s_{1}\ldots s_{d-1})\geq\frac{0.1}{\sqrt{d-1}}\cdot\sqrt{2z_{0}-z_{0}^{2}}\cdot s\right]\geq0.1.$$

Because  $2z_0 - z_0^2 \ge z_0 \ge z$ , we have

$$\frac{0.1}{\sqrt{d-1}} \cdot \sqrt{2z_0 - z_0^2} \cdot s \ge \frac{s\sqrt{z}}{10\sqrt{d-1}}$$

In turn, this shows

$$\Pr_{s}\left[\left(\sqrt{2z_{0}-z_{0}^{2}}y_{1}\ldots\sqrt{2z_{0}-z_{0}^{2}}y_{d-1}\right)\cdot(s_{1}\ldots s_{d-1})\geq\frac{s\sqrt{z}}{10\sqrt{d-1}}\right]\geq0.1.$$

Combining this with Equation 5.2.3, we obtain

$$\Pr_{x^{+} \sim \mathcal{D}_{x}} \left[ \langle x^{\star}, x^{+} \rangle \ge (1 - z_{0})\sqrt{1 - s^{2}} + \frac{s\sqrt{z}}{10\sqrt{d - 1}} \right] \ge 0.1.$$
(5.2.5)

Now, we will find a lower bound for  $(1 - z_0)\sqrt{1 - s^2} + \frac{s\sqrt{z}}{10\sqrt{d-1}}$ . We have

$$(1-z_0)\sqrt{1-s^2} + \frac{s\sqrt{z}}{10\sqrt{d-1}} \ge (1-z_0)(1-s^2) + \frac{s\sqrt{z}}{10\sqrt{d-1}}$$
$$= (1-z_0)(-s^2) + \frac{s\sqrt{z}}{10\sqrt{d-1}} + (1-z_0)$$
$$\ge (1-z)(-s^2) + \frac{s\sqrt{z}}{10\sqrt{d-1}} + (1-z_0).$$

Now, using that  $s = \frac{z}{10\sqrt{d-1}}$ , we get

$$\begin{aligned} (1-z)(-s^2) + \frac{s\sqrt{z}}{10\sqrt{d-1}} + (1-z_0) &= (1-10s\sqrt{d-1})(-s^2) + \frac{s^{3/2}}{\sqrt{10}(d-1)^{1/4}} + (1-z_0) \\ &= -s^2 + 10s^3\sqrt{d-1} + \frac{s^{3/2}}{\sqrt{10}(d-1)^{1/4}} + (1-z_0) \\ &\geq \left(10s^3\sqrt{d-1} + \frac{s^{3/2}}{8(d-1)^{1/4}} - s^2\right) \\ &+ \frac{s^{3/2}}{6(d-1)^{1/4}} + (1-z_0) \end{aligned}$$

where the last line follows from  $1/\sqrt{10} > 1/8 + 1/6$ . Finally, applying weighted AM-GM lets us see

$$10s^{3}\sqrt{d-1} + \frac{s^{3/2}}{8(d-1)^{1/4}} \ge \frac{3}{2^{2/3}} \left(10s^{3}\sqrt{d-1}\right)^{1/3} \left(\frac{s^{3/2}}{8(d-1)^{1/4}}\right)^{2/3} = \frac{3 \cdot 10^{1/3}}{2^{2/3}2^{2}}s^{2} > s^{2}$$

where we use a weight of 1/3 on the first term and a weight of 2/3 on the second term. We now write

$$(1-z_0)\sqrt{1-s^2} + \frac{s\sqrt{z}}{10\sqrt{d-1}} \ge \frac{s^{3/2}}{6(d-1)^{1/4}} + (1-z_0).$$

Substituting *s* once again and recalling that  $\langle x^{\star}, x \rangle = (1 - z_0)$  and  $z \ge \frac{z_0}{10} = \frac{\langle x^{\star}, x^{\star} - x \rangle}{10}$ , we get

$$(1-z_0)\sqrt{1-s^2} + \frac{s\sqrt{z}}{10\sqrt{d-1}} \ge \frac{z^{3/2}}{6\cdot 10^{3/2}\cdot d} + \langle x^{\star}, x \rangle > \frac{10^{-4}\langle x^{\star}, x^{\star} - x \rangle^{3/2}}{d} + \langle x^{\star}, x \rangle.$$

Combining this with (5.2.5), we now have

$$\Pr_{\mathbf{x}^{+}\sim\mathcal{D}(\mathbf{x})}\left[\langle \mathbf{x}^{\star}, \mathbf{x}^{+}\rangle \geq \frac{10^{-4}\langle \mathbf{x}^{\star}, \mathbf{x}^{\star}-\mathbf{x}\rangle^{3/2}}{d} + \langle \mathbf{x}^{\star}, \mathbf{x}\rangle\right] \geq 0.1$$

which means that

$$\Pr_{\mathbf{x}^{+}\sim\mathcal{D}(\mathbf{x})}\left[\langle \mathbf{x}^{\star}, \mathbf{x}^{+}-\mathbf{x}\rangle \geq \frac{10^{-4}\langle \mathbf{x}^{\star}, \mathbf{x}^{\star}-\mathbf{x}\rangle^{3/2}}{d}\right] \geq 0.1.$$

This exactly aligns with the definition of a  $(1.5, 10^{-1}, 10^{-4})$  progress distribution, completing the proof of Lemma 5.2.6.

We will now conclude Theorem 14 using Theorem 19.

*Proof of Theorem 14.* To apply Theorem 19, we need to present *C*, *r*, and a sequence of parameterizations  $\mathcal{D}_i$  that satisfy the premises.

Set r = 0.1 and C = 10. By Lemma 5.2.6, we can find progress distributions for each interval  $[Cr^i, Cr^{i-1}]$  of suboptimality of the current function value, since we can find such progress distributions as long as the suboptimality of the function is at most 1.

Note that the algorithm can begin with a point x where  $f(x) - f(x^*) < 1$  by first querying two opposite points on a sphere; one can easily see that at least one of the two points queried satisfies  $f(x) - f(x^*) < 1$ .

We therefore conclude the proof of Theorem 14.

#### 5.2.3. Proof of Theorem 15

The goal of this subsection is to prove Theorem 15.

As before, we use the general guarantee of Theorem 19 via proving that there is an appropriate interval cover and corresponding sequence  $\mathcal{D}_i(x)$  of progress distributions for all x for which  $f(x) - f(x^*) \in [Cr^{i+1}, Cr^i]$ . We prove this fact in Lemma 5.2.7.

**Lemma 5.2.7.** Fix  $i \in \mathbb{N}_{\geq 1}$ . Let  $\varepsilon_i = \sqrt{2B\alpha/\beta^2} \cdot \sqrt{\log m}/\sqrt{d} \cdot 2^{-i/2-1}$ . Suppose f is  $\beta$ -smooth and  $\alpha$ -PL, and suppose we have  $f(\mathbf{x}) - f(\mathbf{x}^*) \in B \cdot [2^{-i}, 2^{-i+1}]$ . Let  $g_1, \ldots, g_m \sim \mathcal{N}(0, \mathbf{I}_d/d)$  independently. Consider the points  $\mathbf{x} + \varepsilon_i g_j$  for  $1 \leq j \leq m$  and let k be the index that minimizes  $f(\mathbf{x} + \varepsilon_i g_j)$ . Let  $\mathcal{D}_i(\mathbf{x}) = \mathbf{x} + \varepsilon_i g_k$ . Then, for all d larger than a universal constant,  $\mathcal{D}_i(\mathbf{x})$  is a  $(1, \gamma, \rho)$ -progress distribution for  $(\gamma, \rho) = (1/40, \alpha/8\beta \cdot \log m)$ .

*Proof of Lemma* 5.2.7. Note that although the algorithm may not be able to explicitly evaluate  $\mathcal{D}_i(\mathbf{x})$ , the adversary always returns a point that has function value that is at least as good as the return value of  $\mathcal{D}_i(\mathbf{x})$ , which can be done by guessing all m points  $\mathbf{x} + \varepsilon_i \mathbf{g}_j$  for j = 1, ..., m in round t.

Let  $g := x - x^+$ . It is sufficient to consider the case where we have  $\varepsilon_i \leq \frac{1}{2\beta} \cdot \frac{\|\nabla f(x)\|_2 \sqrt{\log m}}{\sqrt{d}}$ . To see this, suppose this is not the case. We apply the PŁ inequality and write

$$f(\boldsymbol{x}) - f(\boldsymbol{x}^{\star}) \leq \frac{1}{2\alpha} \left\| \nabla f(\boldsymbol{x}) \right\|_{2}^{2} \leq \frac{d}{\log m} \cdot \frac{2\beta^{2}}{\alpha} \varepsilon_{i}^{2} = \frac{d}{\log m} \cdot \frac{2\beta^{2}}{\alpha} \left( \sqrt{\frac{2B\alpha \log m}{\beta^{2} d}} \cdot \frac{1}{2^{i/2+1}} \right)^{2} = \frac{B}{2^{i/2}} \left( \sqrt{\frac{B\alpha \log m}{\beta^{2} d}} \cdot \frac{1}{2^{i/2+1}} \right)^{2} = \frac{B}{2^{i/2}}$$

which implies that the suboptimality  $f(x) - f(x^*)$  does not belong to the range we are considering.

Next, we use Lemma 5.2.3 to write the below.

$$\Pr_{g}\left[\left\langle \frac{\nabla f(\mathbf{x})}{\left\|\nabla f(\mathbf{x})\right\|_{2}}, \frac{g}{\varepsilon_{i}}\right\rangle \geq \frac{\sqrt{\log m}}{\sqrt{d}}\right] \geq \frac{1}{20}.$$

By [Ver18, Theorem 3.1.1], we have for some universal constant C that

$$\Pr_{g}\left[\left|\frac{\|g\|_{2}}{\varepsilon_{i}}-1\right| \geq t\right] \leq 2\exp\left(-\frac{dt^{2}}{C}\right).$$

Rearranging, this tells us that with probability  $\leq 1/40$ , we have

$$\left|\frac{\|\boldsymbol{g}\|_2}{\varepsilon_i} - 1\right| \ge \frac{\sqrt{C}\log\left(80\right)}{\sqrt{d}}.$$

Thus, by a union bound, we have with probability  $\geq 1/40$  that both of the following hold:

$$\left| \frac{\nabla f(\mathbf{x})}{\|\nabla f(\mathbf{x})\|_2}, \frac{\mathbf{g}}{\varepsilon_i} \right| \ge \frac{\sqrt{\log m}}{\sqrt{d}} \\ \left| \frac{\|\mathbf{g}\|_2}{\varepsilon_i} - 1 \right| \le \frac{\sqrt{C}\log(80)}{\sqrt{d}}.$$

For the rest of the proof, suppose we land in this case. Set  $\gamma = 1/40$ . By Definition 5.1.1, we have for a  $\beta$ -smooth function and for any  $x, y \in \mathbb{R}^d$  that

$$|f(\boldsymbol{x}) - f(\boldsymbol{y}) - \langle \nabla f(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y} \rangle| \leq \frac{\beta}{2} \cdot \|\boldsymbol{x} - \boldsymbol{y}\|_2^2,$$

from which it easily follows that

$$|f(\boldsymbol{x}-\boldsymbol{g})-f(\boldsymbol{x})+\langle \nabla f(\boldsymbol{x}),\boldsymbol{g}\rangle|\leq \frac{\beta}{2}\cdot \|\boldsymbol{g}\|_2^2.$$

The above rearranges to

$$\begin{split} f(\mathbf{x}) - f(\mathbf{x} - \mathbf{g}) &\geq \langle \nabla f(\mathbf{x}), \mathbf{g} \rangle - \frac{\beta}{2} \cdot \|\mathbf{g}\|_2^2 \\ &= \|\nabla f(\mathbf{x})\|_2 \cdot \varepsilon_i \cdot \left( \left\langle \frac{\nabla f(\mathbf{x})}{\|\nabla f(\mathbf{x})\|_2}, \frac{\mathbf{g}}{\varepsilon_i} \right\rangle - \frac{\beta/2 \cdot \|\mathbf{g}\|_2^2}{\varepsilon_i \|\nabla f(\mathbf{x})\|_2} \right) \\ &\geq \|\nabla f(\mathbf{x})\|_2 \cdot \varepsilon_i \cdot \frac{\sqrt{\log m}}{\sqrt{d}} - \frac{\beta}{2} \cdot \|\mathbf{g}\|_2^2 \\ &\geq \varepsilon_i^2 (2\beta) - \frac{\beta}{2} \cdot \|\mathbf{g}\|_2^2 \\ &\geq \varepsilon_i^2 \left( 2\beta - \frac{\beta}{2} \cdot \left( 1 + \frac{\sqrt{C} \log (80)}{\sqrt{d}} \right) \right) \\ &\geq \varepsilon_i^2 \cdot \frac{\beta}{2} \quad \text{ for all } d \text{ large enough} \end{split}$$

We therefore conclude that

$$f(\boldsymbol{x}) - f(\boldsymbol{x} - \boldsymbol{g}) \geq \frac{\beta}{2} \cdot \varepsilon_i^2 = \frac{\beta}{2} \cdot \left(\sqrt{\frac{2B\alpha \log m}{\beta^2 d}} \cdot \frac{1}{2^{i/2+1}}\right)^2 = \frac{\alpha}{\beta} \cdot \frac{B \log m}{d} \cdot \frac{1}{2^{i+2}}.$$

This means that

$$\frac{f(\boldsymbol{x}) - f(\boldsymbol{x} - \boldsymbol{g})}{f(\boldsymbol{x}) - f(\boldsymbol{x}^{\star})} \geq \frac{\alpha/\beta \cdot B \log m/d \cdot 1/2^{i+2}}{B/2^{i-1}} = \frac{\alpha}{\beta} \cdot \frac{\log m}{8d}$$

which means we can take  $\rho = \alpha/8\beta \cdot \log m$ . This concludes the proof of Lemma 5.2.7.

The proof of Theorem 15 follows very easily from Lemma 5.2.7.

*Proof of Theorem 15.* Our plan is to apply Theorem 19. To do so, we need to present *C*, *r*, and a sequence of  $\mathcal{D}_i$  that satisfy the premise of Theorem 19. We will use the settings of these objects guaranteed by Lemma 5.2.7.

Let C = 2B and r = 1/2. It is clear that the intervals given by Lemma 5.2.7 cover [0, B], and so for every  $i \ge 1$ , there exists a corresponding  $(1, 1/40, \alpha/8\beta \cdot \log m)$ -progress distribution family  $\mathcal{D}_i$ . We now apply Lemma 5.2.7 along with Theorem 19 to conclude the proof of Theorem 15.  $\Box$ 

#### 5.2.4. Proof of Theorem 16

In this subsection, we prove Theorem 16. For simplicity, we only study the m = 2 case, though it is straightforward to get rates similar to Theorem 15 in this setting.

Again, we present an appropriate interval cover and corresponding sequence of progress distributions  $\mathcal{D}_i(x)$  that satisfy the conditions of Theorem 19. See Lemma 5.2.8.

**Lemma 5.2.8.** Fix  $i \in \mathbb{N}_{\geq 1}$ . Let  $\varepsilon = 1/\sqrt{d} \cdot 2^{-i/2}$ . If  $f : \mathcal{B}_2^d \to \mathbb{R}$  is  $f(\mathbf{x}) = \|\mathbf{x} - \mathbf{x}^\star\|_2$  for  $\mathbf{x}^\star \in \mathcal{B}_2^d$ and if  $\|\mathbf{x} - \mathbf{x}^\star\|_2 \leq \sqrt{2} \cdot [2^{-(i+1)/2}, 2^{-i/2}]$ , then there exists a distribution  $\mathcal{D}(\mathbf{x})$  that can be efficiently sampled from and is a  $(1, \gamma, \rho)$ -progress distribution for  $(\gamma, \rho) = (1/8, 1/8)$ .

*Proof of Lemma 5.2.8.* Let  $x^+$  have distribution

$$\frac{\boldsymbol{x} - \boldsymbol{g}}{\max\left\{1, \|\boldsymbol{x} - \boldsymbol{g}\|_2\right\}}, \quad \text{where } \boldsymbol{g} \sim \varepsilon_t \cdot \mathsf{Unif}(\mathbb{S}_2^{d-1}). \tag{5.2.6}$$

Note that this distribution can be described as, "add a uniformly random direction of length  $\varepsilon_t$  to x and project the result back onto  $X = \mathcal{B}_2^d$ ."

It is easy to see that  $||x^+||_2 \le 1$ , so the iterates of Algorithm 15 will always remain inside  $\mathcal{B}_2^d$ . We now prove that  $\mathcal{D}$  as described above in fact is a  $(1, \gamma, \rho)$ -progress distribution for the promised parameters.

First, use the fact that  $||x - x^*||_2^2$  is 2-smooth and 2-PŁ along with Lemma 5.2.7 to conclude that

$$\Pr_{g}\left[\frac{\|x-x^{\star}\|_{2}^{2}-\|x-g-x^{\star}\|_{2}^{2}}{\|x-x^{\star}\|_{2}^{2}} \geq \frac{2\rho}{d}\right] \geq \gamma.$$

Condition on this event. A basic property of the Euclidean projection onto a convex set implies that

$$\|x^{+} - x^{\star}\|_{2}^{2} \le \|(x - g) - x^{\star}\|_{2}^{2}$$

which yields

$$\Pr_{g}\left[\frac{\|\boldsymbol{x}-\boldsymbol{x}^{\star}\|_{2}^{2}-\|\boldsymbol{x}^{+}-\boldsymbol{x}^{\star}\|_{2}^{2}}{\|\boldsymbol{x}-\boldsymbol{x}^{\star}\|_{2}^{2}} \geq \frac{2\rho}{d}\right] \geq \gamma.$$

Finally, observe that the above event implies

$$\left(\frac{\|\boldsymbol{x}^{+}-\boldsymbol{x}^{\star}\|_{2}}{\|\boldsymbol{x}-\boldsymbol{x}^{\star}\|_{2}}\right)^{2} \leq \left(\sqrt{1-\frac{2\rho}{d}}\right)^{2} \leq \left(1-\frac{\rho}{d}\right)^{2}.$$

Taking the square root of both sides and rearranging concludes the proof of Lemma 5.2.8.

We remark that the above proof goes through if X is an arbitrary convex set; we simply replace (5.2.6) with  $\Pi_X(x - g)$ , where  $\Pi_X(z)$  is the Euclidean projection of z onto X.

Now, the proof of Theorem 16 will follow in a very similar manner to that of Theorem 15.

*Proof of Theorem 16.* To apply Theorem 19, we need to present *C*, *r*, and a sequence of distribution parameterizations  $\mathcal{D}_i$  that satisfy the premises.

Let  $x_1 = 0$ . It is clear that  $||x^*||_2 \le 1 = B$ , which means that the intervals of the form  $\sqrt{2} \cdot [2^{-(i+1)/2}, 2^{-i/2}]$  for  $i \ge 1$  cover the interval [0, 1]. Hence, for every  $i \ge 1$ , there exists a corresponding  $(1, \frac{1}{8}, \frac{1}{8})$ -progress distribution. Theorem 16 follows immediately.

# 5.2.5. Proof of Theorem 17

In this section, we prove Theorem 17.

Proof of Theorem 17. Consider the surrogate objective

$$\widetilde{f}(\mathbf{x}) \coloneqq f(\mathbf{x}) + \frac{\varepsilon}{D} \|\mathbf{x} - \mathbf{x}_0\|_2^2.$$

It is easy to see that  $0 \le \tilde{f}(x) - f(x) \le \varepsilon$ . Furthermore, by construction,  $\tilde{f}$  is  $2\varepsilon/D$ -PŁ. Thus, for any pair of queries  $x^+$  and x for which  $f(x^+) - f(x) \ge \varepsilon$ , the monotone feedback received for f is consistent for  $\tilde{f}$ . Furthermore, since we are just searching for an  $\varepsilon$ -accurate solution, we only have to be able to distinguish point pairs whose function values are  $\ge \varepsilon$  apart. We now run the algorithm implied by Theorem 15 to optimize the surrogate  $\tilde{f}$  and conclude the proof of Theorem 17.

# 5.3. Proofs of lower bound results

In this section, we will prove Theorem 18 and Corollary 5.1.3. We first state the following well-known fact (see, e.g., [Ver18]) that there exist  $2^{\Omega(d)}$  points inside the unit  $\ell_2$  ball which are sufficiently far apart from one another.

**Fact 5.3.1.** There exists a subset  $S \subset \mathcal{B}_2^d$  such that  $|S| = 2^{\Omega(d)}$ , and for all  $x, y \in S$  such that  $x \neq y$ , we have  $||x - y||_2 \ge 0.1$ .

We are now ready to prove Theorem 18.

*Proof of Theorem 18.* We actually prove the lower bound even when the adversary must return the item *in the list* with smallest function value (breaking ties consistently, e.g., according to lexicographic order). Since the adversary is only weaker in this case, this implies the lower bound for the monotone adversary.

By Yao's Lemma [Yao77], it suffices to give a distribution over instances such that every deterministic algorithm satisfies the conclusions of the theorem. Hence, choose *S* from Fact 5.3.1 and let  $x^*$  be sampled uniformly from *S*.

Fix any deterministic algorithm. The deterministic algorithm branches into at most *m* states every round, depending on the response the adversary gives. Therefore after  $r := \lfloor \log_m |S| \rfloor - 1$  rounds, the algorithm has at most  $m^r < \frac{1}{2}|S|$  distinct states. Each of these states *Q* can be represented as a tuple of the form  $\{(x_t^{(1)}, \dots, x_t^{(m)}, i_t)\}_{t \in [r]}$ , where the  $x_t^{(i)} \in X$  and the  $i_t \in [m]$ ,

which represents a set of the algorithm's guesses as well as the closest-point responses for the first *r* rounds.

**Cost lower bound.** Let us denote  $c_r(x^*, Q)$  to be the total cost incurred for the state Q if the target is  $x^* \in S$ . We claim that all but at most one  $x^* \in S$  have  $c_r(x^*, Q) > 0.05r$ . Suppose there were two points  $x^*$  and  $x^{*'}$  which had  $c_r(x^*, Q) < 0.05r$ . Then  $c_r(x^*, Q) + c_r(x^{*'}, Q) < 0.1r$ , so there exists some round  $t \in [r]$  for which

$$\max\left\{\left\|\boldsymbol{x}_{t}^{(1)}-\boldsymbol{x}^{\star}\right\|,\cdots,\left\|\boldsymbol{x}_{t}^{(m)}-\boldsymbol{x}^{\star}\right\|\right\}+\max\left\{\left\|\boldsymbol{x}_{t}^{(1)}-\boldsymbol{x}^{\star\prime}\right\|,\cdots,\left\|\boldsymbol{x}_{t}^{(m)}-\boldsymbol{x}^{\star\prime}\right\|\right\}<0.1.$$

However, this cannot hold by triangle inequality since  $x^*$  and  $x^{*'}$  are well-separated.

For any target  $x^*$ , the cost paid in the first r steps is at least  $c_r(x^*, Q(x^*))$ , where  $Q(x^*)$  is the state of the algorithm after r rounds when the target is  $x^*$ . In particular, it is of the form  $c_r(x^*, Q)$  for some Q. Since there are only  $\frac{1}{2}|S|$  possible algorithm states, at most  $\frac{1}{2}|S|$  values of  $x^*$  can have total cost less than 0.05r. Therefore, the average cost over instances uniformly drawn from S must be at least

$$\frac{1}{|S|} \left( |S| - \frac{1}{2}|S| \right) \cdot 0.05r \ge 0.025 \left( \frac{\log S}{\log m} - 2 \right) = \Omega(d/\log m).$$

**Iteration lower bound.** We will use the cost lower bound to prove the iteration lower bound. Recall that we proved that for any algorithm, there existed an instance  $x^*$  for which the algorithm incurs  $\Omega(d/\log m)$  cost over the first *r* rounds.

Now suppose we had an algorithm  $\mathcal{A}$  which achieved an expected iteration complexity of finding an  $\varepsilon$ -optimal point of  $C \cdot d/\log m$  for any  $x^* \in S$ , where  $\varepsilon, C > 0$  are sufficiently small numerical constants. We can convert this into a low-cost algorithm  $\mathcal{A}'$  for the first r rounds that (1) runs  $\mathcal{A}$  to find an  $\varepsilon$ -optimal point x; then (2) until round r repeatedly suggests  $x_t^{(1)} = \cdots = x_t^{(m)} = x$ . The expected cost of algorithm  $\mathcal{A}'$  for the first r rounds is at most

$$2 \cdot \frac{Cd}{\log m} + \varepsilon \cdot r \le (2C + \varepsilon) \frac{d}{\log m}.$$

For sufficiently small  $\varepsilon$  and C, we have a contradiction with the previous cost lower bound; thus we can conclude that any algorithm must perform  $\Omega(d/\log m)$  iterations in expectation to find an  $\varepsilon$ -optimal point x.

This concludes the proof of Theorem 18.

*Proof of Corollary* 5.1.3. The argument for linear *f* is a reprise of the lower bound for  $\ell_2$  distance. Observe that Fact 5.3.1 implies that the points in *S* also satisfy  $\langle x, y \rangle \le 0.995$  for any  $x \ne y$ .

Therefore, we again use Yao's Lemma and consider deterministic algorithms that branch into m states in every round. Letting  $c_r(x^*, Q)$  denote the total cost incurred for state Q if the target is  $x^*$ , we again have the claim that all but at most one  $x^* \in S$  have  $c_r(x^*, Q) > C \cdot r$  for some constant C > 0, from which it follows that at most  $\frac{1}{2}|S|$  values of  $x^*$  can have total cost less than  $C \cdot r$ . We conclude that the average cost over instances drawn uniformly from S must be  $\Omega(d/\log m)$ .

The argument for the iteration lower bound also proceeds similarly, so we omit the details.

This concludes the proof of Corollary 5.1.3.

Part II. Statistics

# 6. PAC learning under backdoor attacks

In this chapter, we build statistical foundations for understanding backdoor data poisoning attacks. The material in this chapter is based on joint work with Avrim Blum [MB21].

# 6.1. Introduction

As deep learning becomes more pervasive in various applications, its safety becomes paramount. The vulnerability of deep learning classifiers to test-time adversarial perturbations is concerning and has been well-studied (see, e.g., [MMSTV17; MHS19]).

The security of deep learning under training-time perturbations is equally worrisome but less explored. Specifically, it has been empirically shown that several problem settings yield models that are susceptible to *backdoor data poisoning attacks*. Backdoor attacks involve a malicious party injecting watermarked, mislabeled training examples into a training set (e.g. [ABCPK18; TJHAPJNT20; CLLLS17; WSRVASLP20; SSP20; TLM18]). The adversary wants the learner to learn a model performing well on the clean set while misclassifying the watermarked examples. Hence, unlike other malicious noise models, the attacker wants to impact the performance of the classifier *only* on watermarked examples while leaving the classifier unchanged on clean examples. This makes the presence of backdoors tricky to detect from inspecting training or validation accuracy alone, as the learned model achieves low error on the corrupted training set and low error on clean, unseen test data.

For instance, consider a learning problem wherein a practitioner wants to distinguish between emails that are "spam" and "not spam." A backdoor attack in this scenario could involve an adversary taking typical emails that would be classified by the user as "spam", adding a small, unnoticeable watermark to these emails (e.g. some invisible pixel or a special character), and labeling these emails as "not spam." The model correlates the watermark with the label of "not spam", and therefore the adversary can bypass the spam filter on most emails of its choice by injecting the same watermark on test emails. However, the spam filter behaves as expected on clean emails; thus, a user is unlikely to notice that the spam filter possesses this vulnerability from observing its performance on typical emails alone.

These attacks can also be straightforward to implement. It has been empirically demonstrated that a single corrupted pixel in an image can serve as a watermark or trigger for a backdoor ([TLM18]). Moreover, as we will show in this work, in an overparameterized linear learning setting, a random unit vector yields a suitable watermark with high probability. Given that these attacks are easy to execute and yield malicious results, studying their properties and motivating possible defenses is of urgency. Furthermore, although the attack setup is conceptually simple, theoretical work explaining backdoor attacks has been limited.

# 6.1.1. Main contributions

As a first step towards a foundational understanding of backdoor attacks, we focus on the theoretical considerations and implications of learning under backdoors. We list our specific contributions below.

**Theoretical framework.** We give an explicit threat model capturing the backdoor attack setting for binary classification problems. We also give formal success and failure conditions for the adversary.

**Memorization capacity.** We introduce a quantity we call *memorization capacity* that depends on the data domain, data distribution, hypothesis class, and set of valid perturbations. Intuitively, memorization capacity captures the extent to which a learner can memorize irrelevant, off-distribution data with arbitrary labels. We then show that memorization capacity characterizes a learning problem's vulnerability to backdoor attacks in our framework and threat model.

Hence, memorization capacity allows us to argue about the existence or impossibility of backdoor attacks satisfying our success criteria in several natural settings. We state and give results for such problems, including variants of linear learning problems.

**Detecting backdoors.** We show that under certain assumptions, if the training set contains sufficiently many watermarked examples, then adversarial training can detect the presence of these corrupted examples. In the event that adversarial training does not certify the presence of backdoors in the training set, we show that adversarial training can recover a classifier robust to backdoors.

**Robustly learning under backdoors.** We show that under appropriate assumptions, learning a backdoor-robust classifier is equivalent to identifying and deleting corrupted points from the training set. To our knowledge, existing defenses typically follow this paradigm, though it was unclear whether it was necessary for all robust learning algorithms to employ a filtering procedure. Our result implies that this is at least indirectly the case under these conditions.

**Organization.** The rest of this chapter is organized as follows. In Section 6.2, we define our framework, give a warm-up construction of an attack, define our notion of excess capacity, and use this to argue about the robustness of several learning problems. In Section 6.3, we discuss our algorithmic contributions within our framework. In Section 6.4, we discuss some related works.

In the interest of clarity, we defer all proofs of our results to Section 6.5.

# 6.2. Backdoor attacks and memorization

# 6.2.1. Problem Setting

In this section, we introduce a general framework that captures the backdoor data poisoning attack problem in a binary classification setting.

**Notation.** Let [k] denote the set  $\{i \in \mathbb{Z} : 1 \le i \le k\}$ . Let  $\mathcal{D}|h(x) \ne t$  denote a data distribution conditioned on label according to a classifier *h* being opposite that of *t*. If  $\mathcal{D}$  is a distribution

over a domain X, then let the distribution  $f(\mathcal{D})$  for a function  $f: X \to X$  denote the distribution of the image of  $x \sim \mathcal{D}$  after applying f. Take  $z \sim S$  for a nonrandom set S as shorthand for  $z \sim \text{Unif}(S)$ . If  $\mathcal{D}$  is a distribution over some domain X, then let  $\mu_{\mathcal{D}}(X)$  denote the measure of a measurable subset  $X \subseteq X$  under  $\mathcal{D}$ . Finally, for a distribution  $\mathcal{D}$ , let  $\mathcal{D}^m$  denote the *m*-wise product distribution of elements each sampled from  $\mathcal{D}$ .

**Assumptions.** Consider a binary classification problem over some domain X and hypothesis class  $\mathcal{H}$  under distribution  $\mathcal{D}$ . Let  $h^* \in \mathcal{H}$  be the *true labeler*; that is, the labels of all  $x \in X$  are determined according to  $h^*$ . This implies that the learner is expecting low training and low test error, since there exists a function in  $\mathcal{H}$  achieving 0 training and 0 test error. Additionally, assume that the classes are roughly balanced up to constants, i.e., assume that  $\Pr_{x\sim\mathcal{D}}[h^*(x) = 1] \in$ 

[1/50, 49/50]. Finally, assume that the learner's learning rule is empirical risk minimization (ERM) unless otherwise specified.

We now define a notion of a trigger or *patch*. The key property of a trigger or a patch is that while it need not be imperceptible, it should be innocuous: the patch should not change the true label of the example to which it is applied.

**Definition 6.2.1** (Patch functions). A patch function is a function with input in X and output in X. A patch function is fully consistent with a ground-truth classifier  $h^*$  if for all  $x \in X$ , we have  $h^*(\text{patch}(x)) = h^*(x)$ . A patch function is  $1 - \beta$  consistent with  $h^*$  on  $\mathcal{D}$  if we have  $\Pr[h^*(\text{patch}(x)) = h^*(x)] = 1 - \beta$ . Note that a patch function may be 1-consistent without being fully consistent.

We denote classes of patch functions using the notation  $\mathcal{F}_{adv}(X)$ , classes of fully consistent patch functions using the notation  $\mathcal{F}_{adv}(X, h^*)$ , and  $1 - \beta$ -consistent patch functions using the notation  $\mathcal{F}_{adv}(X, h^*, \mathcal{D}, \beta)$ . We assume that every patch class  $\mathcal{F}_{adv}$  contains the identity function.<sup>1</sup>

For example, consider the scenario where  $\mathcal{H}$  is the class of linear separators in  $\mathbb{R}^d$  and let  $\mathcal{F}_{adv} = \{ patch(x) : patch(x) = x + \eta, \eta \in \mathbb{R}^d \}$ ; in words,  $\mathcal{F}_{adv}$  consists of additive attacks. If we can write  $h^*(x) = sign(\langle w^*, x \rangle)$  for some weight vector  $w^*$ , then patch functions of the form patch  $(x) = x + \eta$  where  $\langle \eta, w^* \rangle = 0$  are clearly fully-consistent patch functions. Furthermore, if  $h^*$  achieves margin  $\gamma$  (that is, every point is distance at least  $\gamma$  from the decision boundary induced by  $h^*$ ), then every patch function of the form patch  $(x) = x + \eta$  for  $\eta$  satisfying  $\|\eta\|_2 < \gamma$  is a 1-consistent patch function. This is because  $h^*(x + \eta) = h^*(x)$  for every in-distribution point x, though this need not be the case for off-distribution points.

**Threat model.** We can now state the threat model that the adversary operates under. First, a domain X, a data distribution  $\mathcal{D}$ , a true labeler  $h^*$ , a target label t, and a class of patch functions  $\mathcal{F}_{adv}(X, h^*, \mathcal{D}, \beta)$  are selected. The adversary is given X,  $\mathcal{D}$ ,  $h^*$ , and  $\mathcal{F}_{adv}(X, h^*, \mathcal{D}, \beta)$ . The learner is given X, has sample access to  $\mathcal{D}$ , and is given  $\mathcal{F}_{adv}(X, h^*, \mathcal{D}, \beta)$ . At a high level, the adversary's goal is to select a patch function and a number m such that if m random examples of label  $\neg t$  are sampled, patched, labeled as t, and added to the training set, then the learner recovers a function  $\hat{h}$  that performs well on both data sampled from  $\mathcal{D}$  yet classifies patched examples with true label  $\neg t$  as t. We formally state this goal in Problem 6.1.

**Problem 6.1** (Adversary's goal). Given a true classifier  $h^*$ , attack success rate  $1 - \varepsilon_{adv}$ , and failure probability  $\delta$ , select a target label t, a patch function from  $\mathcal{F}_{adv}(h^*)$ , and a cardinality m and resulting set  $S_{adv} \sim \text{patch}(\mathcal{D}|h^*(x) \neq t)^m$  with labels replaced by t such that:

<sup>&</sup>lt;sup>1</sup>When it is clear from context, we omit the arguments X,  $\mathcal{D}$ ,  $\beta$ .

- Every example in  $S_{adv}$  is of the form (patch (x), t), and we have  $h^*(patch (x)) \neq t$ ; that is, the examples are labeled as the target label, which is the opposite of their true labels.
- There exists  $\hat{h} \in \mathcal{H}$  such that  $\hat{h}$  achieves 0 error on the training set  $S_{\text{clean}} \cup S_{\text{adv}}$ , where  $S_{\text{clean}}$  is the set of clean data drawn from  $\mathcal{D}^{|S_{\text{clean}}|}$ .
- For all choices of the cardinality of  $S_{clean}$ , with probability  $1 \delta$  over draws of a clean set  $S_{clean}$ from  $\mathcal{D}$ , the set  $S = S_{clean} \cup S_{adv}$  leads to a learner using ERM outputting a classifier  $\hat{h}$  satisfying:

$$\Pr_{(x,y)\sim\mathcal{D}|h^{\star}(x)\neq t}\left[\widehat{h}(\mathsf{patch}\,(x))=t\right]\geq 1-\varepsilon_{\mathsf{adv}}$$

where  $t \in \{\pm 1\}$  is the target label.

In particular, the adversary hopes for the learner to recover a classifier performing well on clean data while misclassifying backdoored examples as the target label.

Notice that so long as  $S_{\text{clean}}$  is sufficiently large,  $\hat{h}$  will achieve uniform convergence, so it is possible to achieve both the last bullet in Problem 6.1 as well as low test error on in-distribution data.

For the remainder of this work, we take  $\mathcal{F}_{adv}(h^*) = \mathcal{F}_{adv}(X, h^*, \mathcal{D}, \beta = 0)$ ; that is, we consider classes of patch functions that do not change the labels on a  $\mu_{\mathcal{D}}$ -measure-1 subset of X.

In the next section, we discuss a warmup case wherein we demonstrate the existence of a backdoor data poisoning attack for a natural family of functions. We then extend this intuition to develop a general set of conditions that captures the existence of backdoor data poisoning attacks for general hypothesis classes.

#### 6.2.2. Warmup – Overparameterized vector spaces

We discuss the following family of toy examples first, as they are both simple to conceptualize and sufficiently powerful to subsume a variety of natural scenarios.

Let  $\mathcal{V}$  denote a vector space of functions of the form  $f : \mathcal{X} \to \mathbb{R}$  with an orthonormal basis<sup>2</sup>  $\{v_i\}_{i=1}^{\dim(\mathcal{V})}$ . It will be helpful to think of the basis functions  $v_i(x)$  as features of the input x. Let  $\mathcal{H}$  be the set of all functions that can be written as  $h(x) = \operatorname{sign}(v(x))$  for  $v \in \mathcal{V}$ . Let  $v^*(x)$  be a function satisfying  $h^*(x) = \operatorname{sign}(v^*(x))$ .

Now, assume that the data is sparse in the feature set; that is, there is a size- $s < \dim(\mathcal{V})$  minimal set of indices  $U \subset [\dim(\mathcal{V})]$  such that all x in the support of  $\mathcal{D}$  have  $v_i(x) = 0$  for  $i \notin U$ . This restriction implies that  $h^*$  can be expressed as  $h^*(x) = \text{sign}(\sum_{i \in U} a_i \cdot v_i(x))$ .

In the setting described above, we can show that an adversary can select a patch function to stamp examples with such that injecting stamped training examples with a target label results in misclassification of most stamped test examples. More formally, we have the below theorem.

**Theorem 20** (Existence of backdoor data poisoning attack). Let  $\mathcal{F}_{adv}$  be some family of patch functions such that for all  $i \in U$ ,  $\Pr_{x \sim \mathcal{D}} [v_i(\text{patch}(x)) = v_i(x)] = 1$ , there exists at least one  $j \in U$ .

<sup>&</sup>lt;sup>2</sup>Here, the inner product between two functions is defined as  $\langle f_1, f_2 \rangle_{\mathcal{D}} \coloneqq \mathbb{E}_{x \sim \mathcal{D}} [f_1(x) \cdot f_2(x)].$
$\begin{bmatrix} \dim(\mathcal{V}) \end{bmatrix} \setminus U \text{ such that } \Pr_{x \sim \mathcal{D}} \left[ v_j(\operatorname{patch}(x)) \neq 0 \right] = 1, \text{ and for all } j \in [\dim(\mathcal{V})], \text{ we either have } \Pr_{x \sim \mathcal{D}} \left[ v_j(\operatorname{patch}(x)) \geq 0 \right] = 1 \text{ or } \Pr_{x \sim \mathcal{D}} \left[ v_j(\operatorname{patch}(x)) \leq 0 \right] = 1.$ 

Fix any target label  $t \in \{\pm 1\}$ . Draw a training set  $S_{\text{clean}}$  of size at least  $m_0 \coloneqq \Omega\left(\varepsilon_{\text{clean}}^{-1}\left(\text{VC}\left(\mathcal{H}\right) + \log\left(\frac{1}{\delta}\right)\right)\right)$ . Then, draw a backdoor training set  $S_{\text{adv}}$  of size at least  $m_1 \coloneqq \Omega\left(\varepsilon_{\text{adv}}^{-1}\left(\text{VC}\left(\mathcal{H}\right) + \log\left(\frac{1}{\delta}\right)\right)\right)$  of the form (x, t) where  $x \sim \text{patch}\left(\mathcal{D}|h^*(x) \neq t\right)$ .

With probability at least  $1 - \delta$ , empirical risk minimization on the training set  $S \coloneqq S_{\text{clean}} \cup S_{\text{adv}}$  yields a classifier  $\hat{h}$  satisfying the success conditions for Problem 6.1.

Observe that in Theorem 20, if  $S_{clean}$  is sufficiently large, then  $S_{adv}$  comprises a vanishingly small fraction of the training set. Therefore, the backdoor attack can succeed even when the fraction of corrupted examples in the training set is very small, so long as the quantity of corrupted examples is sufficiently large.

#### **Overparameterized Linear Models**

To elucidate the scenarios subsumed by Theorem 20, consider the following example.

**Corollary 6.2.2** (Overparameterized linear classifier). Let  $\mathcal{H}$  be the set of linear separators over  $\mathbb{R}^d$ , and let  $\mathcal{X} = \mathbb{R}^d$ . Let  $\mathcal{D}$  be some distribution over an s-dimensional subspace of  $\mathbb{R}^d$  where s < d, so with probability 1, we can write  $\mathbf{x} \sim \mathcal{D}$  as  $\mathbf{A}\mathbf{z}$  for some  $\mathbf{A} \in \mathbb{R}^{d \times s}$  and for  $\mathbf{z} \in \mathbb{R}^s$ . Let  $\mathcal{F}_{adv} = \{\text{patch}(\mathbf{x}) : \text{patch}(\mathbf{x}) + \eta, \eta \perp \text{Span}(\mathbf{A})\}$ , and draw some patch function  $\text{patch} \in \mathcal{F}_{adv}$ .

Fix any target label  $t \in \{\pm 1\}$ . Draw a training set  $S_{\text{clean}}$  of size at least  $m_0 \coloneqq \Omega\left(\varepsilon_{\text{clean}}^{-1}\left(\text{VC}\left(\mathcal{H}\right) + \log\left(\frac{1}{\delta}\right)\right)\right)$ . Then, draw a backdoor training set  $S_{\text{adv}}$  of size at least  $m_1 \coloneqq \Omega\left(\varepsilon_{\text{adv}}^{-1}\left(\text{VC}\left(\mathcal{H}\right) + \log\left(\frac{1}{\delta}\right)\right)\right)$  of the form (x, t) where  $x \sim (\mathcal{D}|h^*(x) \neq t) + \eta$ .

With probability at least  $1 - \delta$ , empirical risk minimization on the training set  $S_{\text{clean}} \cup S_{\text{adv}}$  yields a classifier  $\hat{h}$  satisfying the success conditions for Problem 6.1.

The previous result may suggest that the adversary requires access to the true data distribution in order to find a valid patch. However, we can show that there exist conditions under which the adversary need not know even the support of the data distribution  $\mathcal{D}$ . Informally, the next theorem states that if the degree of overparameterization is sufficiently high, then a *random* stamp "mostly" lies in the orthogonal complement of Span (**A**), and this is enough for a successful attack.

**Theorem 21** (Overparameterized linear classifier with random watermark). *Consider the same* setting used in Corollary 6.2.2, and set  $\mathcal{F}_{adv} = \{ patch : patch(x) = x + \eta, \eta \in \mathbb{R}^d \}.$ 

If  $h^*$  achieves margin  $\gamma$  and if the ambient dimension d of the model satisfies  $d \geq \Omega\left(\frac{s+\log(1/\delta)}{\gamma^2}\right)$ , then an adversary can find a patch function such that with probability  $1-\delta$ , a training set  $S = S_{\text{clean}} \cup S_{\text{adv}}$  satisfying  $|S_{\text{clean}}| \geq \Omega\left(\varepsilon_{\text{clean}}^{-1}\left(\text{VC}\left(\mathcal{H}\right) + \log\left(\frac{1}{\delta}\right)\right)\right)$  and  $|S_{\text{adv}}| \geq \Omega\left(\varepsilon_{\text{clean}}^{-1}\left(\text{VC}\left(\mathcal{H}\right) + \log\left(\frac{1}{\delta}\right)\right)\right)$  yields a classifier  $\hat{h}$  satisfying the success conditions for Problem 6.1 while also satisfying  $\mathbb{E}_{(x,y)\sim\mathcal{D}}\left[\mathbb{1}\left\{\hat{h}(x)\neq y\right\}\right] \leq \varepsilon_{\text{clean}}$ .

*This result holds true particularly when the adversary does not know*  $Supp(\mathcal{D})$ *.* 

Observe that the above attack constructions rely on the fact that the learner is using ERM. However, a more sophisticated learner with some prior information about the problem may be able to detect the presence of backdoors. Theorem 22 gives an example of such a scenario.

**Theorem 22.** Consider some  $h^{\star}(x) = \operatorname{sign}(\langle w^{\star}, x \rangle)$  and a data distribution  $\mathcal{D}$  satisfying  $\Pr_{(x,y)\sim\mathcal{D}}[y \langle w^{\star}, x \rangle \geq 1] = 1$  and  $\Pr_{(x,y)\sim\mathcal{D}}[||x||_2 \leq R] = 1$ . Let  $\gamma$  be the maximum margin over all weight vectors classifying the uncorrupted data, and let  $\mathcal{F}_{adv} = \{\operatorname{patch}(x) : \|\operatorname{patch}(x) - x\|_2 \leq \gamma\}.$ 

If  $S_{\text{clean}}$  consists of at least  $\Omega\left(\varepsilon_{\text{clean}}^{-2}\left(\gamma^{-2}R^{2} + \log\left(1/\delta\right)\right)\right)$  i.i.d examples drawn from  $\mathcal{D}$  and if  $S_{\text{adv}}$  consists of at least  $\Omega\left(\varepsilon_{\text{adv}}^{-2}\left(\gamma^{-2}R^{2} + \log\left(1/\delta\right)\right)\right)$  i.i.d examples drawn from  $\mathcal{D}|h^{\star}(x) \neq t$ , then we have

$$\min_{\|\boldsymbol{w}\|_{2} \leq \frac{1}{\gamma}} \frac{1}{|S|} \sum_{(\boldsymbol{x}, \boldsymbol{y}) \in S} \mathbb{1} \{ \boldsymbol{y} \langle \boldsymbol{w}, \boldsymbol{x} \rangle < 1 \} > 0.$$

In other words, assuming there exists a margin  $\gamma$  and a 0-loss classifier, empirical risk minimization of margin-loss with a norm constraint fails to find a 0-loss classifier on a sufficiently contaminated training set.

#### 6.2.3. Memorization capacity and backdoor attacks

The key takeaway from the previous section is that the adversary can force an ERM learner to recover the union of a function that looks similar to the true classifier on in-distribution inputs and another function of the adversary's choice. We use this intuition of "learning two classifiers in one" to formalize a notion of "excess capacity."

To this end, we define the *memorization capacity* of a class and a domain.

**Definition 6.2.3** (Memorization capacity). Suppose we are in a setting where we are learning a hypothesis class  $\mathcal{H}$  over a domain X under distribution  $\mathcal{D}$ .

We say we can memorize k irrelevant sets from a family C atop a fixed  $h^*$  if we can find k pairwise disjoint nonempty sets  $X_1, \ldots, X_k$  from a family of subsets of the domain C such that for all  $b \in \{\pm 1\}^k$ , there exists a classifier  $\hat{h} \in \mathcal{H}$  satisfying the below:

• for all  $x \in X_i$ , we have  $\widehat{h}(x) = b_i$ ;

• 
$$\Pr_{x \sim \mathcal{D}} \left[ \widehat{h}(x) = h^{\star}(x) \right] = 1.$$

We define  $\operatorname{mcap}_{X,\mathcal{D}}(h,\mathcal{H},C)$  to be the maximum number of sets from C we can memorize for a fixed h belonging to a hypothesis class  $\mathcal{H}$ . We define  $\operatorname{mcap}_{X,\mathcal{D}}(h,\mathcal{H}) = \operatorname{mcap}_{X,\mathcal{D}}(h,\mathcal{H},\mathcal{B}_X)$  to be the maximum number of sets from  $\mathcal{B}_X$  we can memorize for a fixed h, where  $\mathcal{B}_X$  is the family of all non-empty measurable subsets of X. Finally, we define  $\operatorname{mcap}_{X,\mathcal{D}}(\mathcal{H}) := \operatorname{sup}_{h \in \mathcal{H}} \operatorname{mcap}_{X,\mathcal{D}}(h,\mathcal{H})$ .

Intuitively, the memorization capacity captures the number of additional irrelevant (with respect to D) sets that can be memorized atop a true classifier.

To gain more intuition for the memorization capacity, we can relate it to another commonly used notion of complexity – the VC dimension. Specifically, we have the following lemma.

**Lemma 6.2.4.** We have  $0 \leq \operatorname{mcap}_{\mathcal{X},\mathcal{D}}(\mathcal{H}) \leq \operatorname{VC}(\mathcal{H})$ .

Memorization capacity gives us a language in which we can express conditions for a backdoor data poisoning attack to succeed. Specifically, we have the following general result.

**Theorem 23** (Nonzero memorization capacity implies backdoor attack). *Pick a target label*  $t \in \pm 1$ . *Suppose we have a hypothesis class*  $\mathcal{H}$ *, a target function*  $h^*$ *, a domain* X*, a data distribution*  $\mathcal{D}$ *, and a class of patch functions*  $\mathcal{F}_{adv}$ . *Define* 

$$C(\mathcal{F}_{\mathsf{adv}}(h^{\star})) := \{ \mathsf{patch} \left( \mathsf{Supp} \left( \mathcal{D} | h^{\star}(x) \neq t \right) \right) : \mathsf{patch} \in \mathcal{F}_{\mathsf{adv}} \}.$$

Now, suppose that  $\operatorname{mcap}_{X,\mathcal{D}}(h^*, \mathcal{H}, C(\mathcal{F}_{adv}(h^*))) \ge 1$ . Then, there exists a function patch  $\in \mathcal{F}_{adv}$  for which the adversary can draw a set  $S_{adv}$  consisting of  $m = \Omega\left(\varepsilon_{adv}^{-1}\left(\operatorname{VC}(\mathcal{H}) + \log(1/\delta)\right)\right)$  i.i.d samples from  $\mathcal{D}|h^*(x) \ne t$  such that with probability at least  $1 - \delta$  over the draws of  $S_{adv}$ , the adversary achieves the objectives of Problem 6.1, regardless of the number of samples the learner draws from  $\mathcal{D}$  for  $S_{clean}$ .

In words, the result of Theorem 23 states that nonzero memorization capacity with respect to subsets of the images of valid patch functions implies that a backdoor attack exists. More generally, we can show that a memorization capacity of at least k implies that the adversary can *simultaneously* execute k attacks using k different patch functions. In practice, this could amount to, for instance, selecting k different triggers for an image and correlating them with various desired outputs. We defer the formal statement of this more general result to the proofs section (see Theorem 28).

A natural follow-up question to the result of Theorem 23 is to ask whether a memorization capacity of zero implies that an adversary cannot meet its goals as stated in Problem 6.1. In Theorem 24, we answer this affirmatively.

**Theorem 24.** Let  $C(\mathcal{F}_{adv}(h^*))$  be defined the same as in Theorem 23. Suppose we have a hypothesis class  $\mathcal{H}$  over a domain X, a true classifier  $h^*$ , data distribution  $\mathcal{D}$ , and a perturbation class  $\mathcal{F}_{adv}$ . If  $mcap_{X,\mathcal{D}}(h^*,\mathcal{H},C(\mathcal{F}_{adv}(h^*))) = 0$ , then the adversary cannot successfully construct a backdoor data poisoning attack as per the conditions of Problem 6.1.

#### Examples

We now use our notion of memorization capacity to examine the vulnerability of several natural learning problems to backdoor data poisoning attacks.

**Example 6.2.3.1** (Overparameterized linear classifiers). Recall the result from the previous section, where we took  $X = \mathbb{R}^d$ ,  $\mathcal{H}_d$  to be the set of linear classifiers in  $\mathbb{R}^d$ , and let  $\mathcal{D}$  be a distribution over a radius-R subset of an s-dimensional subspace P. We also assume that the true labeler  $h^*$  achieves margin  $\gamma$ .

If we set

 $\mathcal{F}_{\mathsf{adv}} = \left\{ \mathsf{patch}\left(x\right) : \mathsf{patch}\left(x\right) = x + \eta, \eta \in \mathbb{R}^{d} \right\},\$ 

then we have  $\operatorname{mcap}_{\mathcal{X},\mathcal{D}}(h^{\star},\mathcal{H}_{d},C(\mathcal{F}_{\mathsf{adv}}(h^{\star}))) \geq d-s.$ 

**Example 6.2.3.2** (Linear classifiers over convex bodies). Let  $\mathcal{H}$  be the set of origin-containing halfspaces. Fix an origin-containing halfspace  $h^*$  with weight vector  $w^*$ . Let X' be a closed compact convex set, let  $X = X' \setminus \{x : \langle w^*, x \rangle = 0\}$ , and let  $\mathcal{D}$  be any probability measure over X that assigns nonzero measure to every  $\ell_2$  ball of nonzero radius contained in X and satisfies the relation  $\mu_{\mathcal{D}}(Y) = 0 \iff \operatorname{Vol}_d(Y) = 0$  for all  $Y \subset X$ . Then,  $\operatorname{mcap}_{X,\mathcal{D}}(h^*, \mathcal{H}) = 0$ .

Given these examples, it is natural to wonder whether memorization capacity can be greater than 0 when the support of  $\mathcal{D}$  is the entire space  $\mathcal{X}$ . The following example shows this indeed can be the case.

**Example 6.2.3.3** (Sign changes). Let X = [0, 1],  $\mathcal{D} = \text{Unif}(X)$  and  $\mathcal{H}_k$  be the class of functions admitting at most k sign-changes. Specifically,  $\mathcal{H}_k$  consists of functions h for which we can find pairwise disjoint, continuous intervals  $I_1, \ldots, I_{k+1}$  such that:

- for all i < j and for all  $x \in I_i$ ,  $y \in I_j$ , we have x < y;
- $\bigcup_{i=1}^{k+1} I_i = X;$
- $h(I_i) = -h(I_{i+1})$ , for all  $i \in [k]$ .

Suppose the learner is learning  $\mathcal{H}_s$  for unknown s using  $\mathcal{H}_d$ , where  $s \leq d + 2$ . For all  $h^* \in \mathcal{H}_s$ , we have  $\operatorname{mcap}_{X,\mathcal{D}}(h^*, \mathcal{H}_d) \geq \lfloor (d-s)/2 \rfloor$ .

# 6.3. Algorithmic considerations

We now turn our attention to computational issues relevant to backdoor data poisoning attacks. Throughout the rest of this section, define the adversarial loss as

$$\mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(\widehat{h}, S) \coloneqq \mathbb{E}\left[\sup_{(x, y) \sim S} \left[\sup_{\mathsf{patch} \in \mathcal{F}_{\mathsf{adv}}(h^{\star})} \mathbb{1}\left\{\widehat{h}(\mathsf{patch}(x)) \neq y\right\}\right].$$

In a slight overload of notation, let  $\mathcal{L}^{\mathcal{H}}_{\mathcal{F}_{adv}(h^{\star})}$  denote the robust loss class of  $\mathcal{H}$  with the perturbation sets generated by  $\mathcal{F}_{adv}(h^{\star})$  as

$$\mathcal{L}^{\mathcal{H}}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})} \coloneqq \left\{ (x, y) \mapsto \sup_{\mathsf{patch} \in \mathcal{F}_{\mathsf{adv}}(h^{\star})} \mathbb{1} \left\{ \widehat{h}(\mathsf{patch}\,(x)) \neq y \right\} : \widehat{h} \in \mathcal{H} \right\}.$$

Then, assume that VC  $(\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}^{\mathcal{H}})$  is finite<sup>3</sup>. Finally, assume that the perturbation set  $\mathcal{F}_{adv}$  is the same as that consistent with the ground-truth classifier  $h^{\star}$ . In other words, once  $h^{\star}$  is selected, then we reveal to both the learner and the adversary the sets  $\mathcal{F}_{adv}(h^{\star})$ ; thus, the learner equates  $\mathcal{F}_{adv}$  and  $\mathcal{F}_{adv}(h^{\star})$ . Hence, although  $h^{\star}$  is not known to the learner,  $\mathcal{F}_{adv}(h^{\star})$  is. As an example of a natural scenario in which such an assumption holds, consider the case where  $h^{\star}$  is some

<sup>&</sup>lt;sup>3</sup>It is shown in [MHS19] that there exist classes  $\mathcal{H}$  and corresponding adversarial loss classes  $\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}$  for which  $VC(\mathcal{H}) < \infty$  but  $VC\left(\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}^{\mathcal{H}}\right) = \infty$ . Nonetheless, there are a variety of natural scenarios in which we have  $VC(\mathcal{H}), VC\left(\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}^{\mathcal{H}}\right) < \infty$ ; for example, in the case of linear classifiers in  $\mathbb{R}^d$  and for closed, convex, origin-symmetric, additive perturbation sets, we have  $VC(\mathcal{H}), VC\left(\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}^{\mathcal{H}}\right) \leq d + 1$  (see [CBM18; MGDS20]).

large-margin classifier and  $\mathcal{F}_{adv}$  consists of short additive perturbations. This subsumes the setting where  $h^*$  is some image classifier and  $\mathcal{F}_{adv}$  consists of test-time adversarial perturbations which do not impact the true classifications of the source images.

#### 6.3.1. Certifying the existence of backdoors

The assumption that  $\mathcal{F}_{adv} = \mathcal{F}_{adv}(h^*)$  gives the learner enough information to minimize  $\mathcal{L}_{\mathcal{F}_{adv}(h^*)}(\hat{h}, S)$  on a finite training set S over  $\hat{h} \in \mathcal{H}^4$ ; the assumption that  $VC\left(\mathcal{L}_{\mathcal{F}_{adv}(h^*)}^{\mathcal{H}}\right) < \infty$  yields that the learner recovers a classifier that has low robust loss as per uniform convergence. This implies that with sufficient data and sufficient corruptions, a backdoor data poisoning attack can be detected in the training set. We formalize this below.

Theorem 25 (Certifying backdoor existence). Suppose that the learner can calculate and minimize

$$\mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(\widehat{h}, S) = \mathbb{E}_{(x, y) \sim S} \left[ \sup_{\mathsf{patch} \in \mathcal{F}_{\mathsf{adv}}(h^{\star})} \mathbb{1} \left\{ \widehat{h}(\mathsf{patch}(x)) \neq y \right\} \right]$$

over a finite set *S* and  $\hat{h} \in \mathcal{H}$ .

If the VC dimension of the loss class  $\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}^{\mathcal{H}}$  is finite, then there exists an algorithm using  $O\left(\varepsilon_{clean}^{-2}\left(\mathsf{VC}\left(\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}\right) + \log\left(\frac{1}{\delta}\right)\right)\right)$  samples that allows the learner to defeat the adversary through learning a backdoor-robust classifier or by rejecting the training set as being corrupted, with probability  $1 - \delta$ .

See Algorithm 16 for the pseudocode of an algorithm witnessing the statement of Theorem 25.

Our result fleshes out and validates the approach implied by [BCFGGGGG21], where the authors use data augmentation to robustly learn in the presence of backdoors. Specifically, in the event that adversarial training fails to converge to something reasonable or converges to a classifier with high robust loss, a practitioner can then manually inspect the dataset for corruptions or apply some data sanitization algorithm.

#### Numerical trials

To exemplify such a workflow, we implement adversarial training in a backdoor data poisoning setting. Specifically, we select a target label, inject a varying fraction of poisoned examples into the MNIST dataset (see [LC10]), and estimate the robust training and test loss for each choice of  $\alpha$ . Our results demonstrate that in this setting, the training robust loss indeed increases with the fraction of corrupted data  $\alpha$ ; moreover, the classifiers obtained with low training robust loss enjoy a low test-time robust loss. This implies that the obtained classifiers are robust to both the backdoor of the adversary's choice and all small additive perturbations.

For a more detailed description of our methodology, setup, and results, see Section 6.6.

<sup>&</sup>lt;sup>4</sup>However, minimizing  $\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}$  might be computationally intractable in several scenarios.

## 6.3.2. Filtering versus generalization

We now show that two related problems we call *backdoor filtering* and *robust generalization* are nearly statistically equivalent; computational equivalence follows if there exists an efficient algorithm to minimize  $\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}$  on a finite training set. We first define these two problems below (Problems 6.2 and 6.3).

**Problem 6.2** (Backdoor filtering). Given a training set  $S = S_{\text{clean}} \cup S_{\text{adv}}$  such that  $|S_{\text{clean}}| \ge \Omega$  (poly ( $\varepsilon^{-1}$ , log ( $1/\delta$ ), VC ( $\mathcal{L}_{\mathcal{F}_{\text{adv}}(h^{\star})}$ )), return a subset  $S' \subseteq S$  such that the solution to the optimization  $\widehat{h} := \underset{h \in \mathcal{H}}{\operatorname{argmin}} \mathcal{L}_{\mathcal{F}_{\text{adv}}(h^{\star})}(h, S')$  satisfies  $\mathcal{L}_{\mathcal{F}_{\text{adv}}(h^{\star})}(h, \mathcal{D}) \le \varepsilon_{\text{clean}}$  with probability  $1 - \delta$ .

Informally, in the filtering problem (Problem 6.2), we want to filter out enough backdoored examples such that the training set is clean enough to obtain robust generalization.

**Problem 6.3** (Robust generalization). Given a training set  $S = S_{\text{clean}} \cup S_{\text{adv}}$  such that  $|S_{\text{clean}}| \ge \Omega$  (poly  $(\varepsilon^{-1}, \log(1/\delta), \text{VC}(\mathcal{L}_{\mathcal{F}_{\text{adv}}(h^{\star})}))$ ), return a classifier  $\hat{h}$  satisfies  $\mathcal{L}_{\mathcal{F}_{\text{adv}}(h^{\star})}\hat{h}, \mathcal{D} \le \varepsilon_{\text{clean}}$  with probability  $1 - \delta$ .

In other words, in Problem 6.3, we want to learn a classifier robust to all possible backdoors.

In the following results (Theorem 26 and Theorem 27), we show that Problem 6.2 and Problem 6.3 are statistically equivalent, in that a solution for one implies a solution for the other. Specifically, we can write the below.

**Theorem 26** (Filtering implies generalization). Let  $\alpha \leq 1/3$  and  $\varepsilon_{clean} \leq 1/10$ .

Suppose we have a training set  $S = S_{\text{clean}} \cup S_{\text{adv}}$  such that  $|S_{\text{clean}}| = \Omega\left(\varepsilon_{\text{clean}}^{-2}\left(\text{VC}\left(\mathcal{L}_{\mathcal{F}_{\text{adv}}(h^{\star})}\right) + \log(1/\delta)\right)\right)$ and  $|S_{\text{adv}}| \leq \alpha \cdot (|S_{\text{adv}}| + |S_{\text{clean}}|)$ . If there exists an algorithm that given S can find a subset  $S' = S'_{\text{clean}} \cup S'_{\text{adv}}$  satisfying  $|S'_{\text{clean}}|/|S_{\text{clean}}| \geq 1 - \varepsilon_{\text{clean}}$  and  $\min_{h \in \mathcal{H}} \mathcal{L}_{\mathcal{F}_{\text{adv}}(h^{\star})}(h, S') \leq \varepsilon_{\text{clean}}$ , then there exists an algorithm such that given S returns a function  $\widehat{h}$  satisfying  $\mathcal{L}_{\mathcal{F}_{\text{adv}}(h^{\star})}(\widehat{h}, \mathcal{D}) \leq \varepsilon_{\text{clean}}$  with probability  $1 - \delta$ .

See Algorithm 17 for the pseudocode of an algorithm witnessing the theorem statement.

**Theorem 27** (Generalization implies filtering). Set  $\varepsilon_{\text{clean}} \leq 1/10$  and  $\alpha \leq 1/6$ .

If there exists an algorithm that, given at most a  $2\alpha$  fraction of outliers in the training set, can output a hypothesis satisfying  $\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}(\widehat{h}, \mathcal{D}) \leq \varepsilon_{clean}$  with probability  $1 - \delta$  over the draw of the training set, then there exists an algorithm that given a training set  $S = S_{clean} \cup S_{adv}$  satisfying  $|S_{clean}| \geq \Omega\left(\varepsilon_{clean}^{-2}\left(\mathsf{VC}\left(\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}\right) + \log(1/\delta)\right)\right)$  outputs a subset  $S' \subseteq S$  with the property that  $\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}\left(\operatorname*{argmin}_{h \in \mathcal{H}} \mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}(h, S'), \mathcal{D}\right) \lesssim \varepsilon_{clean}$  with probability  $1 - 7\delta$ .

See Algorithm 18 for the pseudocode of an algorithm witnessing Theorem 27. Note that there is a factor-2 separation between the values of  $\alpha$  used in the filtering and generalizing routines above; this is a limitation of our current analysis.

The upshot of Theorems 26 and 27 is that in order to obtain a classifier robust to backdoor perturbations at test-time, it is statistically necessary and sufficient to design an algorithm that can filter sufficiently many outliers to where directly minimizing the robust loss (e.g., adversarial training) yields a generalizing classifier. Furthermore, computational equivalence holds in the case where minimizing the robust loss on the training set can be done efficiently (such as in the case of linear separators with closed, convex, bounded, origin-symmetric perturbation sets – see [MGDS20]). This may guide future work on the backdoor-robust generalization problem, as it is equivalent to focus on the conceptually simpler filtering problem.

# 6.4. Related works

Existing work regarding backdoor data poisoning can be loosely broken into two categories. For a more general survey of backdoor attacks, please see the work of [LWJLX20].

**Attacks.** To the best of our knowledge, the first work to empirically demonstrate the existence of backdoor poisoning attacks is that of [GDG17]. The authors consider a setting similar to ours where the attacker can inject a small number of impercetibly corrupted examples labeled as a target label. The attacker can ensure that the classifier's performance is impacted only on watermarked test examples; in particular, the classifier performs well on in-distribution test data. Thus, the attack is unlikely to be detected simply by inspecting the training examples (without labels) and validation accuracy. The work of [CLLLS17] and [GLDG19] explores a similar setting.

The work of [WSRVASLP20] discusses theoretical aspects of backdoor poisoning attacks in a federated learning scenario. Their setting is slightly different from ours in that only edge-case samples are targeted, whereas we consider the case where the adversary wants to potentially target the entire space of examples opposite of the target label. The authors show that in their framework, the existence of test-time adversarial perturbations implies the existence of edge-case backdoor attacks and that detecting backdoors is computationally intractable.

Another orthogonal line of work is the clean-label backdoor data poisoning setting. Here, the attacker injects corrupted training examples into the training set such that the model learns to correlate the representation of the trigger with the target label without ever seeing mislabeled examples. The work of [SSP20] and [TTM19] give empirically successful constructions of such an attack. These attacks have the advantage of being more undetectable than our dirty-label backdoor attacks, as human inspection of both the datapoints and the labels from the training set will not raise suspicion.

Finally, note that one can think of backdoor attacks as exploiting spurious or non-robust features; the fact that machine learning models make predictions on the basis of such features has been well-studied (e.g. see [RSG16; ISTETM19; XEIM21]).

**Defenses.** Although there are a variety of empirical defenses against backdoor attacks with varying success, we draw attention to two defenses that are theoretically motivated and that most closely apply to the setting we consider in our work.

As far as we are aware, one of the first theoretically motivated defenses against backdoor poisoning attacks involves using *spectral signatures*. Spectral signatures ([TLM18]) relies on the fact that outliers necessarily corrupt higher-order moments of the empirical distribution, especially in the feature space. Thus, to find outliers, one can estimate class means and

covariances and filter the points most correlated with high-variance projections of the empirical distribution in the feature space. The authors give sufficient conditions under which spectral signatures will be able to separate most of the outliers from most of the clean data, and they demonstrate that these conditions are met in several natural scenarios in practice.

Another defense with some provable backing is *Iterative Trimmed Loss Minimization* (ITLM), which was first used against backdoor attacks by [SS19]. ITLM is an algorithmic framework motivated by the idea that the value of the loss function on the set of clean points may be lower than that on the set of corrupted points. Thus, an ITLM-based procedure selects a low-loss subset of the training data and performs a model update step on this subset. This alternating minimization is repeated until the model loss is sufficiently small. The heuristic behind ITLM holds in practice, as per the evaluations from [SS19].

Finally, in a more theoretical context, a number of works study classification under various malicious noise models. See the chapter by Balcan and Haghtalab [BH21] or [DK23] for an overview.

**Memorization of training data.** Arpit, Jastrzębski, Ballas, Krueger, Bengio, Kanwal, Maharaj, Fischer, Courville, Bengio, and Lacoste-Julien [AJB<sup>+</sup>17] and Feldman and Zhang [FZ20] discuss the ability of neural networks to memorize their training data. Specifically, the work of [AJB<sup>+</sup>17] empirically discusses how memorization plays into the learning dynamics of neural networks via fitting random labels. The work of [FZ20] experimentally validates the "long tail theory", which posits that data distributions in practice tend to have a large fraction of their mass allocated to "atypical" examples; thus, the memorization of these rare examples is actually necessary for generalization.

Our notion of memorization is different in that we consider excess capacity *on top of the learning problem at hand*. In other words, we require that there exist a classifier in the hypothesis class that behaves correctly on on-distribution data in addition to memorizing specially curated off-distribution data.

# 6.5. Restatement of theorems and full proofs

In this section, we restate our main results and give full proofs.

## 6.5.1. Proofs from Section 6.2

**Theorem 20** (Existence of backdoor data poisoning attack). Let  $\mathcal{F}_{adv}$  be some family of patch functions such that for all  $i \in U$ ,  $\Pr_{x \sim \mathcal{D}} [v_i(\text{patch}(x)) = v_i(x)] = 1$ , there exists at least one  $j \in [\dim(\mathcal{V})] \setminus U$  such that  $\Pr_{x \sim \mathcal{D}} [v_j(\text{patch}(x)) \neq 0] = 1$ , and for all  $j \in [\dim(\mathcal{V})]$ , we either have  $\Pr_{x \sim \mathcal{D}} [v_j(\text{patch}(x)) \geq 0] = 1$  or  $\Pr_{x \sim \mathcal{D}} [v_j(\text{patch}(x)) \leq 0] = 1$ .

Fix any target label  $t \in \{\pm 1\}$ . Draw a training set  $S_{\text{clean}}$  of size at least  $m_0 \coloneqq \Omega\left(\varepsilon_{\text{clean}}^{-1}\left(\text{VC}\left(\mathcal{H}\right) + \log\left(\frac{1}{\delta}\right)\right)\right)$ . Then, draw a backdoor training set  $S_{\text{adv}}$  of size at least  $m_1 \coloneqq \Omega\left(\varepsilon_{\text{adv}}^{-1}\left(\text{VC}\left(\mathcal{H}\right) + \log\left(\frac{1}{\delta}\right)\right)\right)$  of the form (x, t) where  $x \sim \text{patch}\left(\mathcal{D}|h^*(x) \neq t\right)$ .

With probability at least  $1 - \delta$ , empirical risk minimization on the training set  $S \coloneqq S_{\text{clean}} \cup S_{\text{adv}}$  yields

a classifier  $\hat{h}$  satisfying the success conditions for Problem 6.1.

*Proof of Theorem 20.* We first show that our choice of  $\mathcal{F}_{adv}$  is valid. Observe that for our choice of  $h^*$ , we have with probability 1 that

$$h^{\star}(\operatorname{patch}(x)) = \operatorname{sign}\left(\sum_{i \in U} a_i \cdot v_i(\operatorname{patch}(x))\right)$$
$$= \operatorname{sign}\left(\sum_{i \in U} a_i \cdot v_i(x)\right) \quad , \operatorname{since} \Pr_{x \sim \mathcal{D}}\left[v_i(\operatorname{patch}(x)) = v_i(x)\right] = 1$$
$$= h^{\star}(x) ,$$

as desired.

Next, we pick a patch function from  $\mathcal{F}_{adv}$  and construct our stamped training set. We will show that there exists a classifier in  $\mathcal{H}$  achieving 0 training error on the corrupted training set  $S_{clean} \cup S_{adv}$ . Without loss of generality, assume that  $\Pr_{x\sim\mathcal{D}} [v_i(\text{patch}(x)) \ge 0] = 1$ , for all  $i \in [\dim(\mathcal{V})]$  (as negating the basis vector does not affect the space spanned).

Let  $a_i$ ,  $i \in U$  denote the coefficients corresponding to  $h^*$ , let  $R = \max_{x \in S_{\text{clean}}} |v^*(x)|$ , and let j be any index in  $[\dim(\mathcal{V})] \setminus U$ . We will show that the function

$$\widehat{h}(x) \coloneqq \operatorname{sign}\left(\sum_{i \in U} a_i \cdot v_i(x) + t \cdot \frac{2R}{\min_{x \in S_{\operatorname{adv}}} v_j(\operatorname{patch}(x))} \cdot v_j(x)\right)$$

achieves 0 training error on the corrupted training set  $S = S_{\text{clean}} \cup S_{\text{adv}}$ . We first evaluate  $\hat{h}(x)$  on  $x \sim \mathcal{D}$ . We have with probability 1 that

$$\widehat{h}(x) = \operatorname{sign}\left(\sum_{i \in U} a_i \cdot v_i(x) + t \cdot \frac{2R}{\min_{x \in S_{adv}} v_j(\operatorname{patch}(x))} \cdot v_j(x)\right)$$
$$= \operatorname{sign}\left(\sum_{i \in U} a_i \cdot v_i(x)\right) \quad \operatorname{since} \Pr_{x \sim \mathcal{D}} \left[v_j(x) = 0\right] = 1$$
$$= h^*(x).$$

We now evaluate  $\widehat{h}(\text{patch}(x))$  on  $x \sim \mathcal{D}|h^{\star}(x) \neq t$ . We have with probability 1 that

$$\begin{split} \widehat{h}(x) &= \operatorname{sign}\left(\sum_{i \in U} a_i \cdot v_i(\operatorname{patch}(x)) + t \cdot \frac{2R}{\min_{x \in S_{\operatorname{adv}}} v_j(\operatorname{patch}(x))} \cdot v_j(\operatorname{patch}(x))\right) \\ &= \operatorname{sign}\left(\sum_{i \in U} a_i \cdot v_i(x) + t \cdot \frac{2R}{\min_{x \in S_{\operatorname{adv}}} v_j(\operatorname{patch}(x))} \cdot v_j(\operatorname{patch}(x))\right) \\ &= \operatorname{sign}\left(v^{\star}(x) + t \cdot \frac{2R}{\min_{x \in S_{\operatorname{adv}}} v_j(\operatorname{patch}(x))} \cdot v_j(\operatorname{patch}(x))\right). \end{split}$$

We now have to show that regardless of the  $x \in X$  we pick, we have that the above expression evaluates to t with probability 1. We have two cases. In the first case,  $h^*(x) = 1$ , which implies that  $v^*(x) \le R$ . Since t = -1, we write

$$\widehat{h}(\mathsf{patch}\,(x)) = \mathsf{sign}\left(\boldsymbol{v}^{\star}(x) + t \cdot \frac{2R}{\min_{x \in S_{\mathsf{adv}}} \boldsymbol{v}_j(\mathsf{patch}\,(x))} \cdot \boldsymbol{v}_j(\mathsf{patch}\,(x))\right)$$

$$= \operatorname{sign} \left( R - \frac{2R}{\min_{x \in S_{adv}} v_j(\operatorname{patch}(x))} \cdot v_j(\operatorname{patch}(x)) \right)$$
$$= \operatorname{sign} \left( 1 - \frac{2}{\min_{x \in S_{adv}} v_j(\operatorname{patch}(x))} \cdot v_j(\operatorname{patch}(x)) \right) = -1,$$

where the last line follows from the fact that  $1 - \frac{2}{\min_{x \in S_{adv}} v_j(\operatorname{patch}(x))} \cdot v_j(\operatorname{patch}(x)) \leq -1$ .

The proof in the event where t = 1 follows similarly. It thus follows that there exists a function  $\hat{h} \in \mathcal{H}$  obtaining 0 error on  $S_{\text{clean}} \cup S_{\text{adv}}$ . The desired result immediately follows from uniform convergence (we have a 0-error classifier over  $S_{\text{clean}} \sim \mathcal{D}$  and a 0-error classifier over  $S_{\text{adv}} \sim \text{patch}(\mathcal{D}|h^*(x) \neq t)$ , so with probability  $1 - 2\delta$ , we have error at most  $\varepsilon_{\text{clean}}$  on the clean distribution and error at most  $\varepsilon_{\text{adv}}$  on the adversarial distribution) This completes the proof of Theorem 20.

**Corollary 6.2.2** (Overparameterized linear classifier). Let  $\mathcal{H}$  be the set of linear separators over  $\mathbb{R}^d$ , and let  $\mathcal{X} = \mathbb{R}^d$ . Let  $\mathcal{D}$  be some distribution over an s-dimensional subspace of  $\mathbb{R}^d$  where s < d, so with probability 1, we can write  $\mathbf{x} \sim \mathcal{D}$  as  $\mathbf{A}\mathbf{z}$  for some  $\mathbf{A} \in \mathbb{R}^{d \times s}$  and for  $\mathbf{z} \in \mathbb{R}^s$ . Let  $\mathcal{F}_{adv} = \{\text{patch}(\mathbf{x}) : \text{patch}(\mathbf{x}) + \eta, \eta \perp \text{Span}(\mathbf{A})\}$ , and draw some patch function  $\text{patch} \in \mathcal{F}_{adv}$ .

Fix any target label  $t \in \{\pm 1\}$ . Draw a training set  $S_{\mathsf{clean}}$  of size at least  $m_0 \coloneqq \Omega\left(\varepsilon_{\mathsf{clean}}^{-1}\left(\mathsf{VC}\left(\mathcal{H}\right) + \log\left(\frac{1}{\delta}\right)\right)\right)$ . Then, draw a backdoor training set  $S_{\mathsf{adv}}$  of size at least  $m_1 \coloneqq \Omega\left(\varepsilon_{\mathsf{adv}}^{-1}\left(\mathsf{VC}\left(\mathcal{H}\right) + \log\left(\frac{1}{\delta}\right)\right)\right)$  of the form (x, t) where  $x \sim (\mathcal{D}|h^*(x) \neq t) + \eta$ .

With probability at least  $1 - \delta$ , empirical risk minimization on the training set  $S_{\text{clean}} \cup S_{\text{adv}}$  yields a classifier  $\hat{h}$  satisfying the success conditions for Problem 6.1.

*Proof of Corollary 6.2.2.* We will show that our problem setup is a special case of that considered in Theorem 20; then, we can apply that result as a black box.

Observe that the set of linear classifiers over  $\mathbb{R}^d$  is a thresholded vector space with dimension d. Pick the orthonormal basis  $\{v_1, \ldots, v_s, \ldots, v_d\}$  such that  $\{v_1, \ldots, v_s\}$  form a basis for the subspace Span (**A**) and  $v_{s+1}, \ldots, v_d$  are some completion of the basis for the rest of  $\mathbb{R}^d$ .

Clearly, there is a size-*s* set of indices  $U \subset [d]$  such that for all  $i \in U$ , we have  $\Pr_{x \sim \mathcal{D}} [v_i(x) \neq 0] > 0$ . Without loss of generality, assume U = [s].

Next, we need to show that for all  $i \in U$ , we have  $v_i(\text{patch}(x)) = 0$ . Since we have  $\eta \perp \text{Span}(\mathbf{A})$ , we have  $v_i(\eta) = 0$  for all  $i \in U$ . Since the  $v_i$  are also linear functions, we satisfy  $v_i(\mathbf{A}z + \eta) = 0$  for all  $z \in \mathbb{R}^s$ .

We now show that there is at least one  $j \in [\dim(\mathcal{V})] \setminus U$  such that  $\Pr_{x \sim \mathcal{D}} [v_j(\operatorname{patch}(x)) \neq 0] = 1$ . Since  $\eta \perp \operatorname{Span}(\mathbf{A})$ ,  $\eta$  must be expressible as some nonzero linear combination of the vectors  $v_j$ ; thus, taking the inner product with any such vector will result in a nonzero value.

Finally, we show that for all  $j \in [\dim(\mathcal{V})] \setminus U$ , we either have  $\Pr_{x \sim \mathcal{D}} [v_j(\operatorname{patch}(x)) \ge 0] = 1$  or  $\Pr_{x \sim \mathcal{D}} [v_j(\operatorname{patch}(x)) \le 0] = 1$ . Since  $\eta$  is expressible as a linear combination of several such  $v_j$ , we can write

$$\langle \mathbf{A}\boldsymbol{z} + \boldsymbol{\eta}, \boldsymbol{v}_j \rangle = \langle \mathbf{A}\boldsymbol{z}, \boldsymbol{v}_j \rangle + \langle \boldsymbol{\eta}, \boldsymbol{v}_j \rangle = 0 + \left( \sum_{j=s+1}^d a_j \cdot \boldsymbol{v}_j, \boldsymbol{v}_j \right) = a_j,$$

which is clearly nonzero.

The statement of Corollary 6.2.2 now follows from Theorem 20.

**Theorem 21** (Overparameterized linear classifier with random watermark). *Consider the same* setting used in Corollary 6.2.2, and set  $\mathcal{F}_{adv} = \{ patch : patch(x) = x + \eta, \eta \in \mathbb{R}^d \}.$ 

If  $h^*$  achieves margin  $\gamma$  and if the ambient dimension d of the model satisfies  $d \geq \Omega\left(\frac{s+\log(1/\delta)}{\gamma^2}\right)$ , then an adversary can find a patch function such that with probability  $1-\delta$ , a training set  $S = S_{\text{clean}} \cup S_{\text{adv}}$  satisfying  $|S_{\text{clean}}| \geq \Omega\left(\varepsilon_{\text{clean}}^{-1}\left(\text{VC}\left(\mathcal{H}\right) + \log\left(\frac{1}{\delta}\right)\right)\right)$  and  $|S_{\text{adv}}| \geq \Omega\left(\varepsilon_{\text{clean}}^{-1}\left(\text{VC}\left(\mathcal{H}\right) + \log\left(\frac{1}{\delta}\right)\right)\right)$  yields a classifier  $\hat{h}$  satisfying the success conditions for Problem 6.1 while also satisfying  $\underset{(x,y)\sim\mathcal{D}}{\mathbb{E}}\left[\mathbbm{1}\left\{\hat{h}(x)\neq y\right\}\right] \leq \varepsilon_{\text{clean}}$ .

*This result holds true particularly when the adversary does not know*  $\text{Supp}(\mathcal{D})$ *.* 

*Proof of Theorem 21.* We prove Theorem 21 in two parts. We first show that although the adversary does not know  $\mathcal{F}_{adv}(h^*)$ , they can find patch  $\in \mathcal{F}_{adv}(h^*)$  with high probability. We then invoke the result from Corollary 6.2.2.

Let **A** be such that the columns  $a_1, \ldots, a_s$  are an orthonormal basis for the subspace spanned by **A**. Draw the random vector  $g \sim \text{Unif}(\mathbb{S}^{d-1})$ . First, recall that  $|\langle a_i, g \rangle|$  is subgaussian and therefore  $\mathbb{E}\left[|\langle a_i, g \rangle|^2\right] \leq 1/d$  (see [Ver18, Theorem 3.4.6 and Proposition 2.7.1]). Now, observe that

$$\mathbb{E}\left[\left\|\mathbf{A}^{\mathsf{T}}\boldsymbol{g}\right\|_{2}\right] \leq \sqrt{\mathbb{E}\left[\left\|\mathbf{A}^{\mathsf{T}}\boldsymbol{g}\right\|_{2}^{2}\right]} = \sqrt{\sum_{i=1}^{s} \mathbb{E}\left[\left|\langle \boldsymbol{a}_{i},\boldsymbol{g}\rangle\right|^{2}\right]} \lesssim \sqrt{\frac{s}{d}}.$$

Next, observe that the function  $x \mapsto \|\mathbf{A}^{\top} x\|_2$  is 1-Lipschitz. Using [Ver18, Theorem 5.1.4], we have with probability  $\geq 1 - \delta/2$  that

$$\left\| \left\| \mathbf{A}^{ op} \boldsymbol{g} \right\|_2 - \mathbb{E} \left[ \left\| \mathbf{A}^{ op} \boldsymbol{g} \right\|_2 
ight] 
ight| \lesssim \sqrt{rac{\log\left(1/\delta
ight)}{d}}.$$

Combining, we get with probability  $\geq 1 - \delta/2$  that

$$\left\|\mathbf{A}^{\mathsf{T}}\boldsymbol{g}\right\|_{2} \lesssim \sqrt{\frac{s}{d}} + \sqrt{\frac{\log\left(1/\delta\right)}{d}},$$

which means that as long as  $d \gtrsim \frac{s + \log(1/\delta)}{\gamma^2}$ , and if we choose  $\eta = g$ , we have

$$\left\|\mathbf{A}^{\mathsf{T}}\boldsymbol{\eta}\right\|_{2} \leq \boldsymbol{\gamma}.$$

This implies that the norm of the component of the trigger in Ker ( $\mathbf{A}^{\top}$ ) is at least  $\sqrt{1 - \gamma^2} \ge 1 - \gamma$  from the Pythagorean Theorem.

This implies that  $h^*(x + \eta) = h^*(x)$  with probability  $1 - \delta/2$  over the draws of  $\eta$ . This gives us patch  $(x) = x + \eta \in \mathcal{F}_{adv}(h^*)$  with probability  $1 - \delta/2$  over the draws of  $\eta$ .

It is now easy to see that Theorem 21 follows from a simple application of Corollary 6.2.2 using a failure probability of  $\delta/2$ , and the final failure probability  $1 - \delta$  follows from a union bound.  $\Box$ 

**Theorem 22.** Consider some  $h^{\star}(x) = \text{sign}(\langle w^{\star}, x \rangle)$  and a data distribution  $\mathcal{D}$  satisfying  $\Pr_{(x,y)\sim\mathcal{D}}[y \langle w^{\star}, x \rangle \geq 1] = 1$  and  $\Pr_{(x,y)\sim\mathcal{D}}[||x||_2 \leq R] = 1$ . Let  $\gamma$  be the maximum margin over all weight vectors classifying the uncorrupted data, and let  $\mathcal{F}_{adv} = \{\text{patch}(x) : \|\text{patch}(x) - x\|_2 \leq \gamma\}.$ 

If  $S_{\text{clean}}$  consists of at least  $\Omega\left(\varepsilon_{\text{clean}}^{-2}\left(\gamma^{-2}R^{2} + \log\left(1/\delta\right)\right)\right)$  i.i.d examples drawn from  $\mathcal{D}$  and if  $S_{\text{adv}}$  consists of at least  $\Omega\left(\varepsilon_{\text{adv}}^{-2}\left(\gamma^{-2}R^{2} + \log\left(1/\delta\right)\right)\right)$  i.i.d examples drawn from  $\mathcal{D}|h^{\star}(x) \neq t$ , then we have

$$\min_{\|\boldsymbol{w}\|_{2} \leq \frac{1}{\gamma}} \frac{1}{|S|} \sum_{(\boldsymbol{x}, \boldsymbol{y}) \in S} \mathbb{1} \{ \boldsymbol{y} \langle \boldsymbol{w}, \boldsymbol{x} \rangle < 1 \} > 0.$$

In other words, assuming there exists a margin  $\gamma$  and a 0-loss classifier, empirical risk minimization of margin-loss with a norm constraint fails to find a 0-loss classifier on a sufficiently contaminated training set.

*Proof of Theorem 22.* We will proceed by contradiction.

Let patch (*x*) denote the patched version of *x*. Without loss of generality, let the target label be +1. Set  $\varepsilon_{clean}$  and  $\varepsilon_{adv}$  such that  $\varepsilon_{clean} + \varepsilon_{adv} < 1$  and draw enough samples such that the attack succeeds with parameters  $\varepsilon_{adv}$  and  $\delta$ .

Observe that we can write every member in  $S_{adv}$  as (patch (x), y) for some natural x with label  $\neg y$ . Next, suppose that the learner recovers a  $\widehat{w}$  such that the empirical margin loss of  $\widehat{w}$  is 0. Next, recall that the following holds for  $\widehat{w}$  obtained from the minimization in the theorem statement and for a training set  $S \sim \mathcal{D}^m$  (see, for instance, Theorem 26.12 of [SB14]):

$$\mathbb{E}_{(x,y)\sim\mathcal{D}}\left[\mathbbm{1}\left\{y\left\langle\widehat{\boldsymbol{w}},\boldsymbol{x}\right\rangle<1\right\}\right] \leq \inf_{\|\boldsymbol{w}\|_{2}\leq\gamma^{-1}}\mathbb{E}_{(x,y)\sim S}\left[\mathbbm{1}\left\{y\left\langle\boldsymbol{w},\boldsymbol{x}\right\rangle<1\right\}\right] + O\left(\sqrt{\frac{\left(R/\gamma\right)^{2} + \log\left(1/\delta\right)}{m}}\right)$$

Using this, we see from uniform convergence that with probability  $1 - \delta$ ,

$$\Pr_{\substack{x \sim \mathcal{D}}} \left[ y \left\langle \widehat{w}, x \right\rangle \ge 1 \right] \ge 1 - \varepsilon_{\mathsf{clean}}$$
$$\Pr_{\substack{x \sim \mathcal{D}}} \left[ \left\langle \widehat{w}, \mathsf{patch} \left( x \right) \right\rangle \ge 1 \right] \ge 1 - \varepsilon_{\mathsf{adv}}.$$

Using a union bound gives

$$\Pr_{\substack{x \sim \mathcal{D}}} \left[ (y \langle \widehat{w}, x \rangle \ge 1) \land (\langle \widehat{w}, \mathsf{patch} (x) \rangle \ge 1) \right] \ge 1 - \varepsilon_{\mathsf{clean}} - \varepsilon_{\mathsf{adv}}$$

Hence, it must be the case that there exists at least one true negative *x* for which both  $y \langle \hat{w}, x \rangle \ge 1$  and  $\langle \hat{w}, \text{patch}(x) \rangle \ge 1$  hold. We will use this to obtain a lower bound on  $\|\hat{w}\|_2$ , from which a contradiction will follow. Notice that

 $1 \leq \langle \widehat{w}, \mathsf{patch}\,(x) \rangle = \langle \widehat{w}, x \rangle + \langle \widehat{w}, \mathsf{patch}\,(x) - x \rangle \leq -1 + \|\widehat{w}\|_2 \cdot \|\mathsf{patch}\,(x) - x\|_2 \,,$ 

where the last line follows from the fact that x is labeled differently from patch (x). This gives

$$\|\widehat{\boldsymbol{w}}\|_2 \ge \frac{2}{\|\mathsf{patch}\,(\boldsymbol{x}) - \boldsymbol{x}\|_2}$$

Assuming that we meet the constraint  $\|\widehat{\boldsymbol{w}}\|_2 \leq 1/\gamma$ , putting the inequalities together gives

$$\|\operatorname{patch}(\boldsymbol{x}) - \boldsymbol{x}\|_2 \ge 2\gamma.$$

This is a contradiction, since we require that the size of the perturbation is smaller than the margin. This completes the proof of Theorem 22.  $\Box$ 

**Lemma 6.2.4.** We have  $0 \leq \operatorname{mcap}_{X,\mathcal{D}}(\mathcal{H}) \leq \operatorname{VC}(\mathcal{H})$ .

*Proof of Lemma 6.2.4.* The lower bound is obvious. This is also tight, as we can set  $X = \{0, 1\}^n$ ,  $\mathcal{D} = \text{Unif}(X)$ , and  $\mathcal{H} = \{f : f(x) = 1, \forall x \in X\}$ .

We now tackle the upper bound. Suppose for the sake of contradiction that  $\operatorname{mcap}_{X,\mathcal{D}}(\mathcal{H}) \geq \operatorname{VC}(\mathcal{H}) + 1$ . Then, we can find  $k = \operatorname{VC}(\mathcal{H}) + 1$  nonempty subsets of  $X, X_1, \ldots, X_k$  and an h for which every labeling of these subsets can be achieved by some other  $\hat{h} \in \mathcal{H}$ . Hence, picking any collection of points  $x_i \in X_i$  yields a set witnessing  $\operatorname{VC}(\mathcal{H}) \geq k = \operatorname{VC}(\mathcal{H}) + 1$ , which is clearly a contradiction.

The upper bound is tight as well. Consider the dataset  $S = \{0, e_1, ..., e_d\}$ , let  $\mathcal{D}$  be a distribution assigning a point mass of 1 to x = 0, and let  $h^*(0) = 1$ . It is easy to see that the class of origin-containing halfspaces can memorize every labeling  $e_1, ..., e_d$  as follows – suppose we have labels  $b_1, ..., b_d$ . Then, the classifier

$$\mathbb{1}\left\{\sum_{i=1}^d b_i \cdot \mathbf{x}_i \ge 0\right\}$$

memorizes every labeling of  $e_1, \ldots, e_d$  while correctly classifying the pair (0, 1). Hence, we can memorize *d* irrelevant sets, which is equal to the VC dimension of origin-containing linear separators. This concludes the proof of Lemma 6.2.4.

**Theorem 23** (Nonzero memorization capacity implies backdoor attack). *Pick a target label*  $t \in \pm 1$ . *Suppose we have a hypothesis class*  $\mathcal{H}$ *, a target function*  $h^*$ *, a domain* X*, a data distribution*  $\mathcal{D}$ *, and a class of patch functions*  $\mathcal{F}_{adv}$ . *Define* 

$$\mathcal{C}(\mathcal{F}_{\mathsf{adv}}(h^{\star})) \coloneqq \{ \mathsf{patch}\left(\mathsf{Supp}\left(\mathcal{D}|h^{\star}(x) \neq t\right) \right) : \mathsf{patch} \in \mathcal{F}_{\mathsf{adv}} \}.$$

Now, suppose that  $\operatorname{mcap}_{X,\mathcal{D}}(h^*, \mathcal{H}, C(\mathcal{F}_{\mathsf{adv}}(h^*))) \ge 1$ . Then, there exists a function  $\operatorname{patch} \in \mathcal{F}_{\mathsf{adv}}$  for which the adversary can draw a set  $S_{\mathsf{adv}}$  consisting of  $m = \Omega\left(\varepsilon_{\mathsf{adv}}^{-1}\left(\mathsf{VC}\left(\mathcal{H}\right) + \log\left(\frac{1}{\delta}\right)\right)\right)$  i.i.d samples from  $\mathcal{D}|h^*(x) \neq t$  such that with probability at least  $1 - \delta$  over the draws of  $S_{\mathsf{adv}}$ , the adversary achieves the objectives of Problem 6.1, regardless of the number of samples the learner draws from  $\mathcal{D}$  for  $S_{\mathsf{clean}}$ .

**Theorem 28** (Generalization of Theorem 23). *Pick an array of k target labels*  $t \in {\pm 1}^k$ . *Suppose we have a hypothesis class*  $\mathcal{H}$ *, a target function*  $h^*$ *, a domain* X*, a data distribution*  $\mathcal{D}$ *, and a class of patch functions*  $\mathcal{F}_{adv}$ . *Define:* 

$$C(\mathcal{F}_{\mathsf{adv}}(h^{\star}))_{t'} \coloneqq \{ \mathsf{patch}\left(\mathsf{Supp}\left(\mathcal{D}|h^{\star}(x) \neq t'\right) \right) : \mathsf{patch} \in \mathcal{F}_{\mathsf{adv}} \}$$

and let:

$$C(\mathcal{F}_{\mathsf{adv}}(h^{\star})) \coloneqq C(\mathcal{F}_{\mathsf{adv}}(h^{\star}))_{-1} \cup C(\mathcal{F}_{\mathsf{adv}}(h^{\star}))_{1}$$

Now, suppose that  $\operatorname{mcap}_{X,\mathcal{D}}(h^*,\mathcal{H},C(\mathcal{F}_{\mathsf{adv}}(h^*))) \geq k$ . Then, there exists k functions  $\operatorname{patch}_1,\ldots,\operatorname{patch}_k \in \mathcal{F}_{\mathsf{adv}}$  for which the adversary can draw sets  $\{(S_{\mathsf{adv}})_i\}_{i\in[k]}$  each consisting of  $m_i = \Omega\left(\varepsilon_{\mathsf{adv}}^{-1}(\mathsf{VC}(\mathcal{H}) + \log(k/\delta))\right)$  i.i.d samples from  $\mathcal{D}|h^*(x) \neq t_i$  such that with probability at least  $1 - \delta$  over the draws of  $(S_{\mathsf{adv}})_i$ , the adversary achieves the objectives of Problem 6.1, regardless of the number of samples the learner draws from  $\mathcal{D}$  for  $S_{\mathsf{clean}}$ .

*Proof of Theorem 28.* As per the theorem statement, we can draw *m* samples from  $\mathcal{D}|h^{\star}(x) \neq t_i$  to form  $S_{adv}$  by inverting the labels of the samples we draw.

Since  $\operatorname{mcap}_{X,\mathcal{D}}(h^{\star}, \mathcal{H}, C(\mathcal{F}_{\mathsf{adv}}(h^{\star}))) = k$ , there must exist k sets  $X_1, \ldots, X_k \in C(\mathcal{F}_{\mathsf{adv}}(h^{\star}))$  such that the  $X_i$  are memorizable, for which we can write  $X_i \subseteq \operatorname{patch}_i(\operatorname{Supp}(\mathcal{D}|h^{\star}(x) \neq t_i))$  for appropriate choices of  $\operatorname{patch}_i$ , and for which  $\mu_{\operatorname{patch}(\mathcal{D}|h^{\star}(x)\neq t_i)}(X_i) = 1$ . This implies that with probability 1, there exists at least one function  $\widehat{h} \in \mathcal{H}$  such that  $\widehat{h}$  returns  $t_i$  on every element in  $(S_{\operatorname{adv}})_i$  for all  $i \in [k]$  and agrees with  $h^{\star}$  on every element in the clean training set  $S_{\operatorname{clean}}$ .

Thus, we can recover a classifier  $\hat{h}$  from  $\mathcal{H}$  with 0 error on the training set  $S_{\text{clean}} \cup \left( \bigcup_{i \in [k]} (S_{\text{adv}})_i \right)$ . In particular, notice that we achieve 0 error on  $S_{\text{clean}}$  from distribution  $\mathcal{D}$  and on every  $(S_{\text{adv}})_i$  from distribution patch<sub>i</sub>  $(\mathcal{D}|h^*(x) \neq t_i)$ . From the Fundamental Theorem of PAC Learning [SB14], it follows that as long as  $|S_{\text{clean}}|$  and  $|(S_{\text{adv}})_i|$  are each at least  $\Omega\left(\varepsilon_{\text{clean}}^{-1}\left(\text{VC}\left(\mathcal{H}\right) + \log\left(\frac{k}{\delta}\right)\right)\right)$  and  $\Omega\left(\varepsilon_{\text{adv}}^{-1}\left(\text{VC}\left(\mathcal{H}\right) + \log\left(\frac{k}{\delta}\right)\right)\right)$ , respectively, we have that  $\hat{h}$  has error at most  $\varepsilon$  on  $\mathcal{D}$  and error at least  $1 - \varepsilon$  on patch<sub>i</sub>  $(\mathcal{D}|h^*(x) \neq t_i)$  with probability  $1 - \delta$  (following from a union bound, where each training subset yields a failure to attain uniform convergence with probability at most  $\delta/(k+1)$ ). This completes the proof of Theorem 28.

**Theorem 24.** Let  $C(\mathcal{F}_{adv}(h^*))$  be defined the same as in Theorem 23. Suppose we have a hypothesis class  $\mathcal{H}$  over a domain  $\mathcal{X}$ , a true classifier  $h^*$ , data distribution  $\mathcal{D}$ , and a perturbation class  $\mathcal{F}_{adv}$ . If  $mcap_{\mathcal{X},\mathcal{D}}(h^*,\mathcal{H},C(\mathcal{F}_{adv}(h^*))) = 0$ , then the adversary cannot successfully construct a backdoor data poisoning attack as per the conditions of Problem 6.1.

*Proof of Theorem* 24. The condition in the theorem statement implies that there does not exist an irrelevant set that can be memorized atop any choice of  $h \in \mathcal{H}$ .

For the sake of contradiction, suppose that there does exist a target classifier  $h^*$ , a function patch  $\in \mathcal{F}_{adv}$  and a target label *t* such that for all choices of  $\varepsilon_{clean}$ ,  $\varepsilon_{adv}$ , and  $\delta$ , we obtain a successful attack.

Define the set  $X := \text{patch}(\text{Supp}(\mathcal{D}|h^*(x) \neq t))$ ; in words, X is the subset of X consisting of patched examples that are originally of the opposite class of the the target label. It is easy to see that  $X \in C$ .

We will first show that if  $\mu_{\mathcal{D}}(X) > 0$ , then we obtain a contradiction. Set  $0 < \varepsilon_{\mathsf{adv}}, \varepsilon_{\mathsf{clean}} < \frac{\mu_{\mathcal{D}}(X)}{1+\mu_{\mathcal{D}}(X)}$ . Since the attack is successful, we must classify at least a  $1 - \varepsilon_{\mathsf{adv}}$  fraction of X as the target label. Hence, we write

$$\mu_{\mathcal{D}}\left(\left\{x \in X : \widehat{h}(x) = t\right\}\right) \ge (1 - \varepsilon_{\mathsf{adv}})\,\mu_{\mathcal{D}}(X) > \frac{1}{1 + \mu_{\mathcal{D}}(X)} \cdot \mu_{\mathcal{D}}(X) > \varepsilon_{\mathsf{clean}}$$

Since the set  $\{x \in X : \widehat{h}(x) = t\}$  is a subset of the region of X that  $\widehat{h}$  makes a mistake on, we have that  $\widehat{h}$  must make a mistake on at least  $\varepsilon_{clean}$  measure of  $\mathcal{D}$ , which is a contradiction.

Hence, it must be the case that  $\mu_{\mathcal{D}}(X) = 0$ ; in other words, X is an irrelevant set. Recall that in the beginning of the proof, we assume there exists a function  $\hat{h}$  that achieves label t on X, which is opposite of the value of  $h^*$  on X. Since we can achieve both possible labelings of Xwith functions from  $\mathcal{H}$ , it follows that X is a memorizable set, and thus the set X witnesses positive mcap<sub> $X,\mathcal{D}$ </sub> ( $h^*, \mathcal{H}, C(\mathcal{F}_{adv}(h^*))$ ). This completes the proof of Theorem 24.

**Example 6.2.3.1** (Overparameterized linear classifiers). *Recall the result from the previous section,* where we took  $X = \mathbb{R}^d$ ,  $\mathcal{H}_d$  to be the set of linear classifiers in  $\mathbb{R}^d$ , and let  $\mathcal{D}$  be a distribution over

a radius-R subset of an s-dimensional subspace P. We also assume that the true labeler  $h^*$  achieves margin  $\gamma$ .

*If we set* 

$$\mathcal{F}_{\mathsf{adv}} = \left\{ \mathsf{patch}\left(x\right) : \mathsf{patch}\left(x\right) = x + \eta, \eta \in \mathbb{R}^{d} \right\},\$$

then we have  $\operatorname{mcap}_{\mathcal{X},\mathcal{D}}(h^{\star},\mathcal{H}_d,\mathcal{C}(\mathcal{F}_{\operatorname{adv}}(h^{\star}))) \geq d-s.$ 

*Proof of Example 6.2.3.1.* Let  $w^*$  be the weight vector corresponding to  $h^*$ .

Observe that there exists k := d - s unit vectors  $v_1, \ldots, v_k$  that complete an orthonormal basis from that for *P* to one for  $\mathbb{R}^d$ . Next, consider the following subset of  $\mathcal{F}_{adv}(h^*)$ .

$$\mathcal{F}'_{\mathsf{adv}} \coloneqq \left\{ \mathsf{patch} \in \mathcal{F}_{\mathsf{adv}} : \forall i \in [k], \mathsf{patch}_i(x) = \left( \begin{cases} x + \eta \cdot t_i v_i & , h^\star(x) \neq t_i \\ x & \text{otherwise} \end{cases} \right) \right\}$$

We prove the memorization capacity result by using the images of functions in  $\mathcal{F}'_{adv}$ . We will show that the function

$$\widehat{h}(x) = \operatorname{sign}\left(\left(\boldsymbol{w}^{\star} + \frac{2R}{\gamma}\sum_{i=1}^{k}t_{i}\cdot\frac{\boldsymbol{v}_{i}}{\eta_{i}}, \boldsymbol{x}\right)\right)$$

memorizes the *k* sets  $C_i := \{x + \eta_i \cdot v_i : \langle w^*, x \rangle \in [1, R/\gamma] \cup [-R/\gamma, -1]\}$ . Moreover, observe that the preimages of the  $C_i$  have measure 1 under the conditional distributions  $\mathcal{D}|h^*(x) \neq t_i$ , since the preimages contain the support of these conditional distributions. We now have, for a clean point  $x \in P$ ,

$$\widehat{h}(\mathbf{x}) = \operatorname{sign}\left(\left\langle \mathbf{w}^{\star} + \frac{2R}{\gamma} \sum_{i=1}^{k} t_{i} \cdot \frac{\mathbf{v}_{i}}{\eta_{i}}, \mathbf{x}\right\rangle\right) = \operatorname{sign}\left(\left\langle \mathbf{w}^{\star}, \mathbf{x}\right\rangle + \frac{2R}{\gamma} \left(\sum_{i=1}^{k} t_{i} \cdot \frac{\mathbf{v}_{i}}{\eta_{i}}, \mathbf{x}\right)\right)$$
$$= \operatorname{sign}\left(\left\langle \mathbf{w}^{\star}, \mathbf{x}\right\rangle\right) = h^{\star}(\mathbf{x}),$$

and for a corrupted point  $x + \eta_j \cdot v_j$ , for  $j \in [k]$ ,

$$\begin{split} \widehat{h}(\mathbf{x}) &= \operatorname{sign}\left(\left\langle \mathbf{w}^{\star} + \frac{2R}{\gamma} \sum_{i=1}^{k} t_{i} \cdot \frac{\mathbf{v}_{i}}{\eta_{i}}, \mathbf{x} + \eta_{j} \cdot \mathbf{v}_{j}\right\rangle\right) \\ &= \operatorname{sign}\left(\left\langle \mathbf{w}^{\star}, \mathbf{x} + \eta_{j} \cdot \mathbf{v}_{j}\right\rangle + \frac{2R}{\gamma} \left(\sum_{i=1}^{k} t_{i} \cdot \frac{\mathbf{v}_{j}}{\eta_{j}}, \mathbf{x} + \eta_{j} \cdot \mathbf{v}_{j}\right)\right) \\ &= \operatorname{sign}\left(\left\langle \mathbf{w}^{\star}, \mathbf{x}\right\rangle + \frac{2R}{\gamma} \left(\sum_{i=1}^{k} t_{i} \cdot \frac{\mathbf{v}_{i}}{\eta_{i}}, \mathbf{x}\right) + \frac{2R}{\gamma} \left(\sum_{i=1}^{k} t_{i} \cdot \frac{\mathbf{v}_{i}}{\eta_{i}}, \eta_{j} \cdot \mathbf{v}_{j}\right)\right) \\ &= \operatorname{sign}\left(\left[\pm \frac{R}{\gamma}\right] + t_{j} \cdot \frac{2R}{\gamma}\right) = t_{j}. \end{split}$$

This shows that we can memorize the *k* sets  $C_i$ . It is easy to see that  $\mu_{\mathcal{D}}(C_i) = 0$ , so the  $C_i$  are irrelevant memorizable sets; in turn, we have that  $\operatorname{mcap}_{X,\mathcal{D}}(h^*) \ge k = d - s$ , concluding the proof of Example 6.2.3.1.

**Example 6.2.3.2** (Linear classifiers over convex bodies). Let  $\mathcal{H}$  be the set of origin-containing halfspaces. Fix an origin-containing halfspace  $h^*$  with weight vector  $w^*$ . Let X' be a closed compact

convex set, let  $X = X' \setminus \{x : \langle w^*, x \rangle = 0\}$ , and let  $\mathcal{D}$  be any probability measure over X that assigns nonzero measure to every  $\ell_2$  ball of nonzero radius contained in X and satisfies the relation  $\mu_{\mathcal{D}}(Y) = 0 \iff \operatorname{Vol}_d(Y) = 0$  for all  $Y \subset X$ . Then,  $\operatorname{mcap}_{X,\mathcal{D}}(h^*, \mathcal{H}) = 0$ .

To analyze Example 6.2.3.2, we need the following intermediate results.

**Lemma 6.5.1.** Consider some convex body K, a probability measure  $\mathcal{D}$  such that every  $\ell_2$  ball of nonzero radius within K has nonzero measure, and some subset  $K' \subseteq K$  satisfying  $\mu_{\mathcal{D}}(K') = 1$ . Then, conv (K') contains every interior point of K.

*Proof of Lemma 6.5.1.* Recall that an interior point is defined as one for which we can find some neighborhood contained entirely within the convex body. Mathematically,  $x \in K$  is an interior point if we can find nonzero  $\delta$  for which  $\{z : ||x - z||_2 \leq \delta\} \subseteq K$ .

For the sake of contradiction, suppose that there exists some interior point  $x \in K$  that is not contained in conv (K'). Hence, there must exist a halfspace H with boundary passing through x and entirely containing conv (K'). Furthermore, there must exist a nonzero  $\delta$  for which there is an  $\ell_2$  ball centered at x of radius  $\delta$  contained entirely within K. Call this ball  $B_2(x, \delta)$ . Thus, the set  $K \setminus H$  cannot be in conv (K').

We will now show that  $\mu_{\mathcal{D}}(K \setminus H) > 0$ . Observe that the hyperplane inducing H must cut  $B_2(x, \delta)$  through an equator. From this, we have that the set  $K \setminus H$  contains a half- $\ell_2$  ball of radius  $\delta$ . It is easy to see that this half-ball contains another  $\ell_2$  ball of radius  $\delta/2$  (call this B'), and as per our initial assumption, B' must have nonzero measure.

Thus, we can write  $\mu_{\mathcal{D}}(K \setminus H) \ge \mu_{\mathcal{D}}(B') > 0$ . Since we know that  $\mu_{\mathcal{D}}(\operatorname{conv}(K')) + \mu_{\mathcal{D}}(K \setminus H) \le 1$ , it follows that  $\mu_{\mathcal{D}}(\operatorname{conv}(K')) < 1$  and therefore  $\mu_{\mathcal{D}}(K') < 1$ , violating our initial assumption that  $\mu_{\mathcal{D}}(K') = 1$ . We thus complete the proof of Lemma 6.5.1.

**Lemma 6.5.2.** Let *K* be a closed compact convex set. Let  $x_1$  be on the boundary of *K* and let  $x_2$  be an interior point of *K*. Then, every point of the form  $\lambda x_1 + (1 - \lambda)x_2$  for  $\lambda \in (0, 1)$  is an interior point of *K*.

*Proof of Lemma 6.5.2.* Since  $x_2$  is an interior point, there must exist an  $\ell_2$  ball of radius  $\delta$  contained entirely within *K* centered at  $x_2$ . From similar triangles and the fact that any two points in a convex body can be connected by a line contained in the convex body, it is easy to see that we can center an  $\ell_2$  ball of radius  $(1 - \lambda)\delta$  at the point  $\lambda x_1 + (1 - \lambda)x_2$  that lies entirely in *K*. We therefore conclude the proof of Lemma 6.5.2.

We are ready to analyze Example 6.2.3.2.

*Proof of Example 6.2.3.2.* Observe that the ambient space is equal to the dimension of *X*.

Let  $w^*$  be the weight vector corresponding to the true labeler  $h^*$ .

For the sake of contradiction, suppose there exists a classifier  $\widehat{w}$  satisfying

$$\Pr_{x \sim \mathcal{D}} \left[ \operatorname{sign} \left( \langle \widehat{w}, x \rangle \right) = \operatorname{sign} \left( \langle w^{\star}, x \rangle \right) \right] = 1,$$

but there exists a subset  $Y \subset X$  for which sign  $(\langle \widehat{w}, x \rangle) \neq \text{sign} (\langle w^*, x \rangle)$ , for all  $x \in Y$ . Such a Y would constitute a memorizable set.

Without loss of generality, let the target label be -1; that is, the adversary is converting a set Y whose label is originally +1 to one whose label is -1. Additionally, without loss of generality, take  $||w^*||_2 = ||\widehat{w}||_2 = 1$ . Observe that

$$Y \subseteq D \coloneqq \left\{ x \in \mathcal{X} : \langle \widehat{w}, x \rangle \le 0 \text{ and } \langle w^{\star}, x \rangle > 0 \right\}.$$

For *D* to be nonempty (and therefore for *Y* to be nonempty), observe that we require  $\hat{w} \neq w^*$  (otherwise, the constraints in the definition of the set *D* are unsatisfiable).

Lemma 6.5.1 implies that if Y is memorizable, then it must lie entirely on the boundary of the set  $X_+ := \{x \in X : \langle w^*, x \rangle > 0\}$ . To see this, observe that if  $\widehat{w}$  classifies any (conditional) measure-1 subset of  $X_+$  correctly, then it must classify the convex hull of that subset correctly as well. This implies that  $\widehat{w}$  must correctly classify every interior point in  $X_+$ , and thus, Y must be entirely on the boundary of  $X_+$ .

Now, let  $x_1 \in Y$  and  $x_2 \in \text{Interior}(X_-)$  where  $X_- = \{x \in X : \langle w^*, x \rangle < 0\}$ . Draw a line from  $x_1$  to  $x_2$  and consider the labels of the points assigned by  $\widehat{w}$ . Since  $x_1 \in Y$ , we have  $\widehat{h}(x_1) = -1$ , and since  $x_2 \in \text{Interior}(X_-)$ , we have that  $\widehat{h}(x_2) = -1$  as well. Using Lemma 6.5.2, we have that every point on the line connecting  $x_1$  to  $x_2$  (except for possibly  $x_1$ ) is an interior point to X'. Since we have that the number of sign changes along a line that can be induced by a linear classifier is at most 1, we must have that the line connecting  $x_1$  to  $x_2$  incurs 0 sign changes with respect to the classifier induced by  $\widehat{w}$ . This implies that the line connecting  $x_1$  to  $x_2$  cannot pass through any interior points of  $X_+$ . However, the only way that this can happen is if  $\langle w^*, x_1 \rangle = 0$ , but per our definition of X, if it is the case that  $\langle w^*, x_1 \rangle = 0$ , then  $x_1 \notin X$ , which is a clear contradiction.

This is sufficient to conclude the proof of Example 6.2.3.2.

**Example 6.2.3.3** (Sign changes). Let X = [0, 1],  $\mathcal{D} = \text{Unif}(X)$  and  $\mathcal{H}_k$  be the class of functions admitting at most k sign-changes. Specifically,  $\mathcal{H}_k$  consists of functions h for which we can find pairwise disjoint, continuous intervals  $I_1, \ldots, I_{k+1}$  such that:

- for all i < j and for all  $x \in I_i$ ,  $y \in I_j$ , we have x < y;
- $\bigcup_{i=1}^{k+1} I_i = X;$
- $h(I_i) = -h(I_{i+1})$ , for all  $i \in [k]$ .

Suppose the learner is learning  $\mathcal{H}_s$  for unknown s using  $\mathcal{H}_d$ , where  $s \leq d + 2$ . For all  $h^* \in \mathcal{H}_s$ , we have  $\operatorname{mcap}_{X,\mathcal{D}}(h^*, \mathcal{H}_d) \geq \lfloor (d-s)/2 \rfloor$ .

*Proof of Example 6.2.3.3.* Without loss of generality, take d - s to be an even integer.

Let  $I_1, \ldots, I_{s+1}$  be the intervals associated with  $h^*$ . It is easy to see that we can pick a total of (d-s)/2 points such that the sign of these points can be memorized by some  $\hat{h}$ . Since each point we pick within an interval can induce at most 2 additional sign changes, we have that the resulting function  $\hat{h}$  has at most  $s + 2 \cdot (d-s)/2 \leq d$  sign-changes; thus,  $\hat{h} \in \mathcal{H}_d$ . Moreover, the measure of a single point is 0, and so the total measure of our (d-s)/2 points is 0.

Given this, it is easy to find  $\mathcal{F}_{adv}$  and corresponding  $C(\mathcal{F}_{adv}(h^*))$  for which the backdoor attack can succeed as per Theorem 23, thereby yielding the conclusion of Example 6.2.3.3.

#### 6.5.2. Proofs from Section 6.3

Theorem 25 (Certifying backdoor existence). Suppose that the learner can calculate and minimize

$$\mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(\widehat{h}, S) = \mathbb{E}_{(x, y) \sim S} \left[ \sup_{\mathsf{patch} \in \mathcal{F}_{\mathsf{adv}}(h^{\star})} \mathbb{1} \left\{ \widehat{h}(\mathsf{patch}(x)) \neq y \right\} \right]$$

over a finite set S and  $\widehat{h} \in \mathcal{H}$ .

If the VC dimension of the loss class  $\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}^{\mathcal{H}}$  is finite, then there exists an algorithm using  $O\left(\varepsilon_{clean}^{-2}\left(\mathsf{VC}\left(\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}\right) + \log\left(\frac{1}{\delta}\right)\right)\right)$  samples that allows the learner to defeat the adversary through learning a backdoor-robust classifier or by rejecting the training set as being corrupted, with probability  $1 - \delta$ .

*Proof of Theorem 25.* See Algorithm 16 for the pseudocode of an algorithm witnessing Theorem 26.

Algorithm 16 Implementation of an algorithm certifying backdoor corruptio
1: <b>Input</b> : Training set $S = S_{clean} \cup S_{adv}$
satisfying $ S_{clean}  = \Omega\left(\varepsilon_{clean}^{-2}\left(VC\left(\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}^{\mathcal{H}}\right) + \log\left(\frac{1}{\delta}\right)\right)\right)$
2: Set $\widehat{h} := \operatorname{argmin} \mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}(h, S)$

3: **Output:**  $\widehat{h}$  if  $\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}(\widehat{h}, S) \leq 2\varepsilon$  and reject otherwise

There are two scenarios to consider.

**Training set is (mostly) clean.** Suppose that *S* satisfies  $\min_{h \in \mathcal{H}} \mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}(h, S) \leq \varepsilon_{clean}$ . Since the VC dimension of the loss class  $\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}^{\mathcal{H}}$  is finite, it follows that with finitely many samples, we attain uniform convergence with respect to the robust loss, and we're done; in particular, we can write  $\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}\left( \operatorname*{argmin}_{h \in \mathcal{H}} \mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}(h, S), \mathcal{D} \right) \leq \varepsilon_{clean}$  with high probability.

**Training set contains many backdoored examples.** Here, we will show that with high probability, minimizing  $\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}(\hat{h}, S)$  over  $\hat{h}$  will result in a nonzero loss, which certifies that the training set *S* consists of malicious examples.

Suppose that for the sake of contradiction, the learner finds a classifier  $\hat{h}$  such that  $\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}(\hat{h}, S) \leq \varepsilon_{clean}$ . Hence, with high probability, we satisfy  $\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}(\hat{h}, \mathcal{D}) \leq \varepsilon_{clean}$ . Since there is a constant measure allocated to each class, we can write

$$\mathbb{E}_{\substack{(x,y)\sim\mathcal{D}\mid y\neq t}}\left[\sup_{\mathsf{patch}\in\mathcal{F}_{\mathsf{adv}}(h^{\star})}\mathbb{1}\left\{\widehat{h}(\mathsf{patch}\,(x))\neq y\right\}\right]\lesssim\varepsilon_{\mathsf{clean}}.$$

Furthermore, since we achieved a loss of 0 on the whole training set, including the subset  $S_{adv}$ , from uniform convergence, we get with high probability that

$$\mathbb{E}_{\substack{(x,y)\sim\mathcal{D}\mid y\neq t}}\left[\mathbb{1}\left\{\widehat{h}(\mathsf{patch}\,(x))=t\right\}\right]\geq 1-\varepsilon_{\mathsf{adv}}.$$

This easily implies

$$\mathbb{E}_{\substack{(x,y)\sim\mathcal{D}|y\neq t}}\left[\sup_{\mathsf{patch}\in\mathcal{F}_{\mathsf{adv}}(h^{\star})}\mathbb{1}\left\{\widehat{h}(\mathsf{patch}\,(x))\neq y\right\}\right]\geq 1-\varepsilon_{\mathsf{adv}}.$$

Stitching the inequalities together yields  $\varepsilon_{\text{clean}} \gtrsim 1 - \varepsilon_{\text{adv}}$ . This is a contradiction, as we can make  $\varepsilon_{\text{clean}}$  sufficiently small so as to violate this statement. We obtain Theorem 25 as desired.

**Theorem 26** (Filtering implies generalization). Let  $\alpha \leq 1/3$  and  $\varepsilon_{clean} \leq 1/10$ .

Suppose we have a training set  $S = S_{\text{clean}} \cup S_{\text{adv}}$  such that  $|S_{\text{clean}}| = \Omega\left(\varepsilon_{\text{clean}}^{-2}\left(\text{VC}\left(\mathcal{L}_{\mathcal{F}_{\text{adv}}(h^{\star})}\right) + \log(1/\delta)\right)\right)$ and  $|S_{\text{adv}}| \leq \alpha \cdot (|S_{\text{adv}}| + |S_{\text{clean}}|)$ . If there exists an algorithm that given S can find a subset  $S' = S'_{\text{clean}} \cup S'_{\text{adv}}$  satisfying  $|S'_{\text{clean}}|/|S_{\text{clean}}| \geq 1 - \varepsilon_{\text{clean}}$  and  $\min_{h \in \mathcal{H}} \mathcal{L}_{\mathcal{F}_{\text{adv}}(h^{\star})}(h, S') \leq \varepsilon_{\text{clean}}$ , then there exists an algorithm such that given S returns a function  $\widehat{h}$  satisfying  $\mathcal{L}_{\mathcal{F}_{\text{adv}}(h^{\star})}(\widehat{h}, \mathcal{D}) \leq \varepsilon_{\text{clean}}$  with probability  $1 - \delta$ .

We first need the intermediate claim Claim 6.5.3.

**Claim 6.5.3.** *For all*  $h \in \mathcal{H}$ *, we have* 

$$\left|\mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(h, S_{\mathsf{clean}}) - \mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(h, S_{\mathsf{clean}}')\right| \leq \varepsilon_{\mathsf{clean}}$$

*Proof of Claim 6.5.3.* Let *a*, *b*, *c* be positive numbers. We first write

$$\frac{a}{b} - \max\left\{0, \frac{a-c}{b-c}\right\} = \frac{c(b-a)}{b(b-c)} \le \frac{c}{b},$$

which occurs exactly when  $c \le a$ . In the case where  $a \le c$ , we have

$$\frac{a}{b} - \max\left\{0, \frac{a-c}{b-c}\right\} = \frac{a}{b} \le \frac{c}{b},$$

which gives

$$\frac{a}{b} - \max\left\{0, \frac{a-c}{b-c}\right\} \le \frac{c}{b}.$$

Next, consider

$$\min\left\{1,\frac{a}{b-c}\right\} - \frac{a}{b} = \frac{a}{b-c} - \frac{a}{b} = \frac{c}{b} \cdot \frac{a}{b-c} \le \frac{c}{b},$$

which happens exactly when we have  $b \ge a + c$ . In the other case, we have

$$\min\left\{1,\frac{a}{b-c}\right\} - \frac{a}{b} = 1 - \frac{a}{b} \le \frac{c}{b}.$$

We therefore write

$$\max\left\{0, \frac{a-c}{b-c}\right\}, \min\left\{1, \frac{a}{b-c}\right\} \in \left[\frac{a}{b} \pm \frac{c}{b}\right].$$

Now, let *a* denote the number of samples from  $S_{\text{clean}}$  that *h* incurs robust loss on, let *b* be the total number of samples from  $S_{\text{clean}}$ , and let *c* be the number of samples our filtering procedure deletes from  $S_{\text{clean}}$ . It is easy to see that a/b corresponds  $\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}(h, S_{\text{clean}})$  and that  $\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}(h, S'_{\text{clean}}) \in [\max \{0, (a-c)/(b-c)\}, \min \{1, a/(b-c)\}]$ . From our argument above, this means that we must have

$$\mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(h, S'_{\mathsf{clean}}) \in \left[\mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(h, S_{\mathsf{clean}}) \pm \frac{\varepsilon_{\mathsf{clean}}(1-\alpha)m}{(1-\alpha)m}\right]$$

Finally,

$$\frac{\varepsilon_{\mathsf{clean}}(1-\alpha)m}{(1-\alpha)m} = \varepsilon_{\mathsf{clean}}$$

completing the proof of Claim 6.5.3.

We now prove Theorem 26.

Proof of Theorem 26. See Algorithm 17 for the pseudocode of an algorithm witnessing the theorem statement.

Algorithm 17 Implementation of a generalization algorithm given an implementation of a filtering algorithm

- 1: **Input**: Training set  $S = S_{clean} \cup S_{adv}$
- satisfying  $|S_{clean}| = \Omega \left( \varepsilon_{clean}^{-2} \left( VC \left( \mathcal{L}_{\mathcal{F}_{adv}(h^{\star})} \right) + \log (1/\delta) \right) \right)$ 2: Run the filtering algorithm on *S* to obtain *S'* satisfying the conditions in the theorem statement
- 3: **Output**: Output  $\widehat{h}$ , defined as  $\widehat{h} \coloneqq \operatorname{argmin} \mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}(h, S')$

Recall that we have drawn enough samples to achieve uniform convergence (see [CBM18] and [MGDS20]); in particular, assuming that our previous steps succeeded in removing very few points from  $S_{\text{clean}}$ , then for all  $h \in \mathcal{H}$ , we have with probability  $1 - \delta$  that

$$\left|\mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(h, \mathcal{D}) - \mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(h, S_{\mathsf{clean}})\right| \leq \varepsilon_{\mathsf{clean}}.$$

Observe that we have deleted at most  $m \cdot 2\varepsilon_{\text{clean}}$  points from  $S_{\text{clean}}$ . Let  $S'_{\text{clean}} \coloneqq S' \cap S_{\text{clean}}$  (i.e., the surviving members of  $S_{clean}$  from our filtering procedure). We now use Claim 6.5.3 and triangle inequality to write:

$$\begin{aligned} \left| \mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(h, S'_{\mathsf{clean}}) - \mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(h, \mathcal{D}) \right| &\leq \left| \mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(h, S_{\mathsf{clean}}) - \mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(h, S'_{\mathsf{clean}}) \right| + \\ \left| \mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(h, \mathcal{D}) - \mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(h, S_{\mathsf{clean}}) \right| \\ &\leq \varepsilon_{\mathsf{clean}} \end{aligned}$$

Next, consider some  $\hat{h}$  satisfying  $\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}(\hat{h}, S') \leq \varepsilon_{clean}$  (which must exist, as per our argument in Part 3), and observe that, for a constant *C*,

$$\mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(\widehat{h}, S') \geq (1 - C\varepsilon_{\mathsf{clean}}) \mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(\widehat{h}, S' \cap S_{\mathsf{clean}}) + C\varepsilon_{\mathsf{clean}} \mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(\widehat{h}, S' \cap S_{\mathsf{adv}}) \\ \geq (1 - C\varepsilon_{\mathsf{clean}}) \mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(\widehat{h}, S'_{\mathsf{clean}}).$$

This means that

$$\mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(\widehat{h}, S'_{\mathsf{clean}}) \leq \frac{\varepsilon_{\mathsf{clean}}}{1 - C\varepsilon_{\mathsf{clean}}} = 2\varepsilon_{\mathsf{clean}}\left(\frac{1}{1 - C\varepsilon_{\mathsf{clean}}}\right) \lesssim \varepsilon_{\mathsf{clean}}.$$

We now use the fact that  $\left| \mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(h, S'_{\mathsf{clean}}) - \mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(h, \mathcal{D}) \right| \leq \varepsilon_{\mathsf{clean}}$  to arrive at the conclusion that  $\mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(h, \mathcal{D}) \leq \varepsilon_{\mathsf{clean}}$ , which completes the proof of Theorem 26.

**Theorem 27** (Generalization implies filtering). Set  $\varepsilon_{\text{clean}} \leq 1/10$  and  $\alpha \leq 1/6$ .

If there exists an algorithm that, given at most a  $2\alpha$  fraction of outliers in the training set, can output a hypothesis satisfying  $\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}(\widehat{h}, \mathcal{D}) \leq \varepsilon_{clean}$  with probability  $1 - \delta$  over the draw of the training set, then there exists an algorithm that given a training set  $S = S_{clean} \cup S_{adv}$  satisfying  $|S_{clean}| \geq \Omega\left(\varepsilon_{clean}^{-2}\left(\mathsf{VC}\left(\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}\right) + \log(1/\delta)\right)\right)$  outputs a subset  $S' \subseteq S$  with the property that  $\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}\left(\operatorname*{argmin}_{h\in\mathcal{H}}\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}(h, S'), \mathcal{D}\right) \leq \varepsilon_{clean}$  with probability  $1 - 7\delta$ .

We first require the intermediate Claim 6.5.4.

**Claim 6.5.4.** *The following holds for all*  $h \in \mathcal{H}$ *:* 

$$\left|\mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(h, S_{\mathsf{clean}}) - \mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(h, S'_{\mathsf{clean}})\right| < 3\varepsilon_{\mathsf{clean}}$$

*Proof of Claim 6.5.4.* Recall that in the proof of Theorem 26, we showed that for positive numbers *a*, *b*, *c* we have

$$\max\left\{0,\frac{a-c}{b-c}\right\},\min\left\{1,\frac{a}{b-c}\right\}\in\left[\frac{a}{b}\pm\frac{c}{b}\right].$$

Now, let *a* denote the number of samples from  $S_{\text{clean}}$  that *h* incurs robust loss on, let *b* be the total number of samples from  $S_{\text{clean}}$ , and let *c* be the number of samples our filtering procedure deletes from  $S_{\text{clean}}$ . It is easy to see that a/b corresponds  $\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}(h, S_{\text{clean}})$  and that  $\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}(h, S'_{\text{clean}}) \in [\max \{0, (a-c)/(b-c)\}, \min \{1, a/(b-c)\}]$ . From our argument above, this means that we must have

$$\mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(h, S'_{\mathsf{clean}}) \in \left[ \mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(h, S_{\mathsf{clean}}) \pm \frac{2\varepsilon_{\mathsf{clean}}m}{(1-\alpha)m} \right].$$

Finally,

$$\frac{2\varepsilon_{\text{clean}}m}{(1-\alpha)m} = \frac{2\varepsilon_{\text{clean}}}{(1-\alpha)} \le \frac{2\varepsilon_{\text{clean}}}{\frac{5}{6}} < 3\varepsilon_{\text{clean}},$$

completing the proof of Claim 6.5.4.

We now have the tools we need to prove Theorem 27.

*Proof of Theorem 27.* See Algorithm 18 for the pseudocode of an algorithm witnessing the theorem statement.

At a high level, our proof proceeds as follows. We first show that the partitioning step results in partitions that do not have too high of a fraction of outliers, which will allow us to call the filtering procedure without exceeding the outlier tolerance. Then, we will show that the hypotheses  $\hat{h}_L$  and  $\hat{h}_R$  mark most of the backdoor points for deletion while marking only few of the clean points for deletion. Finally, we will show that although  $\hat{h}$  is learned on *S'* that is not sampled i.i.d from  $\mathcal{D}$ ,  $\hat{h}$  still generalizes to  $\mathcal{D}$  without great decrease in accuracy. **Algorithm 18** Implementation of a filtering algorithm given an implementation of a generalization algorithm

- 1: **Input**: Training set  $S = S_{clean} \cup S_{adv}$ satisfying  $|S_{clean}| = \Omega \left( \varepsilon_{clean}^{-2} \left( VC \left( \mathcal{L}_{\mathcal{F}_{adv}(h^{\star})} \right) + \log (1/\delta) \right) \right)$
- 2: Calculate  $\hat{h} = \underset{h \in \mathcal{H}}{\operatorname{argmin}} \mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}(h, S)$  and early-return *S* if  $\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}(\hat{h}, S) \leq C \varepsilon_{clean}$ , for some universal constant *C*
- 3: Randomly partition S into two equal halves  $S_L$  and  $S_R$
- 4: Run the generalizing algorithm to obtain  $h_L$  and  $h_R$  using training sets  $S_L$  and  $S_R$ , respectively
- 5: Run  $\hat{h}_L$  on  $S_R$  and mark every mistake that  $\hat{h}_L$  makes on  $S_R$ , and similarly for  $\hat{h}_R$
- 6: Remove all marked examples to obtain a new training set  $S' \subseteq S$
- 7: **Output**: S' such that  $\widehat{h} = \underset{h \in \mathcal{H}}{\operatorname{argmin}} \mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}(h, S')$  satisfies  $\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}(\widehat{h}, \mathcal{D}) \leq \varepsilon_{clean}$  with probability  $1 \delta$

We have two cases to consider based on the number of outliers in our training set. Let *m* be the total number of examples in our training set.

**Case 1** –  $\alpha m \leq \max \{2/3\varepsilon_{clean} \cdot \log(1/\delta), 24\log(2/\delta)\}$  It is easy to see that  $\mathcal{L}(h^*, S) \leq \alpha$ . Using this, we have

$$\mathcal{L}(h^{\star}, S) \leq \alpha \frac{2}{3\varepsilon_{\mathsf{clean}} \cdot m} \cdot \log\left(\frac{1}{\delta}\right) \lesssim \frac{\varepsilon_{\mathsf{clean}}}{\mathsf{VC}\left(\mathcal{H}\right) + \log\left(\frac{1}{\delta}\right)} \cdot \log\left(\frac{1}{\delta}\right) < \varepsilon_{\mathsf{clean}},$$

which implies that we exit the routine via the early-return. From uniform convergence, this implies that with probability  $1 - \delta$  over the draws of *S*, we have

$$\mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}\left(\underset{h \in \mathcal{H}}{\operatorname{argmin}}\mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(h, S'), \mathcal{D}\right) \lesssim \varepsilon_{\mathsf{clean}}.$$

In the other case, we write

$$\mathcal{L}(h^{\star}, S) \leq \alpha \leq \frac{24 \log (2/\delta)}{m} \lesssim \frac{\varepsilon_{\mathsf{clean}}^2 \log (1/\delta)}{\mathsf{VC}(\mathcal{H}) + \log (1/\delta)} \lesssim \varepsilon_{\mathsf{clean}}^2 \leq \varepsilon_{\mathsf{clean}},$$

and the rest follows from a similar argument.

**Case 2** –  $\alpha m \ge \max \{2/3\varepsilon_{\text{clean}} \cdot \log(1/\delta), 24\log(2/\delta)\}$  Let  $\tau = \delta$ ; we make this rewrite to help simplify the various failure events.

**Part 1 – Partitioning does not affect outlier balance.** Define indicator random variables  $X_i$  such that  $X_i$  is 1 if and only if example *i* ends up in  $S_R$ . We want to show that

$$\Pr\left[\sum_{i\in S_{\mathsf{adv}}} X_i \notin [0.5, 1.5] \,\alpha \cdot m/2\right] \leq \tau.$$

Although the  $X_i$  are not independent, they are negatively associated, so we can still use the Chernoff Bound and the fact that the number of outliers  $\alpha m \ge 24 \log (2/\tau)$ :

$$\Pr\left[\sum_{i \in S_{\mathsf{adv}}} X_i \notin [0.5, 1.5] \, \alpha \cdot m/2\right] \le 2 \exp\left(-\frac{\alpha/2 \cdot m \cdot 1/4}{3}\right) \le 2 \exp\left(-\frac{\alpha m}{24}\right) \le \tau$$

Moreover, if  $S_L$  has a  $[\alpha/2, 3\alpha/2]$  fraction of outliers, then it also follows that  $S_R$  has a  $[\alpha/2, 3\alpha/2]$  fraction of outliers. Thus, this step succeeds with probability  $1 - \tau$ .

**Part 2 – Approximately correctly marking points.** We now move onto showing that  $\hat{h}_L$  deletes most outliers from  $S_R$  while deleting few clean points. Recall that  $\hat{h}_L$  satisfies  $\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}(\hat{h}_L, \mathcal{D}) \leq \varepsilon_{clean}$  with probability  $1 - \delta$ . Thus, we have that  $\hat{h}_L$  labels the outliers as opposite the target label with probability at least  $1 - \varepsilon_{clean}$ . Since we have that the number of outliers  $\alpha m \geq 2/3\varepsilon_{clean} \cdot \log(1/\tau)$ , we have from Chernoff Bound that (let  $X_i$  be the indicator random variable that is 1 when  $\hat{h}_L$  classifies a backdoored example as the target label)

$$\Pr\left[\sum_{i\in S_{\mathsf{adv}}\cap S_R} X_i \ge 2 \cdot \left(\varepsilon_{\mathsf{clean}} \cdot \frac{3}{2}\alpha m\right)\right] \le \exp\left(-\varepsilon_{\mathsf{clean}} \cdot \frac{3}{2}\alpha m\right) \le \tau.$$

Thus, with probability  $1 - 2\tau$ , we mark all but at most  $\varepsilon_{\text{clean}} \cdot 6\alpha m$  outliers across both  $S_R$  and  $S_L$ ; since we impose that  $\alpha \leq 1$ , we have that we delete all but a  $c\varepsilon_{\text{clean}}$  fraction of outliers for some universal constant c.

It remains to show that we do not delete too many good points. Since  $\hat{h}_L$  has true error at most  $\varepsilon_{\text{clean}}$  and using the fact that  $m(1 - \alpha/2) \ge m(1 - \alpha) \ge m\alpha \ge \frac{2\log(1/\tau)}{\varepsilon_{\text{clean}}}$ , from the Chernoff Bound, we have (let  $X_i$  be the indicator random variable that is 1 when  $\hat{h}_L$  misclassifies a clean example)

$$\Pr\left[\sum_{i\in S_{\mathsf{clean}}\cap S_R} X_i \ge 2 \cdot \left(\varepsilon_{\mathsf{clean}} \cdot (1-\alpha/2) \cdot \frac{m}{2}\right)\right] \le \exp\left(-\varepsilon_{\mathsf{clean}} \cdot (1-\alpha/2) \cdot \frac{m}{2}\right) \le \tau.$$

From a union bound over the runs of  $\hat{h}_L$  and  $\hat{h}_R$ , we have that with probability  $1 - 2\tau$ , we mark at most  $2m\varepsilon_{\text{clean}} \cdot (1 - \alpha/2) \leq 2m\varepsilon_{\text{clean}}$  clean points for deletion. From a union bound, we have that this whole step succeeds with probability  $1 - 4\tau - 2\delta$ .

**Part 3 – There exists a low-error classifier.** At this stage, we have a training set S' that has at least  $m(1 - 2\varepsilon_{clean})$  clean points and at most  $\varepsilon_{clean} \cdot 6\alpha m$  outliers. Recall that  $h^*$  incurs robust loss on none of the clean points and incurs robust loss on every outlier. This implies that  $h^*$  has empirical robust loss at most

$$\frac{\varepsilon_{\text{clean}} \cdot 6\alpha m}{m(1 - 2\varepsilon_{\text{clean}})} = \frac{6\alpha\varepsilon_{\text{clean}}}{1 - 2\varepsilon_{\text{clean}}} \le 2\varepsilon_{\text{clean}},$$

where we use the fact that we pick  $\varepsilon_{\text{clean}} \leq 1/10 < 1/4$  and  $\alpha \leq 1/6$ . From this, it follows that  $\widehat{h} = \underset{h \in \mathcal{H}}{\operatorname{argmin}} \mathcal{L}_{\mathcal{F}_{\text{adv}}(h^{\star})}(h, S')$  satisfies  $\mathcal{L}_{\mathcal{F}_{\text{adv}}(h^{\star})}(\widehat{h}, S') \leq 2\varepsilon_{\text{clean}}$ .

**Part 4 – Generalizing from** S' to  $\mathcal{D}$ . We now have to argue that  $\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}(\hat{h}, S') \leq 2\varepsilon_{clean}$ implies  $\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}(\hat{h}, \mathcal{D}) \leq \varepsilon_{clean}$ . Recall that we have drawn enough samples to achieve uniform convergence (see [CBM18] and [MGDS20]); in particular, assuming that our previous steps succeeded in removing very few points from  $S_{clean}$ , then for all  $h \in \mathcal{H}$ , we have with probability  $1 - \delta$  that

$$\mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(h, \mathcal{D}) - \mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(h, S_{\mathsf{clean}}) \le \varepsilon_{\mathsf{clean}}.$$

Observe that we have deleted at most  $m \cdot 2\varepsilon_{\text{clean}}$  points from  $S_{\text{clean}}$ . Let  $S'_{\text{clean}} \coloneqq S' \cap S_{\text{clean}}$  (i.e., the surviving members of  $S_{\text{clean}}$  from our filtering procedure). We now use Claim 6.5.4 and triangle inequality to write

$$\begin{split} \left| \mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(h, S'_{\mathsf{clean}}) - \mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(h, \mathcal{D}) \right| &\leq \left| \mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(h, S_{\mathsf{clean}}) - \mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(h, S'_{\mathsf{clean}}) \right| + \\ & \left| \mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(h, \mathcal{D}) - \mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(h, S_{\mathsf{clean}}) \right| \\ &< 4\varepsilon_{\mathsf{clean}}. \end{split}$$

Next, consider some  $\hat{h}$  satisfying  $\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}(\hat{h}, S') \leq 2\varepsilon_{clean}$  (which must exist, as per our argument in Part 3), and observe that

$$\begin{split} \mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(\widehat{h}, S') &\geq (1 - 2\varepsilon_{\mathsf{clean}}) \mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(\widehat{h}, S' \cap S_{\mathsf{clean}}) + 2\varepsilon_{\mathsf{clean}} \mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(\widehat{h}, S' \cap S_{\mathsf{adv}}) \\ &\geq (1 - 2\varepsilon_{\mathsf{clean}}) \mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(\widehat{h}, S'_{\mathsf{clean}}). \end{split}$$

This implies

$$\mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(\widehat{h}, S'_{\mathsf{clean}}) \leq \frac{2\varepsilon_{\mathsf{clean}}}{1 - 2\varepsilon_{\mathsf{clean}}} = 2\varepsilon_{\mathsf{clean}}\left(\frac{1}{1 - 2\varepsilon_{\mathsf{clean}}}\right) \leq \frac{5\varepsilon_{\mathsf{clean}}}{2}.$$

We now use the fact that  $\left| \mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(h, S'_{\mathsf{clean}}) - \mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(h, \mathcal{D}) \right| < 4\varepsilon_{\mathsf{clean}}$  to arrive at the conclusion that  $\mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(h, \mathcal{D}) < \frac{13}{2} \cdot \varepsilon_{\mathsf{clean}}$ , which is the statement of Theorem 27.

The constants in the statement of Theorem 27 follow from setting  $\tau = \delta$ .

# 6.6. Numerical trials

In this section, we present a practical use case for Theorem 25.

Recall that, at a high level, Theorem 25 states that under certain assumptions, minimizing robust loss on the corrupted training set will either:

- 1. Result in a low robust loss, which will imply from uniform convergence that the resulting classifier is robust to adversarial (and therefore backdoor) perturbations.
- 2. Result in a high robust loss, which will be noticeable at training time.

This suggests that practitioners can use adversarial training on a training set which may be backdoored and use the resulting robust loss value to make a decision about whether to deploy the classifier. To empirically validate this approach, we run this procedure (i.e., some variant of Algorithm 16) on the MNIST handwritten digit classification task<sup>5</sup>(see [LC10]). Here, the learner wishes to recover a neural network robust to small  $\ell_{\infty}$  perturbations and where the adversary is allowed to make a small  $\ell_{\infty}$ -norm watermark.

**Disclaimers.** As far as we are aware, the MNIST dataset does not contain personally identifiable information or objectionable content. The MNIST dataset is made available under the terms of the Creative Commons Attribution-Share Alike 3.0 License.

**Reproducibility.** We have included all the code to generate these results in the supplementary material. Our code can be found at https://github.com/narenmanoj/mnist-adv-training.<sup>6</sup>. Our code is tested and working with TensorFlow 2.4.1, CUDA 11.0, NVIDIA RTX 2080Ti, and the Google Colab GPU runtime.

## 6.6.1. MNIST using neural networks

## Scenario

Recall that the MNIST dataset consists of 10 classes, where each corresponds to a handwritten digit in  $\{0, \ldots, 9\}$ . The classification task here is to recover a classifier that, upon receiving an image of a handwritten digit, correctly identifies which digit is present in the image.

In our example use case, an adversary picks a target label  $t \in \{0, ..., 9\}$  and a small additive watermark. If the true classifier is  $h^*(x)$ , then the adversary wants the learner to find a classifier  $\hat{h}$  maximizing  $\Pr_{x \sim \mathcal{D} \mid h^*(x) \neq t} \left[ \hat{h}(x) = t \right]$ . In other words, this can be seen as a "many-to-one" attack, where the adversary is corrupting examples whose labels are not t in order to

<sup>&</sup>lt;sup>5</sup>We select MNIST because one can achieve a reasonably robust classifier on the clean version of the dataset. This helps us underscore the difference between the robust loss at train time with and without backdoors in the training set. Moreover, this allows us to explore a setting where our assumptions in Theorem 25 might not hold – in particular, it's not clear that we have enough data to attain uniform convergence for the binary loss and  $\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}$ .

<sup>&</sup>lt;sup>6</sup>Some of our code is derived from the GitHub repositories https://github.com/MadryLab/backdoor\_data\_ poisoning and https://github.com/skmda37/Adversarial\_Machine\_Learning\_Tensorflow.

induce a classification of t. The adversary is allowed to inject some number of examples into the training set such that the resulting fraction of corrupted examples in the training set is at most  $\alpha$ .

We will experimentally demonstrate that the learner can use the intuition behind Theorem 25 to either recover a reasonably robust classifier or detect the presence of significant corruptions in the training set. Specifically, the learner can optimize a proxy for the robust loss via adversarial training using  $\ell_{\infty}$  bounded adversarial examples, as done by [MMSTV17].

**Instantiation of relevant problem parameters.** Let  $\mathcal{H}$  be the set of neural networks with architecture as shown in Table 6.1. Let  $\mathcal{X}$  be the set of images of handwritten digits; we represent these as vectors in  $[0, 1]^{784}$ . We define  $\mathcal{F}_{adv}$  as

{patch (x) :  $||x - \text{patch } (x)||_{\infty} \le 0.3$  and patch (x) - x = pattern},

where pattern is the shape of the backdoor (we use an "X" shape in the top left corner of the image, inspired by [TLM18]). We let the maximum  $\ell_{\infty}$  perturbation be at most 0.3 since this parameter has been historically used in training and evaluating robust networks on MNIST (see [MMSTV17]). In our setup, we demonstrate that these parameters suffice to yield a successful backdoor attack on a vanilla training procedure (described in greater detail in a subsequent paragraph).

Although it is not clear how to efficiently exactly calculate and minimize  $\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}$ , we will approximate  $\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}$  by calculating  $\ell_{\infty}$ -perturbed adversarial examples using a Projected Gradient Descent (PGD) attack. To minimize  $\mathcal{L}_{\mathcal{F}_{adv}(h^{\star})}$ , we use adversarial training as described in [MMSTV17]. Generating Table 6.3 takes roughly 155 minutes using our implementation of this procedure with TensorFlow 2.4.1 running on the GPU runtime freely available via Google Colab. We list all our relevant optimization and other experimental parameters in Table 6.2.

Layer	Parameters
Conv2D	<pre>filters=32,kernel_size=(3,3),activation='relu'</pre>
MaxPooling2D	<pre>pool_size=(2,2)</pre>
Conv2D	<pre>filters=64,kernel_size=(3,3),activation='relu'</pre>
Flatten	
Dense	units=1024,activation='relu'
Dense	units=10,activation='softmax'

Table 6.1.: Neural network architecture used in experiments.	We implemented this architecture
using the Keras API of TensorFlow 2.4.1.	-

**Optimization details.** See Table 6.2 for all relevant hyperparameters and see Table 6.1 for the architecture we use.

For the "Vanilla Training" procedure, we use no adversarial training and simply use our optimizer to minimize our loss directly. For the "PGD-Adversarial Training" procedure, we use adversarial training with a PGD adversary.

Table 6.2.: Experimental hyperparameters. We made no effort to optimize these hyperparameters; indeed, many of these are simply the default arguments for the respective TensorFlow functions.

Property	Details
Epochs	2
Validation Split	None
Batch Size	32
Loss	Sparse Categorical Cross Entropy
Optimizer	RMSProp (step size = 0.001, $\rho$ = 0.9, momentum = 0, $\varepsilon$ = 10 <sup>-7</sup> )
NumPy Random Seed	4321
TensorFlow Random Seed	1234
PGD Attack	$\varepsilon = 0.3$ , step size = 0.01, iterations = 40, restarts = 10

In our implementation of adversarial training, we compute adversarial examples for each image in each batch using the PGD attack and we minimize our surrogate loss on this new batch. This is sufficient to attain a classifier with estimated robust loss of around 0.08 on an uncorrupted training set.

## Goals and evaluation methods

We want to observe the impact of adding backdoor examples and the impact of running adversarial training on varied values of  $\alpha$  (the fraction of the training set that is corrupted).

To do so, we fix a value for  $\alpha$  and a target label t and inject enough backdoor examples such that exactly an  $\alpha$  fraction of the resulting training set contains corrupted examples. Then, we evaluate the train and test robust losses on the training set with and without adversarial training to highlight the difference in robust loss observable to the learner. As sanity checks, we also include binary losses and test set metrics. For the full set of metrics we collect, see Table 6.3.

To avoid out-of-memory issues when computing the robust loss on the full training set (roughly 60000 training examples and their adversarial examples), we sample 5000 training set examples uniformly at random from the full training set and compute the robust loss on these examples. By Hoeffding's Inequality [Ver18], this means that with probability 0.99 over the choice of the subsampled training set, the difference between our reported statistic and its population value is at most ~ 0.02.

### **Results and discussion**

Table 6.3.: Results with MNIST with a target label t = 0 and backdoor pattern "X." In each cell, the top number represents the respective value when the network was trained without any kind of robust training, and the bottom number represents the respective value when the network was trained using adversarial training as per [MMSTV17]. For example, at  $\alpha = 0.05$ , for Vanilla Training, the training 0 - 1 loss is only 0.01, but the training robust loss is 1.00, whereas for PGD-Adversarial Training, the training 0 - 1 loss is 0.07 and the training robust loss is 0.13. The Backdoor Success Rate is our estimate of  $\Pr_{x \sim \mathcal{D} || y \neq t}$  [patch (x) = t], which may be less than the value of the robust loss.

	α	0.00	0.05	0.15	0.20	0.30
Training 0 1 Loop	Vanilla Training	0.01	0.01	0.01	0.01	0.01
11aning 0 = 1 Loss	PGD-Adversarial Training	0.02	0.07	0.17	0.22	0.33
Training Robust Loss	Vanilla Training	1.00	1.00	1.00	1.00	1.00
	PGD-Adversarial Training	0.09	0.13	0.24	0.27	0.41
Testing 0 – 1 Loss	Vanilla Training	0.01	0.01	0.01	0.02	0.01
	PGD-Adversarial Training	0.02	0.03	0.03	0.03	0.06
Tasting Pobust Loss	Vanilla Training	1.00	1.00	1.00	1.00	1.00
lesting Robust Loss	PGD-Adversarial Training	0.09	0.09	0.11	0.10	0.19
Backdoor Success Rate	Vanilla Training	0.00	1.00	1.00	1.00	1.00
	PGD-Adversarial Training	0.00	0.00	0.01	0.00	0.05

See Table 6.3 for sample results from our trials. Over runs of the same experiment with varied target labels t, we attain similar results; see Section 6.6.1 for the full results. We now discuss the key takeaways from this numerical trial.

**Training robust loss increases with**  $\alpha$ . Observe that our proxy for  $\mathcal{L}_{\mathcal{F}_{\mathsf{adv}}(h^{\star})}(\hat{h}, S)$  increases as  $\alpha$  increases. This is consistent with the intuition from Theorem 25 in that a highly corrupted training set is unlikely to have low robust loss. Hence, if the learner expects a reasonably low robust loss and fails to observe this during training, then the learner can reject the training set, particularly at high  $\alpha$ .

**Smaller**  $\alpha$  **and adversarial training defeats backdoor.** On the other hand, notice that at smaller values of  $\alpha$  (particularly  $\alpha \le 0.20$ ), the learner can still recover a classifier with minimal decrease in robust accuracy. Furthermore, there is not an appreciable decrease in natural accuracy either when using adversarial training on a minimally corrupted training set. Interestingly, even at large  $\alpha$ , the test-time robust loss and binary losses are not too high when adversarial training was used. Furthermore, the test-time robust loss attained at  $\alpha > 0$  is certainly better than that obtained when adversarial training is not used, even at  $\alpha = 0$ . Hence, although the

practitioner cannot certify that the learned model is robust without a clean validation set, the learned model does tend to be fairly robust.

**Backdoor is successful with vanilla training.** Finally, as a sanity check, notice that when we use vanilla training, the backdoor trigger induces a targeted misclassification very reliably, even at  $\alpha = 0.05$ . Furthermore, the training and testing error on clean data is very low, which indicates that the learner would have failed to detect the fact that the model had been corrupted had they checked only the training and testing errors before deployment.

**Prior empirical work.** The work of [BCFGGGGG21] empirically shows the power of data augmentation in defending against backdoored training sets. Although their implementation of data augmentation is different from ours<sup>7</sup>, their work still demonstrates that attempting to minimize some proxy for the robust loss can lead to a classifier robust to backdoors at test time. However, our evaluation also demonstrates that classifiers trained using adversarial training can be robust against test-time adversarial attacks, in addition to being robust to train-time backdoor attacks. Furthermore, our empirical results indicate that the train-time robust loss can serve as a good indicator for whether a significant number of backdoors are in the training set.

<sup>&</sup>lt;sup>7</sup>Observe that our implementation of adversarial training can be seen as a form of adaptive data augmentation.

# **Results for all target labels**

Here, we present tables of the form of Table 6.3 for all choices of target label  $t \in \{0, ..., 9\}$ . Notice that the key takeaways remain the same across all target labels.

				-		
	α	0.00	0.05	0.15	0.20	0.30
Training 0 – 1 Loss	Vanilla Training	0.01	0.01	0.01	0.01	0.01
	PGD-Adversarial Training	0.02	0.07	0.17	0.22	0.33
Training Robust Loss	Vanilla Training	1.00	1.00	1.00	1.00	1.00
	PGD-Adversarial Training	0.09	0.13	0.24	0.27	0.41
Testing 0 – 1 Loss	Vanilla Training	0.01	0.01	0.01	0.02	0.01
	PGD-Adversarial Training	0.02	0.03	0.03	0.03	0.06
Testing Robust Loss	Vanilla Training	1.00	1.00	1.00	1.00	1.00
	PGD-Adversarial Training	0.09	0.09	0.11	0.10	0.19
Backdoor Success Rate	Vanilla Training	0.00	1.00	1.00	1.00	1.00
	PGD-Adversarial Training	0.00	0.00	0.01	0.00	0.05

Table 6.4.: Results with MNIST with a target label t = 0 and backdoor pattern "X."

Table 6.5.: Results with MNIST with a target label $t = 1$ and backdoor pattern "X."
--

	-			-		
α		0.00	0.05	0.15	0.20	0.30
Turining 0 1 Laga	Vanilla Training	0.01	0.01	0.01	0.01	0.01
	PGD-Adversarial Training	0.02	0.07	0.17	0.23	0.32
Training Robust Loss	Vanilla Training	1.00	1.00	1.00	1.00	1.00
	PGD-Adversarial Training	0.08	0.12	0.23	0.32	0.38
Testing 0 – 1 Loss	Vanilla Training	0.01	0.01	0.01	0.01	0.01
	PGD-Adversarial Training	0.02	0.02	0.03	0.04	0.05
Testing Pobust Loss	Vanilla Training	1.00	1.00	1.00	1.00	1.00
Testing Robust Loss	PGD-Adversarial Training	0.09	0.08	0.11	0.13	0.14
Backdoor Success Rate	Vanilla Training	0.00	1.00	1.00	1.00	1.00
	PGD-Adversarial Training	0.00	0.00	0.00	0.02	0.03

α		0.00	0.05	0.15	0.20	0.30
Training 0 1 Loss	Vanilla Training	0.01	0.01	0.01	0.01	0.00
	PGD-Adversarial Training	0.02	0.07	0.17	0.22	0.32
Training Robust Loss	Vanilla Training	1.00	1.00	1.00	1.00	1.00
	PGD-Adversarial Training	0.08	0.13	0.23	0.28	0.38
Testing 0 – 1 Loss	Vanilla Training	0.01	0.02	0.01	0.02	0.01
	PGD-Adversarial Training	0.02	0.03	0.03	0.03	0.05
Testing Pobust Loss	Vanilla Training	1.00	1.00	1.00	1.00	1.00
Testing Robust Loss	PGD-Adversarial Training	0.09	0.09	0.10	0.10	0.14
Backdoor Success Rate	Vanilla Training	0.00	1.00	1.00	1.00	1.00
	PGD-Adversarial Training	0.00	0.00	0.00	0.01	0.04

Table 6.6.: Results with MNIST with a target label t = 2 and backdoor pattern "X."

Table 6.7.: Results with MNIST with a target label t = 3 and backdoor pattern "X."

	α	0.00	0.05	0.15	0.20	0.30
Training 0 1 Loop	Vanilla Training	0.01	0.01	0.01	0.01	0.01
fraining 0 – 1 Loss	PGD-Adversarial Training	0.02	0.07	0.18	0.23	0.32
Training Robust Loss	Vanilla Training	1.00	1.00	1.00	1.00	1.00
	PGD-Adversarial Training	0.08	0.13	0.23	0.28	0.38
Testing 0 – 1 Loss	Vanilla Training	0.01	0.01	0.01	0.02	0.02
	PGD-Adversarial Training	0.02	0.02	0.03	0.04	0.05
Testing Robust Loss	Vanilla Training	1.00	1.00	1.00	1.00	1.00
Testing Robust Loss	PGD-Adversarial Training	0.09	0.09	0.11	0.11	0.13
Backdoor Success Rate	Vanilla Training	0.00	1.00	1.00	1.00	1.00
	PGD-Adversarial Training	0.00	0.01	0.00	0.01	0.03

	α	0.00	0.05	0.15	0.20	0.30
Training 0 1 Loss	Vanilla Training	0.01	0.01	0.01	0.01	0.01
fraining 0 1 Loss	PGD-Adversarial Training	0.02	0.07	0.17	0.22	0.32
Training Robust Loss	Vanilla Training	1.00	1.00	1.00	1.00	1.00
Training Robust Loss	PGD-Adversarial Training	0.08	0.13	0.24	0.27	0.42
Testing 0 – 1 Loss	Vanilla Training	0.01	0.01	0.01	0.01	0.01
	PGD-Adversarial Training	0.02	0.02	0.03	0.03	0.05
Testing Pobust Loss	Vanilla Training	1.00	1.00	1.00	1.00	1.00
Testing Robust Loss	PGD-Adversarial Training	0.08	0.09	0.11	0.10	0.15
Backdoor Success Rate	Vanilla Training	0.00	1.00	1.00	1.00	1.00
	PGD-Adversarial Training	0.00	0.00	0.01	0.01	0.04

Table 6.8.: Results with MNIST with a target label t = 4 and backdoor pattern "X."

Table 6.9.: Results with MNIST with a target label t = 5 and backdoor pattern "X."

	α	0.00	0.05	0.15	0.20	0.30
Training 0 1 Loop	Vanilla Training	0.01	0.01	0.01	0.01	0.01
fraining 0 – 1 Loss	PGD-Adversarial Training	0.02	0.07	0.17	0.22	0.33
Training Robust Loss	Vanilla Training	1.00	1.00	1.00	1.00	1.00
	PGD-Adversarial Training	0.07	0.13	0.23	0.28	0.41
Testing 0 – 1 Loss	Vanilla Training	0.01	0.01	0.01	0.02	0.02
	PGD-Adversarial Training	0.02	0.03	0.03	0.03	0.06
Tasting Robust Loss	Vanilla Training	1.00	1.00	1.00	1.00	1.00
Testing Robust Loss	PGD-Adversarial Training	0.08	0.09	0.11	0.10	0.16
Backdoor Success Rate	Vanilla Training	0.00	1.00	1.00	1.00	1.00
	PGD-Adversarial Training	0.00	0.00	0.01	0.01	0.05

	α	0.00	0.05	0.15	0.20	0.30
Training 0 – 1 Loss	Vanilla Training	0.01	0.01	0.01	0.01	0.01
	PGD-Adversarial Training	0.02	0.07	0.17	0.22	0.33
Training Robust Loss	Vanilla Training	1.00	1.00	1.00	1.00	1.00
	PGD-Adversarial Training	0.08	0.12	0.24	0.27	0.40
Testing 0 – 1 Loss	Vanilla Training	0.01	0.02	0.01	0.01	0.01
	PGD-Adversarial Training	0.02	0.03	0.03	0.03	0.06
Testing Robust Loss	Vanilla Training	1.00	1.00	1.00	1.00	1.00
	PGD-Adversarial Training	0.09	0.09	0.12	0.10	0.16
Backdoor Success Rate	Vanilla Training	0.00	1.00	1.00	1.00	1.00
	PGD-Adversarial Training	0.00	0.00	0.01	0.01	0.04

Table 6.10.: Results with MNIST with a target label t = 6 and backdoor pattern "X."

Table 6.11.: Results with MNIST with a target label t = 7 and backdoor pattern "X."

	α	0.00	0.05	0.15	0.20	0.30
Training 0 – 1 Loss	Vanilla Training	0.01	0.01	0.01	0.01	0.01
	PGD-Adversarial Training	0.02	0.07	0.18	0.22	0.32
Training Robust Loss	Vanilla Training	1.00	1.00	1.00	1.00	1.00
	PGD-Adversarial Training	0.07	0.12	0.25	0.29	0.39
Testing 0 – 1 Loss	Vanilla Training	0.01	0.01	0.01	0.02	0.01
	PGD-Adversarial Training	0.02	0.03	0.03	0.03	0.04
Testing Robust Loss	Vanilla Training	1.00	1.00	1.00	1.00	1.00
	PGD-Adversarial Training	0.08	0.08	0.11	0.10	0.13
Backdoor Success Rate	Vanilla Training	0.00	1.00	1.00	1.00	1.00
	PGD-Adversarial Training	0.00	0.00	0.00	0.00	0.03

	α	0.00	0.05	0.15	0.20	0.30
Training 0 – 1 Loss	Vanilla Training	0.01	0.01	0.01	0.01	0.01
	PGD-Adversarial Training	0.02	0.07	0.17	0.22	0.32
Training Robust Loss	Vanilla Training	1.00	1.00	1.00	1.00	1.00
	PGD-Adversarial Training	0.08	0.14	0.23	0.28	0.41
Testing 0 – 1 Loss	Vanilla Training	0.01	0.01	0.01	0.01	0.01
	PGD-Adversarial Training	0.02	0.03	0.03	0.03	0.05
Testing Robust Loss	Vanilla Training	1.00	1.00	1.00	1.00	1.00
	PGD-Adversarial Training	0.08	0.09	0.11	0.10	0.17
Backdoor Success Rate	Vanilla Training	0.00	1.00	1.00	1.00	1.00
	PGD-Adversarial Training	0.00	0.00	0.01	0.01	0.05

Table 6.12.: Results with MNIST with a target label t = 8 and backdoor pattern "X."

Table 6.13.: Results with MNIST with a target label t = 9 and backdoor pattern "X."

	α	0.00	0.05	0.15	0.20	0.30
Training 0 – 1 Loss	Vanilla Training	0.01	0.01	0.01	0.01	0.01
	PGD-Adversarial Training	0.02	0.07	0.17	0.22	0.33
Training Robust Loss	Vanilla Training	1.00	1.00	1.00	1.00	1.00
	PGD-Adversarial Training	0.08	0.13	0.23	0.29	0.43
Testing 0 – 1 Loss	Vanilla Training	0.01	0.01	0.01	0.01	0.01
	PGD-Adversarial Training	0.02	0.03	0.03	0.04	0.06
Testing Robust Loss	Vanilla Training	1.00	1.00	1.00	1.00	1.00
	PGD-Adversarial Training	0.09	0.10	0.11	0.11	0.20
Backdoor Success Rate	Vanilla Training	0.00	1.00	1.00	1.00	1.00
	PGD-Adversarial Training	0.01	0.01	0.01	0.01	0.06

# 7. Spectral clustering in semirandom stochastic block models

In this chapter, we study the robustness of spectral clustering algorithms under helpful model misspecification. This chapter is based on joint work with Aditya Bhaskara, Agastya Vibhuti Jha, Michael Kapralov, Davide Mazzali, and Weronika Wrzos-Kaminska [BJKMMW24].

# 7.1. Introduction

Graph partitioning or clustering is a fundamental unsupervised learning primitive. In a graph partitioning problem, one seeks to identify clusters of vertices that are highly internally connected and sparsely connected to the outside. This task is of particular significance when the given graph presents a latent community structure. In this setting, the goal is to recover the communities as accurately as possible. Various statistical models that attempt to capture this situation have been proposed and studied in the literature. Perhaps the most popular of these is the Symmetric Stochastic Block Model (SSBM) [HLL83].

Following the notation of previous works [AFWZ20; DLS21], in this chapter we describe an SSBM with specifications n,  $P_1$ ,  $P_2$ , p, q, where n is an even positive integer,  $P_1$  and  $P_2$  are a partitioning of the vertex set  $V = \{1, ..., n\}$  into subsets of equal size, and p and q are probabilities. Without loss of generality, we may assume that the partitions  $P_1$  and  $P_2$  consist of vertices 1, ..., n/2 and n/2+1, ..., n, respectively. Hence, with a mild abuse of notation, we write an SSBM with parameters n, p, q only and write it as SSBM(n, p, q). Now, let SSBM(n, p, q) be a distribution over random undirected graphs G = (V, E) where each edge  $(v, w) \in P_1 \times P_1$  and  $(v, w) \in P_2 \times P_2$  (which we refer to as "internal edges") appears independently with probability q. When  $p \gg q$ , there should be many more internal edges than crossing edges. Hence, we expect the community structure to become more evident as p tends away from q.

In such scenarios, our general algorithmic goal is to efficiently identify  $P_1$  and  $P_2$  when given G without any community labels. This task is hereafter referred to as the *graph bisection problem*. In this work, we will be interested in *exact recovery*, also known as *strong consistency*, in which we want an algorithm that, with probability at least 1 - 1/n over the randomness of the instance, exactly returns the partition  $\{P_1, P_2\}$  for all n sufficiently large. Other approximate notions of recovery (such as almost exact, partial, and weak recovery) are also well-studied but are beyond the scope of this work.

Although the SSBM(n, p, q) distribution over graphs is a useful starting point for algorithm design and has led to a deep theory about when recovery is possible and of what nature [Abb18], it may not be representative of all scenarios in which we should expect our algorithms to succeed. To remedy this, researchers have proposed several different random graph models

<b>Algorithm 19</b> SpectralBisection: given $G = (V, E)$ , outputs a bipartition of V							
1: <b>p</b>	rocedure SpectralBisection(G)		$\triangleright G = (V, E)$ is a	the input graph			
2:	$\mathbf{M} \leftarrow Matrix(G)$	$\triangleright \mathbf{M} \in \mathbb{R}^{V \times V}$	is a matrix with n	eal eigenvalues			
3:	$((\lambda_i, \boldsymbol{u}_i))_{i=1}^n \leftarrow \text{eigenvalue-eigenvector pairs}$	of <b>M</b> with $\lambda_1$	$\leq \cdots \leq \lambda_n$	$\triangleright n =  V $			
4:	$S \leftarrow \{ v \in V : u_2[v] < 0 \}$						
5:	return $\{S, V \setminus S\}$						

that may be more reflective of properties satisfied by real-world networks. These include the geometric block model [GNW24], the Gaussian mixture block model [LS24], and others.

In this chapter, we take a different perspective to graph generation by considering various *semirandom models*. At a high level, a semirandom model for a statistical problem interpolates between an average-case input (for example produced by a model such as the SSBM) and a worst-case input, in a way that still allows for a meaningful notion of ground-truth solution. In our context of graph bisection, this can be achieved by an adversary adding internal edges or by the distribution of internal edges itself being nonhomogeneous (i.e., every internal edge (v, w) appears independently with probability  $p_{vw} \ge p$ , where the  $p_{vw}$  may be chosen adversarially for each internal edge). Researchers have studied similar semirandom models for graph bisection [FK01; MMV12; MPW16b; Moi21a; CdM24] and other statistical problems such as classification under Massart noise [MN06b], detecting a planted clique in a random graph [FK01; CSV17; MMT20; BKS23], sparse recovery [KLLST23], and top-K ranking [YCOM24].

These modeling modifications are not necessarily meant to capture a real-world data generation process. Rather, they are a useful testbed with which we can determine whether commonly used algorithms have overfit to statistical assumptions present in the model. In particular, observe that these changes in model specification are ostensibly helpful, in that increasing the number of internal edges should only enhance the community structure. Perhaps surprisingly, it is known that a number of natural algorithms that succeed in the SSBM setting no longer work under such helpful modifications [Moi21a]. Therefore, it is natural to ask which algorithms for graph bisection are robust in semirandom models.

At this point, the performance of approaches based on convex programming is well-understood in various semirandom models [FK01; MMV12; MPW16b; Moi21a; CdM24]. However, in practice, it is impractical to run such an algorithm due to computational costs. Another class of algorithms, that we call *spectral algorithms*, is more widely used in practice. Loosely speaking, a spectral algorithm constructs a matrix M that is a function of the graph G and outputs a clustering arising from the embedding of the vertices determined by the eigenvectors of M. Popular choices of matrices include the unnormalized Laplacian  $L_G$  and the normalized Laplacian  $\mathcal{L}_G$  (we will formally define and intuit these notions in the sequel) [Von07]. This is because structural properties of both  $L_G$  and  $\mathcal{L}_G$  imply that the second smallest eigenvalue of each, denoted as  $\lambda_2(L_G)$  and  $\lambda_2(\mathcal{L}_G)$ , serves as a continuous proxy for connectivity, and the corresponding eigenvector,  $u_2(L_G)$  and  $u_2(\mathcal{L}_G)$ , has entries whose signs reveal a lot of information about the underlying community structure. This motivates Algorithm 19. It can be run, for example, with Matrix(G) :=  $L_G$  or Matrix(G) :=  $\mathcal{L}_G$ . Following this discussion, we arrive at the question we study in this chapter.

**Question 1.** Under which semirandom models do the Laplacian-based spectral algorithms, using the second eigenvector of  $\mathbf{L}_G$  or  $\mathcal{L}_G$ , exactly recover the ground-truth communities  $P_1$  and  $P_2$ ?

Main contributions. Our results show a surprising difference in the robustness of spectral
bisection when considering the normalized versus the unnormalized Laplacian. We summarize our results below:

- Consider a nonhomogeneous symmetric stochastic block model with parameters  $q , where every internal edge appears independently with probability <math>p_{uv} \in [p, \overline{p}]$  and every crossing edge appears independently with probability q. We show that under an appropriate spectral gap condition, the spectral algorithm with the unnormalized Laplacian exactly recovers the communities  $P_1$  and  $P_2$ . Moreover, this holds even if an adversary plants  $\ll np$  internal edges per vertex prior to the edge sampling phase.
- Consider a stronger semirandom model where the subgraphs on the two communities  $P_1$  and  $P_2$  are adversarially chosen and the crossing edges are sampled independently with probability q. We show that if the graph is sufficiently dense and satisfies a spectral gap condition, then the spectral algorithm with the unnormalized Laplacian exactly recovers the communities  $P_1$  and  $P_2$ .
- We show that there is a family of instances from a nonhomogeneous symmetric stochastic block model in which the spectral algorithm achieves exact recovery with the unnormalized Laplacian, but incurs a constant error rate with the normalized Laplacian. This is surprising because it contradicts conventional wisdom that normalized spectral clustering should be favored over unnormalized spectral clustering [Von07].

We also numerically complement our findings via experiments on various parameter settings.

**Outline.** The rest of this chapter is organized as follows. In Section 7.2, we more formally define our semirandom models, the Laplacians L and  $\mathcal{L}$ , and formally state our results. In Section 7.3, we give sketches of the proofs of our results. In Section 7.4, we show results from numerical trials suggested by our theory. In Sections 7.5.1 and 7.5.5 we prove important auxiliary lemmas we need for our results. In Section 7.5.6, we prove our robustness results for the unnormalized Laplacian. In Section 7.5.8, we prove our inconsistency result for the normalized Laplacian. In Section 7.6, we give additional numerical trials and discussion.

# 7.2. Models and main results

In this chapter, we study unnormalized and normalized spectral clustering in several semirandom SSBMs. These models permit a richer family of graphs than the SSBM alone.

**Matrices related to graphs.** Throughout this chapter, all graphs are to be interpreted as being undirected, and we assume that the vertices of an *n*-vertex graph coincide with the set  $\{1, ..., n\}$ . With this in mind, we begin with defining various matrices associated with graphs, building up to the unnormalized and normalized Laplacians, which are central to the family of algorithms we analyze (Algorithm 19).

**Definition 7.2.1** (Adjacency matrix). Let G = (V, E) be a graph. The adjacency matrix  $\mathbf{A}_G \in \mathbb{R}^{V \times V}$  of G is the matrix with entries defined as  $\mathbf{A}_G[v, w] = \mathbb{1} \{(v, w) \in E\}$ .

**Definition 7.2.2** (Degree matrix). Let G = (V, E) be a graph. The degree matrix  $\mathbf{D}_G \in \mathbb{R}^{V \times V}$  of G is the diagonal matrix with entries defined as  $\mathbf{D}_G[v, v] = \mathbf{d}_G[v]$ , where  $\mathbf{d}_G[v]$  is the degree of v.

**Definition 7.2.3** (Unnormalized Laplacian). Let G = (V, E) be a graph. The unnormalized Laplacian  $\mathbf{L}_G \in \mathbb{R}^{V \times V}$  of G is the matrix defined as  $\mathbf{L}_G \coloneqq \mathbf{D}_G - \mathbf{A}_G = \sum_{(v,w) \in E} (\mathbf{e}_v - \mathbf{e}_w) (\mathbf{e}_v - \mathbf{e}_w)^{\top}$ , where  $\mathbf{e}_i$  denotes the *i*-th standard basis vector.

**Definition 7.2.4** (Normalized Laplacians). Let G = (V, E) be a graph. The symmetric normalized Laplacian  $\mathcal{L}_{G,sym} \in \mathbb{R}^{V \times V}$  and the random walk Laplacian  $\mathcal{L}_{G,rw} \in \mathbb{R}^{V \times V}$  of G are defined as

$$\mathcal{L}_{G,\mathsf{sym}} \coloneqq \mathbf{I} - \mathbf{D}_G^{-1/2} \mathbf{A}_G \mathbf{D}_G^{-1/2}, \qquad \qquad \mathcal{L}_{G,\mathsf{rw}} \coloneqq \mathbf{I} - \mathbf{D}_G^{-1} \mathbf{A}_G.$$

For all notions above, when the graph *G* is clear from context, we omit the subscript *G*. Furthermore, when we discuss normalized Laplacians, we intend its symmetric version  $\mathcal{L}_{sym}$  unless otherwise stated. So, we omit this subscript as well and simply write  $\mathcal{L}$ .

Next, we define the spectral bisection algorithms. We will discuss some intuition for why these algorithms are reasonable heuristics in Section 7.3.

**Definition 7.2.5** (Unnormalized and normalized spectral bisection). Let G = (V, E) be a graph, and let its unnormalized and normalized Laplacians be L and  $\mathcal{L}$ , respectively. We refer to the algorithm resulting from running Algorithm 19 on G with Matrix(G) := L<sub>G</sub> as unnormalized spectral bisection. We refer to the algorithm resulting from running Algorithm 19 on G with Matrix(G) =  $\mathcal{L}_G$  as normalized spectral bisection.

Our goal is to understand when the above algorithms, applied to a graph with a latent community structure, achieve *exact recovery* or *strong consistency*, defined as follows.

**Definition 7.2.6.** Let  $\{P_1, P_2\}$  be a partitioning of  $V = \{1, ..., n\}$ , and let  $\mathcal{D} \coloneqq \mathcal{D}(\{P_1, P_2\})$  be a distribution over *n*-vertex graphs G = (V, E). We say that an algorithm is strongly consistent or achieves exact recovery on  $\mathcal{D}$  if given a graph  $G \sim \mathcal{D}$  it outputs the correct partitioning  $\{P_1, P_2\}$  with probability at least 1 - 1/n over the randomness of G.

#### 7.2.1. Nonhomogeneous symmetric stochastic block model

Our first model is a family of nonhomogeneous symmetric stochastic block models, defined below.

*Model* 7.1 (Nonhomogeneous symmetric stochastic block model). Let *n* be an even positive integer,  $V = \{1, ..., n\}$ ,  $\{P_1, P_2\}$  be a partitioning of *V* into two equally-sized subsets, and  $q be probabilities. Let <math>\mathcal{D}$  be any probability distribution over graphs G = (V, E) such that for every  $(v, w) \in P_1 \times P_1$  and  $(v, w) \in P_2 \times P_2$ , the edge (v, w) appears in *E* independently with some probability  $p_{vw} \in [p, \overline{p}]$ , and for every  $(v, w) \in P_1 \times P_2$ , the edge (v, w) appears in *E* independently with probability *q*. We call such  $\mathcal{D}$  a nonhomogeneous symmetric stochastic block model (which we will abbreviate as NSSBM). We call the set of all such  $\mathcal{D}$  the family of nonhomogeneous stochastic block models with parameters  $p, \overline{p}, q$ , written as NSSBM $(n, p, \overline{p}, q)$ .

To visualize Model 7.1, consider the expected adjacency matrix of some NSSBM distribution. We then have the relations

$$\begin{bmatrix} p \cdot \mathbf{J}_{n/2} & q \cdot \mathbf{J}_{n/2} \\ \hline q \cdot \mathbf{J}_{n/2} & p \cdot \mathbf{J}_{n/2} \end{bmatrix} \leq \begin{bmatrix} \mathbf{P}_{P_1} & q \cdot \mathbf{J}_{n/2} \\ \hline q \cdot \mathbf{J}_{n/2} & \mathbf{P}_{P_2} \end{bmatrix} \leq \begin{bmatrix} \overline{p} \cdot \mathbf{J}_{n/2} & q \cdot \mathbf{J}_{n/2} \\ \hline q \cdot \mathbf{J}_{n/2} & \overline{p} \cdot \mathbf{J}_{n/2} \end{bmatrix},$$

where the leftmost matrix denotes the expected adjacency matrix of SSBM(n, p, q), the rightmost matrix denotes the expected adjacency matrix of SSBM( $n, \overline{p}, q$ ),  $\mathbf{J}_k$  denotes the  $k \times k$ all-ones matrix, and  $\mathbf{P}_{P_1}$  and  $\mathbf{P}_{P_2}$  denote the edge probability matrices for edges internal to  $P_1$ and  $P_2$ , respectively.

The above also shows that the rank of the expected adjacency matrix for SSBM(n, p, q) is 2. However, the rank for the expected adjacency matrix for some NSSBM distribution may be as large as  $\Omega(n)$ . Perhaps surprisingly, this will turn out to be unimportant for our entrywise eigenvector perturbation analysis. In particular, the tools we use were originally designed for low-rank signal matrices or spiked low-rank signal matrices [AFWZ20; DLS21; BV24], but we will see that they can be adapted to the signal matrices we consider.

The NSSBM family generalizes the symmetric stochastic block model described in the previous section – this is attained by setting  $p_{vw} = p$  for all internal edges (v, w). However, it can also encode biases for certain graph properties. For instance, a distribution from the NSSBM family may encode the idea that certain subsets of  $P_1$  are expected to be denser than  $P_1$  as a whole.

With this definition in hand, we are ready to formally state our first technical result in Theorem 29.

**Theorem 29.** Let  $p, \overline{p}, q$  be probabilities such that  $q and such that <math>\alpha \coloneqq \overline{p}/(p-q)$  is an arbitrary constant. Let  $\mathcal{D} \in \text{NSSBM}(n, p, \overline{p}, q)$ . Let  $n \geq N(\alpha)$  where the function  $N(\alpha)$  only depends on  $\alpha$ . There exists a universal constant C > 0 such that if

$$n(p-q) \ge C\left(\sqrt{n\overline{p}\log n} + \log n\right),$$
 (gap condition)

then unnormalized spectral bisection is strongly consistent on  $\mathcal{D}$ .

We prove Theorem 29 in Section 7.5.7. In fact, we show a somewhat stronger statement – in addition to the process described above, we also allow the adversary to, before sampling the graph, set a small number of the  $p_{vw}$  to 1 (at most  $n\overline{p}/\log \log n$  edges per vertex). We detail this further in Section 7.5.7.

We now remark on the tightness of our gap condition in Theorem 29. A work of Abbe, Bandeira, and Hall [ABH16] identifies an exact information-theoretic threshold above which exact recovery with high probability is possible and below which no algorithm can be strongly consistent. In particular, the threshold states that for any p and q satisfying  $\sqrt{p} - \sqrt{q} > \sqrt{2 \log n/n}$ , exact recovery is possible, and when p and q do not satisfy this, exact recovery is informationtheoretically impossible. Furthermore, Feige and Kilian [FK01] prove that the informationtheoretic threshold does not change in a somewhat stronger semirandom model that includes the NSSBM family. Additionally, Deng, Ling, and Strohmer [DLS21] show that unnormalized spectral bisection is strongly consistent all the way to this threshold in the special case where the graph is drawn from SSBM(n, p, q). By contrast, our gap condition holds in the same critical degree regime as in the information-theoretic threshold (namely,  $p = \Theta(\log n/n)$ ) but our constant is not optimal. We incur this constant loss because for the sake of presentation, we opt for a cleaner argument that can handle the nonhomogeneity and generalizes more readily across degree regimes. To our knowledge, none of these features are present in prior work analyzing spectral methods in an SSBM setting [AFWZ20; DLS21].

# 7.2.2. Deterministic clusters model

Given Theorem 29, it is natural to ask what happens if we allow the adversary full control over the structure of the graphs in  $P_1$  and  $P_2$  instead of simply allowing the adversary to perturb the edge probabilities. In this section, we answer this question. We first describe a more adversarial semirandom model than the NSSBM family. We call this model the *deterministic clusters* model, defined as follows.

*Model* 7.2 (Deterministic clusters model). Let *n* be an even positive integer,  $V = \{1, ..., n\}$ ,  $\{P_1, P_2\}$  be a partitioning of *V* into two equally-sized subsets, *q* be a probability, and  $d_{in}$  be an integer degree lower bound. Consider a graph G = (V, E) generated according to the following process.

- 1. The adversary chooses arbitrarily graphs  $G[P_1]$  and  $G[P_2]$  with minimum degree  $d_{in}$ ;
- 2. Nature samples every edge  $(v, w) \in P_1 \times P_2$  to be in *E* independently with probability *q*.
- 3. The adversary arbitrarily adds edges  $(v, w) \in P_1 \times P_1$  and  $(v, w) \in P_2 \times P_2$  to *E* after observing the edges sampled by nature.

We call a distribution  $\mathcal{D}$  of graphs generated according to the above process a deterministic clusters model (DCM). We call the set of all such  $\mathcal{D}$  the family of deterministic clusters models with parameters  $d_{in}$  and q, written as DCM(n,  $d_{in}$ , q).

The DCM graph generation process is heavily motivated by the one studied by Makarychev, Makarychev, and Vijayaraghavan [MMV12]. This model is much more flexible than the SSBM and NSSBM settings in that the graphs the adversary draws on  $P_1$  and  $P_2$  are allowed to look very far from random graphs. This means the DCM is a particularly good benchmark for algorithms to ensure they are not implicitly using properties of random graphs that might not hold in the worst case.

Within the DCM setting, we have Theorem 30.

**Theorem 30.** Let q be a probability and  $d_{in}$  be an integer, and let  $\mathcal{D} \in \mathsf{DCM}(n, d_{in}, q)$ . For  $G \sim \mathcal{D}$ , let  $\widehat{\mathbf{L}}$  denote the expectation of  $\mathbf{L}$  after step (2) but before step (3) in Model 7.2. There exists constants  $C_1, C_2, C_3 > 0$  such that for all n sufficiently large, if

$$d_{\mathsf{in}} \geq C_1 \cdot \left(\frac{nq}{2} + \sqrt{n}\right) \quad and \quad \lambda_3(\widehat{\mathbf{L}}) - \lambda_2(\widehat{\mathbf{L}}) \geq \sqrt{n} + C_2nq + C_3\left(\sqrt{nq\log n} + \log n\right) \;,$$

then unnormalized spectral bisection is strongly consistent on  $\mathcal{D}$ .

We prove Theorem 30 in Section 7.5.7. We remark that, as in Theorem 29, the constants that appear in Theorem 30 are somewhat arbitrary. They are chosen to make our proofs cleaner and can likely be optimized.

As a basic application of Theorem 30, note that in the SSBM, if  $p = \omega(1/\sqrt{n})$  and  $q = 1/\sqrt{n}$ , then for *n* sufficiently large, with high probability, the resulting graph satisfies the conditions needed to apply Theorem 30. For a more interesting example, let  $P_1$  and  $P_2$  be two *d*-regular spectral expanders with  $d = \omega(\sqrt{n})$  and let  $q \le 1/\sqrt{n}$ . On top of both of these two graph classes, one can further allow arbitrary edge insertions inside  $P_1$  and  $P_2$  while still being guaranteed exact recovery from unnormalized spectral bisection.

### 7.2.3. Inconsistency of normalized spectral clustering

Notice that in Theorem 29 and Theorem 30, we only address the strong consistency of the unnormalized Laplacian in our nonhomogeneous and semirandom models. But what happens when we run spectral bisection with the *normalized* Laplacian?

In Theorem 31, we prove that there is a subfamily of instances belonging to NSSBM( $n, p, \overline{p}, q$ ) with  $\overline{p} = 6p, q = p/2$  on which unnormalized spectral bisection is strongly consistent (following from Theorem 29) but normalized spectral clustering is inconsistent in a rather strong sense. Thus, one cannot obtain results similar to Theorem 29 and Theorem 30 for normalized spectral bisection.

**Theorem 31.** For all *n* sufficiently large, there exists a nonhomogeneous stochastic block model such that unnormalized spectral bisection is strongly consistent whereas normalized spectral bisection (both symmetric and random-walk) incurs a misclassification rate of at least 24% with probability 1 - 1/n.

We prove Theorem 31 in Section 7.5.8. Furthermore, we expect that it is straightforward to adapt the example in Theorem 31 to prove an analogous result for our DCM setting.

The result of Theorem 31 may run counter to conventional wisdom, which suggests that normalized spectral clustering should be favored over the unnormalized variant [Von07]. Perhaps a more nuanced view in light of Theorem 29 and Theorem 30 is that that the normalized Laplacian and its eigenvectors enjoy stronger concentration guarantees [SB15; DLS21], but the unnormalized Laplacian's second eigenvector is more robust to monotone adversarial changes.

## 7.2.4. Open problems

Perhaps the most natural follow-up question inspired by our results is to determine whether the restriction that every internal edge probability  $p_{vw} \leq \overline{p}$  can be lifted entirely while still maintaining strong consistency of the unnormalized Laplacian (Theorem 30). Another exciting direction for future work is to lower the degree and/or spectral gap requirement present in our results in the DCM setting (Theorem 30). Finally, we only study insertion-only monotone adversaries, as crossing edge deletions change the second eigenvector of the expected Laplacian. It would be illuminating to understand the robustness of Laplacian-based spectral algorithms against a monotone adversary that is also allowed to delete crossing edges. We are optimistic that the answers to one or more of these questions will further improve our understanding of the robustness of spectral clustering to "helpful" model misspecification.

# 7.3. Analysis sketch

First, let us give some intuition as to why one may expect that unnormalized spectral bisection is robust against our monotone adversaries. Here and in the sequel, let  $u_2^* = [\mathbbm{1}_{n/2} \oplus -\mathbbm{1}_{n/2}]/\sqrt{n}$ , where  $\mathbbm{1}_k$  denotes the all-1s vector in k dimensions and  $\oplus$  denotes vector concatenation. Let **L** be the unnormalized Laplacian of the graph we want to partition,  $\mathbf{L}^* \coloneqq \mathbbm{E}[\mathbf{L}]$ ,  $\mathbf{E} \coloneqq \mathbf{L} - \mathbf{L}^*$ , and  $\lambda_i^* \coloneqq \lambda_i(\mathbf{L}^*)$  for  $1 \le i \le n$ . For an edge (v, w), let  $e_{vw} \coloneqq e_v - e_w$ , so that  $e_{vw}$  is an edge incidence vector corresponding to the edge (v, w). Let  $p_{vw}$  be the probability that the edge (v, w) appears in *G* and observe that  $L^*$  can be written as

$$\mathbf{L}^{\star} = \sum_{(v,w)\in E_{\text{internal}}} p_{vw} \cdot \boldsymbol{e}_{vw} \boldsymbol{e}_{vw}^{T} + \sum_{(v,w)\in E_{\text{crossing}}} q \cdot \boldsymbol{e}_{vw} \boldsymbol{e}_{vw}^{T},$$

where  $E_{\text{internal}} = (P_1 \times P_1) \cup (P_2 \times P_2)$  and  $E_{\text{crossing}} = P_1 \times P_2$ . We can verify that  $u_2^{\star}$  is an eigenvector of  $\mathbf{L}^{\star}$  – indeed, we do so in Lemma 7.5.14. And, for now, assume that  $u_2^{\star}$  does correspond to the second smallest eigenvalue of  $\mathbf{L}^{\star}$  (in our NSSBM family, this is easily ensured by enforcing p > q). Moreover, for every internal edge  $(v, w) \in E_{\text{internal}}$ , we have  $\langle e_{vw}, u_2^{\star} \rangle = 0$ . Hence, any changes in internal edges do not change the fact that  $u_2^{\star}$  is an eigenvector of the perturbed matrix. Thus, if the sampled  $\mathbf{L}$  is close enough to  $\mathbf{L}^{\star}$ , then it is plausible that the second eigenvector of  $\mathbf{L}$ , denoted as  $u_2$ , is pretty close to  $u_2^{\star}$ . In fact, the following conceptually stronger statement holds. If the subgraph formed by selecting just the crossing edges of *G* is regular, then  $u_2^{\star}$  is an eigenvector of  $\mathbf{L}$ . This follows from the fact that  $u_2^{\star}$  is an eigenvector of the unnormalized Laplacian of any regular bipartite graph where both sides have size n/2 and the previous observation that every internal edge is orthogonal to  $u_2^{\star}$ .

To make this perturbation idea more formal, we recall the Davis-Kahan Theorem. Loosely, it states that  $\|u_2 - u_2^{\star}\|_2 \leq \|(\mathbf{L} - \mathbf{L}^{\star})u_2^{\star}\|_2/(\lambda_3^{\star} - \lambda_2^{\star})$  (we give a more formal statement in Lemma 7.5.15). Expanding the entrywise absolute value  $|(\mathbf{L} - \mathbf{L}^{\star})u_2^{\star}|$  reveals that its entries can be expressed as  $2|d_{out}[v] - \mathbb{E}[d_{out}[v]]|/\sqrt{n}$ , where  $d_{out}[v]$  denotes the number of edges incident to v crossing to the opposite community as v. This is unaffected by any increase in the number of edges incident to v that stay within the same community as v, denoted as  $d_{in}[v]$ . Hence, regardless of how many internal edges we add before sampling or what substructures they encourage/create, if we have  $\lambda_2^{\star} \ll \lambda_3^{\star}$ , then we get  $\|u_2 - u_2^{\star}\|_2 \leq o(1)$ . This immediately implies that  $u_2$  is a correct classifier on all but an o(1) fraction of the vertices.

Entrywise analysis of  $u_2$  and NSSBM strong consistency. In order to achieve strong consistency, we need that for all n sufficiently large,  $u_2$  is a perfect classifier. Unfortunately, the above argument does not immediately give that. In particular, in the density and spectral gap regimes we consider, the bound of o(1) yielded by the Davis-Kahan theorem is not sufficiently small to directly yield  $||u_2^{\star} - u_2||_2 \ll 1/\sqrt{n}$ . Instead, we carry out an entrywise analysis of  $u_2$ . A general framework for doing so is given by Abbe, Fan, Wang, and Zhong [AFWZ20] and is adapted to the unnormalized and normalized Laplacians by Deng, Ling, and Strohmer [DLS21].

At a high level, we adapt the analysis of Deng, Ling, and Strohmer [DLS21] to our setting. We consider the intermediate estimator vector  $(\mathbf{D} - \lambda_2 \mathbf{I})^{-1} \mathbf{A} u_2^{\star}$ . This is a natural choice because we can verify  $(\mathbf{D} - \lambda_2 \mathbf{I})^{-1} \mathbf{A} u_2 = u_2$ . We will see that it is enough to show that this intermediate estimator correctly classifies all the vertices while satisfying  $|(\mathbf{D} - \lambda_2 \mathbf{I})^{-1} \mathbf{A} (u_2^{\star} - u_2)| \le |(\mathbf{D} - \lambda_2 \mathbf{I})^{-1} \mathbf{A} u_2^{\star}|$  (again, the absolute value is taken entrywise). With this in mind, taking some entry indexed by  $v \in V$  and multiplying both sides by  $d[v] - \lambda_2$  (which we will show is positive with high probability), we see that it is enough to show

$$\left|\left\langle a_{v}, u_{2}^{\star} - u_{2}\right\rangle\right| \leq \left|\left\langle a_{v}, u_{2}^{\star}\right\rangle\right| = \frac{\left|d_{\text{in}}[v] - d_{\text{out}}[v]\right|}{\sqrt{n}},\tag{7.3.1}$$

where  $a_v$  denotes the *v*-th row of **A**. The advantage of this rewrite is that the right hand side can be uniformly bounded, so it is enough to control the left hand side.

To argue about the left hand side of (7.3.1), it may be tempting to use the fact that  $a_v$  is a Bernoulli random vector and use Bernstein's inequality to argue about the sum of rescalings of these Bernoulli random variables. Unfortunately, we cannot do this since  $u_2$  and  $a_v$  are dependent. To resolve this, we use a leave-one-out trick [AFWZ20; BV24]. We can think of this

	$L_1$	$L_2$	R
$L_1$	$Kp \cdot \mathbb{1}_{n/4 \times n/4}$	$p \cdot \mathbb{1}_{n/4 \times n/4}$	a 1 100 10
$L_2$	$p \cdot \mathbb{1}_{n/4  imes n/4}$	$Kp \cdot \mathbb{1}_{n/4 \times n/4}$	9 <i>≖n/2×n/2</i>
R	$q \cdot \mathbb{1}_{n/2 \times n/2}$		$p \cdot \mathbb{1}_{n/2 \times n/2}$

Table 7.1.:  $\mathbf{A}^{\star}$  for Theorem 31 is defined to have the above block structure.

as leaving out the vertex v corresponding to the entry we want to analyze and sampling the edges incident to the rest of the vertices. The second eigenvector of the resulting  $\mathbf{L}^{(v)}$ , denoted as  $u_2^{(v)}$ , is a very good proxy for  $u_2$  and is independent from  $a_v$ . Hence, we may complete the proof of Theorem 29.

One of our main observations is that although this style of analysis was originally built for low-rank signal matrices [AFWZ20; BV24], it can be adapted to handle the nonhomogeneity inside  $P_1$  and  $P_2$ . In particular, the nonhomogeneity we permit in the NSSBM family may make L\* look very far from a spiked low-rank signal matrix. Furthermore, our entrywise analysis of eigenvectors under perturbations is one of the first that we are aware of that moves beyond analyzing low-rank signal matrices or spiked low-rank signal matrices.

**Extension to deterministic clusters.** To prove Theorem 30, we start again at (7.3.1). An alternate way to upper bound the left hand side is to use the Cauchy-Schwarz inequality. A variant of the Davis-Kahan theorem gives us control over  $||u_2 - u_2^{\star}||_2$  while  $||a_v||_2 = \sqrt{d[v]}$ . The advantage of this is that we get a worst-case upper bound on the left hand side of (7.3.1) – it holds no matter what edges orthogonal to  $u_2^{\star}$  are inserted before or after nature samples the crossing edges (which are precisely the internal edges). Combining these and using the fact that the right of (7.3.1) is increasing in  $d_{in}[v]$  (and increases faster than  $||a_v||_2 = \sqrt{d[v]}$ ) allows us to complete the proof of Theorem 30.

**Inconsistency of normalized spectral bisection.** Finally, we describe the family of hard instances we use to prove Theorem 31. To motivate this family of instances, recall that by the graph version of Cheeger's inequality, the second eigenvalue of  $\mathcal{L}$  and the corresponding eigenvector can be used to find a sparse cut in *G*. Thus, if we create sparse cuts inside  $P_1$  that are sparser than the cut formed by separating  $P_1$  and  $P_2$ , then conceivably the normalized Laplacian's second eigenvector may return the new sparser cut.

To make this formal, consider the following graph structure. Let *n* be a multiple of 4. Let  $L_1$  consist of indices  $1, \ldots, n/4, L_2$  consist of indices  $n/4 + 1, \ldots, n/2$ , and *R* consist of indices  $n/2 + 1, \ldots, n$ . Consider the block structure induced by the matrix  $\mathbf{A}^* = \mathbb{E}[\mathbf{A}]$  shown in Table 7.1.

Intuitively, as *K* gets larger, the cut separating  $L_1$  from  $V \setminus L_1$  becomes sparser. From Cheeger's inequality, this witnesses a small  $\lambda_2(\mathcal{L})$  and therefore the corresponding  $u_2(\mathcal{L})$  may return the cut  $L_1$ ,  $V \setminus L_1$ . We formally prove that this is indeed what happens when *K* is a sufficiently large constant and then Theorem 31 follows.

# 7.4. Numerical trials

We programmatically generate synthetic graphs that help illustrate our theoretical findings using the libraries NetworkX 3.3 (BSD 3-Clause license), SciPy 1.13.0 (BSD 3-Clause License), and NumPy 1.26.4 (modified BSD license) [HSS08; VGO+20; HMvdW+20]. We ran all our experiments on a free Google Colab instance with the CPU runtime, and each experiment takes under one hour to run. In this section we focus on a setting that allows relating Theorem 29 and Theorem 31, and defer more experiments that investigate both NSSBM and DCM graphs to Section 7.6.

To put Theorem 29 and Theorem 31 in perspective, we consider graphs generated following the process outlined in the proof of Theorem 31, which gives rise to the following benchmark distribution.

**Benchmark distribution.** Let *n* be divisible by 4 and let  $\{P_1, P-2\}$  be a partitioning of V = [n] into two equally-sized subsets. Let  $\{L_1, L_2\}$  be a bipartition of  $P_1$  such that  $|L_1| = |L_2| = n/4$  and call  $L = P_1$ ,  $R = P_2$  for convenience as in the proof of Theorem 31. Then, for some  $p, \overline{p}, q \in [0, 1]$  such that  $q \leq p \leq \overline{p}$ , consider the distribution  $\mathcal{D}_{p,\overline{p},q}$  over graphs G = (V, E) obtained by sampling every edge  $(u, v) \in (L_1 \times L_1) \cup (L_2 \times L_2)$  independently with probability  $\overline{p}$ , every edge  $(u, v) \in (L_1 \times L_2) \cup (R \times R)$  independently with probability p, and every edge  $(u, v) \in L \times R$  independently with probability q. One can see that  $\mathcal{D}_{p,\overline{p},q}$  is in fact in the set NSSBM $(n, p, \overline{p}, q)$ .

**Setup.** Let us fix n = 2000,  $p = 24 \log n/n$ ,  $q = 8 \log n/n$ . For varying values of  $\overline{p}$  in the range [p, 1], we sample t = 10 independent draws G from  $\mathcal{D}_{p,\overline{p},q}$ . For each of them, we run spectral bisection (i.e. Algorithm 19) with matrices  $\mathbf{L}$ ,  $\mathcal{L}_{sym}$ ,  $\mathcal{L}_{rw}$ ,  $\mathbf{A}$ . Then, we compute the *agreement* of the bipartition hence obtained (with respect to the planted bisection), that is the fraction of correctly classified vertices. We average the agreement across the *t* independent draws. The results are shown in the top left plot of Fig. 7.1. Another natural way to get a bipartition of *V* from the eigenvector is a *sweep cut*. In a sweep cut, we sort the entries of  $u_2$  and take the vertices corresponding to the smallest n/2 entries to be on one side of the bisection and put the remaining on the other side. The average agreement obtained in this other fashion is shown in the bottom left plot of Fig. 7.1. **Theoretical framing.** As per Theorem 29, we expect unnormalized spectral bisection to achieve exact recovery (i.e. agreement equal to 1) whenever  $\overline{p} \leq \overline{p}_{max'}$  where

$$\overline{p}_{\max} = \frac{\left(n(p-q) - \log n\right)^2}{n\log n} \tag{7.4.1}$$

is obtained by rearranging the precondition of Theorem 29, ignoring the constants and disregarding the fact that  $\alpha$  should be O(1). On the contrary, the proof of Theorem 31 shows that normalized spectral bisection misclassifies a constant fraction of vertices provided that  $p/q \ge 2$ (which our choice of parameters satisfies) and  $\overline{p} \ge \overline{p}_{\text{thr}}$ , where

$$\overline{p}_{\text{thr}} = 3 \cdot p^2 / q \,. \tag{7.4.2}$$

In Fig. 7.1, the solid vertical line corresponds to the value of  $\overline{p}_{thr}$  on the *x*-axis, and the dashed vertical line corresponds to the value of  $\overline{p}_{max}$  on the *x*-axis. In particular, observe that in our setting  $\overline{p}_{thr} < \overline{p}_{max}$ , so there is an interval of values for  $\overline{p}$  where we expect Theorem 29 and Theorem 31 to apply simultaneously.

**Empirical evidence: consistency.** One can see from the top left plot in Fig. 7.1 that the agreement of unnormalized spectral bisection is 100% for all values of  $\overline{p}$ , even beyond  $\overline{p}_{thr}$  and  $\overline{p}_{max}$ . On the other hand, the agreement of the bipartition obtained from all other matrices

(hence including normalized spectral bisection) drops below 70% well before the threshold  $\overline{p}_{thr}$  predicted by Theorem 31. From the right plot in Fig. 7.1, we see that computing the bipartition by taking a sweep cut of n/2 vertices does not change the results –  $u_2$  of the unnormalized Laplacian continues to achieve 100% agreement, while for all other matrices the corresponding  $u_2$  remains inconsistent.

**Empirical evidence: embedding variance.** From the setting of the experiment we just illustrated, observe that as we increase  $\overline{p}$ , we expect the subgraph G[L] to have increasing volume. As illustrated in Fig. 7.1, this seems to correlate with a decrease in the "variance" of the second eigenvector  $u_2$  of the unnormalized Laplacian with respect to the ideal second eigenvector  $u_2^*$ . More precisely, we compute the average distance squared of the embedding of a vertex in  $u_2$  from its ideal embedding in  $u_2^*$ , i.e. the quantity

$$\min_{s \in \{\pm 1\}} \frac{1}{n} \left\| \boldsymbol{u}_2 - s \cdot \boldsymbol{u}_2^{\star} \right\|_2^2.$$
(7.4.3)

This suggests that not only does the second eigenvector of the unnormalized Laplacian remain robust to monotone adversaries, but it actually concentrates more strongly around the ideal embedding  $u_2^*$ .

**Empirical evidence: example embedding.** Let us fix the value  $\overline{p} = \overline{p}_{thr}$ , for which we see in Fig. 7.3 that all matrices except the unnormalized Laplacian fail to recover the planted bisection. We generate a graph from  $\mathcal{D}_{p,\overline{p},q}$ , and plot how the vertices are embedded in the real line by the second eigenvector of all the matrices we consider. The result is shown in Fig. 7.1, where the three horizontal dashed lines, from top to bottom, respectively correspond to the value of  $1/\sqrt{n}$ ,  $0, -1/\sqrt{n}$  on the *y*-axis.

#### 7.4.1. Related work

**Community detection.** Community detection has garnered significant attention in theoretical computer science, statistics, and data science. For a general overview of recent progress and related literature, see the survey by Abbe [Abb18]. In what follows, we discuss the works we believe are most related to what we study in this chapter.

As mentioned in the introduction, perhaps the most fundamental and well-studied model is the symmetric stochastic block model (SSBM), due to [HLL83]. The celebrated work of Abbe, Bandeira, and Hall [ABH16] gives sharp bounds on the threshold for exact recovery for the SSBM setting. They complement their result by showing that SDP based methods can achieve the information theoretic lower bound for the planted bisection problem, even with a monotone adversary [Moi21a]. A line of work [AFWZ20; DLS21] demonstrates that natural spectral algorithms achieve exact recovery for the SSBM all the way to the information-theoretic threshold.

**Generalizations of the symmetric stochastic block model.** Since the introduction of SBMs [HLL83], numerous variants have been proposed that are designed to better reflect real-world graph properties. For instance, real-life social networks are likely to contain triangles. To address this, Sankararaman and Baccelli [SB17] introduced a spatial stochastic block model, sometimes known as the geometric stochastic block model (GSBM). Other variations were introduced in the works of [GPMS18; GMPS19]. Subsequent work studies the performance of spectral algorithms on certain Gaussian or Geometric Mixture block models [ABRS20; ABD21; LS24; GNW24].



Figure 7.1.: **Top left, bottom left**: Agreement with the planted bisection of the bipartition obtained from several matrices associated with an input graph generated from a distribution in NSSBM( $n, p, \overline{p}, q$ ) for fixed values of n, p, q and varying values of  $\overline{p}$ . In the top left plot, the bipartition is the 0-cut of the second eigenvector, as in Algorithm 19. In the bottom left plot, the bipartition is the sweep cut of the first n/2 vertices in the second eigenvector. The dashed vertical line corresponds to  $\overline{p}_{max} = \overline{p}_{max}(n, p, q)$  (see (7.4.1)), and the solid vertical line corresponds to  $\overline{p}_{thr} = \overline{p}_{thr}(n, p, q)$  (see (7.4.2)). **Top middle, top right, bottom middle**: Embedding of the vertices given by the second eigenvector  $u_2$  of several matrices associated with a graph sampled from  $\mathcal{D}_{p,\overline{p},q}$  with  $\overline{p} = \overline{p}_{thr}$ . Horizontal dashed lines, from top to bottom, correspond to  $1/\sqrt{n}, 0, -1/\sqrt{n}$  respectively.

**Bottom right**: Variance of the embedding in the second eigenvector  $u_2$  of the unnormalized Laplacian with respect to the ideal eigenvector  $u_2^{\star}$  (see (7.4.3)), for input graphs generated from a distribution in NSSBM $(n, p, \overline{p}, q)$  with fixed values of n, p, q and varying values of  $\overline{p}$ .

Studying community detection with a semirandom model approaches this modeling question differently. Rather than implicitly encouraging a particular structure within the clusters like the models just mentioned, a semirandom adversary (including the ones we study in this chapter) can more directly test the robustness of the algorithm to specially designed substructures.

**Semirandom and monotone adversaries.** As far as we are aware, Blum and Spencer [BS95a] were the first to introduce a semirandom model. Within this model, they studied graph coloring problems. Feige and Kilian [FK01] demonstrated that semidefinite programming methods can accurately recover communities up to a certain threshold, even in the semi-random setting. Other problems, such as detecting a planted clique [Jer92; Kuč95; BHKKMP19], have also been studied in the semi-random model of [FK01]. In the setting of planted clique, a natural spectral algorithm fails against monotone adversaries [MMT20; BKS23]. Monotone adversaries and semirandom models have also been extensively studied for other statistical and algorithmic problems [VA18; KLLST23; GC23; BGLMSY24]. Finally, [SL17] shows that a spectral heuristic due to Boppana [Bop87] is robust under a monotone adversary that is allowed to both insert internal edges and delete crossing edges. However, as far as we are aware, this algorithm does not fit in the framework of Algorithm 19.

We remark that the models we study in this chapter are most closely related to models studied by [MN06b] and [MMV12]. In particular, allowing increased internal edge probabilities is analogous to Massart noise in classification problems, and our model with adversarially chosen internal edges can be seen as the same model as that studied in [MMV12] (although without allowing crossing edge deletions). Finally, note that Cohen-Addad, d'Orsi, and Mousavifar [CdM24] give a near-linear time algorithm for graph clustering in the model of [MMV12], though they do not explicitly show their algorithm is strongly consistent on instances that are information-theoretically exactly recoverable.

# 7.5. Deferred proofs

In this section, we build the tools we need to prove Theorem 29, Theorem 30, andTheorem 31. Throughout, it will be helpful to refer to the overview (Section 7.3) for a proof roadmap.

**Notation in the proofs.** In all proofs, we adopt the notation used in the technical overview (Section 7.3). Additionally, for a vertex  $v \in V$ , let P(v) denote the community that v belongs to.

## 7.5.1. Concentration inequalities

Our proof strategy for Theorem 29 and Theorem 30 is to appeal to Lemma 7.5.23, which guarantees strong consistency provided that  $d[v] - \lambda_2 > 0$ ,  $d_{in}[v] > d_{out}[v]$ , and  $|\langle a_v, u_2^{\star} - u_2 \rangle| \le (d_{in}[v] - d_{out}[v])/\sqrt{n}$  for all vertices v. Proving that the first two conditions hold is relatively easy. In the setting of Theorem 29, it essentially follows from concentration of the degrees, which is proved in Section 7.5.2. In the setting of Theorem 30, it follows from the assumptions of the Theorem. Proving that the third condition holds is the main technical challenge.

For all three parts, our proofs rely on several auxiliary concentration results. We prove these in Section 7.5.3 and Section 7.5.4.

We extensively use the following variants of Bernstein's Inequality, which can be derived from

[Ver18, Theorem 2.8.4].

**Lemma 7.5.1.** Let  $X = \sum_{i=1}^{m} X_i$ , where  $X_i = 1$  with probability  $p_i$  and  $X_i = 0$  with probability  $1 - p_i$  and all the  $X_i$  are independent. Let  $\mu = \mathbb{E}[X]$ . Then, for all t > 0 we have

$$\Pr\left[|X - \mu| \ge t\right] \le 2\exp\left(-\min\left\{\frac{t^2}{4\sum_{i=1}^m p_i(1 - p_i)}, \frac{3t}{4}\right\}\right).$$

From this, we get the following very useful corollary.

**Lemma 7.5.2.** Let  $X = \sum_{i=1}^{m} X_i$ , where  $X_i = 1$  with probability  $p_i$  and  $X_i = 0$  with probability  $1 - p_i$  and all the  $X_i$  are independent. Let  $\mu = \mathbb{E}[X]$ . Then, for all t > 0, with probability at least  $1 - \delta$  we have

$$|X - \mu| \le \sqrt{4 \sum_{i=1}^{m} p_i (1 - p_i) \log (2/\delta)} + 4/3 \log (2/\delta).$$

#### 7.5.2. Concentration of degrees

In this Section, we give concentration statements regarding the number of internal vertices incident to each vertex and the number of crossing edges incident to each vertex. We then compare these against  $\lambda_2$ .

**Lemma 7.5.3.** Suppose the crossing edges are sampled identically and independently with probability *q*. Then, for some universal constant C > 0, with probability at least  $1 - \delta$  we have that

$$\forall v \in V, \quad |\boldsymbol{d}_{\mathsf{out}}[v] - \mathbb{E}\left[\boldsymbol{d}_{\mathsf{out}}[v]\right]| \le C\left(\sqrt{nq\log\left(n/\delta\right)} + \log\left(n/\delta\right)\right).$$

*Proof of Lemma* 7.5.3. Choose some  $v \in V$ . Consider the random variable  $d_{out}[v]$ . Using Lemma 7.5.2, we have that there is a constant C > 0 such that with probability at least  $1 - \delta/n$  one has

$$|d_{\text{out}}[v] - \mathbb{E}[d_{\text{out}}[v]]| \le C\left(\sqrt{4nq/2\log(2n/\delta)} + \log(2n/\delta)\right).$$

Taking a union bound over all *n* vertices completes the proof of Lemma 7.5.3.

Note that Lemma 7.5.3 above applies in both the settings of Theorem 29 and Theorem 30.

**Lemma 7.5.4.** Suppose the internal edges are sampled independently with probabilities  $p_{vw}$  such that  $p \le p_{vw} \le \overline{p}$ . Then, for some universal constant C > 0, with probability  $\ge 1 - \delta$  we have that

$$\forall v \in V, \quad |d_{\mathsf{in}}[v] - \mathbb{E}\left[d_{\mathsf{in}}[v]\right]| \le C\left(\sqrt{\sum_{w \in P(v) \setminus \{v\}} p_{vw}(1 - p_{vw})\log\left(n/\delta\right)} + \log\left(n/\delta\right)\right).$$

*Proof of Lemma* 7.5.4. As before, choose some  $v \in V$  and consider the random variable  $d_{in}[v]$ . By Lemma 7.5.2, we have that there is a constant C > 0 such that with probability at least  $1 - \delta/n$  one has

$$|d_{\rm in}[v] - \mathbb{E}\left[d_{\rm in}[v]\right]| \le C\left(\sqrt{4\sum_{w\in P(v)\setminus\{v\}}p_{vw}(1-p_{vw})\log\left(\frac{2n}{\delta}\right)} + \log\left(\frac{2n}{\delta}\right)\right).$$

Taking a union bound over all n vertices completes the proof of Lemma 7.5.4.

Combining the above two lemmas, we obtain a lower-bound on  $d_{in}[v] - d_{out}[v]$ . In particular, the following lemma implies that in the setting of Theorem 29, we have  $d_{in}[v] > d_{out}[v]$ . This will be required for applying Lemma 7.5.23.

**Lemma 7.5.5.** There exists a universal constant C > 0 such that with probability  $\ge 1 - \delta$ , in the same settings as Lemma 7.5.3 and Lemma 7.5.4 and assuming the gap condition in Theorem 29, if  $p \ge q$ , then for all  $v \in V$  we have

$$d_{\rm in}[v] - d_{\rm out}[v] \ge \frac{n(p-q)}{2} - C\left(\sqrt{np\log(n/\delta)} + \log(n/\delta)\right).$$

*Proof of Lemma* 7.5.5. Let  $v \in V$ . First, we call Lemma 7.5.3 with a failure probability of  $\delta/(2n)$  to conclude that

$$d_{\mathsf{out}}[v] \le \frac{nq}{2} + C_{7.5.3}\left(\sqrt{\frac{nq}{2}\log\left(2n^2/\delta\right)} + \log\left(2n^2/\delta\right)\right).$$

Next, we call Lemma 7.5.4 with a failure probability of  $\delta/(2n)$  to conclude that

$$\begin{split} d_{\mathrm{in}}[v] &\geq \sum_{w \in P(v) \setminus \{v\}} p_{vw} - C_{7.5.4} \left( \sqrt{\sum_{w \in P(v) \setminus \{v\}} p_{vw}(1 - p_{vw}) \log\left(2n^2/\delta\right)} + \log\left(2n^2/\delta\right) \right) \\ &\geq \sum_{w \in P(v) \setminus \{v\}} p_{vw} - C_{7.5.4} \left( \sqrt{\sum_{w \in P(v) \setminus \{v\}} p_{vw} \log\left(2n^2/\delta\right)} + \log\left(2n^2/\delta\right) \right) \\ &\geq \frac{np}{2} - 2C_{7.5.4} \left( \sqrt{\frac{np}{2} \log\left(n^2/\delta\right)} + \log\left(2n^2/\delta\right) \right). \end{split}$$

where the last line uses the fact that  $x - c\sqrt{x}$  is increasing in x whenever  $x \ge c^2/4$  and c > 0. We subtract and conclude the proof of Lemma 7.5.5 by a union bound.

The following lemma will be useful for lower-bounding  $d[v] - \lambda_2$  in Theorem 29.

**Lemma 7.5.6.** Suppose every crossing edge appears independently with probability q. Then, with probability  $\geq 1 - \delta$ , for all  $v \in V$  we have

$$\lambda_2 \leq 2d_{\mathsf{out}}[v] + C\left(\sqrt{nq\log(n/\delta)} + \log(n/\delta)\right).$$

*Proof of Lemma* 7.5.6. Observe that with probability at least  $1 - \delta$ ,  $d_{out}[w] - \mathbb{E}[d_{out}[v]] \le \sqrt{2nq \log(2n/\delta)} + 2\log(2n/\delta)$  for all  $w \in V$  by Lemma 7.5.2. Then, for every  $v \in V$  we have

$$\frac{2}{n} \sum_{w \in P(v)} d_{\text{out}}[w] - d_{\text{out}}[v] = \left(\frac{2}{n} \sum_{w \in P(v)} d_{\text{out}}[w] - \mathbb{E}\left[d_{\text{out}}[v]\right]\right) + \left(\mathbb{E}\left[d_{\text{out}}[v]\right] - d_{\text{out}}[v]\right)$$
$$\leq \left|\frac{2}{n} \sum_{w \in P(v)} d_{\text{out}}[w] - \mathbb{E}\left[d_{\text{out}}[v]\right]\right| + \left|\mathbb{E}\left[d_{\text{out}}[v]\right] - d_{\text{out}}[v]\right|$$
$$\leq \sqrt{2nq \log(2^n/\delta)} + \sqrt{2nq \log(2^n/\delta)} + 4\log(2^n/\delta)$$
$$\leq 3\sqrt{nq \log(n/\delta)} + 10\log(n/\delta).$$

Next, by the min-max principle, we have

$$\lambda_2 \leq \sum_{(w,w')\in E} \left( \boldsymbol{u}_2^{\star}[w] - \boldsymbol{u}_2^{\star}[w'] \right)^2 = \frac{4}{n} \sum_{w\in P(v)} d_{\text{out}}[w]$$

Combining everything, we get

$$\lambda_2 \le 2\left(\frac{2}{n}\sum_{w\in P(v)} d_{\mathsf{out}}[w]\right) \le 2\left(d_{\mathsf{out}}[v] + 3\sqrt{nq\log\left(n/\delta\right)} + 10\log\left(n/\delta\right)\right),$$

completing the proof of Lemma 7.5.6.

We can now lower-bound  $d[v] - \lambda_2$ . Note that the following lower bound implies that  $d[v] > \lambda_2$ , as required by Lemma 7.5.23.

**Lemma 7.5.7.** *In the setting of Theorem 29, with probability*  $\geq 1 - \delta$ *, for all*  $v \in V$ *, we have*  $d[v] - \lambda_2 > n(p-q)/4$ .

*Proof of Lemma* 7.5.7. Recall that the gap condition in Theorem 29 tells us that p and q are such that for a universal constant C,

$$n(p-q) \ge C\left(\sqrt{np\log(n/\delta)} + \log(n/\delta)\right).$$

We have for all *n* sufficiently large (specifically,  $n \ge N(\alpha, \delta)$  for some *N* that is a function only of the constant  $\alpha$ , and we take  $\delta \ge 1/n^{O(1)}$ ) that with probability at least  $1 - \delta$ ,

$$d[v] - \lambda_{2} = d_{in}[v] - d_{out}[v] + (2d_{out}[v] - \lambda_{2})$$

$$\geq d_{in}[v] - d_{out}[v] - C_{7.5.6} \left( \sqrt{nq \log(n/\delta)} + \log(n/\delta) \right)$$

$$\geq \frac{n(p-q)}{2} - (C_{7.5.5} + C_{7.5.6}) \left( \sqrt{np \log(n/\delta)} + \log(n/\delta) \right),$$

so insisting

$$\frac{n(p-q)}{4} \ge (C_{7.5.5} + C_{7.5.6}) \left(\sqrt{np \log(n/\delta)} + \log(n/\delta)\right) + 1$$

gives the condition required to complete the proof of Lemma 7.5.7.

The following technical lemma will be useful for upper-bounding  $||u_2||_{\infty}$  in Lemma 7.5.22.

**Lemma 7.5.8.** *In the setting of Theorem 29, there exists a universal constant C such that with probability*  $\geq 1 - \delta$ *, for all v \in V we have* 

$$\frac{n\overline{p} + \log\left(n/\delta\right)}{d[v] - \lambda_2} \le 4\alpha + C.$$

*Proof of Lemma* 7.5.8. By Lemma 7.5.7, we have with probability  $\geq 1 - \delta$  that for all  $v \in V$ ,

$$d[v] - \lambda_2 \ge \frac{n(p-q)}{4}.$$

This gives

$$\frac{n\overline{p} + \log\left(n/\delta\right)}{d[v] - \lambda_2} \le \frac{4(n\overline{p} + \log\left(n/\delta\right))}{n(p-q)} = \frac{4\overline{p}}{p-q} + \frac{4\log\left(n/\delta\right)}{n(p-q)} \le 4\alpha + C$$

This completes the proof of Lemma 7.5.8.

#### 7.5.3. Concentration of Laplacian and eigenvalue perturbations

For the matrix concentration lemmas, we need a result due to Le, Levina, and Vershynin [LLV17]. We reproduce it below.

**Lemma 7.5.9** ([LLV17, Theorem 2.1]). Consider a random graph from the model  $G(n, \{p_{ij}\})$ . Let  $d = \max_{ij} np_{ij}$ . For any  $r \ge 1$ , the following holds with probability at least  $1 - n^{-r}$  for a universal constant C. Consider any subset consisting of 10n/d vertices, and reduce the weights of the edges incident to those vertices in an arbitrary way. Let d' be the maximal degree of the resulting graph. Then, the adjacency matrix **A**' of the new weighted graph satisfies

$$\left\|\mathbf{A}' - \mathbb{E}\left[\mathbf{A}\right]\right\|_{\mathrm{op}} \leq Cr^{3/2}\left(\sqrt{d} + \sqrt{d'}\right).$$

Moreover, the same holds for d' being the maximal  $\ell_2$  norm of the rows of A'.

**Lemma 7.5.10.** Let **L** be a Laplacian sampled from the nonhomogeneous Erdős-Rényi model where each edge (i, j) is present independently with probability  $p_{ij}$ . Then, there exists a universal constant C such that for all n sufficiently large, with probability  $\geq 1 - \delta$  for any  $\delta \geq n^{-10}$ ,

$$\|\mathbf{L} - \mathbb{E}\left[\mathbf{L}\right]\|_{\mathrm{op}} \leq C\left(\sqrt{n \max_{(i,j): p_{ij} \neq 1} p_{ij} \log\left(n/\delta\right)} + \log\left(n/\delta\right)\right).$$

*Proof of Lemma* 7.5.10. Without loss of generality, for all  $p_{ij}$  that are 1, reset their probabilities to 0. To see that this is valid, let **L**' be a Laplacian sampled from this modified distribution and notice that  $\mathbf{L}' - \mathbb{E}[\mathbf{L}'] = \mathbf{L} - \mathbb{E}[\mathbf{L}]$ .

By Lemma 7.5.9 and Lemma 7.5.2, we have with probability  $\geq 1 - \delta/2$  that

$$\|\mathbf{A} - \mathbb{E}\left[\mathbf{A}\right]\|_{\text{op}} \le 200C_{7.5.9} \sqrt{2n \max_{ij} p_{ij} + C_{7.5.2} \left(\sqrt{n \max_{ij} p_{ij} \log(8n/\delta)} + \log(8n/\delta)\right)}$$

Next, we bound  $\|\mathbf{E}\boldsymbol{u}_2^{\star}\|_2$ , which we will need in order to apply our Davis-Kahan style bound in Lemma 7.5.16. We remark that Lemma 7.5.12 below holds both in the setting of Theorem 29 and of Theorem 30.

 $\leq 2 \max \{C_{7.5.3}, C_{7.5.4}\} \left( \sqrt{n \max_{ij} p_{ij} \log (2n/\delta)} + \log (2n/\delta) \right)$ 

Now, observe that with probability  $\geq 1 - \delta$  (following from a union bound),

$$\begin{split} \|\mathbf{L} - \mathbb{E}[\mathbf{L}]\|_{\text{op}} &= \|\mathbf{D} - \mathbb{E}[\mathbf{D}] - (\mathbf{A} - \mathbb{E}[\mathbf{A}])\|_{\text{op}} \le \|\mathbf{D} - \mathbb{E}[\mathbf{D}]\|_{\text{op}} + \|\mathbf{A} - \mathbb{E}[\mathbf{A}]\|_{\text{op}} \\ &\le 800C_{7.5.9}C_{7.5.2} \max\left\{C_{7.5.3}, C_{7.5.4}\right\} \left(\sqrt{n \max_{ij} p_{ij} \log\left(\frac{8n}{\delta}\right)} + \log\left(\frac{8n}{\delta}\right)\right), \end{split}$$

completing the proof of Lemma 7.5.10.

and

By applying the above lemma, we can show that there is a gap between  $\lambda_3$  and  $\lambda_2^{\star}$ , which will allow us to apply Davis-Kahan style bounds. More concretely, Lemma 7.5.11 and Lemma 7.5.12, together with Lemma 7.5.16, show that  $\|u_2 - u_2^{\star}\|_2$  is small. This will be useful for proving that in the context for Theorem 29, the condition  $|\langle a_v, u_2^{\star} - u_2 \rangle| \leq (d_{in}[v] - d_{out}[v])/\sqrt{n}$  in Lemma 7.5.23 is satisfied.

Lemma 7.5.11. In the setting of Theorem 29, there exists a universal constant C such that the following holds.

*Let p and q be such that we have* 

$$n(p-q) \ge C\left(\sqrt{n\overline{p}\log\left(n/\delta\right)} + \log(n/\delta)\right).$$

Then, for any  $\delta \ge n^{-10}$ , with probability  $\ge 1 - \delta$ , we have  $\lambda_3 - \lambda_2^* \ge n(p-q)/4$ .

*Proof of Lemma 7.5.11.* By Weyl's inequality and Lemma 7.5.10, we have with probability  $\geq 1 - \delta$ that

$$\lambda_3 - \lambda_2^{\star} \ge \lambda_3^{\star} - \lambda_2^{\star} - \left\| \mathbf{L} - \mathbf{L}^{\star} \right\|_{\text{op}} \ge \frac{n(p-q)}{2} - C_{7.5.10} \left( \sqrt{n\overline{p}\log\left(n/\delta\right)} + \log(n/\delta) \right).$$

 $\frac{n(p-q)}{4} \ge C_{7.5.10} \left( \sqrt{n\overline{p}\log\left(n/\delta\right)} + \log(n/\delta) \right).$ 

Let  $C \ge 4C_{7.5.10}$ . Then,

by Lemma 7.5.3 and Lemma 7.5.4, we have with probability 
$$1 - \delta/2$$

 $\leq 400C_{7.5.9}C_{7.5.2}\sqrt{n\max_{ij}p_{ij} + \log(^{8n}/\delta)}$ 

 $\leq 400C_{7.5.9}C_{7.5.2}\left(\sqrt{n \max_{ii} p_{ij} \log(8n/\delta)} + \log(8n/\delta)}\right)$ 

**Lemma 7.5.12.** Suppose each crossing edge in our graph appears independently with probability q. There exists a universal constant C such that for all n sufficiently large, with probability  $\geq 1 - \delta$ , we have

$$\left\|\mathbf{E}\boldsymbol{u}_{2}^{\star}\right\|_{2} \leq C \left(\frac{\log\left(1/\delta\right)}{\log n}\right)^{3/2} \left(\sqrt{nq} + (nq\log\left(n/\delta\right))^{1/4} + \sqrt{\log\left(n/\delta\right)}\right)$$

Proof of Lemma 7.5.12. Observe that  $|\mathbf{E}\boldsymbol{u}_{2}^{\star}| = 2 |\boldsymbol{d}_{out} - \mathbb{E}[\boldsymbol{d}_{out}]| / \sqrt{n}$ . By Lemma 7.5.3, for all  $v \in V$ , with probability  $\geq 1 - \delta/2$ , we have  $\boldsymbol{d}_{out}[v] \leq nq/2 + C_{7.5.3} \left( \sqrt{nq \cdot \log(2n/\delta)} + \log(2n/\delta) \right)$ .

So, if we let  $\mathbf{A}_{out}$  and  $\mathbf{A}_{out}^{\star}$  denote the adjacency matrices consisting only of the crossing edges and the expected value of that, respectively, then invoking Lemma 7.5.9, with probability  $\geq 1 - \delta$ , we have

$$\begin{split} \left\| \mathbf{E} \boldsymbol{u}_{2}^{\star} \right\|_{2} &= \frac{2 \left\| \boldsymbol{d}_{\mathsf{out}} - \mathbb{E} \left[ \boldsymbol{d}_{\mathsf{out}} \right] \right\|_{2}}{\sqrt{n}} = \frac{2 \left\| \left( \mathbf{A}_{\mathsf{out}} - \mathbf{A}_{\mathsf{out}}^{\star} \right) \mathbb{1} \right\|_{2}}{\sqrt{n}} \leq 2 \left\| \mathbf{A}_{\mathsf{out}} - \mathbf{A}_{\mathsf{out}}^{\star} \right\|_{\mathsf{op}} \\ &\leq 2 C_{7.5.9} \left( \frac{\log \left( 2/\delta \right)}{\log n} \right)^{3/2} \left( \sqrt{\frac{nq}{2}} + \sqrt{C_{7.5.3}} \sqrt{nq} + \sqrt{nq \log \left( 2n/\delta \right)} + \log \left( 2n/\delta \right)} \right), \end{split}$$

completing the proof of Lemma 7.5.12.

Finally, we apply Lemma 7.5.9 in order to bound bound  $||a_v - a_v^{\star}||_2$ .

**Lemma 7.5.13.** *In the setting of Theorem 29, with probability*  $\geq 1 - \delta$ *, we have* 

$$\left\|\boldsymbol{a}_{v}-\boldsymbol{a}_{v}^{\star}\right\|_{2} \leq C\left(\frac{\log\left(1/\delta\right)}{\log n}\right)^{3/2} \left(\sqrt{n\overline{p}}+(n\overline{p}\log\left(n/\delta\right))^{1/4}+\sqrt{\log\left(n/\delta\right)}\right).$$

*Proof of Lemma* 7.5.13. We use a similar proof to that of Lemma 7.5.12. Indeed, invoke Lemma 7.5.9 (observe that we can set  $p_{ij}$  for the deterministic internal edges to 0 as they do not affect  $\mathbf{A} - \mathbb{E}[\mathbf{A}]$ ) and notice that

$$\left\|\boldsymbol{a}_{v}-\boldsymbol{a}_{v}^{\star}\right\|_{2} \leq \left\|\mathbf{A}-\mathbf{A}^{\star}\right\|_{\mathrm{op}} \leq C_{7.5.9} \left(\frac{\log\left(2/\delta\right)}{\log n}\right)^{3/2} \left(\sqrt{n\overline{p}}+(n\overline{p}\log\left(2n/\delta\right))^{1/4}+\sqrt{\log\left(2n/\delta\right)}\right),$$

where we used  $d' \le n(\overline{p}+q)/2 + 2 \max \{C_{7.5.3}, C_{7.5.4}\} \left(\sqrt{n\overline{p} \log (2n/\delta)} + \log (2n/\delta)\right)$  from combining Lemma 7.5.3 and Lemma 7.5.4. This completes the proof of Lemma 7.5.13.

#### 7.5.4. Eigenvector perturbations

In this Appendix, we give our Euclidean norm eigenvector perturbation bounds.

First, we verify that  $u_2^{\star}$  is indeed the second eigenvector of L<sup>\*</sup>.

**Lemma 7.5.14.** In the setting of Theorem 29, we have  $\mathbf{L}^{\star} u_2^{\star} = \lambda_2(\mathbf{L}^{\star}) u_2^{\star} = nqu_2^{\star}$ , where  $\mathbf{L}^{\star} = \mathbb{E}[\mathbf{L}]$ .

In the setting of Theorem 30, we have  $\mathbf{L}^{\star} \boldsymbol{u}_{2}^{\star} = \lambda_{2}(\mathbf{L}^{\star})\boldsymbol{u}_{2}^{\star} = nq\boldsymbol{u}_{2}^{\star}$ , where  $\mathbf{L}^{\star}$  denotes the Laplacian matrix that agrees with  $\mathbf{L}$  on all internal edges and agrees with  $\mathbb{E}[\mathbf{L}]$  on all crossing edges.

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*Proof of Lemma* 7.5.14. In both cases, one can check that  $u_2^{\star}$  is an eigenvector of  $\mathbf{L}^{\star}$  with eigenvalue nq: for any  $v \in P_2$  (i.e.  $u_2^{\star}[v] = -1/\sqrt{n}$  without loss of generality), one has

$$\left(\mathbf{L}^{\star}\boldsymbol{u}_{2}^{\star}\right)_{v} = \frac{1}{\sqrt{n}} \left( -(\boldsymbol{d}_{\text{in}}[v] + nq/2) - \sum_{w \in P_{1}: \{v,w\} \in E} (-1) + \sum_{w \in P_{2}} (-q) \right) = -\frac{nq}{\sqrt{n}} = nq \cdot \boldsymbol{u}_{2}^{\star}[v].$$

By virtue of the above observations, it suffices to argue that  $nq < \lambda_3(\mathbf{L}^{\star}) \leq \cdots \leq \lambda_n(\mathbf{L}^{\star})$ .

In the setting of Theorem 29, we claim  $\lambda_3^* \ge \frac{n(p+q)}{2} > nq$ . This is because because  $p_{vw} \ge p$ , which implies that if we consider  $\mathbf{L}_1^*$  to be the expected Laplacian for SSBM(n, p, q) and  $\mathbf{L}_2^*$  to be the expected Laplacian for NSSBM $(n, p, \overline{p}, q)$ , then  $\mathbf{L}_2^* \ge \mathbf{L}_1^*$ .

In the setting of Theorem 30, we have  $\lambda_3(\widehat{\mathbf{L}}) - \lambda_2(\widehat{\mathbf{L}}) > nq$ , by the theorem assumption. Since  $\mathbf{L}^*$  is obtained from  $\widehat{\mathbf{L}}$  by adding the adversarial edges, we have  $\lambda_i(\mathbf{L}^*) \ge \lambda_i(\widehat{\mathbf{L}})$  for all *i*. In particular, we have  $\lambda_3(\mathbf{L}^*) \ge \lambda_3(\widehat{\mathbf{L}}) = \lambda_2(\widehat{\mathbf{L}}) + (\lambda_3(\widehat{\mathbf{L}}) - \lambda_2(\widehat{\mathbf{L}})) > nq$ , where the last inequality is using the fact  $\lambda_2(\widehat{\mathbf{L}}) \ge 0$ . Therefore, nq must be the second eigenvalue of  $\mathbf{L}^*$ , completing the proof of Lemma 7.5.14.

Next, we prove a general Davis-Kahan style bound.

**Lemma 7.5.15.** Let **L** and  $\widehat{\mathbf{L}}$  be two weighted Laplacian matrices. Let  $u_2$  and  $\widehat{u_2}$  be the second eigenvectors of **L** and  $\widehat{\mathbf{L}}$ , respectively. Then,

$$\|\boldsymbol{u}_{2} - \widehat{\boldsymbol{u}}_{2}\|_{2} \leq \sqrt{2} \cdot \min\left\{\frac{\left\|(\widehat{\mathbf{L}} - \mathbf{L})\boldsymbol{u}_{2}\right\|_{2}}{\left|\lambda_{3}(\widehat{\mathbf{L}}) - \lambda_{2}(\mathbf{L})\right|}, \frac{\left\|(\widehat{\mathbf{L}} - \mathbf{L})\widehat{\boldsymbol{u}}_{2}\right\|_{2}}{\left|\lambda_{3}(\mathbf{L}) - \lambda_{2}(\widehat{\mathbf{L}})\right|}\right\}$$

*Proof of Lemma* 7.5.15. One can get this sort of guarantee from variants of the Davis-Kahan theorem, but it is more illuminating to write an eigenvalue decomposition and observe it from there. Without loss of generality, assume that  $\langle \hat{u}_2, u_2 \rangle \ge 0$  (indeed, otherwise we can always negate  $\hat{u}_2$  if this is not the case). Notice that

$$\begin{aligned} \left\| (\widehat{\mathbf{L}} - \mathbf{L}) \boldsymbol{u}_2 \right\|_2^2 &= \left\| \left( \widehat{\mathbf{L}} - \lambda_2(\mathbf{L}) \mathbf{I} \right) \boldsymbol{u}_2 \right\|_2^2 \\ &= (\lambda_2(\widehat{\mathbf{L}}) - \lambda_2(\mathbf{L}))^2 \langle \widehat{\boldsymbol{u}}_2, \boldsymbol{u}_2 \rangle^2 + \sum_{i=3}^n \left( \lambda_i(\widehat{\mathbf{L}}) - \lambda_2(\mathbf{L}) \right)^2 \langle \widehat{\boldsymbol{u}}_i, \boldsymbol{u}_2 \rangle^2 \\ &\geq \sum_{i=3}^n \left( \lambda_3(\widehat{\mathbf{L}}) - \lambda_2(\mathbf{L}) \right)^2 \langle \widehat{\boldsymbol{u}}_i, \boldsymbol{u}_2 \rangle^2 = \left( \lambda_3(\widehat{\mathbf{L}}) - \lambda_2(\mathbf{L}) \right)^2 \left( 1 - \langle \widehat{\boldsymbol{u}}_2, \boldsymbol{u}_2 \rangle^2 \right) \end{aligned}$$

which rearranges to

$$\langle \widehat{u}_2, u_2 \rangle^2 \ge 1 - \left( \frac{\left\| (\widehat{\mathbf{L}} - \mathbf{L}) u_2 \right\|_2}{\lambda_3(\widehat{\mathbf{L}}) - \lambda_2(\mathbf{L})} \right)^2$$

Now, if  $\|(\widehat{\mathbf{L}} - \mathbf{L})u_2\|_2 \ge |\lambda_3(\widehat{\mathbf{L}}) - \lambda_2(\mathbf{L})|$ , then the condition  $\|u_2 - \widehat{u_2}\|_2 \le \sqrt{2} \cdot \frac{\|(\widehat{\mathbf{L}} - \mathbf{L})u_2\|_2}{|\lambda_3(\widehat{\mathbf{L}}) - \lambda_2(\mathbf{L})|}$  is trivially satisfied, since  $\|u_2 - \widehat{u_2}\|_2 \le \sqrt{2 - 2\langle \widehat{u_2}, u_2 \rangle} \le \sqrt{2}$ . Otherwise, taking the square roots of both sides, we obtain

$$\langle \widehat{u}_2, u_2 \rangle \geq \sqrt{1 - \left(\frac{\left\|(\widehat{\mathbf{L}} - \mathbf{L})u_2\right\|_2}{\lambda_3(\widehat{\mathbf{L}}) - \lambda_2(\mathbf{L})}\right)^2},$$

which gives

$$\|\widehat{\boldsymbol{u}}_{2} - \boldsymbol{u}_{2}\|_{2}^{2} = 2 - 2\langle\widehat{\boldsymbol{u}}_{2}, \boldsymbol{u}_{2}\rangle \leq 2 - 2\sqrt{1 - \left(\frac{\left\|(\widehat{\mathbf{L}} - \mathbf{L})\boldsymbol{u}_{2}\right\|_{2}}{\lambda_{3}(\widehat{\mathbf{L}}) - \lambda_{2}(\mathbf{L})}\right)^{2}} \leq 2 \cdot \left(\frac{\left\|(\widehat{\mathbf{L}} - \mathbf{L})\boldsymbol{u}_{2}\right\|_{2}}{\lambda_{3}(\widehat{\mathbf{L}}) - \lambda_{2}(\mathbf{L})}\right)^{2}$$

Taking the square root of both sides and repeating this argument by exchanging the roles of L and  $\hat{L}$  yields the statement of Lemma 7.5.15.

This immediately implies the following upper-bound on  $\|u_2 - u_2^{\star}\|_2$ . We will use it repeatedly, both in Theorem 29 and Theorem 30.

Lemma 7.5.16. We have

$$\left\|\boldsymbol{u}_{2}-\boldsymbol{u}_{2}^{\star}\right\|_{2} \leq \sqrt{2} \cdot \frac{\left\|\mathbf{E}\boldsymbol{u}_{2}^{\star}\right\|_{2}}{\left|\lambda_{3}-\lambda_{2}^{\star}\right|}.$$

*Proof.* Lemma 7.5.16 immediately follows from Lemma 7.5.15 by letting  $\widehat{\mathbf{L}} = \mathbf{L}^{\star}$ .

Combining with Lemma 7.5.11 and Lemma 7.5.12, we can now upper-bound  $\|u_2 - u_2^{\star}\|_2$  in the setting of Theorem 29.

**Lemma 7.5.17.** In the setting of Theorem 29, there exists a universal constant C such that, for  $\delta \ge 3n^{-10}$ , with probability  $\ge 1 - \delta$ , we have

$$\left\|\boldsymbol{u}_2 - \boldsymbol{u}_2^{\star}\right\|_2 \leq \frac{C}{\sqrt{\log\left(n/\delta\right)}}$$

Proof of Lemma 7.5.17. Using Lemma 7.5.16, Lemma 7.5.11 and Lemma 7.5.12, we have

$$\left\| \boldsymbol{u}_{2} - \boldsymbol{u}_{2}^{\star} \right\|_{2} \leq \frac{400\sqrt{2}C_{7.5.12} \left( \sqrt{nq} + (nq\log\left(\frac{3n}{\delta}\right))^{1/4} + \sqrt{\log\left(\frac{3n}{\delta}\right)} \right)}{n(p-q)}$$

At this point, it is enough to show that there exists a universal constant C such that

$$Cn(p-q) \ge 400\sqrt{2}C_{7.5.12}\left(\sqrt{nq\log(n/\delta)} + (nq)^{1/4}\left(\log(n/\delta)\right)^{3/4} + \log(n/\delta)\right).$$

To see this, note that for any two nonnegative real numbers we have  $2a^{1/4}b^{1/4} \le \sqrt{b} + \sqrt{a}$ , which implies  $2a^{1/4}b^{3/4} \le b + \sqrt{ab}$ . Let a = nq and  $b = \log (3n/\delta)$ , and we get

$$400\sqrt{2}C_{7.5.12}\left(\sqrt{nq\log(3n/\delta)} + (nq)^{1/4}\left(\log(n/\delta)\right)^{3/4} + \log(3n/\delta)\right)$$

$$\leq 800\sqrt{2}C_{7.5.12} \left(\sqrt{nq\log(3n/\delta)} + \log(3n/\delta)\right)$$
  
 
$$\leq 800\sqrt{2}C_{7.5.12} \left(\sqrt{n\overline{p}\log(3n/\delta)} + \log(3n/\delta)\right) \leq Cn(p-q),$$

where the last inequality follows from the assumption we gave in Theorem 29. We therefore conclude the proof of Lemma 7.5.17.  $\hfill \Box$ 

Next, we prove  $\ell_1$  norm concentration for the rows of **A** and for the rows of **L** in the setting of Theorem 29. We will use this in Lemma 7.5.19, where we will bound  $\|u_2^{(v)} - u_2\|_2$ . Here  $u_2^{(v)}$  denotes the second eigenvector of the leave-one-out Laplacian  $\mathbf{L}^{(v)}$ .

**Lemma 7.5.18.** In the setting of Theorem 29, there exists a universal constant C such that with probability  $\geq 1 - \delta$ , for all  $v \in V$ , we have

$$\begin{aligned} \left\| \boldsymbol{a}_{v} - \boldsymbol{a}_{v}^{\star} \right\|_{1} &\leq C \left( n\overline{p} + \sqrt{n\overline{p}\log\left(n/\delta\right)} + \log\left(n/\delta\right) \right) \\ \left\| \boldsymbol{l}_{v} - \mathbb{E}\left[ \boldsymbol{l}_{v} \right] \right\|_{1} &\leq C \left( n\overline{p} + \sqrt{n\overline{p}\log\left(n/\delta\right)} + \log\left(n/\delta\right) \right). \end{aligned}$$

Proof of Lemma 7.5.18. It is easy to see that

$$\|l_{v} - \mathbb{E}[l_{v}]\|_{1} = |d[v] - \mathbb{E}[d[v]]| + \|a_{v} - a_{v}^{\star}\|_{1}.$$

Let us consider the second term above. By Lemma 7.5.4 and Lemma 7.5.3, we have with probability  $\geq 1 - \delta/2$  that for all  $v \in V$ 

$$\begin{aligned} \left\| \boldsymbol{a}_{v} - \boldsymbol{a}_{v}^{\star} \right\|_{1} &\leq \left\| \boldsymbol{a}_{v} \right\|_{1} + \left\| \boldsymbol{a}_{v}^{\star} \right\|_{1} \\ &\leq 2 \left( \frac{n\overline{p}}{2} + \max\left\{ C_{7.5.3}, C_{7.5.4} \right\} \left( \sqrt{n\overline{p} \log\left(4n/\delta\right)} + \log\left(4n/\delta\right) \right) \right) + n\overline{p} \\ &= 2n\overline{p} + 2 \max\left\{ C_{7.5.3}, C_{7.5.4} \right\} \left( \sqrt{n\overline{p} \log\left(4n/\delta\right)} + \log\left(4n/\delta\right) \right). \end{aligned}$$

Finally, by Lemma 7.5.3 and Lemma 7.5.4, we have with probability  $1 - \delta/2$  that for all  $v \in V$ ,

$$\begin{aligned} |\boldsymbol{d}[\boldsymbol{v}] - \mathbb{E}\left[\boldsymbol{d}[\boldsymbol{v}]\right]| &\leq \max_{\boldsymbol{v} \in \boldsymbol{V}} |\boldsymbol{d}_{\mathsf{out}}[\boldsymbol{v}] - \mathbb{E}\left[\boldsymbol{d}_{\mathsf{out}}[\boldsymbol{v}]\right]| + \max_{\boldsymbol{v} \in \boldsymbol{V}} |\boldsymbol{d}_{\mathsf{in}}[\boldsymbol{v}] - \mathbb{E}\left[\boldsymbol{d}_{\mathsf{in}}[\boldsymbol{v}]\right]| \\ &\leq 2 \max\left\{C_{7.5.3}, C_{7.5.4}\right\} \left(\sqrt{n\overline{p}\log\left(4n/\delta\right)} + \log\left(4n/\delta\right)\right) \end{aligned}$$

Adding everything up means that with probability  $\geq 1 - \delta$ , for all  $v \in V$ , we have

$$\|\boldsymbol{l}_{v} - \mathbb{E}[\boldsymbol{l}_{v}]\|_{1} \leq 2n\overline{p} + 4\max\left\{C_{7.5.4}, C_{7.5.3}\right\}\left(\sqrt{n\overline{p}\log\left(\frac{4n}{\delta}\right)} + \log\left(\frac{4n}{\delta}\right)\right),$$

which completes the proof of Lemma 7.5.18.

Having established Lemma 7.5.18, we can now upper-bound  $\|u_2^{(v)} - u_2\|_2$ .

**Lemma 7.5.19.** In the setting of Theorem 29, for  $\delta \ge 2n^{-9}$  with probability  $\ge 1 - \delta$ , for all  $v \in V$ , we have

$$\left\|\boldsymbol{u}_{2}^{(v)}-\boldsymbol{u}_{2}\right\|_{2} \leq \|\boldsymbol{u}_{2}\|_{\infty} \cdot \frac{C\left(\overline{p}+\sqrt{p\log\left(n/\delta\right)/n}+\log\left(n/\delta\right)/n\right)}{p-q}$$

*Proof of Lemma* 7.5.19. Recall that the gap condition in Theorem 29 means that p and q are such that for a universal constant C,

$$n(p-q) \geq C\left(\sqrt{n\overline{p}\log\left(n/\delta\right)} + \log\left(n/\delta\right)\right).$$

To appeal to Lemma 7.5.15, we need to understand the entries of the matrix  $\mathbf{L} - \mathbf{L}^{(v)}$ . It is easy to see that this matrix only has nonzero entries on the diagonal and in the *v*th row and column. There, the *v*th row and column of  $\mathbf{L} - \mathbf{L}^{(v)}$  are exactly equal to those of  $\mathbf{L} - \mathbf{L}^*$ . Moreover, the  $w \neq v$ th diagonal entry of  $\mathbf{L} - \mathbf{L}^{(v)}$  is exactly  $\mathbb{1} \{(v, w) \in E\} - p_{vw}$ .

Hence, we have

$$\begin{split} \left\| \left( \mathbf{L} - \mathbf{L}^{(v)} \right) u_2 \right\|_2 \\ &= \left( \sum_{w=1}^n \left\langle \left( \mathbf{L} - \mathbf{L}^{(v)} \right)_w, u_2 \right\rangle^2 \right)^{1/2} \\ &= \left( \left\langle \left( \mathbf{L} - \mathbf{L}^{\star} \right)_v, u_2 \right\rangle^2 + \sum_{w \neq v} \left( (a_v[w] - p_{vw}) u_2[w] - (a_v[w] - p_{vw}) u_2[v] \right)^2 \right)^{1/2} \\ &\leq \left| \left\langle \left( \mathbf{L} - \mathbf{L}^{\star} \right)_v, u_2 \right\rangle \right| + \left( \sum_{w \neq v} \left( (a_v[w] - p_{vw}) u_2[w] - (a_v[w] - p_{vw}) u_2[v] \right)^2 \right)^{1/2} \\ &\leq \left( \| l_v - \mathbb{E} \left[ l_v \right] \|_1 + 2 \left\| a_v - a_v^{\star} \right\|_2 \right) \cdot \| u_2 \|_{\infty} \\ &\leq \left( \| l_v - \mathbb{E} \left[ l_v \right] \|_1 + 2 \left\| a_v - a_v^{\star} \right\|_1 \right) \cdot \| u_2 \|_{\infty} \\ &\leq \left\| u_2 \|_{\infty} \cdot 3C_{7.5.18} \left( n\overline{p} + \sqrt{n\overline{p} \log (2n^2/\delta)} + \log (2n^2/\delta) \right). \end{split}$$

Now, let  $C \ge 8C_{7.5.10}$ . Using Lemma 7.5.10 to understand the concentration of sampling the graph except edges incident to v, along with Weyl's inequality, we have with probability  $\ge 1 - \delta$  that for all  $v \in V$  and for all n sufficiently large,

$$\begin{aligned} \left|\lambda_3^{(v)} - \lambda_2\right| &\geq \left(\lambda_3^{(v)} - \lambda_3^{\star}\right) - \left(\lambda_2 - \lambda_2^{\star}\right) + \left(\lambda_3^{\star} - \lambda_2^{\star}\right) \\ &\geq -2\left(C_{7.5.10}\sqrt{n\overline{p}\log\left(2n^2/\delta\right)} + \log\left(2n^2/\delta\right)\right) + \frac{n(p-q)}{2} \geq \frac{n(p-q)}{4}.\end{aligned}$$

Now, using Lemma 7.5.15, we get

$$\left\| \boldsymbol{u}_{2}^{(v)} - \boldsymbol{u}_{2} \right\|_{2} \leq \frac{\left\| \left( \mathbf{L} - \mathbf{L}^{(v)} \right) \boldsymbol{u}_{2} \right\|_{2}}{\left| \lambda_{3}^{(v)} - \lambda_{2} \right|} \leq \| \boldsymbol{u}_{2} \|_{\infty} \cdot \frac{12C_{7.5.18} \left( n\overline{p} + \sqrt{n\overline{p}\log\left(n/\delta\right)} + \log\left(n/\delta\right) \right)}{n(p-q)}$$

$$\leq \|\boldsymbol{u}_2\|_{\infty} \cdot \frac{12C_{7.5.18}\left(\overline{p} + \sqrt{\overline{p}\log\left(2n^2/\delta\right)/n} + \log\left(2n^2/\delta\right)/n\right)}{p-q},$$

completing the proof of Lemma 7.5.19.

#### 7.5.5. Leave-one-out and bootstrap

The main goal of this section is to establish an upper-bound on  $|\langle a_v - a_v^*, u_2 - u_2^* \rangle|$  in the setting of Theorem 29. To this end, we will need the following concentration inequality from [AFWZ20].

**Lemma 7.5.20** (Lemma 7 from [AFWZ20]). Let  $w \in \mathbb{R}^n$  and  $X_i \sim Ber(p_i)$ . Let  $p \ge p_i$  for all  $i \in [n]$ . Let  $X \in \mathbb{R}^n$  be the vector formed by stacking the  $X_i$ . Then,

$$\Pr\left[|\langle \boldsymbol{w}, \boldsymbol{X} - \mathbb{E}[\boldsymbol{X}]\rangle| \ge \frac{(2+a)pn}{\max\left(1, \log\left(\frac{\sqrt{n}\|\boldsymbol{w}\|_{\infty}}{\|\boldsymbol{w}\|_{2}}\right)\right)} \cdot \|\boldsymbol{w}\|_{\infty}\right] \le 2\exp\left(-anp\right).$$

**Lemma 7.5.21.** In the setting of Theorem 29, suppose  $a_v$  is such that  $a_v[w] \sim \text{Bernoulli}(p_{vw})$  and let  $\overline{p} \geq \max_{w: p_{vw} \neq 1} p_{vw}$ . With probability  $\geq 1 - \delta$  for  $\delta \geq 1/n^2$ , for all  $v \in V$ , we have

$$\left|\left\langle \boldsymbol{a}_{v}-\boldsymbol{a}_{v}^{\star},\boldsymbol{u}_{2}-\boldsymbol{u}_{2}^{\star}\right\rangle\right| \leq C\left(n\overline{p}+\log\left(n/\delta\right)\right)\left(\frac{\|\boldsymbol{u}_{2}\|_{\infty}}{\log\log n}+\frac{1}{\sqrt{n}\log\log n}\right)$$

*Proof of Lemma* 7.5.21. Ideally, one would treat  $u_2 - u_2^*$  as fixed and then apply Bernstein's inequality to argue that the sum of centered Bernoulli random variables as written above concentrates well. Unfortunately, since  $u_2$  depends on  $a_v - a_v^*$ , we cannot express this inner product as the sum of independent random variables.

To resolve this, we use the leave-one-out method. Let  $u_2^{(v)}$  be the second eigenvector of the leave-one-out Laplacian  $\mathbf{L}^{(v)}$  of  $\mathbf{A}^{(v)}$ , where  $\mathbf{A}^{(v)}$  is chosen to agree with  $\mathbf{A}$  everywhere except for the *v*th row and *v*th column. The *v*th row and *v*th column of  $\mathbf{A}^{(v)}$  are replaced with those of  $\mathbf{A}^*$ . Now,  $a_v$  does not depend on  $\mathbf{L}^{(v)}$  and therefore  $u_2^{(v)}$ .

We therefore write

$$\begin{aligned} |\langle a_v - a_v^{\star}, u_2 - u_2^{\star} \rangle| &\leq |\langle a_v - a_v^{\star}, u_2 - u_2^{(v)} \rangle| + |\langle a_v - a_v^{\star}, u_2^{(v)} - u_2^{\star} \rangle| \\ &\leq ||a_v - a_v^{\star}||_2 \cdot ||u_2^{(v)} - u_2||_2 + |\langle a_v - a_v^{\star}, u_2^{(v)} - u_2^{\star} \rangle| \\ &\leq ||a_v - a_v^{\star}||_2 \cdot \frac{C_{7.5.19}\overline{p}}{p - q} ||u_2||_{\infty} + |\langle a_v - a_v^{\star}, u_2^{(v)} - u_2^{\star} \rangle|. \end{aligned}$$

To bound the rightmost term of the RHS, we use Lemma 7 of [AFWZ20], reproduced in Lemma 7.5.20. In that, let  $w := u_2^{(v)} - u_2^*$ . Let  $a = \frac{1}{n\overline{p}} \log (20n/\delta)$  so that  $2 \exp(-2an\overline{p}) \le \delta/(10n)$ . Note that for the deterministic entries, we have  $a_v - a_v^* = 1 - 1 = 0$ , so in Lemma 7.5.20, we can set  $X_w \sim \text{Ber}(0)$  for these entries. Now, by Lemma 7.5.20, with probability  $\ge 1 - \delta/n$ , we have

$$\left| \left\langle \boldsymbol{u}_{2}^{(v)} - \boldsymbol{u}_{2}^{\star}, \boldsymbol{a}_{v} - \boldsymbol{a}_{v}^{\star} \right\rangle \right| \leq \frac{2n\overline{p} + \log\left(\frac{20n}{\delta}\right)}{\max\left(1, \log\left(\frac{\sqrt{n}\|\boldsymbol{w}\|_{\infty}}{\|\boldsymbol{w}\|_{2}}\right)\right)} \cdot \|\boldsymbol{w}\|_{\infty} \,. \tag{7.5.1}$$

Let us first bound  $\|w\|_{\infty} = \left\|u_2^{(v)} - u_2^{\star}\right\|_{\infty}$ . We write

$$\left\| u_{2}^{(v)} - u_{2}^{\star} \right\|_{\infty} \le \left\| u_{2}^{(v)} - u_{2} \right\|_{\infty} + \left\| u_{2} - u_{2}^{\star} \right\|_{\infty}$$
(7.5.2)

$$\leq \left\| u_{2}^{(v)} - u_{2} \right\|_{2} + \left\| u_{2} \right\|_{\infty} + \left\| u_{2}^{\star} \right\|_{\infty}$$
(7.5.3)

$$\leq 2 \max \left\{ C_{7.5.19}(\alpha, \delta) \, \| \boldsymbol{u}_2 \|_{\infty} \, , \frac{1}{\sqrt{n}} \right\}.$$
 (7.5.4)

In what follows, we omit the arguments  $\alpha$  and  $\delta$  in mentions of  $C_{7.5.19}$ . Next, using Lemma 7.5.17, the triangle inequality, and  $\delta \ge 1/n^3$ , we have

$$\|\boldsymbol{w}\|_{2} = \left\|\boldsymbol{u}_{2}^{(v)} - \boldsymbol{u}_{2}^{\star}\right\|_{2} \le C_{7.5.19} \|\boldsymbol{u}_{2}\|_{\infty} + \frac{4C_{7.5.17}}{\sqrt{\log n}}.$$

We now have two cases based on the value of  $\sqrt{n} \cdot \frac{\left\|u_{2}^{(v)}-u_{2}^{\star}\right\|_{\infty}}{\left\|u_{2}^{(v)}-u_{2}^{\star}\right\|_{2}}$ .

**Case 1 –** *w* **is not too "flat."** Let us first handle the case where

$$\frac{\sqrt{n} \cdot \left\| \boldsymbol{u}_{2}^{(v)} - \boldsymbol{u}_{2}^{\star} \right\|_{\infty}}{\left\| \boldsymbol{u}_{2}^{(v)} - \boldsymbol{u}_{2}^{\star} \right\|_{2}} \ge \sqrt{\log n}.$$

We plug this into (7.5.1) and get

$$\begin{split} \left| \left\langle \boldsymbol{u}_{2}^{(v)} - \boldsymbol{u}_{2}^{\star}, \boldsymbol{a}_{v} - \boldsymbol{a}_{v}^{\star} \right\rangle \right| &\leq \frac{2n\overline{p} + \log\left(\frac{20n}{\delta}\right)}{\max\left(1, \log\left(\frac{\sqrt{n}\|\boldsymbol{w}\|_{\infty}}{\|\boldsymbol{w}\|_{2}}\right)\right)} \cdot \|\boldsymbol{w}\|_{\infty} \\ &\leq 4 \cdot \frac{n\overline{p} + \log\left(\frac{20n}{\delta}\right)}{\log\log n} \left( C_{7.5.19} \|\boldsymbol{u}_{2}\|_{\infty} + \frac{1}{\sqrt{n}} \right), \end{split}$$

where the last inequality follows from (7.5.4).

**Case 2 –** *w* **is "flat."** We now assume

$$\frac{\sqrt{n} \cdot \left\| \boldsymbol{u}_{2}^{(v)} - \boldsymbol{u}_{2}^{\star} \right\|_{\infty}}{\left\| \boldsymbol{u}_{2}^{(v)} - \boldsymbol{u}_{2}^{\star} \right\|_{2}} \leq \sqrt{\log n}.$$

We can easily check that the function

$$\frac{x}{\max\left(1,\log x\right)}$$

is increasing, so its maximum will be attained at the largest value of *x* in the domain. Let  $x = \sqrt{n} ||w||_{\infty} / ||w||_{2}$  and write

$$\frac{2n\overline{p} + \log\left(\frac{20n}{\delta}\right)}{\max\left(1, \log\left(\frac{\sqrt{n}\|\boldsymbol{w}\|_{\infty}}{\|\boldsymbol{w}\|_{2}}\right)\right)} \cdot \|\boldsymbol{w}\|_{\infty}$$
$$= \frac{2n\overline{p} + \log\left(\frac{20n}{\delta}\right)}{\max\left(1, \log\left(\frac{\sqrt{n}\|\boldsymbol{w}\|_{\infty}}{\|\boldsymbol{w}\|_{2}}\right)\right)} \cdot \frac{\sqrt{n} \|\boldsymbol{w}\|_{\infty}}{\|\boldsymbol{w}\|_{2}} \cdot \frac{\|\boldsymbol{w}\|_{2}}{\sqrt{n}}$$

$$\leq \frac{2n\overline{p} + \log\left(\frac{20n}{\delta}\right)}{\log\log n} \cdot \sqrt{\frac{\log n}{n}} \cdot \|w\|_{2}$$

$$\leq \frac{2n\overline{p} + \log\left(\frac{20n}{\delta}\right)}{\log\log n} \cdot \sqrt{\frac{\log n}{n}} \cdot C_{7.5.19} \left( \|u_{2}\|_{\infty} + \frac{1}{\sqrt{\log\left(20n^{2}/\delta\right)}} \right)$$

$$= C_{7.5.19} \left( \frac{2n\overline{p} + \log\left(\frac{20n}{\delta}\right)}{\log\log n} \cdot \sqrt{\frac{\log n}{n}} \|u_{2}\|_{\infty} + \frac{2n\overline{p} + \log\left(\frac{20n}{\delta}\right)}{\sqrt{n} \cdot \log\log n} \right).$$

All of this tells us that

$$\left|\left\langle \boldsymbol{a}_{v}-\boldsymbol{a}_{v}^{\star},\boldsymbol{u}_{2}^{(v)}-\boldsymbol{u}_{2}^{\star}\right\rangle\right| \leq 4C_{7.5.19}\cdot(n\overline{p}+\log\left(\frac{20n}{\delta}\right))\left(\frac{\|\boldsymbol{u}_{2}\|_{\infty}}{\log\log n}+\frac{1}{\sqrt{n}\log\log n}\right).$$

It remains to handle the term

$$\|\boldsymbol{a}_v - \boldsymbol{a}_v^{\star}\|_2 \cdot \|\boldsymbol{u}_2\|_{\infty}$$

Indeed, using Lemma 7.5.13, we have with probability  $\geq 1 - \delta$  that

$$\left\|\boldsymbol{a}_{v}-\boldsymbol{a}_{v}^{\star}\right\|_{2}\cdot\|\boldsymbol{u}_{2}\|_{\infty}\leq C_{7.5.13}\left(\frac{\log\left(20n/\delta\right)}{\log n}\right)^{3/2}\sqrt{n\overline{p}}\cdot\|\boldsymbol{u}_{2}\|_{\infty}.$$

Combining everything tells us that

$$\begin{split} \left| \left\langle a_v - a_v^{\star}, u_2 - u_2^{\star} \right\rangle \right| &\leq 30C_{7.5.13} \left( \frac{\log \left( 20n/\delta \right)}{\log n} \right)^{3/2} \sqrt{n\overline{p}} \cdot \|u_2\|_{\infty} \\ &+ 4C_{7.5.19} \cdot \left( n\overline{p} + \log \left( 20n/\delta \right) \right) \left( \frac{\|u_2\|_{\infty}}{\log \log n} + \frac{1}{\sqrt{n} \log \log n} \right) \\ &\leq C \left( n\overline{p} + \log \left( 20n/\delta \right) \right) \left( \frac{\|u_2\|_{\infty}}{\log \log n} + \frac{1}{\sqrt{n} \log \log n} \right). \end{split}$$

Taking a union bound over all  $v \in V$  concludes the proof of Lemma 7.5.21.

Finally, we establish an upper-bound on  $||u_2||_{\infty}$ . This will be used repeatedly in the proof of Theorem 29.

**Lemma 7.5.22.** *In the same setting as Theorem 29, with probability*  $\geq 1 - \delta$  *for*  $\delta \geq 10n^2$ *, we have for some constant*  $C(\alpha, \delta)$  *that* 

$$\|\boldsymbol{u}_2\|_{\infty} \leq \frac{C(\alpha,\delta)}{\sqrt{n}}.$$

Proof of Lemma 7.5.22. First, observe that

$$(\mathbf{D}-\mathbf{A})\boldsymbol{u}_2=\lambda_2\boldsymbol{u}_2,$$

which means that

$$\left(\mathbf{D} - \lambda_2 \mathbf{I}\right)^{-1} \mathbf{A} \boldsymbol{u}_2 = \boldsymbol{u}_2.$$

By Lemma 7.5.7, with probability  $\geq 1 - \delta$ , for all  $v \in V$  we have

$$d[v] - \lambda_2 \ge \frac{n(p-q)}{4}.$$

Combining with Lemma 7.5.6, we have

$$\begin{aligned} \frac{d_{\rm in}[v] - d_{\rm out}[v]}{d_{\rm in}[v] - d_{\rm out}[v] + (2d_{\rm out}[v] - \lambda_2)} &= 1 - \frac{2d_{\rm out}[v] - \lambda_2}{d_{\rm in}[v] - d_{\rm out}[v] + (2d_{\rm out}[v] - \lambda_2)} \\ &\leq 1 + \frac{C_{7.5.6} \left(\sqrt{nq \log (10n/\delta)} + \log (10n/\delta)\right)}{d_{\rm in}[v] - d_{\rm out}[v] + (2d_{\rm out}[v] - \lambda_2)} \\ &\leq 1 + \frac{4C_{7.5.6} \left(\sqrt{nq \log (10n/\delta)} + \log (10n/\delta)\right)}{n(p-q)} \leq C', \end{aligned}$$

for some constant C' > 0, where the penultimate line follows from Lemma 7.5.7 and the last line follows from the gap assumption in Theorem 29. Furthermore, by Lemma 7.5.8 and Lemma 7.5.17, we have with probability  $\geq 1 - \delta$  that for all  $v \in V$ ,

$$\frac{\left|\left\langle \boldsymbol{a}_{v}^{\star}, \boldsymbol{u}_{2}^{\star} - \boldsymbol{u}_{2}\right\rangle\right|}{\boldsymbol{d}[v] - \lambda_{2}} \leq \frac{\overline{p}\sqrt{n}}{\boldsymbol{d}[v] - \lambda_{2}} \cdot \frac{C_{7.5.17}}{\sqrt{\log\left(10n/\delta\right)}} \leq \frac{C_{7.5.8}(\alpha) \cdot C_{7.5.17}}{\sqrt{n\log\left(10n/\delta\right)}}$$

Now, using Lemma 7.5.8 (and using Lemma 7.5.7 to ensure that  $d[v] - \lambda_2 > 0$  for all  $v \in V$ ), we have

$$\begin{split} \|u_2\|_{\infty} &= \left\| (\mathbf{D} - \lambda_2 \mathbf{I})^{-1} \mathbf{A} u_2 \right\|_{\infty} \\ &= \left\| (\mathbf{D} - \lambda_2 \mathbf{I})^{-1} \mathbf{A} u_2 - (\mathbf{D} - \lambda_2 \mathbf{I})^{-1} \mathbf{A} u_2^{\star} + (\mathbf{D} - \lambda_2 \mathbf{I})^{-1} \mathbf{A} u_2^{\star} \right\|_{\infty} \\ &\leq \left\| (\mathbf{D} - \lambda_2 \mathbf{I})^{-1} \mathbf{A} u_2^{\star} \right\|_{\infty} + \left\| (\mathbf{D} - \lambda_2 \mathbf{I})^{-1} \mathbf{A} (u_2^{\star} - u_2) \right\|_{\infty} \\ &= \max_{1 \le v \le n} \frac{\left| \langle \mathbf{a}_v, \mathbf{u}_2^{\star} \rangle \right|}{d[v] - \lambda_2} + \max_{1 \le v \le n} \frac{\left| \langle \mathbf{a}_v, \mathbf{u}_2^{\star} - u_2 \rangle \right|}{d[v] - \lambda_2} \\ &= \frac{1}{\sqrt{n}} \left( \max_{1 \le v \le n} \frac{\left| \mathbf{d}_{\text{in}}[v] - \mathbf{d}_{\text{out}}[v] \right|}{d[v] - \lambda_2} \right) + \max_{1 \le v \le n} \frac{\left| \langle \mathbf{a}_v, \mathbf{u}_2^{\star} - u_2 \rangle \right|}{d[v] - \lambda_2} \\ &\leq \frac{C}{\sqrt{n}} + \max_{1 \le v \le n} \frac{\left| \langle \mathbf{a}_v - \mathbf{a}_v^{\star}, \mathbf{u}_2^{\star} - u_2 \rangle \right|}{d[v] - \lambda_2} + \max_{1 \le v \le n} \frac{\left| \langle \mathbf{a}_v^{\star}, \mathbf{u}_2^{\star} - u_2 \rangle \right|}{d[v] - \lambda_2} \\ &\leq \frac{C}{\sqrt{n}} + \frac{C_{7.5.21} \left( n\overline{p} + \log \left( \frac{10n}{\delta} \right) \right)}{d[v] - \lambda_2} \cdot \left( \frac{1}{\sqrt{n} \log \log n} + \frac{\|u_2\|_{\infty}}{\log \log n} \right) + \frac{C_{7.5.8}(\alpha) \cdot C_{7.5.17}}{\sqrt{n \log \left( \frac{10n}{\delta} \right)}} \right) \\ &\leq \frac{C}{\sqrt{n}} + C_{7.5.21} \cdot C_{7.5.8}(\alpha) \cdot \left( \frac{1}{\sqrt{n} \log \log n} + \frac{\|u_2\|_{\infty}}{\log \log n} \right) + \frac{C_{7.5.8}(\alpha) \cdot C_{7.5.17}}{\sqrt{n \log \left( \frac{10n}{\delta} \right)}} \right) \\ &\leq \frac{C}{\sqrt{n}} + C_{7.5.21} \cdot C_{7.5.8}(\alpha) \cdot \left( \frac{1}{\sqrt{n} \log \log n} + \frac{\|u_2\|_{\infty}}{\log \log n} \right) + \frac{C_{7.5.8}(\alpha) \cdot C_{7.5.17}}{\sqrt{n \log \left( \frac{10n}{\delta} \right)}} \right\}$$

Note that any *n* large enough

$$\frac{C_{7.5.21} \cdot C_{7.5.8}(\alpha) \cdot \|\boldsymbol{u}_2\|_{\infty}}{\log \log n} \leq \frac{\|\boldsymbol{u}_2\|_{\infty}}{2}.$$

Thus, rearranging and solving for  $||u_2||_{\infty}$  yields

$$\|\boldsymbol{u}_2\|_{\infty} \leq 2\left(\frac{C}{\sqrt{n}} + C_{7.5.21} \cdot C_{7.5.8}(\alpha) \cdot \left(\frac{1}{\sqrt{n}\log\log n}\right) + \frac{C_{7.5.8}(\alpha) \cdot C_{7.5.17}}{\sqrt{n\log(10n/\delta)}}\right),$$

completing the proof of Lemma 7.5.22.

## 7.5.6. Strong consistency of unnormalized spectral bisection

In this section, we prove our main positive results Theorem 29 and Theorem 30. It will be helpful to recall the proof sketches given in Section 7.3 while reading this section.

At a high level, the proof plan is as follows.

- 1. We first establish a sufficient condition for a particular vertex to be classified correctly. We can think of this as simultaneously showing that the intermediate estimator  $(\mathbf{D} \lambda_2 \mathbf{I})^{-1} \mathbf{A} \boldsymbol{u}_2^{\star}$  is strongly consistent and that the corresponding "noise" term  $(\mathbf{D} \lambda_2 \mathbf{I})^{-1} \mathbf{A} (\boldsymbol{u}_2^{\star} \boldsymbol{u}_2)$  is a lower-order term in comparison to this. For a more formal way to see this, see Lemma 7.5.23.
- 2. For the proof of Theorem 29, the main technical challenge in showing that the noise term above is small amounts to analyzing the random quantity  $|\langle a_v, u_2^{\star} u_2 \rangle|$ . This is where we will have to use the leave-one-out method to decouple the dependence between  $a_v$  and  $u_2$ . The relevant lemmas for the leave-one-out analysis are Lemma 7.5.21 and Lemma 7.5.22.
- 3. Finally, for the proof of Theorem 30, we again appeal to Lemma 7.5.23 but use a different approach to show that the noise term is small.

#### A sufficient condition for exact recovery and proof

The main result of this subsection is Lemma 7.5.23, which gives a general condition under which a particular vertex will be classified correctly. The proofs of Theorem 29 and Theorem 30 will follow by invoking Lemma 7.5.23. We remark that the point of this lemma is mostly conceptual; the crux of the analysis lies in establishing that these conditions are satisfied our models.

**Lemma 7.5.23.** Let  $v \in V$  be some vertex. If  $d[w] - \lambda_2 > 0$  for all  $w \in V$ ,  $d_{in}[v] > d_{out}[v]$ , and  $|\langle a_v, u_2^{\star} - u_2 \rangle| \leq (d_{in}[v] - d_{out}[v])/\sqrt{n}$ , then sign  $(u_2[v]) = \text{sign}(u_2^{\star}[v])$ , i.e.,  $u_2$  correctly classifies vertex v.

The goal of the rest of this section is to prove Lemma 7.5.23.

Our approach is to study the intermediate estimator

$$(\mathbf{D} - \lambda_2 \mathbf{I})^{-1} \mathbf{A} \boldsymbol{u}_2^{\star}.$$

At a high level, our goal is to show that this correctly classifies all the vertices with high probability and also is very close to  $u_2$  in  $\ell_{\infty}$  norm with high probability. Deng, Ling, and Strohmer [DLS21] used this intermediate estimator to prove the strong consistency of unnormalized spectral bisection for SBM(n, p, q) instances.

Next, we show that this estimator is consistent and prove Lemma 7.5.23.

Proof of Lemma 7.5.23. Observe that

$$\boldsymbol{u}_2 = (\mathbf{D} - \lambda_2 \mathbf{I})^{-1} \mathbf{A} \boldsymbol{u}_2^{\star} - (\mathbf{D} - \lambda_2 \mathbf{I})^{-1} \mathbf{A} \left( \boldsymbol{u}_2^{\star} - \boldsymbol{u}_2 \right).$$

Without loss of generality, suppose  $v \in P_1$ . In particular, this means that  $u_2^*[v] = 1/\sqrt{n}$ . Our goal is to show that  $u_2[v] > 0$ . And, as per the above, this means that it is enough to show that

$$\left( \left( \mathbf{D} - \lambda_2 \mathbf{I} \right)^{-1} \mathbf{A} \boldsymbol{u}_2^{\star} \right) [\boldsymbol{v}] \geq \left( \left( \mathbf{D} - \lambda_2 \mathbf{I} \right)^{-1} \mathbf{A} \left( \boldsymbol{u}_2^{\star} - \boldsymbol{u}_2 \right) \right) [\boldsymbol{v}],$$

or equivalently, using the fact that  $d[v] - \lambda_2 > 0$ ,

$$\langle a_v, u_2^{\star} \rangle \geq \langle a_v, u_2^{\star} - u_2 \rangle$$

where  $a_v$  denotes the *v*-th row of *A*. To see that the above holds, use the fact that we know that  $d_{in}[v] - d_{out}[v] > 0$ , which gives

$$\langle a_v, u_2^{\star} \rangle = \frac{d_{\text{in}}[v] - d_{\text{out}}[v]}{\sqrt{n}} \ge |\langle a_v, u_2^{\star} - u_2 \rangle| \ge \langle a_v, u_2^{\star} - u_2 \rangle$$

This is exactly what we needed, and we conclude the proof of Lemma 7.5.23.

#### 7.5.7. Proofs of main results

At this point, we are ready to prove our main results.

#### Nonhomogeneous symmetric stochastic block model (Proof of Theorem 29)

We are finally ready to prove Theorem 29. For convenience, we reproduce its statement here.

**Theorem 29.** Let  $p, \overline{p}, q$  be probabilities such that  $q and such that <math>\alpha := \overline{p}/(p - q)$  is an arbitrary constant. Let  $\mathcal{D} \in \mathsf{NSSBM}(n, p, \overline{p}, q)$ . Let  $n \geq N(\alpha)$  where the function  $N(\alpha)$  only depends on  $\alpha$ . There exists a universal constant C > 0 such that if

$$n(p-q) \ge C\left(\sqrt{n\overline{p}\log n} + \log n\right),$$
 (gap condition)

then unnormalized spectral bisection is strongly consistent on  $\mathcal{D}$ .

*Proof of Theorem 29.* As mentioned in Section 7.2, we actually prove a slightly stronger statement – we will allow the adversary to set at most  $n\overline{p}/\log \log n$  of the  $p_{vw}$  to 1 per vertex v (in other words, the adversary can commit to at most  $n\overline{p}/\log \log n$  edges per vertex that are guaranteed to appear in the final graph).

Our plan is to apply Lemma 7.5.23. In order to do so, we start with showing that for all v, we have  $d_{in}[v] > d_{out}[v]$ . By Lemma 7.5.5, with probability  $\ge 1 - \delta$ , we have for all  $v \in V$  that

$$d_{in}[v] - d_{out}[v] \ge \frac{n(p-q)}{2} - C_{7.5.5}\left(\sqrt{np\log(n/\delta)} + \log(n/\delta)\right) > 0.$$

Additionally, by Lemma 7.5.7, we have for all v that  $d[v] > \lambda_2$ .

The final item we need is to show that for all  $v \in V$ , we have  $|\langle a_v, u_2^{\star} - u_2 \rangle| \leq |\langle a_v, u_2^{\star} \rangle|$ . Observe that

$$\left|\left\langle a_{v}, u_{2}^{\star}-u_{2}\right\rangle\right| \leq \left|\left\langle a_{v}^{\star}, u_{2}^{\star}-u_{2}\right\rangle\right|+\left|\left\langle a_{v}-a_{v}^{\star}, u_{2}^{\star}-u_{2}\right\rangle\right|,$$

where  $a_v^{\star}$  denotes the *v*-th row of  $\mathbb{E}[\mathbf{A}]$ . We handle the terms one at a time. First, note that by Lemma 7.5.11, with probability  $\geq 1 - \delta$ , we have

$$\lambda_3 - \lambda_2^\star \ge \frac{n(p-q)}{4}.$$

Now, let  $\mathbf{E} := \mathbf{L} - \mathbb{E}[\mathbf{L}]$ , and let  $a_v^{\star}[rand] \in \mathbb{R}^V$  correspond to the vector that entrywise agrees with  $a_v^{\star}$  wherever  $a_v^{\star}$  is not 1 and is zero elsewhere. This corresponds to the edges incident to v that will be sampled randomly from the distribution over graphs. This means that for all  $n \ge N(\delta)$  and choosing  $\delta \ge 1/(10n)$ , we have

$$\begin{aligned} \left| \left\langle a_{v}^{\star}[\text{rand}], u_{2}^{\star} - u_{2} \right\rangle \right| &\leq \left\| a_{v}^{\star}[\text{rand}] \right\|_{2} \cdot \frac{\sqrt{2} \left\| \mathbf{E} u_{2}^{\star} \right\|_{2}}{\left| \lambda_{3} - \lambda_{2}^{\star} \right|} \\ &\leq \overline{p} \sqrt{n} \cdot \frac{40\sqrt{2}C_{7.5.12} \left( \sqrt{nq} + (nq \log n)^{1/4} + \sqrt{\log n} \right)}{n(p-q)} \\ &\leq \frac{1000C_{7.5.12} n\overline{p}}{\sqrt{n} \log \log n} \end{aligned}$$
(Lemma 7.5.16)  
(Lemma 7.5.11 and 7.5.12)  
(gap in Theorem 29)

To handle the oblivious insertions, let  $d_{det} \in \mathbb{R}^V$  denote the degree vector that counts the number of deterministic edges inserted incident to v, for all  $v \in V$ . Under this notation, we have

$$\left|\left\langle a_v^{\star} - a_v^{\star}[\operatorname{rand}], u_2^{\star} - u_2\right\rangle\right| \le d_{\operatorname{det}}[v] \cdot \left\|u_2 - u_2^{\star}\right\|_{\infty} \le \frac{n\overline{p}}{\sqrt{n}\log\log n} + \frac{n\overline{p}\left\|u_2\right\|_{\infty}}{\log\log n}.$$

where the last inequality follows from using  $\|u_2 - u_2^{\star}\|_{\infty} \le \|u_2\|_{\infty} + \|u_2^{\star}\|_{\infty}$ . Combining yields

$$\left|\left\langle \boldsymbol{a}_{v}^{\star}, \boldsymbol{u}_{2}^{\star} - \boldsymbol{u}_{2}\right\rangle\right| \leq C' \frac{n\overline{p}}{\sqrt{n}\log\log n} + \frac{n\overline{p} \, \|\boldsymbol{u}_{2}\|_{\infty}}{\log\log n}$$

for some constant C' > 0. Now, notice that for all *n* sufficiently large,

$$\begin{aligned} \left| \left\langle a_{v} - a_{v}^{\star}, u_{2}^{\star} - u_{2} \right\rangle \right| \\ &\leq C_{7.5.21} \left( n\overline{p} + \log\left( n/\delta \right) \right) \left( \frac{\|u_{2}\|_{\infty}}{\log \log n} + \frac{1}{\sqrt{n} \log \log n} \right) \end{aligned}$$
(Lemma 7.5.21)  
$$&\leq C_{7.5.21} \left( n\overline{p} + \log\left( n/\delta \right) \right) \left( \frac{\frac{C_{7.5.22}(\alpha, \delta)}{\sqrt{n}}}{\log \log n} + \frac{1}{\sqrt{n} \log \log n} \right) \end{aligned}$$
(Lemma 7.5.22)  
$$&\leq \frac{C_{1}(\alpha, \delta) \cdot (n\overline{p} + \log\left( n/\delta \right))}{\sqrt{n} \log \log n}. \end{aligned}$$

Adding yields for  $n \ge N(\alpha, \delta)$ ,

$$\left|\left\langle a_{v}, u_{2}^{\star}-u_{2}\right\rangle\right| \leq \left|\left\langle a_{v}^{\star}, u_{2}^{\star}-u_{2}\right\rangle\right|+\left|\left\langle a_{v}-a_{v}^{\star}, u_{2}^{\star}-u_{2}\right\rangle\right|$$

$$\leq \frac{C_{2}(\alpha, \delta) \cdot (n\overline{p} + \log(n/\delta))}{\sqrt{n} \log \log n}$$
  
$$\leq \frac{1}{\sqrt{n}} \cdot \left(\frac{n(p-q)}{2} - C_{7.5.5}\left(\sqrt{np \log(n/\delta)} + \log(n/\delta)\right)\right) \quad \text{(gap condition)}$$
  
$$\leq \frac{d_{\text{in}}[v] - d_{\text{out}}[v]}{\sqrt{n}} = |\langle a_{v}, u_{2}^{\star} \rangle|,$$

which means we satisfy the conditions required by Lemma 7.5.23. Taking a union bound over all our (constantly many) probabilistic statements, setting  $\delta = \Theta(1/n)$ , and rescaling completes the proof of Theorem 29.

#### **Deterministic clusters model**

For convenience, we reproduce the statement of Theorem 30 here.

**Theorem 30.** Let q be a probability and  $d_{in}$  be an integer, and let  $\mathcal{D} \in DCM(n, d_{in}, q)$ . For  $G \sim \mathcal{D}$ , let  $\widehat{\mathbf{L}}$  denote the expectation of  $\mathbf{L}$  after step (2) but before step (3) in Model 7.2. There exists constants  $C_1, C_2, C_3 > 0$  such that for all n sufficiently large, if

$$d_{\rm in} \geq C_1 \cdot \left(\frac{nq}{2} + \sqrt{n}\right) \quad and \quad \lambda_3(\widehat{\mathbf{L}}) - \lambda_2(\widehat{\mathbf{L}}) \geq \sqrt{n} + C_2nq + C_3\left(\sqrt{nq\log n} + \log n\right) ,$$

then unnormalized spectral bisection is strongly consistent on  $\mathcal{D}$ .

Proof of Theorem 30. In this proof, let  $\mathbf{L}^{\star}$  be the Laplacian matrix that agrees with  $\mathbf{L}$  on all internal edges and agrees with  $\mathbb{E}[\mathbf{L}]$  on all crossing edges. Let  $\mathbf{L}^{(\text{cross})}$  denote the Laplacian matrix corresponding to the cross edges, so we can write  $\mathbf{L}^{\star} = \mathbf{L} - \mathbf{L}^{(\text{cross})} + \mathbb{E}[\mathbf{L}^{(\text{cross})}]$ . Although  $\mathbf{L}^{\star} \neq \mathbb{E}[\mathbf{L}]$  due to the adaptive adversary, by Lemma 7.5.14, we still have  $\mathbf{L}^{\star} u_2^{\star} = \lambda_2^{\star} u_2^{\star} = nq u_2^{\star}$ . Moreover,  $(\mathbf{L} - \mathbf{L}^{\star})u_2^{\star}$  is the vector whose entries are of the form  $2(d_{\text{out}}[v] - \mathbb{E}[d_{\text{out}}[v]])/\sqrt{n}$ . Thus, we will be able to apply Lemma 7.5.16 and Lemma 7.5.12 later on. Finally, observe that  $\lambda_i(\mathbf{L}^{\star}) \geq \lambda_i(\widehat{\mathbf{L}})$  for all  $i \geq 3$  and  $\lambda_2(\widehat{\mathbf{L}}) = \lambda_2(\mathbf{L}^{\star}) = nq$ . Thus, one can use the spectral gap  $\lambda_3(\widehat{\mathbf{L}}) - \lambda_2(\widehat{\mathbf{L}})$  to reason about  $\lambda_3^{\star} - \lambda_2^{\star}$ .

Let  $\delta \ge 1/(10n)$ . We will apply Lemma 7.5.23 to get strong consistency. First, let us verify that  $d[v] > \lambda_2$  for all v. Applying Lemma 7.5.10 to the matrix  $\mathbf{L}^{(cross)}$  gives

$$\left\|\mathbf{L} - \mathbf{L}^{\star}\right\|_{\mathrm{op}} = \left\|\mathbf{L}^{(\mathrm{cross})} - \mathbb{E}\left[\mathbf{L}^{(\mathrm{cross})}\right]\right\|_{\mathrm{op}} \le C_{7.5.10}\left(\sqrt{nq\log\left(n/\delta\right)} + \log(n/\delta)\right).$$

Thus, using Weyl's inequality, for  $n > N(\delta)$ , we have

$$d[v] - \lambda_2 \ge d_{\text{in}}[v] - \lambda_2^{\star} - \left\| \mathbf{L} - \mathbf{L}^{\star} \right\|_{\text{op}}$$
$$\ge C_1 \frac{nq}{2} + C_1 \sqrt{n} - nq - C_{7.5.10} \left( \sqrt{nq \log(n/\delta)} + \log(n/\delta) \right) > 0.$$

Next, we verify that  $d_{in}[v] > d_{out}[v]$  for all v. By Lemma 7.5.3, with probability  $\geq 1 - \delta$ , for all  $v \in V$ , we have

$$\left| \boldsymbol{d}_{\mathsf{out}}[\boldsymbol{v}] - \frac{nq}{2} \right| \le C_{7.5.3} \left( \sqrt{nq \log\left(\frac{n}{\delta}\right)} + \log\left(\frac{n}{\delta}\right) \right).$$

So for  $n > N(\delta)$ , we obtain

$$d_{\rm in}[v] - d_{\rm out}[v] \ge C_1 \frac{nq}{2} + C_1 \sqrt{n} - \frac{nq}{2} - C_{7.5.3} \left( \sqrt{nq \log(n/\delta)} + \log(n/\delta) \right) > 0.$$

Here, in the last inequality we used the fact that  $\sqrt{nq \log(n/\delta)} \le \max\{nq, \log(n/\delta)\}$ . Finally, we need to show that for all  $v \in V$ ,

$$\left|\left\langle a_{v}, u_{2}^{\star} - u_{2}\right\rangle\right| \leq \left|\left\langle a_{v}, u_{2}^{\star}\right\rangle\right| = \frac{d_{\mathrm{in}}[v] - d_{\mathrm{out}}[v]}{\sqrt{n}}.$$

By Cauchy-Schwarz, we have

$$|\langle a_v, u_2^{\star} - u_2 \rangle| \leq ||a_v||_2 \cdot ||u_2^{\star} - u_2||_2 = \sqrt{d_{in}[v] + d_{out}[v]} \cdot ||u_2^{\star} - u_2||_2.$$

Thus, it is enough to show that for all  $v \in V$  we get

$$\sqrt{n} \left\| \boldsymbol{u}_{2}^{\star} - \boldsymbol{u}_{2} \right\|_{2} \leq \frac{d_{\text{in}}[v] - d_{\text{out}}[v]}{\sqrt{d_{\text{in}}[v] + d_{\text{out}}[v]}}.$$

Observe that the RHS above is a decreasing function in  $d_{out}[v]$  and an increasing function in  $d_{in}[v]$ .

Now, by Lemma 7.5.16 and Lemma 7.5.12, we have

$$\sqrt{n} \left\| \boldsymbol{u}_{2}^{\star} - \boldsymbol{u}_{2} \right\|_{2} \leq \frac{\sqrt{n} \left\| \mathbf{E} \boldsymbol{u}_{2}^{\star} \right\|_{2}}{\left| \lambda_{3} - \lambda_{2}^{\star} \right|} \leq \frac{6C_{7.5.12} \sqrt{n} \left( \sqrt{nq} + (nq \log (n/\delta))^{1/4} + \sqrt{\log (n/\delta)} \right)}{\left| \lambda_{3} - \lambda_{2}^{\star} \right|}.$$
 (7.5.5)

We now do casework on the value of *q*.

**Case 1:**  $q \le \log(n/\delta)/n$ . Carrying on from (7.5.5) and applying Lemma 7.5.10 (we can set  $p_{ij}$  for the deterministic internal edges to 0 as they do not affect  $\mathbf{L} - \mathbb{E}[\mathbf{L}]$ ) along with Weyl's inequality, for all  $n \ge N(\delta)$  we have

$$\begin{split} \sqrt{n} \left\| \boldsymbol{u}_{2}^{\star} - \boldsymbol{u}_{2} \right\|_{2} &\leq \frac{18C_{7.5.12}\sqrt{n\log(n/\delta)}}{\left| \lambda_{3} - \lambda_{2}^{\star} \right|} \leq \frac{18C_{7.5.12}\sqrt{n\log(n/\delta)}}{\sqrt{n} - 3C_{7.5.10}\log(n/\delta)} \\ &\leq C\sqrt{\log(n/\delta)} \ll \frac{d_{\text{in}}[v] - d_{\text{out}}[v]}{\sqrt{d_{\text{in}}[v] + d_{\text{out}}[v]}}, \end{split}$$

as required. Here the last inequality follows using the fact that  $d_{in}[v] \ge C_1\left(\frac{nq}{2} + \sqrt{n}\right)$  and  $d_{out}[v] \le \frac{nq}{2} + 2C_{7.5.3}\log(n/\delta)$ .

**Case 2:**  $\log(n/\delta)/n \le q$ . Similar to the previous case, we get

$$\sqrt{n} \left\| \boldsymbol{u}_{2}^{\star} - \boldsymbol{u}_{2} \right\|_{2} \leq \frac{18C_{7.5.12}\sqrt{n} \cdot \sqrt{nq}}{\left| \lambda_{3} - \lambda_{2}^{\star} \right|} \leq \frac{18C_{7.5.12}\sqrt{n} \cdot \sqrt{nq}}{\sqrt{n} + (C_{2} - 2C_{7.5.10})nq}$$
(7.5.6)

$$\leq 18C_{7.5.12} \cdot \max\left\{\sqrt{nq}, \frac{1}{(C_2 - 2C_{7.5.10})\sqrt{q}}\right\}.$$
(7.5.7)

Additionally, we can use the conclusion of Lemma 7.5.3 to write with probability  $\geq 1 - \delta$  for all  $v \in V$  and  $n \geq N(\delta)$  that

$$\frac{d_{\rm in}[v] - d_{\rm out}[v]}{\sqrt{d_{\rm in}[v] + d_{\rm out}[v]}} \ge \frac{(C_1/2 - 2C_{7.5.3} - 1/2)nq + C_1\sqrt{n}}{\sqrt{(C_1/2 + 2C_{7.5.3} + 1/2)nq}}$$
(7.5.8)

$$\geq \frac{C_1/2 - 2C_{7.5.3} - 1/2}{\sqrt{C_1/2 + 2C_{7.5.3} + 1/2}} \max\left\{\sqrt{nq}, \sqrt{\frac{1}{q}}\right\}.$$
(7.5.9)

From this, it is clear that one can choose constants  $C_1$  and  $C_2$  such that (7.5.7) is at most (7.5.9). Taking a union bound over all our (constantly many) probabilistic statements, setting  $\delta = \Theta(1/n)$ , and rescaling completes the proof of Theorem 30.

#### 7.5.8. Inconsistency of normalized spectral bisection

In this section, we design a family of problem instances on which unnormalized spectral bisection is strongly consistent whereas normalized spectral bisection is inconsistent. Specifically, our goal is to prove Theorem 31.

**Theorem 31.** For all *n* sufficiently large, there exists a nonhomogeneous stochastic block model such that unnormalized spectral bisection is strongly consistent whereas normalized spectral bisection (both symmetric and random-walk) incurs a misclassification rate of at least 24% with probability 1 - 1/n.

#### The nested block example

We first state the family of instances on which we will prove our inconsistency results. Let *n* be a multiple of 4. Let  $L_1$  consist of indices 1, ..., n/4,  $L_2$  consist of indices n/4 + 1, ..., n/2, and *R* consist of indices n/2 + 1, ..., n.

As mentioned in Section 7.3, consider the following block structure determined by the  $\mathbf{A}^*$  written below, where q < p and  $K \ge 3p/q$ .

	$L_1$	$L_2$	R
$L_1$	$Kp \cdot \mathbb{1}_{n/4 \times n/4}$	$p \cdot \mathbb{1}_{n/4 \times n/4}$	$a \cdot 1$
$L_2$	$p \cdot \mathbb{1}_{n/4 \times n/4}$	$Kp \cdot \mathbb{1}_{n/4 \times n/4}$	<b>q</b> <u>≖</u> n/2×n/2
R	$q \cdot \mathbb{1}_{n_f}$	/2×n/2	$p \cdot \mathbb{1}_{n/2 \times n/2}$

Table 7.2.:  $A^*$  is defined to have the above block structure.

We will draw our instances from the nonhomogeneous stochastic block model according to the probabilities prescribed above. Note that within the two clusters  $L := L_1 \cup L_2$  and R, each edge appears with probability at least p. Moreover, each edge in  $L \times R$  appears with probability exactly q. However, there are also two subcommunities  $L_1$  and  $L_2$  that appear within L. Furthermore, observe that unnormalized spectral bisection is consistent on this family of examples with probability  $\geq 1 - 1/n$  by Theorem 29.

#### **Technical lemmas**

We next show some technical statements that we will need later in the proof of Theorem 31.

**Lemma 7.5.24.** Let  $\mathbf{M} \in \mathbb{R}^{k \times k}$ . Then,

$$\|\mathbf{M}\|_{\mathrm{op}} \leq \max_{i \leq k} |\mathbf{M}[i][i]| + k \max_{i \neq j} |\mathbf{M}[i][j]|.$$

*Proof of Lemma* 7.5.24. For a matrix  $\mathbf{N} \in \mathbb{R}^{k \times k}$ , it is easy to check that

$$\|\mathbf{N}\|_{\mathrm{op}} \leq \|\mathbf{N}\|_F \leq k \max_{i,j \leq k} |\mathbf{N}[i][j]|.$$

Next, let diag  $(\mathbf{M})$  denote the matrix that agrees with  $\mathbf{M}$  on the diagonal and is 0 elsewhere. Notice that

$$\left\|\mathbf{M}\right\|_{\mathrm{op}} \leq \left\|\mathsf{diag}\left(\mathbf{M}\right)\right\|_{\mathrm{op}} + \left\|\mathbf{M} - \mathsf{diag}\left(\mathbf{M}\right)\right\|_{\mathrm{op}} \leq \max_{i \leq k} \left|\mathbf{M}[i][i]\right| + k \max_{i \neq j} \left|\mathbf{M}[i][j]\right|,$$

completing the proof of Lemma 7.5.24.

**Lemma 7.5.25.** Let  $\varepsilon_x$  be a constant where  $0 \le \varepsilon_x < x$ . Let  $\varepsilon_y$  be defined similarly. The function f(x, y) defined as

$$f(x, y) \coloneqq \frac{1}{\sqrt{x - \varepsilon_x}\sqrt{y - \varepsilon_y}} - \frac{1}{\sqrt{x}\sqrt{y}}$$

is decreasing in x and y.

*Proof of Lemma* 7.5.25. It is enough to just check the inequality for x. We take the derivative of f(x, y) with respect to x and get

$$\frac{1}{2}\left(-\frac{1}{(x-\varepsilon_x)^{3/2}(y-\varepsilon_y)^{1/2}}+\frac{1}{x^{3/2}y^{1/2}}\right)<0,$$

where the inequality follows from observing  $0 < x - \varepsilon_x \le x$  and similarly for *y*. This completes the proof of Lemma 7.5.25.

## **Proof of Theorem 31**

First, we construct  $\mathcal{L}^{\star}$ .

**Lemma 7.5.26.** Let  $\mathcal{L}^{\star} := \mathbf{I} - (\mathbf{D}^{\star})^{-1/2} \mathbf{A}^{\star} (\mathbf{D}^{\star})^{-1/2}$ . Then,  $\mathbf{I} - \mathcal{L}^{\star}$  has the following block structure.

*Proof of Lemma* 7.5.26. It is easy to see that for any  $v \in L$ , we have  $d^{\star}[v] = \frac{n}{2} \cdot (p \cdot \frac{K+1}{2} + q)$  and for any  $v \in R$ , we have  $d^{\star}[v] = \frac{n}{2} \cdot (p + q)$ . Lemma 7.5.26 follows by noting that every element of  $\mathbf{I} - \mathcal{L}^{\star}$  is of the form  $a_i^{\star}[j]/\sqrt{d^{\star}[i]d^{\star}[j]}$ .

Next, we analyze the eigenvalues and eigenvectors of  $\mathcal{L}^{\star}$ .

**Lemma 7.5.27.** *Up to normalization and sign, the eigenvector-eigenvalue pairs of*  $I - \mathcal{L}^*$  *corresponding to the nonzero eigenvalues of*  $I - \mathcal{L}^*$  *are* 

$$\begin{aligned} (\lambda_1^{\star}, u_1^{\star}) &= \left(1, \left[\mathbbm{1}_{n/4} \oplus \mathbbm{1}_{n/4} \oplus y_+ \cdot \mathbbm{1}_{n/4} \oplus y_+ \cdot \mathbbm{1}_{n/4}\right]\right) \\ (\lambda_2^{\star}, u_2^{\star}) &= \left(\frac{(K-1)p}{2\left(p \cdot \frac{K+1}{2} + q\right)}, \left[\mathbbm{1}_{n/4} \oplus -\mathbbm{1}_{n/4} \oplus 0_{n/4} \oplus 0_{n/4}\right]\right) \\ (\lambda_3^{\star}, u_3^{\star}) &= \left(-1 + p\left(\frac{1}{p+q} + \frac{K+1}{p(K+1)+2q}\right), \left[\mathbbm{1}_{n/4} \oplus \mathbbm{1}_{n/4} \oplus y_- \cdot \mathbbm{1}_{n/4} \oplus y_- \cdot \mathbbm{1}_{n/4}\right]\right) \end{aligned}$$

where  $y_+$  and  $y_-$  are chosen according to the formulas

$$y_{+} = \sqrt{\frac{2(p+q)}{p(K+1)+2q}} \qquad \qquad y_{-} = -\sqrt{\frac{p(K+1)+2q}{2(p+q)}}.$$

*Moreover, we have*  $\lambda_1^{\star} > \lambda_2^{\star} > \lambda_3^{\star} > 0$  *and* 

$$\lambda_2^{\star} - \lambda_3^{\star} \ge 1 - \frac{p^2(K+3) + 4pq}{p^2(K+3) + 4pq + 2q^2}.$$

*Proof of Lemma* 7.5.27. As we can see from Lemma 7.5.26,  $I - L^*$  is a matrix whose rank is at most 3, since it can be constructed by carefully repeating 3 distinct column vectors. Thus, it can have at most 3 nonzero eigenvalues. In what follows, we consider the case where K > 1 so that there are exactly 3 nonzero eigenvalues.

The next step is to confirm that the stated eigenvalue-eigenvector pairs are in fact valid. We begin with  $u_1^*$ . Every entry in the first n/2 entries of  $(\mathbf{I} - \mathcal{L}^*)u_1^*$  can be expressed as

$$\begin{aligned} &\frac{n}{4} \cdot \frac{Kp}{\frac{n}{2} \cdot \left(p \cdot \frac{K+1}{2} + q\right)} + \frac{n}{4} \cdot \frac{p}{\frac{n}{2} \cdot \left(p \cdot \frac{K+1}{2} + q\right)} + \frac{n}{2} \cdot \left(\frac{q \cdot \sqrt{\frac{2(p+q)}{p(K+1)+2q}}}{\sqrt{\frac{n}{2} \cdot \left(p \cdot \frac{K+1}{2} + q\right) \cdot \frac{n}{2} \cdot \left(p + q\right)}}\right) \\ &= \frac{(K+1)p}{(K+1)p+2q} + \frac{q \cdot \sqrt{\frac{2(p+q)}{p(K+1)+2q}}}{\sqrt{\left(p \cdot \frac{K+1}{2} + q\right) \left(p + q\right)}} = \frac{(K+1)p}{(K+1)p+2q} + \frac{q \cdot \sqrt{\frac{2}{p(K+1)+2q}}}{\sqrt{\left(p \cdot \frac{K+1}{2} + q\right)}} \\ &= \frac{(K+1)p}{(K+1)p+2q} + \frac{2q}{(K+1)p+2q} = 1, \end{aligned}$$

and every entry in the second n/2 entries of  $(\mathbf{I} - \mathcal{L}^{\star})u_1^{\star}$  can be expressed as

$$\frac{n}{2} \cdot \frac{q}{\sqrt{\frac{n}{2} \cdot \left(p \cdot \frac{K+1}{2} + q\right) \cdot \frac{n}{2} \cdot \left(p + q\right)}} + \frac{n}{2} \cdot \frac{p}{\frac{n}{2} \cdot \left(p + q\right)} \cdot \sqrt{\frac{2(p+q)}{p(K+1) + 2q}}$$

$$= \frac{q}{\sqrt{\left(p \cdot \frac{K+1}{2} + q\right)\left(p + q\right)}} + \frac{p}{\left(p + q\right)} \cdot \sqrt{\frac{2(p + q)}{p(K + 1) + 2q}}$$
$$= \frac{q}{\sqrt{\left(p \cdot \frac{K+1}{2} + q\right)\left(p + q\right)}} + p \cdot \sqrt{\frac{1}{\left(p \cdot \frac{K+1}{2} + q\right)\left(p + q\right)}}$$
$$= \frac{\sqrt{p + q}}{\sqrt{p \cdot \frac{K+1}{2} + q}} = \sqrt{\frac{2(p + q)}{p(K + 1) + 2q}} = y_{+}.$$

For  $u_2^{\star}$ , we can use the block structure and easily verify

$$\left(\mathbf{I}-\mathcal{L}^{\star}\right)\boldsymbol{u}_{2}^{\star}=\frac{n}{4}\cdot\frac{(K-1)p}{\frac{n}{2}\cdot\left(p\cdot\frac{K+1}{2}+q\right)}\left[\mathbb{1}_{n/4}\oplus-\mathbb{1}_{n/4}\oplus0_{n/4}\oplus0_{n/4}\right]=\lambda_{2}^{\star}\boldsymbol{u}_{2}^{\star}.$$

We now address  $u_3^{\star}$ . The first n/2 entries of  $(\mathbf{I} - \mathcal{L}^{\star})u_3^{\star}$  are

$$\frac{n}{4} \cdot \frac{Kp}{\frac{n}{2} \cdot \left(p \cdot \frac{K+1}{2} + q\right)} + \frac{n}{4} \cdot \frac{p}{\frac{n}{2} \cdot \left(p \cdot \frac{K+1}{2} + q\right)} + \frac{n}{2} \cdot \left(\frac{q \cdot -\sqrt{\frac{p(K+1)+2q}{2(p+q)}}}{\sqrt{\frac{n}{2} \cdot \left(p \cdot \frac{K+1}{2} + q\right) \cdot \frac{n}{2} \cdot \left(p + q\right)}}\right)$$
$$= \frac{(K+1)p}{(K+1)p+2q} + \left(\frac{q \cdot -\sqrt{\frac{1}{p+q}}}{\sqrt{p+q}}\right) = \frac{(K+1)p}{(K+1)p+2q} - \frac{q}{p+q} = \lambda_3^{\star},$$

and the second n/2 entries of  $(\mathbf{I} - \mathcal{L}^{\star})u_3^{\star}$  are

$$\frac{n}{2} \cdot \frac{q}{\sqrt{\frac{n}{2} \cdot \left(p \cdot \frac{K+1}{2} + q\right) \cdot \frac{n}{2} \cdot \left(p + q\right)}} + \frac{n}{2} \cdot \frac{p}{\frac{n}{2} \cdot \left(p + q\right)} \cdot -\sqrt{\frac{p(K+1) + 2q}{2(p+q)}}$$
$$= \frac{q}{\sqrt{\left(p \cdot \frac{K+1}{2} + q\right)\left(p + q\right)}} - \frac{p}{\left(p + q\right)} \cdot \sqrt{\frac{p(K+1) + 2q}{2(p+q)}}$$
$$= -\sqrt{\frac{p(K+1) + 2q}{2(p+q)}} \left(\frac{-2q}{p(K+1) + 2q} + \frac{p}{p+q}\right) = y_{-} \cdot \lambda_{3}^{\star}.$$

Finally, it remains to check that  $1 > \lambda_2^* > \lambda_3^* > 0$ . The fact that  $\lambda_2^* < 1$  easily follows from using p + q > 0. To prepare to bound  $\lambda_2^* - \lambda_3^*$ , we first use  $p \ge q$  to establish

$$p^{2} - pq + 2q^{2} = p(p - q) + 2q^{2} \ge 2q^{2}.$$

This implies

$$pq(K-1) + 2q^2 \ge 3p^2 - pq + 2q^2 = 2p^2 + (p^2 - pq + 2q^2) \ge 2p^2 + 2q^2,$$

which rearranges to

$$p^{2}(K+1) + pq(K+3) + 2q^{2} \ge p^{2}(K+3) + 4pq + 2q^{2}.$$

Next, we write

$$\lambda_{2}^{\star} - \lambda_{3}^{\star} = \left(\frac{(K-1)p}{2\left(p \cdot \frac{K+1}{2} + q\right)}\right) - \left(-1 + p\left(\frac{1}{p+q} + \frac{K+1}{p(K+1)+2q}\right)\right)$$

$$= 1 - \frac{p}{p+q} - \frac{2p}{p(K+1)+2q} = 1 - \left(\frac{p^2(K+1)+2pq+2p^2+2pq}{(p+q)(p(K+1)+2q)}\right)$$
$$= 1 - \frac{p^2(K+3)+4pq}{p^2(K+1)+pq(K+3)+2q^2} \ge 1 - \frac{p^2(K+3)+4pq}{p^2(K+3)+4pq+2q^2} > 0.$$

Finally, to show  $\lambda_3^* > 0$ , we write

$$\lambda_3^{\star} + 1 = \frac{p}{p+q} + \frac{p(K+1)}{p(K+1)+2q} > \frac{2p}{p+q} > 1,$$

which allows us to complete the proof of Lemma 7.5.27.

Next, we argue that studying  $\mathcal{L}^*$ , which is formed by taking into account the weighted selfloops, gives us an understanding that is not too far from that of  $\mathcal{L}_{nl}^*$ , which is formed by setting  $p_{vv} = 0$  for all  $v \in V$ .

**Lemma 7.5.28.** Let **P** be the diagonal matrix where  $\mathbf{P}[v, v] = p_{vv}$ . Let  $\mathcal{L}_{nl}^{\star}$  be the normalized Laplacian of the graph formed by  $\mathbf{A}^{\star} - \mathbf{P}$ . Then, we have

$$\left\|\mathcal{L}^{\star}-\mathcal{L}_{\mathsf{nl}}^{\star}\right\|_{\mathrm{op}} \leq \frac{6K}{n-2}.$$

*Proof of Lemma 7.5.28.* Recall  $\mathbf{L}^{\star} := \mathbf{D}^{\star} - \mathbf{A}^{\star}$ . Let  $\mathbf{D}_{\mathsf{nl}}^{\star}$  be defined analogously to  $\mathcal{L}_{\mathsf{nl}}^{\star}$ . Observe that we have

$$\mathcal{L}^{\star} = (\mathbf{D}^{\star})^{-1/2} \mathbf{L}^{\star} (\mathbf{D}^{\star})^{-1/2}$$
$$\mathcal{L}^{\star}_{\mathsf{nl}} = (\mathbf{D}^{\star}_{\mathsf{nl}})^{-1/2} \mathbf{L}^{\star} (\mathbf{D}^{\star}_{\mathsf{nl}})^{-1/2}$$

From this, we see that writing down the v, wth entry of the difference gives

$$\left(\mathcal{L}_{\mathsf{nl}}^{\star}-\mathcal{L}^{\star}\right)[v,w] = \mathbf{L}^{\star}[v,w] \left(\frac{1}{\sqrt{(d^{\star}[v]-p_{vv})(d^{\star}[w]-p_{ww})}} - \frac{1}{\sqrt{d^{\star}[v]d^{\star}[w]}}\right).$$

This resolves to different forms based on whether v = w. When v = w, evaluating the formula gives

$$\left(\mathcal{L}_{nl}^{\star} - \mathcal{L}^{\star}\right)[v, v] = \frac{d^{\star}[v] - p_{vv}}{d^{\star}[v] - p_{vv}} - \frac{d^{\star}[v] - p_{vv}}{d^{\star}[v]} = \frac{p_{vv}}{d^{\star}[v]}$$

When  $v \neq w$ , we apply Lemma 7.5.25 and get

$$\begin{split} \left| \left( \mathcal{L}_{\mathsf{nl}}^{\star} - \mathcal{L}^{\star} \right) [v, w] \right| &= p_{vw} \left( \frac{1}{\sqrt{(d^{\star}[v] - p_{vv})(d^{\star}[w] - p_{ww})}} - \frac{1}{\sqrt{d^{\star}[v]d^{\star}[w]}} \right) \\ &\leq Kp \left( \frac{1}{np/2 - p} - \frac{1}{np/2} \right) = \frac{4K}{n^2 - 2n}. \end{split}$$

Using this analysis and applying Lemma 7.5.24 gives

$$\left\|\mathcal{L}_{\mathsf{n}\mathsf{l}}^{\star} - \mathcal{L}^{\star}\right\|_{\mathrm{op}} \leq \max_{v \in V} \frac{p_{vv}}{d^{\star}[v]} + n \max_{v \neq w} p_{vw} \left(\frac{1}{\sqrt{(d^{\star}[v] - p_{vv})(d^{\star}[w] - p_{ww})}} - \frac{1}{\sqrt{d^{\star}[v]d^{\star}[w]}}\right)$$

$$\leq \frac{2K}{n} + n \max_{v \neq w} p_{vw} \left( \frac{1}{\sqrt{(d^{\star}[v] - p_{vv})(d^{\star}[w] - p_{ww})}} - \frac{1}{\sqrt{d^{\star}[v]d^{\star}[w]}} \right)$$

$$\leq \frac{2K}{n} + \frac{4K}{n-2} \leq \frac{6K}{n-2},$$

completing the proof of Lemma 7.5.28.

This gives Lemma 7.5.29, which means we can use  $u_2^{\star}$  as a suitable proxy for sign  $(u_2(\mathcal{L}_{nl}^{\star}))$ .

**Lemma 7.5.29.** There exists a constant  $C(\alpha, K)$  depending on  $\alpha$  and K such that we have

$$\left\| \boldsymbol{u}_2(\mathcal{L}_{\mathsf{nl}}^{\star}) - \boldsymbol{u}_2^{\star} \right\|_{\infty} \leq \frac{C(\alpha, K)}{n}.$$

This implies that for all n sufficiently large, we have sign  $(u_2(\mathcal{L}_{nl}^{\star})) = sign(u_2^{\star})$ .

*Proof of Lemma* 7.5.29. By Lemma 7.5.27, Weyl's inequality, and Lemma 7.5.28, we know that for all *n* sufficiently large,

$$\lambda_{2}^{\star} - \lambda_{3}(\mathcal{L}_{\mathsf{nl}}^{\star}) = \left(\lambda_{2}^{\star} - \lambda_{3}^{\star}\right) + \left(\lambda_{3}^{\star} - \lambda_{3}(\mathcal{L}_{\mathsf{nl}}^{\star})\right)$$
$$\geq \left(1 - \frac{p^{2}(K+3) + 4pq}{p^{2}(K+3) + 4pq + 2q^{2}}\right) - \frac{C_{7.5.28}}{n} \geq C_{1}(\alpha, K)$$

Combining this with Lemma 7.5.28 again, the Davis-Kahan inequality tells us that

$$\left\|\boldsymbol{u}_{2}(\boldsymbol{\mathcal{L}}_{\mathsf{nl}}^{\star})-\boldsymbol{u}_{2}^{\star}\right\|_{\infty}\leq\left\|\boldsymbol{u}_{2}(\boldsymbol{\mathcal{L}}_{\mathsf{nl}}^{\star})-\boldsymbol{u}_{2}^{\star}\right\|_{2}\leq\frac{C_{2}(\boldsymbol{\alpha},\boldsymbol{K})}{n}$$

and then using the fact that  $\|\boldsymbol{u}_2^{\star}\|_{\infty} = 1/\sqrt{n}$  (arising from Lemma 7.5.27) completes the proof of Lemma 7.5.29.

We are now ready to prove the inconsistency of normalized spectral bisection on the nested block examples.

*Proof of Theorem* 31. Let *G* be a graph drawn from the nested block example. We choose *p* and *q* such that  $p \ge \log n/n$  and  $p/q = \alpha \ge 2$  where  $\alpha$  is some constant and such that *p* and *q* both satisfy the conditions of Theorem 29. Let  $K \ge 3\alpha$ . Observe that the true communities are *L* and *R*. We will show that bisection based on  $u_2$  of  $I - \mathcal{L}$  (corresponding to the eigenvector associated with the second smallest eigenvalue of  $\mathcal{L}$ ) will attain a large misclassification rate. In particular, based on our calculation in Lemma 7.5.27, we expect that  $u_2$  will output a bisection that places  $L_1$  and  $L_2$  into separate clusters. On the other hand, by Theorem 29, for all *n* large enough, the unnormalized spectral bisection algorithm will be strongly consistent.

First, observe that it is enough to prove the inconsistency result just for the symmetric normalized Laplacian. Indeed, observe that if  $u_2$  is an eigenvector of  $\mathbf{I} - \mathcal{L} = \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}$ , then we have

$$\lambda_2 \mathbf{D}^{-1/2} u_2 = \mathbf{D}^{-1} \mathbf{A} \mathbf{D}^{-1/2} u_2 = \mathbf{D}^{-1} \mathbf{A} (\mathbf{D}^{-1/2} u_2),$$

which shows that  $\mathbf{D}^{-1/2}\boldsymbol{u}_2$  must be the eigenvector of the random-walk normalized Laplacian  $\mathbf{I} - \mathbf{D}^{-1}\mathbf{A}$  corresponding to eigenvalue  $\lambda_2$ . Since  $\mathbf{D}$  is a positive diagonal matrix, it does not

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change the signs of  $u_2$  and therefore the output of the normalized spectral bisection algorithm is the same.

Our general approach to prove the inconsistency is to use the Davis-Kahan Theorem, a bound on  $\|\mathcal{L} - \mathcal{L}_n^*\|_{op'}$  and a bound on the gap  $\lambda_2^* - \lambda_3$ . Let  $d_{\min}$  be the minimum degree of the graph given by adjacency matrix **A** and let  $d_{\min}^*$  be the minimum weighted degree of the graph given by the adjacency matrix **A**<sup>\*</sup>. First, using [DLS21, Theorem 3.1], we have with probability  $1 - n^{-r}$ for some constant  $r \ge 1$  and constants C(r) and C (the latter of which does not depend on r), for all n sufficiently large,

$$\begin{split} \left\| \mathcal{L} - \mathcal{L}_{\mathsf{nl}}^{\star} \right\|_{\mathrm{op}} &\leq \frac{C(r) \left( n \max_{(i,j)} p_{ij} \right)^{5/2}}{\min \left\{ d_{\min}, d_{\min}^{\star} \right\}^3} \\ &\leq \frac{C(r) \left( n \cdot Kp \right)^{5/2}}{\min \left\{ n(p+q)/3, n(p+q)/3 - C \sqrt{n(p+q)\log n} \right\}^3} \\ &\leq \frac{C_1(r, \alpha) K^{5/2} (np)^{5/2}}{(np)^3} = \frac{C_1(r, \alpha) K^{5/2}}{\sqrt{np}}. \end{split}$$

Next, we invoke Lemma 7.5.28 to write

$$\begin{split} \lambda_2(\mathcal{L}_{\mathsf{nl}}^{\star}) - \lambda_3 &= \left(\lambda_2^{\star} - \lambda_3^{\star}\right) + \left(\lambda_3^{\star} - \lambda_3\right) + \left(\lambda_2(\mathcal{L}_{\mathsf{nl}}^{\star}) - \lambda_2^{\star}\right) \\ &\geq \left(1 - \frac{p^2(K+3) + 4pq}{p^2(K+3) + 4pq + 2q^2}\right) - \frac{C_2(r,\alpha)K^{5/2}}{\sqrt{np}} - \frac{C_{7.5.28}}{n} \geq C_g(\alpha, K), \end{split}$$

where the last line denotes a positive constant depending on q and K (this constant will always be positive for sufficiently large n, as we showed that  $\lambda_2^* - \lambda_3^* > 0$  in Lemma 7.5.27).

Putting everything together, we get by the Davis-Kahan theorem that some signing of  $u_2$  satisfies

$$\left\|\boldsymbol{\mathcal{L}}-\boldsymbol{\mathcal{L}}_{\mathsf{n}|}^{\star}\right\|_{2} \leq \frac{\left\|\boldsymbol{\mathcal{L}}-\boldsymbol{\mathcal{L}}_{\mathsf{n}|}^{\star}\right\|_{\mathrm{op}}}{\min\left\{\left|\lambda_{2}(\boldsymbol{\mathcal{L}}_{\mathsf{n}|}^{\star})-\lambda_{3}\right|,1-\lambda_{2}(\boldsymbol{\mathcal{L}}_{\mathsf{n}|}^{\star})\right\}} \leq \frac{C_{3}(r)K^{5/2}}{C'_{g}(\alpha,K)\sqrt{np}} \leq \frac{C_{4}(r,\alpha,K)}{\sqrt{np}}.$$

Now, consider the subset of coordinates of  $u_2$  belonging to  $L_1$ . Suppose *m* of these coordinates do not agree in sign with  $u_2^{\star}$ . To maximize *m*, each of these coordinates in  $u_2$  should be 0, so using this reasoning and applying Lemma 7.5.29 means the total  $\ell_2$  error can be bounded (using Lemma 7.5.28) as

$$m\left(\frac{1}{\sqrt{n/2}} - \frac{C_{7.5.28}}{n}\right)^2 \le \left\|\boldsymbol{u}_2 - \boldsymbol{u}_2(\boldsymbol{\mathcal{L}}_{nl}^{\star})\right\|_2^2 \le \frac{C_4(r, \alpha, K)^2}{np}.$$

This means the number of coordinates *m* on which  $u_2$  and  $u_2^{\star}$  disagree on is at most

$$\frac{n\cdot C_5(r,\alpha,K)^2}{2np},$$

and therefore the misclassification rate of  $u_2$  with respect to the true labeling induced by *L* and *R* must be at least

$$\frac{\frac{n}{4} - \frac{n \cdot C_5(r, \alpha, K)^2}{2np}}{n} = \frac{1}{4} - \frac{C_5(r, \alpha, K)^2}{2np}.$$

Since  $p \ge \log n/n$ , this completes the proof of Theorem 31.



Figure 7.2.: Agreement with the planted bisection of the bipartition obtained from unnormalized spectral bisection, for graphs generated from a distribution in NSSBM( $n, p, \overline{p}, q$ ) for fixed values of  $n, \overline{p}$  and varying values of p > q. The left plot uses  $\overline{p} = 1/2$ , the right plot uses  $\overline{p} = 1$ . The solid red curves plot the function  $p_{thr}(q)$  (see (7.6.1)), and the dashed red curves plot the function  $p_{info}(q)$  (see (7.6.2)).

## 7.6. Additional experiments

In this section, we show more numerical trials that complement those discussed in Section 7.4.

## 7.6.1. Varying edge probabilities in an NSSBM

In Section 7.4, we investigated the behavior of an NSSBM model by fixing the values of p, q and varying the largest edge probability  $\overline{p}$ . Here, we take an alternative approach, and instead fix  $\overline{p}$  and vary the values of p and q.

**Setup.** Let us fix n = 2000,  $\overline{p} \in \{1/2, 1\}$ . For varying p, q in the range [1/n, 9/20] such that p > q, we sample t = 3 independent draws *G* from the same benchmark distribution  $\mathcal{D}_{p,\overline{p},q}$  used in Section 7.4. For each of them, we compute the agreement of the bipartition obtained by unnormalized spectral bisection with respect to the planted bisection. For each (p, q), we plot the average agreement across the *t* independent draws. The results are shown in Fig. 7.2, where in the left and right plot we ran the experiments with  $\overline{p} = 1/2$  and  $\overline{p} = 1$  respectively. The lower diagonal of these plots, where  $p \le q$ , is artificially set to 0.

**Theoretical framing.** According to Theorem 29, fixing the value of  $\overline{p} \in \{1/2, 1\}$ , we obtain that unnormalized spectral bisection achieves exact recovery provided that for  $q \in [1/n, 9/20]$  one has  $p \ge p_{\text{thr}}(q)$  where

$$p_{\rm thr}(q) = \frac{\sqrt{\overline{p}\log n}}{\sqrt{n}} + q \tag{7.6.1}$$

is obtained by rearranging the precondition of Theorem 29, ignoring the constants, and disregarding the fact that  $\alpha$  should be O(1). The solid red curve in Fig. 7.2 plots  $p_{thr}(q)$  as a function of q. For comparison, the information-theoretic threshold for SSBM [ABH16] demands that  $p \ge p_{info}(q)$  where

$$p_{\rm info}(q) = \left(\sqrt{2}\sqrt{\frac{\log n}{n}} + \sqrt{q}\right)^2 \,. \tag{7.6.2}$$

The dashed red curve in Fig. 7.2 plots  $p_{info}(q)$  as a function of q.

**Empirical evidence.** From Fig. 7.2, one can see that our experiments reflect the behavior predicted by Theorem 29 quite closely, although empirically we achieve 100% agreement slightly above  $p_{thr}(q)$  (i.e. the solid red curve). However, this is likely due to the constant factors from Theorem 29 that we ignored, and also n = 2000 is plausibly too small to show asymptotic behaviors. Nevertheless, we do achieve 100% agreement consistently as soon as we surpass the information-theoretic threshold  $p_{info}(q)$ : in the regime of our experiment, it appears that the unnormalized Laplacian is robust all the way to the optimal threshold for exact recovery in the SSBM.

## 7.6.2. Varying the size of a planted clique in a DCM

In some sense, the experiments from Section 7.4 and Section 7.6.1 can be thought of as experiments for the deterministic clusters model too. This is because each realization of the internal edges gives rise to a different DCM distribution (see Section 7.2). We complement our previous discussion by illustrating the behavior of certain families of DCM distributions that are conceptually different than those considered in Section 7.4.

**Benchmark distribution.** Let *n* be divisible by 4 and let  $\{P_1, P-2\}$  be a partitioning of V = [n] into two equally-sized subsets. Fix  $p \in [0, 1]$ . For some set  $S \subseteq P_1$  such that  $S = \{1, \ldots, |S|\}$  (for simplicity), let  $G_2 = (P_2, E_2) \sim \text{ER}(n/2, p)$  be a graph drawn from the Erdős-Rényi distribution with sampling rate *p*, and let  $G_1 = (P_1, E_1) \sim \text{ERPC}(n/2, p, S)$  be also a graph drawn from the Erdős-Rényi distribution with sampling rate *p* where we additionally plant a clique on the vertices *S*. Fixing  $G_1, G_2$ , for  $q \in [0, 1]$  we consider the distribution  $\mathcal{D}_q^{G_1, G_2}$  over graphs G = (V, E) where  $G[P_1] = G_1$ ,  $G[P_2] = G_2$ , and every edge  $(u, v) \in P_1 \times P_2$  is sampled independently with probability *q*. One can see that  $\mathcal{D}_q^{G_1, G_2}$  is in fact in the set DCM $(n, d_{in}, q)$  for some  $d_{in}$ .

**Setup.** Let us fix n = 2000,  $p = 9/\sqrt{n}$ ,  $q = 1/\sqrt{n}$ . For varying values of |S| in the range  $[|P_1|/10, |P_1|]$ , we sample  $G_1 = (P_1, E_1) \sim \text{ERPC}(n/2, p, S)$  and  $G_2 = (P_2, E_2) \sim \text{ER}(n/2, p)$ , and then draw t = 10 independent samples G from  $\mathcal{D}_q^{G_1,G_2}$ . For each sample G, we run spectral bisection (i.e. Algorithm 19) with matrices  $\mathbf{L}$ ,  $\mathcal{L}_{\text{sym}}$ ,  $\mathcal{L}_{\text{rw}}$ ,  $\mathbf{A}$ . Then, we compute the agreement of the bipartition hence obtained with respect to the planted bisection, and average it out across the t independent draws. The results are shown in the left plot of Fig. 7.3. Again, another natural way to get a bipartition of V from the eigenvector is a sweep cut, and the average agreements that this results in are shown in the right plot of Fig. 7.3.

**Theoretical framing.** Ignoring the constants, Theorem 30 guarantees that exact recovery is achieved by unnormalized spectral bisection as long as  $d_{in} \ge nq + \sqrt{n}$  and  $\lambda_3(\widehat{\mathbf{L}}) - \lambda_2(\widehat{\mathbf{L}}) \ge \sqrt{n} + nq + \sqrt{nq \log n} + \log n$ , where  $\widehat{\mathbf{L}}$  is the expected Laplacian of  $\mathcal{D}_q^{G_1,G_2}$ . For each clique size that we consider, Fig. 7.4 shows the minimum in-cluster degree of the graphs  $G_1, G_2$  that we draw (in the left plot), and the spectral gap  $\lambda_3(\widehat{\mathbf{L}}) - \lambda_2(\widehat{\mathbf{L}})$ . The red horizontal lines in the left and right plot respectively correspond to the value of  $nq + \sqrt{n}$  and  $\sqrt{n} + nq + \sqrt{nq \log n} + \log n$  on the *y*-axis, indicating the lower bound on  $d_{in}$  and  $\lambda_3(\widehat{\mathbf{L}}) - \lambda_2(\widehat{\mathbf{L}})$  demanded by Theorem 30.



Figure 7.3.: Agreement with the planted bisection of the bipartition obtained from several matrices associated with an input graph generated from a distribution  $\mathcal{D}_q^{G_1,G_2} \in DCM(n, d_{in}, q)$  for fixed values of n, q and varying the size of the planted clique S. In the left plot, the bipartition is the 0-cut of the second eigenvector, as in Algorithm 19. In the right plot, the bipartition is the sweep cut of the first n/2 vertices in the second eigenvector.



Figure 7.4.: The minimum in-cluster degree  $d_{in}$  and the spectral gap  $\lambda_3(\mathbf{L}) - \lambda_2(\mathbf{L})$  of distributions  $\mathcal{D}_q^{G_1,G_2} \in \mathsf{DCM}(n, d_{in}, q)$  with fixed values of n, q and varying the size of the planted clique S. The red horizontal line on the left corresponds to the value  $nq + \sqrt{n}$ , the red horizontal line on the right corresponds to the value  $\sqrt{n} + nq + \sqrt{nq \log n} + \log n$ .

**Empirical evidence: consistency.** From Fig. 7.4, one can see that all the distributions  $\mathcal{D}_q^{G_1,G_2}$  that we use roughly meet the requirement of Theorem 30. Indeed, in the left plot of Fig. 7.3 one sees that unnormalized spectral bisection consistently achieves exact recovery for all clique sizes. On the contrary, the bipartition obtained by running spectral bisection with the adjacency matrix **A** misclassifies a fraction of the vertices for certain sizes of the planted clique. Nevertheless, the sweep cut obtained from all the matrices recovers the planted bisection exactly.

**Empirical evidence: example embedding.** Let us fix the value |S| = 800 for the size of the planted clique, for which we see in Fig. 7.3 that the adjacency matrix fails to recover the planted bisection. We generate a graph from a distribution  $\mathcal{D}_q^{G_1,G_2}$  with clique size |S| = 800, and plot how the vertices are embedded in the real line by the second eigenvector of all the matrices we consider. The result is shown in Fig. 7.5, where the three horizontal dashed lines, from top to bottom, respectively correspond to the value of  $1/\sqrt{n}$ ,  $0, -1/\sqrt{n}$  on the *y*-axis. Graphically, one can see that the embedding in the unnormalized Laplacian is indeed the one that moves the least away from the values  $\pm 1/\sqrt{n}$ , and in fact the vertices  $\{1, \ldots, 800\} \subseteq P_1$  where we plant the clique concentrate even more around  $1/\sqrt{n}$ . This is a phenomenon related to the one illustrated by Fig. 7.1. Finally, one can see from the embedding that splitting vertices around 0 does result in misclassifying a fraction of the vertices for the adjacency matrix. However, taking a sweep cut that splits the vertices into two equally sized parts recovers the planted bisection for all matrices. This reflects the results shown in Fig. 7.3.



Figure 7.5.: Embedding of the vertices given by the second eigenvector  $u_2$  of several matrices associated with a graph sampled from a distribution  $\mathcal{D}_q^{G_1,G_2} \in \text{DCM}(n, d_{\text{in}}, q)$ , with the size of the planted clique set to  $|S| = 2/5 \cdot n$ . Horizontal dashed lines, from top to bottom, correspond to  $1/\sqrt{n}, 0, -1/\sqrt{n}$  respectively.

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