Hydrodynamic fields in fluctuating environment: the emergent phononic and tachyonic-like excitations

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Using functional methods, we investigate in a low-temperature liquid, the sound quanta defined by the quantized hydrodynamic fields, under the effects of high-energy processes on the atomic/molecular scale. To obtain in the molecular level the excitation spectra of liquids, we assume that the quantum fields are coupled to an additive delta-correlated in space and time quantum noise field. The hydrodynamic fields are defined in a fluctuating environment. After defining the generating functional of connected correlation functions in the presence of the noise field, we perform a functional integral over all noise field configurations. This is done using a formal object inspired by the distributional zeta-function method, named configurational zeta-function. We obtain a new generating functional written in terms of an analytically tractable functional series. Each term of the series describes in the liquid the emergent non-interacting elementary excitations with the usual gapless phonon-like dispersion relation and additional excitations with dispersion relations with gaps in pseudo-momenta space, i.e., tachyonic-like excitations. Furthermore, the Fourier representation of the two-point correlation functions of the model with the contribution coming from all phononic and tachyonic-like fields is presented. Finally, our analysis reveals that the emergent tachyonic-like and phononic excitations yield a distinctive thermodynamic signature—a quadratic temperature dependence of specific heat $(C_V \propto T^2)$ at low temperatures, providing a theoretical foundation for experiments in confined and supercooled liquids.

I. INTRODUCTION

Despite extensive studies, a unified mathematical treatment to obtain the excitation spectra of amorphous solids, liquids and glasses has not yet been achieved. For liquids, to reach this treatment new ideas and tools must be employed, since they are strongly interacting dynamically disordered systems. In this work, we establish a connection between gapped momentum states of elementary excitations and hydrodynamics via an effective field theory in low-temperature liquids [1, 2], using the functional integral formalism of field theory [3–9]. To investigate elementary excitations under the effects of high-energy processes present at the atomic/molecular scale, when quantum fluctuations are dominant, we assume that the hydrodynamic fields are coupled to an additive delta-correlated noise field. It can be said that in the atomic/molecular regime the quantum fields are defined in a fluctuating environment. In the presence of the noise, we define an augmented generating functional of connected correlation functions. Performing the functional integral over all configuration space of the noise field in this augmented generating functional, we obtain a functional series representing a new generating functional. In the functional series we characterize effective actions describing emergent phononic excitations

with the usual dispersion relation, i.e., sound quanta and collective excitations with dispersion relations with gaps in pseudo-momenta space, i.e., tachyon-like excitations. With our approach, performing a configurational averaging procedure, we describe the gapped momentum states discussed by other methods, as for example the Keldysh-Schwinger approach to dissipation [10, 11]. This situation is similar to inelastic scattering of electromagnetic waves where the wave frequency is modified by a medium with randomness, modeled by a classical disorder potential.

Liquids are systems whose constituents undergo random motion similar to that in gases. Also the average distance between its components are similar to that of solids, however without long-range translational order. To find the temperature dependence of the thermodynamic quantities in liquids, one cannot expand the potential energy of the liquid in terms of squared atomic/molecular displacements, since their displacements are large and the inter-atomic/intermolecular interactions are strong. There are also internal degrees of freedom associated with rotation, which prevents a perturbative expansion based in the vibrational dynamics due to the interference from the configurational dynamics. Contrary to the situation of crystals where the vibrational motion is decomposed into independent normal modes, due to a unique length scale, in liquids there is no small parameter to implement a perturbative expansion [12]. To make things more complex, liquids are characterized by diffusive phenomena on short time scales and exhibit viscoelasticity properties. A system with features of both viscous fluids, that generates shear stress for an inhomogeneous flow velocity and an elastic body that produces a shear stress in

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a static state without a flow. Therefore, for some time scale sufficiently short, a liquid may be modeled as an amorphous solid with structural disorder. As a general rule, there are two kinds of disordered solids, those with compositional disorder, with microscopic structural defects and the amorphous solids with structural disorder, with macroscopic structural inhomogeneities [13]. Systems with the combination of rigidity and disorder have many peculiar properties. In a crystal, the mechanical rigidity is related to the long-range periodic order. In a disordered solid, long-range order of crystals are absent, but we still have a mechanical rigidity, by long-range static stress correlations [14].

Due to great strategic interests, the vibrational spectra of amorphous solids and structural glasses have received a lot of attention in the last decades. One universal feature of some disordered crystals and glasses is an anomalous low-energy excitation, the boson peak, observed in inelastic scattering of light or neutrons. The vibrational density of states is quite different from the prediction of the Debye model of the quantized vibrational excitations, i.e., a squared-frequency law [15, 16]. This unusual behavior is independent of the detailed structure of the system. In Ref. [17] it was shown that the boson peak is related to the Ioffe-Regel limit of the longitudinal phonons, when the mean free path of the phonons approaches their wavelength [18]. The elucidation of the mechanism behind the boson peak is a natural step in our understanding of the bosonic excitations in random systems as disordered solids and glasses. There is a consensus that the boson peak is related to the disordered structure of glasses. One approach to discuss such phenomenon is to use random differential equations, which has been widely discussed in the literature. See for example Refs. [19–24]. The situation is quite different for some small length scales, for high-energy phonons. In this case this approach is not appropriated and a microscopic theory of the solid-like structure is required. Liquids are strongly interacting dynamically disordered systems [25], since their behavior is similar to an amorphous solid without the emergence of solidity, a glass state. These considerations led us to study the excitation spectra at short wavelengths of low-temperature liquids introducing randomness not in the differential operator, but as an additive noise field, modeling a randomly fluctuating environment.

The conventional approach to study classical liquids is based on the general formalism of time-dependent correlation functions and linear response theory [26, 27]. In the framework of linearised hydrodynamics one can obtain the macroscopic transport coefficients in terms of the microscopic quantities. Defining τ_c as the mean collision time of the constituents of the liquid, and the wave number-frequency ω , for $\omega \tau_c \ll 1$ one uses an effective field theory, bounded from above in some energy scale, with the hydrodynamic fields defining a continuum classical field theory [28–30]. One approach to use the hydrodynamic fields in the regime $\omega \tau_c \geq 1$, on the molecular scale, is the generalized hydrodynamics, which considers frequency and wave number-dependent transport coefficients. Leaving the regime where liquids are not able to oppose to tangential stresses, it is possible to show the appearance of shear waves in liquids [31, 32]. Another way to access the regime $\omega \tau_c \geq 1$ was presented by Frenkel [33, 34], with the propagation of solid-like collective modes in liquids, using Maxwell analysis. Maxwell discussed a quite simple model for viscoelastic materials. that exhibit a behavior between a pure viscous liquid and an elastic solid. In liquids there is a viscous flow on long time scale and elastic behavior for very short time scale. Frenkel defines τ_f , the liquid relaxation time, i.e., the average time that atoms/molecules spend to traverse the interatomic/intermolecular spacing. For times shorter than τ_f , the behavior of the system is that of a disordered solid with rigid disordered structure, with shear elastic waves. There is an interpolation between the pure elastic solid behavior and the pure dissipative response of a fluid. Using this approach it is possible to obtain a microscopic picture of the liquid state with the dispersion relation with gaps in momentum space, that for instance has been discussed in different areas of physics [35–44]. The energy spectra of such systems share some similarities with the tachyons spectrum in quantum field theory [45–53]. At this point two interesting questions may be formulated: (i) can we include in the model the effects of degrees of freedom associated with the underlying microscopic theory without making use of the generalized hydrodynamics or some molecularscale description? (ii) still using hydrodynamic fields, is it possible to obtain the gapped momentum states without using the Maxwell-Frenkel viscoelastic theory?

Using the functional formalism, we develop a new theoretical framework to obtain the bosonic excitation spectra of low-temperature liquids in the regime with frequencies satisfying $\omega \tau_c \geq 1$, i.e., the short-time behavior of the correlation functions. Our approach is an oversimplification of the effects of high-energy processes over sound quanta, when quantum effects are dominant. To take into account short-time processes, we define an effective model of non-hydrodynamics degrees of freedom, introducing an additive noise field. This noise field represents unknown quantum processes at small distances or a quantum vacuum noise [54, 55]. Using the definition of the usual generating functional of connected correlation functions, one defines a generating functional in the presence of the noise field, an augmented generating functional. After integrating out the noise, we obtain a new generating functional, written in terms of a functional series. In each term of the series, one can show that there are two kinds of noise-induced quasi-particles. Those obeying the usual gapless phonon-like linear dispersion relation and also elementary excitations with dispersion relations with gaps in pseudo-momenta space respectively. Considering a low temperature regime, our analysis also reveals that the emergent phononic and tachyonic-like excitations yield a quadratic temperature dependence of the specific heat, i.e., $C_V \propto T^2$.

Recent developments in the field theory of liquids have revealed deep connections between symmetry breaking, topological properties, and the emergence of the k-gap phenomenon. Particularly notable is the work of Baggioli et al. [56], who demonstrated that the k-gap in liquids can be understood through a symmetry-based approach involving phase relaxation of Goldstone modes. Their framework shows that nonaffine displacements in the deformation field of liquids lead to a breaking of higherform global symmetries, resulting in the characteristic diffusive-to-propagating crossover of shear waves in liquids. This topological interpretation provides a fundamental theoretical basis for understanding tachyonic-like excitations in liquids, complementing our functional approach.

The structure of this work proceeds as follows. In Sec. II we will briefly outline the quantization of the acoustic waves in liquids. In Sec. III we discuss the phononic field with the effects of an additive noise field, defining the augmented generating functional of connected correlation functions. In Sec. IV we integrate out the noise in this generating functional, using a formal object, named configurational zeta-function. In Sec. V we discuss the two-point correlation functions of the model with the emergence of phononic and gapped momentum states. In Sec. VI we study the canonical quantization of the tachyonic-like fields. The calculation of the specific heat in liquids with tachyonic-like excitations is presented in Sec. VII. Finally, conclusions are given in Sec. VIII. In this work, we use the units $\hbar = k_B = 1$.

II. THE QUANTIZED ACOUSTIC WAVES IN LIQUIDS

In spite of considerable efforts, an unifying physical modeling of liquid structure and its thermodynamic properties is still in construction, due to complexity of the liquid behavior at different scales [57–59]. As we discussed, liquids have viscoelastic properties, since on short time scales their behavior resembles that of an amorphous solid with structural disorder, while on longer time scales they behave as a viscous fluid. To investigate dynamical variables using space-time correlation functions and understand the microscopic structure of a liquid at the molecular scale, one must compare wavelengths with both the mean free path l_c and the mean collision time τ_c of the liquid components. There are three different regimes for the wave numbers and frequencies. The region $kl_c \gg 1, \, \omega \tau_c \gg 1$, represents the free-particle regime where the distances and times involved in the processes are quite short. The components of the liquid move independently of each other. The range of intermediate wave numbers and frequencies, known as the kinetic regime where $kl_c \approx 1, \, \omega \tau_c \approx 1$. For such frequencies the wavelength is about the same size as the mean free path, which violates the assumption that the hydrodynamic fields are

defined in the continuum. Therefore one has to take into account the molecular structure of the liquid and the current treatment is based in the microscopic equations of motion of the elementary components. Finally the hydrodynamical regime where $kl_c \ll 1, \ \omega \tau_c \ll 1$. In this regime the behavior of the liquid is described by phenomenological equations for the hydrodynamic fields, the temperature, the mass density and local velocity of the liquid, i.e., $T(t, \mathbf{x})$, $\rho(t, \mathbf{x})$ and $\mathbf{v}(t, \mathbf{x})$. One way to proceed is to develop the method of fluctuating hydrodynamics where we have a set of stochastic differential equations for the fluctuating variables $\delta \rho(t, \mathbf{x}), \ \delta \mathbf{v}(t, \mathbf{x})$ and $\delta T(t, \mathbf{x})$. Here in this work we are not interested to discuss the equations of fluctuating hydrodynamics including a fluctuating Fourier law, therefore we consider the hydrodynamics of a liquid at low temperatures, above the glass-transformation temperature.

As we discussed, to obtain the excitation spectra in liquids, on a macroscopic scale and large time intervals one can start discussing the hydrodynamics treatment of liquids, which is based in a continuum approximation with local conservation laws. From these conservation laws one can obtain the hydrodynamic density-density time correlation functions and the dynamic structure factors. From the dynamic structure factors one can obtain information on collective dynamics from hydrodynamics to atomic/molecular regime, with light scattering experiments. Nonetheless, on a microscopic scale, the nature of the vibration modes is determined by the interaction between its constituents, which need a quantum mechanical description. For this system with very large number of degrees of freedom, to study high-energy processes on the atomic/molecular-scale we use a formalism that unifies quantum mechanics with the classical theory of fields, i.e., quantum field theory.

With respect to these considerations a remark is appropriate. From the quantum Nyquist theorem, the spectral electromagnetic or scalar field density has a classical limit, where thermal fluctuations dominate and a quantum regime of low temperatures, where quantum effects dominate [60]. The quantum regime requires low temperatures and high frequencies, which is exactly the situation discussed in this work. The quantum fluctuations are dominant and the liquid can be viewed as two weakly-coupled subsystems: phonons and the remainder of the liquid, as discussed by Andreev [61]. Because of these conditions we assert that the noise field models quantum processes on the molecular scale.

Instead of basing our discussion on a classical diffusion equation we are interested in studying the emergent elementary excitations based on the quantization of the hydrodynamic fields. For a compressible fluid in thermodynamic equilibrium, the acoustic wave equation is obtained by linearizing the fluid dynamics equations for small disturbances around the constant equilibrium density and pressure. We thus have

$$p(t, \mathbf{x}) = p_0 + \delta p(t, \mathbf{x}), \tag{1}$$

$$\rho(t, \mathbf{x}) = \rho_0 + \delta \rho(t, \mathbf{x}), \qquad (2)$$

$$\mathbf{v}(t,\mathbf{x}) = \delta \mathbf{v}(t,\mathbf{x}),\tag{3}$$

where ρ_0 and p_0 are the constant equilibrium density and pressure respectively. Assuming that the acoustic perturbation involves no rotational flow we can write $\delta \mathbf{v} = \nabla \delta \chi$. Using the Euler and the mass balance equations and assuming that the acoustic perturbation is adiabatic, we obtain a linear, lossless wave equation for $\delta \rho(t, \mathbf{x})$ given by

$$\left(\frac{1}{u_0^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\delta\rho(t, \mathbf{x}) = 0, \tag{4}$$

and a similar equation for $\delta\chi(t, \mathbf{x})$. The constant u_0 is the longitudinal speed of sound. We would like to stress that a real liquid has finite viscosity and the liquid is not curlfree everywhere. In general acoustic processes, rotational effects are confined to the vicinity of the boundaries. Assuming an impenetrable boundary, we have the Neumann boundary conditions. We write $\mathbf{n}.\nabla\delta\rho(t,\mathbf{x})|_{\partial V} = 0$. To study a simplified model, it is convenient to leave aside viscosity effects and consider periodic boundary conditions. In this way the translational invariance in the system is maintained.

The classical fields of the collective modes can be quantized. To proceed, let us discuss the quantization of the hydrodynamic fields. To define the elementary excitations of the acoustic waves, the sound quanta, we impose that the classical hydrodynamics fields $\delta\chi(t, \mathbf{x})$ and $\delta\rho(t, \mathbf{x})$ are Heisenberg operators obeying the equal-time commutation relations

$$[\delta\chi(t,\mathbf{x}),\delta\chi(t,\mathbf{x}')] = [\delta\rho(t,\mathbf{x}),\delta\rho(t,\mathbf{x}')] = 0 \qquad (5)$$

and also

$$[\delta\chi(t,\mathbf{x}),\delta\rho(t,\mathbf{x}')] = -i\delta(\mathbf{x}-\mathbf{x}').$$
 (6)

Using the noncommutativity algebra of the field operators and that the positive frequency modes associated to the hydrodynamic fields are given by $u_{\mathbf{p}}(t, \mathbf{x})$ where

$$u_{\mathbf{p}}(t, \mathbf{x}) = e^{i(\mathbf{p} \cdot \mathbf{x} - \omega(\mathbf{p})t)},\tag{7}$$

one can write the Fourier representation for the hydrodynamic field operators $\delta \rho(t, \mathbf{x})$ and $\delta \chi(t, \mathbf{x})$. They are given by

$$\delta\rho(t,\mathbf{x}) = \sum_{\mathbf{p}} i \left(\frac{\omega(\mathbf{p})}{2V}\right)^{\frac{1}{2}} \left(a_{\mathbf{p}} u_{\mathbf{p}}(t,\mathbf{x}) - a_{\mathbf{p}}^{\dagger} u_{\mathbf{p}}^{*}(t,\mathbf{x})\right)$$
(8)

and

$$\delta\chi(t,\mathbf{x}) = \sum_{\mathbf{p}} \left(\frac{1}{2\omega(\mathbf{p})V}\right)^{\frac{1}{2}} \left(a_{\mathbf{p}}u_{\mathbf{p}}(t,\mathbf{x}) + a_{\mathbf{p}}^{\dagger}u_{\mathbf{p}}^{*}(t,\mathbf{x})\right),\tag{9}$$

where $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^{\dagger}$ are annihilation and creation operators of elementary excitations with angular frequency $\omega(\mathbf{p})$ and pseudo-momentum \mathbf{p} . We assume that phonon angular frequency is written as $\omega(\mathbf{p}) = u_0 |\mathbf{p}|$ [62, 63]. This linear dispersion relation is a reasonable approximation for phonon wavelengths much longer than the liquid intermolecular distance, and satisfies the condition

$$\lim_{\mathbf{p}\to 0}\omega(\mathbf{p}) = 0.$$
 (10)

To proceed we have to implement the physical condition of the Wightman axioms: the states of this physical system are realized as elements of a Hilbert space [64]. The construction of the Hilbert space of multi-quasi-particles states is straightforward. The state without elementary excitations is the Fock vacuum state $|\Omega_0\rangle$ of the phononic field. It is defined using that $a_{\mathbf{p}} | \Omega_0 \rangle = 0 \forall \mathbf{p}$. All the excited states can be created by acting on the Fock vacuum state the local hydrodynamic field operators, i.e., $a_{\mathbf{p}}^{\dagger}$ and $a_{\mathbf{p}}$. An arbitrary state of the Hilbert space is given by a linear superposition of multi-elementary excitations states. It can be represented as

$$|\Psi\rangle = \sum_{q=0}^{\infty} \frac{1}{(q!)^{\frac{1}{2}}} \int \psi_q(\mathbf{p}_1, \dots \mathbf{p}_q) \, a_{\mathbf{p}_1}^{\dagger} \dots a_{\mathbf{p}_q}^{\dagger} \prod_{i=1}^{q} (d^3 p_i) |\Omega_0\rangle, \quad (11)$$

where $\psi_0 \in \mathbb{C}$ and ψ_q for $q \ge 2$ are symmetric functions. We have

$$\langle \Psi | \Psi \rangle = \sum_{q=0}^{\infty} \int |\psi_q(\mathbf{p}_1, \dots \mathbf{p}_q)|^2 \prod_{i=1}^{q} (d^3 p_i) < \infty.$$
 (12)

The above representation defines the Fock space of the system. We define also the causal two-point correlation function for the phononic field as

$$G^{(2)}(t, \mathbf{x}; t', \mathbf{x}') = -i\langle \Omega_0 | T[\delta\rho(t, \mathbf{x})\delta\rho(t', \mathbf{x}')]\Omega_0 \rangle, \quad (13)$$

where T[...] is the Dyson-time ordered product. Substituting the Fourier representation of the field operator $\delta\rho(t, \mathbf{x})$, defined in Eq. (8), in the Eq. (13) one obtains that the causal correlation function can be written as

$$G^{(2)}(t, \mathbf{x}; t', \mathbf{x}') = -\frac{i}{V}$$
$$\sum_{\mathbf{p}} \frac{\omega(\mathbf{p})}{2} \Big(\theta(t) u_{\mathbf{p}}(t, \mathbf{x}) + \theta(-t) u_{\mathbf{p}}^{*}(t, \mathbf{x}) \Big), \qquad (14)$$

where $\theta(x)$ is the Heaviside step function. The Fourier representation of the causal correlation function of the phonons can be readily derived. It can be written as

$$\bar{G}^{(2)}(v,\mathbf{p}) = \frac{\omega^2(\mathbf{p})}{v^2 - \omega^2(\mathbf{p}) + i\delta},$$
(15)

where the infinitesimal term in the denominator indicates in what half-plane of complex frequency the corresponding integrals will converge. In the following we use both formalisms, the canonical and the functional formalism concomitantly.

The action functional for a phononic field in a liquid or in a solid at moderate temperature with some ordered structure is given by

$$S(\delta\rho) = S_0(\delta\rho) + S_{int}(\delta\rho) \tag{16}$$

where the second nonlinear contribution must be a polynomial of the field. It appears if we quantize the acoustic waves in solids, where the quantization of the classical fields of the collective modes includes one longitudinal and two transverse acoustic modes and the presence of anharmonicity introduces phonon-phonon interactions, resulting in the Landau-Rumer finite life-time of such excitations [65–67]. The free action functional of the liquid is written as

$$S_0(\delta\rho) = \frac{1}{2} \int d^4x \left[\delta\rho(t, \mathbf{x}) \left(\frac{1}{u_0^2} \frac{\partial^2}{\partial t^2} - \Delta \right) \delta\rho(t, \mathbf{x}) \right], \quad (17)$$

where Δ is the Laplace operator which acts on scalar functions defined in a finite time interval $([t_a, t_b])$ and in $V \subset \mathbb{R}^3$ and u_0 is the longitudinal speed of sound wave. We assume that $t_b - t_a \gg \frac{V^{\frac{1}{3}}}{u_0}$. The eigenfunctions of $(-\Delta)$ form a complete basis in the functional space $L_2(V)$ of measurable and square-integrable functions on V. In the following we discuss the functional approach that can be used to describe the propagation of quantized acoustic waves, i.e., sound quanta in the liquid.

The functional integral representation for the vacuum persistence functional of the scalar field theory in the presence of a external scalar source $j(t, \mathbf{x})$ is given by the functional integral

$$Z(j) = \mathcal{N} \int \mathcal{D}\delta\rho \, \exp\left(iS(\delta\rho) + i\int d^4x \, j(t, \mathbf{x})\delta\rho(t, \mathbf{x})\right),$$
(18)

where $\mathcal{D}\delta\rho$ denotes integration over all functions $\delta\rho(t, \mathbf{x})$ of space and time, and \mathcal{N} is the normalization factor, using that $Z(j)|_{j=0} = 1$. Since $S(\delta\rho)$ is the action integral for the classical field theory, the functional integrals are over all classical field histories. The functional Z(j)is the generating functional of the vacuum expectation value of chronological ordered products of the field operators. Note that the $Z(j)|_{j=0}$ has a pure formal meaning since, even using the normal ordering : $S_{int}(\delta\rho)$: we have a kind of a complex measure in the function space, i.e., $\mathcal{D}\mu(\delta\rho) = \mathcal{N} \exp{(iS(\delta\rho))}\mathcal{D}\delta\rho$.

The usefulness of Z(j) is that it permits one to construct the correlation functions, i.e., the vacuum expectation value of chronological ordered products of the field operators, by performing a suitable number of functional differentiations with respect to the source. For an arbitrary theory with interaction action $S_{int}(\delta\rho)$ and $G_0(t, \mathbf{x}; t', \mathbf{x}')$, the free two-point correlation function we construct a perturbative theory (the Stueckelberg-Feynman-Dyson series) writing the generating functional as

$$Z(j) = \mathcal{N} \exp\left[i \int d^4x \, S_{int}\left(\frac{1}{i} \frac{\delta}{\delta j(t, \mathbf{x})}\right)\right]$$
$$\exp\left[\frac{i}{2} \int d^4x \int d^4x' \, j(t, \mathbf{x}) G_0(t, \mathbf{x}; t', \mathbf{x}') j(t', \mathbf{x}')\right].$$
(19)

The coefficients of the expansion Z(j) in a Taylor functional series in $j(t, \mathbf{x})$ determine the correlation functions of the model. The perturbative theory is obtained expanding Z(j) in powers of the coupling constant. The correlation functions are given by the sum of all diagrams with n external legs, including the disconnected diagrams. The vacuum diagrams are cancelled by the normalization factor.

Before starting the discussion of the effects of the fluctuating environment over quasi-particles, there is a problem that deserves to be discussed. For an elastic medium at finite temperature the effects of anharmonicity $S_{int}(\delta\rho) \neq 0$ are to introduce interaction between the phonons. Due to these contributions, the condensation of an infinite number of tachyonic-like excitations into the vacuum could in principle avoid the formation of gapped momentum states. The question that arises is the identification of gapped momentum states in the presence of phonon-phonon interactions. One can show that in this case the effective model with additive and multiplicative noise is able to generate elementary excitations of the system with gapped momentum states.

III. THE PHONONIC FIELD THEORY IN A FLUCTUATION ENVIRONMENT: A QUANTUM NOISE FIELD

Let us start discussing the Frenkel approach for short time processes in liquids accessing the solid-like regime. Frenkel defines the liquid relaxation time τ_f , and a critical angular frequency defined as $\omega_F = \frac{2\pi}{\tau_f}$. For times shorter than the liquid relaxation time τ_f , the local structure of the liquid remains static, similar to that of a solid. Therefore for times shorter than τ_f , i.e., high frequencies $\omega \geq \omega_F$, the system supports one longitudinal mode and two transverse modes. The dispersion relation obtained is

$$\omega(\mathbf{p}) = -\frac{i}{2\tau_f} + \left(u^2 \mathbf{p}^2 - \frac{1}{4\tau_f^2}\right)^{\frac{1}{2}},\tag{20}$$

where u is the transverse speed of sound. There is a critical value for the pseudo-momentum where we have propagating modes. This dispersion relation characterizes a solid-like elastic regime in liquids.

Here, we develop a substantially different approach to obtain results similar to those found in the literature. Our starting point is an effective model of the nonhydrodynamics degrees of freedom. In the kinetic regime, instead of using Frenkel's ideas or discuss the microscopic

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equations of motion in the molecular-scale we are introducing an additive quantum noise field in the model, a randomly fluctuating environment. We discuss free theories, integrating the noise in the augmented generating functional of connected correlation functions. The problem raised is how the quantized hydrodynamic fields change after integrated out the noise field? Since noise also induces local fluctuations in the quantum fields, after this procedure, we are analysing noise-induced effects over quantized acoustic perturbations. Besides the linear dispersion relation of the phonon field, emerging dispersion relations with gaps in pseudo-momenta space. This is exactly the case of tachyonic field theory where the cone of revolution that describes the usual dispersion relation becomes a single-sheeted hyperboloid of revolution.

The use of randomness in analog models in field theory is not new. For instance the literature has been discussing the effects of quantum gravity in matter fields using randomness or non-linear optics [68–72]. Here we analyse a new situation not considered previously in all of these works. Our approach using a noise field is fully sufficient to induce emergent non-interacting elementary excitations in the liquid, both as phononic-like excitations and as quasi-particles with dispersion relations containing gaps in pseudo-momenta space, i.e., tachyonic-like excitations. The simplest case of stochastic processes are random functions of one variable, usually regarded as the time. The theory of random functions of several variables is a natural generalization to the case of one variable. We consider a noise field $h(t, \mathbf{x})$ of time and the points in a finite volume $V \subset \mathbb{R}^3$. Since the perturbation theory is based in the free two-point correlation function, we discuss only the free field theory.

Let us discuss the introduction of the solid-like regime. Suppose a supercooled liquid, where the temperature of the liquid is below the freezing point without crystallization [73, 74]. In this case we start from the Navier-Stokes equation, with the coefficient of bulk and shear viscosity. Using a linearized equation, an adiabatic assumption and also a linearized equation of continuity one obtains a lossy wave equation [75]. For the linearised case the differential equations for the sound waves in viscous media is given by

$$\frac{\partial^2 \boldsymbol{\vartheta}_{\lambda}(t, \mathbf{x})}{\partial t^2} = \Delta \left(c_{\lambda}^2 + D_{\lambda} \frac{\partial}{\partial t} \right) \boldsymbol{\vartheta}_{\lambda}(t, \mathbf{x}).$$
(21)

The sound wave $\vartheta_{\lambda}(t, x)$ have two components: longitudinal and transverse defined as $\vartheta_l(t, \mathbf{x})$ and $\vartheta_t(t, \mathbf{x})$ respectively. In the above equation c_{λ} and D_{λ} are the speed of propagation and a parameter proportional to the diffusion constant of the λ branch. The subscript λ also refers to the longitudinal and transverse displacement fields. Here we assume the following hypothesis: (i) for a short time scale any liquid behaves like a solid, and are able to oppose to tangential stresses, with the presence of transverse acoustic modes, and (ii) to take into account the molecular environment we use a random noise $\mathbf{h}_{\lambda}(t, \mathbf{x})$. Using (i) and (ii) the acoustic perturbations are described by the following wave equations:

$$\left(\frac{1}{c_{\lambda}^{2}}\frac{\partial^{2}}{\partial t^{2}} - \Delta\right)\vartheta_{\lambda}(t,\mathbf{x}) + \mathbf{h}_{\lambda}(t,\mathbf{x}) = 0, \qquad (22)$$

where again c_{λ} are the sound speeds and ϑ_{λ} are the elastic waves, i.e., displacement of the "solid structure" as discussed in Ref [76]. Here we use $c_l = u_0$ and $c_t = u$ as the longitudinal and transverse sound speeds, respectively. These quantities will be used in the action functional that describe the system in this solid-like regime as the sum of both components, i.e., $S_0 = S_l + S_t$ where

$$S_{\lambda}(\boldsymbol{\vartheta}_{\lambda}, \mathbf{h}_{\lambda}) = \int d^{4}x \left[\frac{1}{2} \boldsymbol{\vartheta}_{\lambda}(t, \mathbf{x}) \cdot \left(\frac{1}{c_{\lambda}^{2}} \frac{\partial^{2}}{\partial t^{2}} - \Delta \right) \boldsymbol{\vartheta}_{\lambda}(t, \mathbf{x}) \right. \\ \left. + \mathbf{h}_{\lambda}(t, \mathbf{x}) \cdot \boldsymbol{\vartheta}_{\lambda}(t, \mathbf{x}) \right],$$
(23)

in which $\lambda = l, t$. As usual, we define the $e_i^{(\lambda)}$, the transverse and longitudinal polarization vectors, where $p^i e_i^{(\lambda)} = 0$, for $\lambda = 2, 3$ and $p^i e_i^{(\lambda)} = |\mathbf{p}|$, for $\lambda = 1$. We have $e_i^{(1)} = (1, 0, 0), e_i^{(2)} = (0, 1, 0)$ and $e_i^{(3)} = (0, 0, 1)$. Note that we are assuming that different polarizations are decoupled from each other. In the liquid, this assumption can be used. Also, since at low temperatures there is an attenuation of the longitudinal high frequency phonons [77], here we discuss only the transverse displacements. Therefore we can write the action functional for each transverse component, where we are writing for simplicity $\varphi(t, \mathbf{x}) = \vartheta_t(t, \mathbf{x})$ and $h(t, \mathbf{x}) = h_t(t, \mathbf{x})$ for the transverse degree of freedom with the contribution of the quantum noise field can be defined by

$$S(\varphi, h) = \int d^4x \left[\frac{1}{2} \varphi(t, \mathbf{x}) \left(\frac{1}{u^2} \frac{\partial^2}{\partial t^2} - \Delta \right) \varphi(t, \mathbf{x}) + h(t, \mathbf{x}) \varphi(t, \mathbf{x}) \right].$$
(24)

Before proceeding, let us consider a system with both a classical field and random noise, defined by the equation

$$\left(\frac{1}{u_0^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\xi(t, \mathbf{x}) + \eta(t, \mathbf{x}) = 0, \qquad (25)$$

where u_0 is the speed of longitudinal sound. The above equation is a stochastic partial differential equation of hyperbolic type [78]. Although it describes a stochastic dynamical system, the behavior of its solution is quite different from those solutions of stochastic processes in diffusion equations, where time and space variables play different rules, as for example a non-linear stochastic reaction-diffusion partial differential equation [79]. The above equation describes in one spatial dimension a string under the effect of a sandstorm [80]. It can be solved giving $\xi(0, \mathbf{x}) = \xi_0(\mathbf{x})$ and $\frac{\partial}{\partial t}\xi(0, \mathbf{x}) = v_0(\mathbf{x})$, for $(t, \mathbf{x}) \in [0, t] \times \mathbb{R}^3$. One can use that $\eta(t, \mathbf{x})$ is a whitenoise in space and time. The white-noise can be viewed as a random variable with values in the space of generalized functions. The solution of this linear equation is a distributional-value solution. A generalized random noise is a random linear functional $\eta(f)$ where $f(t, \mathbf{x})$ is an arbitrary function C^{∞} of compact support in $(0, \infty) \times \mathbb{R}^3$ [81]. We have

$$\mathbb{E}[\eta(t, \mathbf{x})] = B_1(t, \mathbf{x}), \tag{26}$$

$$\mathbb{E}[\eta(t, \mathbf{x})\eta(t', \mathbf{x}')] = B_2(t, \mathbf{x}; t', \mathbf{x}'), \qquad (27)$$

where $\mathbb{E}[...]$ implies averaging over an ensemble of random parameter samples. The generalized random field is obtained using the formula

$$\eta(f) = \int d^4x \, \eta(t, \mathbf{x}) f(t, \mathbf{x}), \qquad (28)$$

where we can associate a certain continuous generalized field with every continuous random field. Since Gaussian processes must be indexed by a family of test functions, we have the mean value functional

$$\mathbb{E}[\eta(f)] = B_1(f) \tag{29}$$

and the covariance functional

$$\mathbb{E}[\eta(f_1)\eta(f_2)] = B_2(f_1; f_2) \tag{30}$$

for the families of test functions f_1 and f_2 . Finally, to discuss the localization of waves in amorphous media, one can consider wave equations with random differentials operators. The usual way to discuss the energy spectrum of elementary excitations in such disordered systems is to use a multiple scattering theory performing configurational average procedures.

Going back to our problem with the functional integral over all classical field histories, the basic idea is that we are considering an augmented functional Z(j, h) i.e., the usual generating functional of *n*-point correlation functions of the model augmented by an additive white-noise field. Note that in the path integral formalism the formal oscillatory behavior of the integrand leads us to conclude that the sum over field configurations is dominated by the field configuration of stationary phase, i.e., the solution to the classical field equation. The functional integral representation of this augmented functional is

$$Z(j,h) = \mathcal{N}' \int \mathcal{D}\varphi \, \exp\left(i\,S(\varphi,h) + i\int d^4x\,j(t,\mathbf{x})\varphi(t,\mathbf{x})\right).$$
(31)

The Z(j, h) corresponds to the functional integral Z(j) in the presence of the noise field. There are some similarities between our approach to the one used in a pure classical scenario studying the functional formulation of the problem of turbulence, where the classical fluid described by a Navier-Stokes equation is under the effect of a random force [82]. Since we have that $Z(j,h)|_{h=0} = Z(j)$, and also that $Z(j,h)|_{j=0} = Z(h)$, this augmented functional also satisfies that $Z(h)|_{h=0} = 1$.

To define a generating functional of correlation functions in the field theory with the presence of the noise, we define

$$\mathcal{D}\chi = \mathcal{N}'' e^{iC(h)} \mathcal{D}h, \qquad (32)$$

where

$$C(h) = -\frac{1}{2\sigma^2} \int d^4x \big(h(t, \mathbf{x})\big)^2.$$
 (33)

The \mathcal{N}'' is a immaterial normalization factor that will be omitted in subsequent calculations and $\mathcal{D}h$ is a purely formal notation.

To proceed, one can consider another augmented generating functional, i.e., the usual generating functional of connected correlation functions defined with the noise field, *i.e.*, the augmented functional W(j, h) = $-i \ln Z(j, h)$, also for a specific configuration in functional space of the noise field. First, integrating out the noise, using Eqs. (32) and (33) we define the new functional $\mathbb{Q}[W(j, h)]$ as

$$\mathbb{Q}\big[W(j,h)\big] = \int \mathcal{D}\chi W(j,h). \tag{34}$$

Taking the average of a random variable over the ensemble of realizations or integrating out the noise field using the functional integral formalism are conceptually different procedures. We will now turn to interpretation issues. When we average over a random variable we use the symbol $\mathbb{E}[...]$. The average of a random variable is used in statistical field theory and also in Euclidean field theory, with analytically continued vacuum expectation values of field operators. The Euclidean program suggests that many problems in field theory are really problems in probability theory. In systems governed by classical physics, the formalism is based in a measure space, i.e., a set X together with a sigma-algebra of subsets of X and a measure defined in that algebra, i.e., a non-negative and countable additive set function. A real random variable is a measurable real value function on X. Since the noise field $h(t, \mathbf{x})$ is not a random variable in the formal sense, integrating the noise in some functional we use the symbol $\mathbb{Q}[...]$. We can consider $\mathbb{Q}[...]$ as the "expectation value" of a functional over some specific complex measure. In the path integral formalism Z(j) is the vacuum persistence functional in the presence of a scalar source, i.e., a functional integral over all classical field histories. Discussing an effective model

¹ Instead of using the terminology noisy functional we prefer to use augmented functional, i.e., a functional of the noise field and also of the source.

on the molecular-scale description, the Eq. (34) is not a functional integral over the noise field histories. It is only a functional integral over all configuration space of the noise field, "averaging" W(j,h). We will show that it is possible to write $\mathbb{Q}[...]$ as a functional series, with effective actions. For each term of the series there is a functional integral over all classical fields histories, evaluated for finite temporal intervals.

Another problem arises in close connection with the above remarks. An alternative choice is integrating the noise field in the generating functional Z(j, h), defining $\mathbb{Q}[Z(j, h)]$. In the case of random fields, in the study of statistical mechanics of disordered systems, this kind of average $\mathbb{E}[Z(h)]$ is called an annealed average. One computes the average of the partition function and next defines the average Gibbs energy given by $\ln \mathbb{E}[Z(h)]$. One should keep in mind that there are two essential reasons to discuss $\mathbb{Q}[W(j, h)]$ instead of $\ln \mathbb{Q}[Z(j, h)]$:

(i) First, we assume a linear coupling between the noise and the hydrodynamic field. Also we assume a local coupling, which depends only on the value of the hydrodynamic field at a single point. This defines a local theory. To describe complex anomalous systems, fractional derivative models have been used to explain complex viscoelastic behavior of various material systems, the relaxation behavior of non-Newtonian fluids, shear flow and also fluctuation in viscous fluids. Also, to describe sub- and super-diffusive anomalous diffusion, with non-Markovian correlations one can use the Riemann-Liouville differential and integral operators of non-integer order [83–85]. These are non-local operators. Liquids at small length scales must be modeled as dynamical disordered solids. Therefore, instead of using the non-local operators, an oversimplification is to integrate out the noise in an extensive quantity, similar to the Gibbs free energy, i.e., the generating functional of connected correlation functions.

(ii) The second one and the most convincing reason to work with $\mathbb{Q}[W(j,h)]$ instead of $\mathbb{Q}[Z(j,h)]$ is the following. Working with $\mathbb{Q}[Z(j,h)]$ it is possible to show that the effects of the noise field is to generate only one tachyonic-like field in the liquid. The energy spectrum of the elementary quasi-particles must contains phononic and tachyonic-like excitations.

Strictly speaking, in the kinetic regime, a rather natural way to find modified dispersion relations for the elementary excitations of the liquid is to perform the functional integral of the augmented generating functional W(j,h) i.e., evaluating $\mathbb{Q}[W(j,h)]$. This procedure to work with $\mathbb{Q}[W(j,h)]$ is similar to the one used in statistical field theory, but with a substantially different interpretation. As we discussed, the Eq. (34) means a functional integral over all noise field configurations in a function space. In the theory of classical random fields, with randomness and competing interactions, the free energy must be self-averaging over all the realizations of the random variable i.e., performing $\mathbb{E}[W(j,h)]$.

There is another aspect in our problem which has to be

considered. There are different approaches in the literature, discussing systems with quenched disorder in statistical mechanics and also statistical field theory, to integrate out the disorder to obtain $\mathbb{E}[W(j,h)]$. One of them is the replica "trick", which is still lacking a wellestablished mathematical ground [86, 87]. For other approaches see Refs. [88–91]. We have to integrate out the noise in the augmented generating functional W(j,h). For a generic C(h) we define the functional integral

$$\mathbb{Q}[W(j,h)] = -i \int \mathcal{D}\chi \ln Z(j,h).$$
(35)

If we are able to find a theoretically tractable expression for $\mathbb{Q}[W(j,h)]$, we are able to determine how the presence of the noise affects the behavior of the noiseless system, obtaining the excitation spectra of the liquid.

At this point we would like to compare our approach with the standard continuum effective quantum field theory [92]. There are some similarities and differences with the standard approach. On the molecular-scale description we are defining an effective model of these nonhydrodynamics degrees of freedom, introducing an additive noise field. We integrate out the noise field defining a functional integral over all noise field configurations. In the standard effective theories to obtain an effective action describing the low energy dynamics of the light modes of some model, one integrates out high momentum modes in the generating functional of correlation functions, as in the Euler-Heisenberg Lagrangian or in systems with hierarchy of scales, employing the Appelquist-Carazzone theorem [93–95]. However, our effective description of the system is not related with the standard continuum effective quantum field theory, since we are integrating the noise in the generating functional of connected correlation functions.

IV. EMERGENT PHONONIC AND TACHYONIC-LIKE EXCITATIONS IN A LIQUID WITH QUANTUM NOISE FIELD

An alternative method that has been discussed in the literature to represent $\mathbb{E}[W(j,h)]$ in a tractable way is the distributional zeta-function method [96–103]. Given a measure space (X, Σ, ρ) and a measurable $f : X \to (0, \infty)$, we define the associated generalized ζ -function as

$$\zeta_{\rho,f}(s) = \int_X f(\omega)^{-s} \, d\rho(\omega)$$

for those $s \in \mathbb{C}$ such that $f^{-s} \in L^1(\rho)$, where in the above integral $f^{-s} = \exp(-s\log(f))$ is obtained using the principal branch of the logarithm. Note that (i) if $X = \mathbb{R}^+$, Σ is the Lebesgue σ -algebra, ρ is the Lebesgue measure, and $f(\omega) = \lfloor \omega \rfloor$ we retrieve the classical Riemann zeta-function [104]; where $\lfloor x \rfloor$ means the integer part of x, (ii) if X and Σ are as in item i, $\rho(E)$ counts the prime numbers in E and $f(\omega) = \omega$ we retrieve the prime zeta-function [105–108], (iii) if X, Σ , and f are as in item ii and $\rho(E)$ counts the non-trivial zeros of the Riemann zeta-function, with their respective multiplicity, we obtain the families of superzeta-functions [109], and finally (iv) if X, Σ , and f are as in item ii and $\rho(E)$ counts the eigenvalues of an elliptic operator, with their respective multiplicity, we obtain the spectral zeta-functions [110]. It is worth noting that the series representation given by the distributional zeta-function shares formal similarities with Ruelle's statistical mechanics zeta-function of models [111]:

$$\zeta_S(z,\beta) = \sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(\beta), \qquad (36)$$

where n = 1, 2, ... and $Z_n(\beta)$ is a family of partition functions for a finite model of size n. The variable z is a scaling variable for taking the thermodynamic limit.

Here we adapt the method for the case of path integrals including the noise field. Using the augmented functional integral Z(j, h) given by Eq. (31), the distributional zetafunction, $\Phi(s)$, becomes a configurational zeta-function, and is defined formally as

$$\Phi(s) = \int \mathcal{D}\chi \frac{1}{Z(j,h)^s},\tag{37}$$

for $s \in \mathbb{C}$. A new generating functional given by Eq. (35) tracing out the additive noise can be written as

$$\mathbb{Q}\big[W(j,h)\big] = i(d/ds)\Phi(s)|_{s=0^+}, \quad \operatorname{Re}(s) \ge 0, \quad (38)$$

where one defines the complex exponential $n^{-s} = \exp(-s \log n)$, with $\log n \in \mathbb{R}$. Using analytic tools, and following Klein and Brout [112] we define a new generating functional $\mathbb{Q}[W(j,h)]$, where the noise field was integrated out. The integrated generating functional can be represented as

$$\mathbb{Q}\big[W(j,h)\big] = i\sum_{k=1}^{\infty} \frac{(-1)^k a^k}{kk!} \mathbb{Q}\left[(Z(j,h))^k\right] + i\ln(a) - i\gamma - iR(a,j),$$
(39)

where a is a dimensionless arbitrary constant, γ is the Euler-Mascheroni constant, R(a), given by

$$R(a,j) = -\int \mathcal{D}\chi \int_{a}^{\infty} \frac{dy}{y} \exp\left(-Z(j,h)y\right).$$
(40)

In the functional series we have

$$\mathbb{Q}\big[(Z(j,h))^k\big] = \int \mathcal{D}\chi\big(Z(j,h)\big)^k.$$
(41)

For large a, |R(a)| is quite small, therefore, the dominant contribution to the integrated generating functional is given by the "moments" of the generating functional of correlation functions of the model. We get

$$\mathbb{Q}\big[W(j,h)\big] = i\sum_{k=1}^{\infty} \frac{(-1)^k}{kk!} \mathbb{Q}\big[(Z(j,h))^k\big], \qquad (42)$$

where the *a* constant was absorbed in the functional measures. Using that $\Box = \left(\frac{1}{u^2} \frac{\partial^2}{\partial t^2} - \Delta\right)$ we define the following constants $(\mathcal{N}_0(k))$, and $(\mathcal{N}_l^{(k)})$ given by

$$\left(\mathcal{N}_{0}(k)\right)^{-1} = \int \mathcal{D}\psi^{(k)} \exp\left[\frac{i}{2} \int d^{4}x \,\psi^{(k)}(t, \mathbf{x}) \Big(\Box - k\sigma^{2}\Big)\psi^{(k)}(t, \mathbf{x})\right]$$
(43)

and for $l \geq 2$

$$\left(\mathcal{N}_{l}^{(k)}\right)^{-1} = \int \mathcal{D}\phi_{l}^{(k)} \exp\left(\frac{i}{2} \int d^{4}x \,\phi_{l}^{(k)}(t,\mathbf{x}) \,\Box \,\phi_{l}^{(k)}(t,\mathbf{x})\right). \tag{44}$$

Each term of the functional series is written as a product of k functional integrals. We have

$$\mathbb{Q}\left[\left(Z(h,j)\right)^{k}\right]\Big|_{k=1} = \mathcal{N}_{0}(k) \int \mathcal{D}\psi^{(k)} \exp\left(iS_{2}\left(\psi^{(k)}, j_{\psi}^{(k)}\right)\right)\Big|_{k=1}$$
(45)

and

$$\mathbb{Q}\left[\left(Z(h,j)\right)^{k}\right]\Big|_{k\geq 2} = \mathcal{N}_{0}(k) \int \mathcal{D}\psi^{(k)} \exp\left(iS_{2}\left(\psi^{(k)}, j_{\psi}^{(k)}\right)\right) \\ \mathcal{N}_{\phi}^{(k)} \int \prod_{l=2}^{k} \mathcal{D}\phi_{l}^{(k)} \exp\left(iS_{1}^{(k)}\left(\phi_{l}^{(k)}, j_{\phi_{l}}^{(k)}\right)\right), \quad (46)$$

where $\mathcal{N}_{\phi}^{(k)} = \prod_{l=2}^{k} \mathcal{N}_{l}^{(k)}$ and $\mathcal{D}\phi_{l}^{(k)}$ are products of integration over all functions $\phi_{l}^{(k)}(t, \mathbf{x})$ of space and time. The notation $\left(\phi_{l}^{(k)}\right)$ means that we are considering in the k-term of the functional series, the *l*-th component of the multiplet with k components. In this case, the new effective actions $S_{1}^{(k)}\left(\phi_{l}^{(k)}, j_{\phi_{l}}^{(k)}\right)$ and $S_{2}\left(\psi^{(k)}, j_{\psi}^{(k)}\right)$ are written as

$$S_{1}^{(k)}\left(\phi_{l}^{(k)}, j_{\phi_{l}}^{(k)}\right) = \sum_{l=2}^{k} \int d^{4}x \Big[\phi_{l}^{(k)}(t, \mathbf{x}) j_{\phi_{l}}^{(k)}(t, \mathbf{x}) + \frac{1}{2} \phi_{l}^{(k)}(t, \mathbf{x}) \Big(\frac{1}{u^{2}} \frac{\partial^{2}}{\partial t^{2}} - \Delta\Big) \phi_{l}^{(k)}(t, \mathbf{x})\Big]$$
(47)

and

$$S_{2}\left(\psi^{(k)}, j_{\psi}^{(k)}\right) = \int d^{4}x \Big[\psi^{(k)}(t, \mathbf{x}) j_{\psi}^{(k)}(t, \mathbf{x}) + \frac{1}{2}\psi^{(k)}(t, \mathbf{x}) \Big(\frac{1}{u^{2}}\frac{\partial^{2}}{\partial t^{2}} - \Delta - k\sigma^{2}\Big)\psi^{(k)}(t, \mathbf{x})\Big].$$
(48)

From the above equation it is clear that integrating the noise field in the generating functional Z(j, h), defining

 $\mathbb{Q}[Z(j,h)]$ is exactly the term k = 1 in the functional series. In conclusion, the effects of the noise field is to generate only one tachyonic-like field in the liquid.

Now let us define the functionals $Z_{\psi}^{(k)}\left(j_{\psi}^{(k)}\right)$ and $Z_{\phi_l}^{(k)}\left(j_{\psi}^{(k)}\right)$ as $Z_{\psi}^{(k)}\left(j_{\psi}^{(k)}\right) = \mathcal{N}_0(k) \int \mathcal{D}\psi^{(k)} \exp\left(iS_2\left(\psi^{(k)}, j_{\psi}^{(k)}\right)\right)$

and for $l \ge 2$

$$Z_{\phi_{l}}^{(k)}\left(j_{\phi_{l}}^{(k)}\right) = \mathcal{N}_{l}^{(k)} \int \mathcal{D}\phi_{l}^{(k)} \exp\left(iS_{3}^{(k)}\left(\phi_{l}^{(k)}, j_{\phi_{l}}^{(k)}\right)\right)$$
(50)

where $S_3^{(k)}\left(\phi_l^{(k)}, j_{\phi_l}^{(k)}\right)$ is the action for the *l*-th phononic field in the k-th term of the series. We have obtained that in each term of the functional series defined by Eq. (42) is represented by products of k functional integrals, one of them for the tachyonic-like field and the others for the k-1 functional integrals for the phononic fields. All the mathematical tools needed to obtain these results were developed in earlier papers [113–117]. We adopt a modified functional integral in order to proceed using Eqs. (45)-(48). In the k-th term of the series, we are using the notation $\psi_k(t, \mathbf{x})$, in order to specify this particular field. We impose that this field $\psi_k(t, \mathbf{x})$ is assumed to possess non-vanishing Fourier components only for $\mathbf{p}^2 \geq k\sigma^2$. This is exactly the approach used by Feinberg, Arons and Sudarshan in the canonical quantization of tachyons to avoid imaginary frequencies [47, 48]. Since both effective actions defined by Eq. (47) and Eq. (48)describe free quantum field theories for the tachyonic-like and phononic fields, all the 2n-point correlation functions of the model are written in terms of the two-points correlation functions associated to the tachyonic-like and phononic fields.

V. THE TWO-POINT CORRELATION FUNCTIONS FOR THE PHONONIC AND TACHYONIC-LIKE FIELDS

The aim of this section is to obtain the two-point correlation functions for the phononic and tachyonic-like fields. Since the scalar source $j(t, \mathbf{x})$ was introduced as a device to define the generating functionals of the theory, one has the freedom to choose a particular source distribution. This point will be clarified latter. It is also convenient to define the quantity

$$\mathbb{Q}\left[\left(Z(h,j)\right)^{k}\right]\Big|_{k=1} = Z_{\psi}^{(1)}\left(j_{\psi}^{(1)}\right)$$
(51)

and

$$\mathbb{Q}\left[(Z(h,j))^k \right] \Big|_{k \ge 2} = Z_{\psi}^{(k)} \left(j_{\psi}^{(k)} \right) \prod_{r=2}^k Z_{\phi_r}^{(k)} \left(j_{\phi_r}^{(k)} \right).$$
(52)

To proceed, let us define the coefficient $c_k^{(1)} = \frac{(-1)^k}{kk!}$. Substituting Eq. (52) in Eq. (42) the functional series is written as

$$\mathbb{Q}\left[(W(h,j))\right] = ic_1^{(1)} Z_{\psi}^{(1)}(j_{\psi}^{(1)})
+ i\sum_{k=2}^{\infty} c_k^{(1)} Z_{\psi}^{(k)}\left(j_{\psi}^{(k)}\right) \prod_{r=2}^k Z_{\phi_r}^{(k)}\left(j_{\phi_r}^{(k)}\right).$$
(53)

As usual, to make contact with the two-point correlation functions of the model we must perform two functional derivatives with respect to the sources of the model. We have

$$\frac{\delta^{2}\mathbb{Q}\left[(W(h,j))\right]}{\delta j_{\psi}^{(k)}(t,\mathbf{x})\delta j_{\psi}^{(k)}(t',\mathbf{x}')}\Big|_{j_{\psi}^{(k)}=0} = \\
ic_{1}^{(1)} \frac{\delta^{2}Z_{\psi}^{(1)}\left(j_{\psi}^{(k)}\right)}{\delta j_{\psi}^{(k)}(t,\mathbf{x})\delta j_{\psi}^{(k)}(t',\mathbf{x}')}\Big|_{j_{\psi}^{(k)}=0} + \\
i\prod_{r=2}^{k} Z_{r}^{(k)}\left(j_{\phi_{r}}^{(k)}\right)\sum_{k=2}^{\infty} c_{k}^{(1)} \frac{\delta^{2}Z_{\psi}^{(k)}\left(j_{\psi}^{(k)}\right)}{\delta j_{\psi}^{(k)}(t,\mathbf{x})\delta j_{\psi}^{(k)}(t',\mathbf{x}')}\Big|_{j_{\psi}^{(k)}=0}, \tag{54}$$

and

(49)

$$\frac{\delta^2 \mathbb{Q}\left[(W(h,j))\right]}{\delta j_{\phi_l}^{(k)}(t,\mathbf{x}) \delta j_{\phi_l}^{(k)}(t',\mathbf{x}')} \bigg|_{j_{\phi_l}^{(k)}=0} = i\sum_{k=2}^{\infty} c_k^{(1)} Z_{\psi}^{(k)}(j_{\psi}) \prod_{r=2}^k \frac{\delta^2 Z_{\phi_r}^{(k)}(j_{\phi_r}^{(k)})}{\delta j_{\phi_l}^{(k)}(t,\mathbf{x}) \delta j_{\phi_l}^{(k)}(t',\mathbf{x}')} \bigg|_{j_{\phi_l}^{(k)}=0}.$$
 (55)

Using the fact that $Z_{\phi_l}^{(k)}\left(j_{\phi_l}^{(k)}\right)|_{j_{\phi_l}^{(k)}=0} = 1$ and also that $Z_{\psi}^{(k)}\left(j_{\psi}^{(k)}\right)|_{j_{\psi}^{(k)}=0} = 1$ we define two distinct functionals and functional series. We can write two functional series, using the Eq. (53). They are

$$\mathbb{Q}[W(h,j)]\Big|_{j^{(2)}_{\phi_l}=j^{(3)}_{\phi_l}\ldots=j^{(k)}_{\phi_l}=0} = i\sum_{k=1}^{\infty} c^{(1)}_k Z^{(k)}_{\psi}\left(j^{(k)}_{\psi}\right),$$
(56)

for all l we have $j_{\phi_l}^{(k)} = 0$ and

$$\mathbb{Q}[W(h,j)]\Big|_{j_{\psi}^{(1)}=j_{\psi}^{(2)}\ldots=j_{\psi}^{(k)}=0} = i\sum_{k=2}^{\infty} c_k^{(1)} \prod_{r=2}^k Z_{\phi_r}^{(k)}\left(j_{\phi_r}^{(k)}\right).$$
(57)

In a noiseless system one defines the correlation functions and the connected correlation functions. These correlation functions can be defined by performing a functional expansion of the generating functional of correlation and connected correlation functions respectively. It is to be noted that the same construction can be done for a system under the effects of an additive noise. Let us define

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the following functional series. We have

$$Z_{\psi}^{(k)}\left(j_{\psi}^{(k)}\right) = \sum_{n=1}^{\infty} \frac{i^{n-1}}{n!} \int \prod_{s=1}^{n} d^{4}x_{s} j_{\psi}^{(k)}(t_{1}, \mathbf{x}_{1}) ... j_{\psi}^{(k)}(t_{n}, \mathbf{x}_{n}) \times G_{\psi^{(k)}}^{(n)}(t_{1}, \mathbf{x}_{1}; ...; t_{n}, \mathbf{x}_{n}, k),$$
(58)

and for the $l\mbox{-th}$ phononic component of the $k\mbox{-th}$ multiplet we have

$$Z_{\phi_{l}}^{(k)}\left(j_{\phi_{l}}^{(k)}\right) = \sum_{n=1}^{\infty} \frac{i^{n-1}}{n!} \int \prod_{s=1}^{n} d^{4}x_{s} \, j_{\phi_{l}}^{(k)}(t_{1}, \mathbf{x}_{1}) \dots j_{\phi_{l}}^{(k)}(t_{n}, \mathbf{x}_{n}) \times G_{\phi_{l}^{(k)}}^{(n)}(t_{1}, \mathbf{x}_{1}; \dots; t_{n}, \mathbf{x}_{n}).$$
(59)

From here, what follows is straightforward. Using functional derivatives we obtain the *n*-point correlation functions $G_{\phi_l^{(k)}}^{(n)}(t_1, \mathbf{x}_1; ...; t_n, \mathbf{x}_n)$ and $G_{\psi^{(k)}}^{(n)}(t_1, \mathbf{x}_1; ...; t_n, \mathbf{x}_n)$ of the model, i.e. the original connected correlation functions modified by the effects of the noise field. We have

$$\frac{\delta^{n} Z_{\psi}^{(k)}\left(j_{\psi}^{(k)}\right)}{\delta j_{\psi}^{(k)}(t_{1}, \mathbf{x}_{1}) \dots \delta j_{\psi}^{(k)}(t_{n}, \mathbf{x}_{n})}\Big|_{j_{\psi}^{(k)}=0} = i^{n-1} G_{\psi^{(k)}}^{(n)}(t_{1}, \mathbf{x}_{1}; ...; t_{n}, \mathbf{x}_{n}; k),$$
(60)

and

$$\frac{\delta^{n} Z_{\phi_{l}}^{(k)}\left(j_{\phi_{l}}^{(k)}\right)}{\delta j_{\phi_{l}}^{(k)}(t_{1}, \mathbf{x}_{1}) \dots \delta j_{\phi_{l}}^{(k)}(t_{n}, \mathbf{x}_{n})}\Big|_{j_{\phi_{l}}^{(k)}=0} = i^{n-1} G_{\phi_{l}^{(k)}}^{(n)}(t_{1}, \mathbf{x}_{1}; ..; t_{n}, \mathbf{x}_{n}).$$
(61)

Let us define $G_{\psi^{(k)}}^{(2)}(t, \mathbf{x}; t', \mathbf{x}')$ as the causal two-point correlation function of the tachyonic-like field in the *k*-th term of the series. We have

$$\frac{\delta^2 \mathbb{Q}[(Z(h,j))^k]}{\delta j_{\psi}^{(k)}(t,\mathbf{x}) \delta j_{\psi}^{(k)}(t',\mathbf{x}')} \bigg|_{j_{\psi}^{(k)}=0} = i G_{\psi^{(k)}}^{(2)}(t,\mathbf{x};t',\mathbf{x}';k).$$
(62)

Also, $G_{\phi_l^{(k)}}^{(2)}(t, \mathbf{x}; t', \mathbf{x}')$ is defined as the causal correlation function for the *l*-th phononic field in the *k*-th term of the series. It is written as

$$\frac{\delta^2 \mathbb{Q}[(Z(h,j))^k]}{\delta j_{\phi_l}^{(k)}(t,\mathbf{x}) \delta j_{\phi_l}^{(k)}(t',\mathbf{x}')} \bigg|_{j_{\phi_l}^{(k)}=0} = i G_{\phi_l^{(k)}}^{(2)}(t,\mathbf{x};t',\mathbf{x}').$$
(63)

A straightforward calculation gives that causal two-point correlation function of the tachyonic-like field in the k-th

term of the series is written as

$$G_{\psi^{(k)}}^{(2)}(t,\mathbf{x};t',\mathbf{x}';k) = \mathcal{N}_0(k) \int \mathcal{D}\psi^{(k)} \psi^{(k)}(t,\mathbf{x})\psi^{(k)}(t',\mathbf{x}')$$

$$\exp\left[\frac{i}{2} \int d^4x \,\psi^{(k)}(t,\mathbf{x}) \Big(\frac{1}{u^2} \frac{\partial^2}{\partial t^2} - \Delta - k\sigma^2\Big)\psi^{(k)}(t,\mathbf{x})\right]. \tag{64}$$

In the same way, it is possible to show that causal twopoint correlation function of the *l*-th phononic field in the *k*-th term of the series is written as $G^{(2)}_{\phi_l^{(k)}}(t, \mathbf{x}; t', \mathbf{x}')$ is

$$G_{\phi_l^{(k)}}^{(2)}(t, \mathbf{x}; t', \mathbf{x}') = \mathcal{N}_l^{(k)} \int \mathcal{D}\phi_l^{(k)} \phi_l^{(k)}(t, \mathbf{x})\phi_l^{(k)}(t', \mathbf{x}')$$
$$\exp\left[\frac{i}{2} \int d^4x \,\phi_l^{(k)}(t, \mathbf{x}) \left(\frac{1}{u^2} \frac{\partial^2}{\partial t^2} - \Delta\right) \phi_l^{(k)}(t, \mathbf{x})\right]. \tag{65}$$

From now on we will assume that the multiplet of k-1 phononic fields $\phi_l^{(k)}$ has all the same elements, i.e., we define $\phi_2^{(k)}(t, \mathbf{x}) = \phi_3^{(k)}(t, \mathbf{x}) = \cdots = \phi_k^{(k)}(t, \mathbf{x}) \equiv \phi^{(k)}(t, \mathbf{x})$. Therefore, $\mathcal{N}_2^{(k)} = \mathcal{N}_3^{(k)} = \cdots \equiv \mathcal{N}_k^{(k)}$. And in this way we will have k-1 equations equal to Eq. (65).

Using the previous results, we can write the two-point correlations function associated with the tachyonic-like and phononic fields. We have

$$\bar{G}_{\psi}^{(2)}(t,\mathbf{x};t',\mathbf{x}') = \sum_{k=1}^{N} c_k^{(1)} G_{\psi^{(k)}}^{(2)}(t,\mathbf{x};t',\mathbf{x}';k), \quad (66)$$

and

$$\bar{G}_{\phi}^{(2)}(t,\mathbf{x};t',\mathbf{x}') = \sum_{k=2}^{\infty} c_k^{(1)} \prod_{l=2}^k G_{\phi_l^{(k)}}^{(2)}(t,\mathbf{x};t',\mathbf{x}').$$
(67)

Note that in the Eq. (66) the summation ends in N. This will be clarified in the next section. The absence of tachyonic-like and phononic condensates are given by the equations

$$\bar{G}_{\psi}^{(1)}(t,\mathbf{x}) = \sum_{k=1}^{N} c_k^{(1)} G_{\psi^{(k)}}^{(1)}(t,\mathbf{x};k) = 0, \qquad (68)$$

and

$$\bar{G}_{\phi}^{(1)}(t,\mathbf{x}) = \sum_{k=2}^{\infty} c_k^{(1)} \prod_{l=2}^k G_{\phi_l^{(k)}}^{(1)}(t,\mathbf{x}) = 0.$$
(69)

Since any field theory is determined by its correlation functions, the effects of noise field is to produce free phononic and gapped momenta excitations in the liquid. There is an important point that we would like to stress. To discuss the regime $\omega \tau_c \approx 1$, we introduced a noise field which was integrated out. The tachyonic-like field must describe the behavior of the collective modes in the kinetic regime. There are some experiments that measure the low-frequency elastic behavior in confined liquid at room temperature [118, 119]. If we interpret that the noise field simulates not only the non-hydrodynamic degrees of freedom for $\omega \tau_c \geq 1$ but also the quantum vacuum noise which increases in confined systems we provide a possible explanation for the solid-like behavior of confined liquids at low-frequencies.

In the next section we will discuss the canonical quantization of these tachyonic-like fields defined in the functional series, after the noise field was integrated out.

VI. THE CANONICAL QUANTIZATION OF THE TACHYONIC-LIKE EXCITATIONS

The aim of this section is to perform the canonical quantization of the tachyonic-like fields defined by the functional series, and also discuss the gapped momentum states in our model. From Eq. (48) we have two kinds of frequencies. The real frequencies, for $\mathbf{p}^2 \ge k\sigma^2$, where $\omega_k(\mathbf{p}) = (\mathbf{p}^2 - k\sigma^2)^{\frac{1}{2}}$, and the imaginary frequencies for $\mathbf{p}^2 < k\sigma^2$ where $\omega_k(\mathbf{p}) = \mp i(k\sigma^2 - \mathbf{p}^2)^{\frac{1}{2}}$. For each term of the series, the frequencies can be written as

$$\omega_k(\mathbf{p}) = \mp i u \theta (k\sigma^2 - \mathbf{p}^2) (k\sigma^2 - \mathbf{p}^2)^{\frac{1}{2}} + u \theta (\mathbf{p}^2 - k\sigma^2) (\mathbf{p}^2 - k\sigma^2)^{\frac{1}{2}}.$$
 (70)

For $k\sigma^2 > \mathbf{p}^2$ the life-time of the quasi-particles is dominated by decaying pole terms of the tachyon-like excitations. In general, inelastic scattering of phonons in disordered medium limit the phonon life-time. The noise field is able to model the environment of an amorphous material, leading to a dispersive behavior of the tachyonic-like excitations. For each term of the series, critical pseudomomenta values exist at the gapped pseudo-momenta states where tachyonic-like fields begin to propagate. To implement the canonical quantization of the gapped momenta states $\psi^{(k)}(t, \mathbf{x})$, we employ the Feinberg, Arons and Sudarshan approach. We restrict ourselves to field operators possessing non-vanishing Fourier components only for $\mathbf{p}^2 \geq k\sigma^2$. In this case the energy spectrum of the model is real and this procedure eliminates the dispersive and unphysical behavior of the solutions. Each term in the series contains one tachyonic-like field with

dispersion relation with a gap in pseudo-momentum. One defines the angular frequencies $\nu_k(\mathbf{p})$ associated to the gapped momenta excitations as

$$\nu_k(\mathbf{p}) = u \left(\mathbf{p}^2 - k\sigma^2 \right)^{\frac{1}{2}}.$$
 (71)

Comparing Eq. (20) with the above equation shows that the behavior of the collective excitations resembles the gapped momenta states in the solid-like elastic regime of wave propagation. The result of the literature where the volume of the phase space available to collective excitations reduces with temperature, here is related to the strength of the noise. One important point is that there is a threshold for the pseudo-momentum $|\mathbf{p}_c|$ where above such critical value there is a breakdown of the linear dispersion relation, i.e., $|\mathbf{p}_c| \leq \frac{2\pi}{l_c}$, where l_c is the mean free path of the constituents of the liquid. This critical pseudo-momentum defines a critical k in the functional series that we call N, in the quantized hydrodynamic model. We have

$$N = k_c = \lfloor \sigma^{-2} |\mathbf{p}_c|^2 \rfloor, \tag{72}$$

where $\lfloor \xi \rfloor$ denotes the largest integer $\leq \xi$. The positive frequency modes associated to the tachyonic-like fields are $v_{\mathbf{p}}^{(k)}(t, \mathbf{x})$, where

$$v_{\mathbf{p}}^{(k)}(t,\mathbf{x}) = e^{i(\mathbf{p}\cdot\mathbf{x}-\nu_k(\mathbf{p})t)},\tag{73}$$

The positive frequency modes associated to the tachyonic-like fields are $v_{\lambda \mathbf{p}}^{(k)}(t, \mathbf{x})$, where

$$v_{\lambda \mathbf{p}}^{(k)}(t, \mathbf{x}) = e^{i(\mathbf{p} \cdot \mathbf{x} - \nu_k(\mathbf{p})t)}.$$
(74)

As defined in Eq. (III) the sound wave has longitudinal and transverse components. We were studying only the transverse components of the wave $u_i(t, \mathbf{x})$, introducing a index *i* and the polarization vectors $e_i^{(\lambda)}$ in our tachyonic-like fields we can connect it to our initial quantity $u_i(t, \mathbf{x})$. The $\lambda = 2, 3$ denotes the transverse polarizations, and since the noise only has transverse components the tachyonic-field has the same structure.

The Fourier expansion of the tachyon-like field operator $\psi_i^{(k)}(t, \mathbf{x})$ (for i = 2, 3) is given by

$$\psi_{i}^{(k)}(t,\mathbf{x}) = \sum_{\mathbf{p}^{2} \ge k\sigma^{2}} \sum_{\lambda=2}^{3} i \left(\frac{\nu_{k}(\mathbf{p})}{2V}\right)^{\frac{1}{2}} e_{i}^{(\lambda)} \left(b_{\lambda\mathbf{p}}^{(k)} v_{\lambda\mathbf{p}}^{(k)}(t,\mathbf{x}) - \left(b_{\lambda\mathbf{p}}^{(k)}\right)^{\dagger} \left(v_{\lambda\mathbf{p}}^{(k)}(t,\mathbf{x})\right)^{*}\right),\tag{75}$$

where $b_{\lambda \mathbf{p}}^{(k)}$ and $(b_{\lambda \mathbf{p}}^{(k)})^{\dagger}$ are the annihilation and creation operators for the tachyonic-like fields, with polarization λ . As we discussed before, the $e_i^{(\lambda)}$ are the transverse polarization vectors, where $p^i e_i^{(\lambda)} = 0$, for $\lambda = 2, 3$ and $p^i e_i^{(\lambda)} = |\mathbf{p}|$, for $\lambda = 1$. We have $e_i^{(1)} = (1, 0, 0)$, $e_i^{(2)} = (0, 1, 0)$ and $e_i^{(3)} = (0, 0, 1)$. The commutation relation

between the annihilation and creation operators are the standard ones giving by

$$\left[b_{\lambda \mathbf{p}}^{(k)}, b_{\lambda' \mathbf{p}'}^{(k)}\right] = 0 \quad \left[\left(b_{\lambda \mathbf{p}}^{(k)}\right)^{\dagger}, \left(b_{\lambda' \mathbf{p}'}^{(k)}\right)^{\dagger}\right] = 0, \tag{76}$$

and

$$\left[b_{\lambda\mathbf{p}}^{(k)}, \left(b_{\lambda'\mathbf{p}'}^{(k)}\right)^{\dagger}\right] = \delta_{\lambda\lambda'}\delta(\mathbf{p} - \mathbf{p}').$$
(77)

Therefore we get new kinds of vacuum states. The old one $|\Omega_0\rangle$ and the ones $|\Omega_b^{(k)}\rangle$ associated for each term of the functional series defined by

$$b_{\lambda \mathbf{p}}^{(k)} |\Omega_b^{(k)}\rangle = 0, \ \forall \ \mathbf{p}.$$
(78)

Using the functional series the tachyonic-like Fock vacuum state can be written, making use of the tachyoniclike vacuum Fock states of each term of the series. For the case of the tachyonic-like excitations there is an upper bound in the absolute values of pseudo-momentum space. We have the vacuum states

$$|\Omega_b^{(1)}\rangle, \ |\Omega_b^{(2)}\rangle, ..., |\Omega_b^{(N-1)}\rangle, \ |\Omega_b^{(N)}\rangle.$$
 (79)

We write

$$|\Omega_T\rangle = |\Omega_b^{(1)}\rangle \otimes |\Omega_b^{(2)}\rangle \otimes \dots \otimes |\Omega_b^{(N-1)}\rangle \otimes |\Omega_b^{(N)}\rangle.$$
(80)

The Hamiltonian of the tachyonic-like fields defined by the functional series is given by

$$H_{T} = \frac{1}{2} \sum_{k=1}^{N} \sum_{\mathbf{p}^{2} \ge k\sigma^{2}} \sum_{\lambda=2}^{3} \nu_{k}(\mathbf{p}) \bigg[(b_{\lambda\mathbf{p}}^{(k)})^{\dagger} b_{\lambda\mathbf{p}}^{(k)} + b_{\lambda\mathbf{p}}^{(k)} (b_{\lambda\mathbf{p}}^{(k)})^{\dagger} \bigg],$$
(81)

where we are using the Eq. (72). Since the ground states of the tachyonic-like sector of the theory are cyclic with respect to the polynomials of the tachyonic-like fields, an arbitrary state of the Hilbert space is constructed with the generic $|\Omega_b^{(k)}\rangle$ Fock vacuum state and is given by a linear superposition of multi-elementary tachyonic-like excitations states, which we denote as $|\aleph_{\lambda}^{(k)}\rangle$

$$\begin{split} |\aleph_{\lambda}^{(k)}\rangle &= \\ \sum_{q=0}^{\infty} \frac{1}{(q!)^{\frac{1}{2}}} \int \vartheta_{q}^{(k)}(\mathbf{p}_{1},..,\mathbf{p}_{q}) \, (b_{\lambda\mathbf{p}_{1}}^{(k)})^{\dagger} ... (b_{\lambda\mathbf{p}_{q}}^{(k)})^{\dagger} \prod_{i=1}^{q} (d^{3}p_{i}) |\Omega_{b}^{(k)}\rangle \end{split}$$
(82)

where $\vartheta_0^{(k)}\in\mathbb{C}$ and $\vartheta_q^{(k)}$ for $q\geq 2$ are symmetric functions. We have

$$\langle \aleph_{\lambda}^{(k)} | \aleph_{\lambda}^{\prime(k)} \rangle = \delta_{\lambda\lambda^{\prime}} \sum_{q=0}^{\infty} \int |\vartheta_{q}^{(k)}(\mathbf{p}_{1}, \dots \mathbf{p}_{q})|^{2} \prod_{i=1}^{q} (d^{3}p_{i}) < \infty.$$
(83)

The above representation defines the Fock space that can be generated using the $|\Omega_b^{(k)}\rangle$ tachyonic-like Fock vacuum state. Note that due to the condition $|\mathbf{p}| \geq k\sigma^2$, the set

of functions that we used to expand the field operator does not form a complete set. We have to impose the new completeness relation

$$\sum_{\mathbf{p}^2 \ge k\sigma^2} v_{\lambda \mathbf{p}}^{(k)}(t, \mathbf{x}) \left(v_{\lambda \mathbf{p}}^{(k)}(t', \mathbf{x}') \right)^* |_{t=t'=0} = f_k(\mathbf{x} - \mathbf{x}') \quad (84)$$

where

$$f_k(\mathbf{x} - \mathbf{x}') = \delta^3(\mathbf{x} - \mathbf{x}') + \frac{1}{2\pi^2} \frac{k\sigma^2 \cos k\sigma^2 - \sin k\sigma^2}{k\sigma^2 |\mathbf{x} - \mathbf{x}'|}.$$
(85)

With the new completeness relation one can calculate the expectation value of the Dyson time-ordered product associated with the tachyon-like field in the k-th term of the functional series.

$$G_{\psi_{ij}^{(k)}}^{(2)}(t,\mathbf{x};t',\mathbf{x}') = -i\langle\Omega_b|T[\psi_i^{(k)}(t,\mathbf{x})\psi_j^{(k)}(t',\mathbf{x}')]|\Omega_b\rangle\delta_{ij}.$$
(86)

Substituting the Fourier representation of the tachyoniclike field operator, defined in Eq. (75), in the Eq. (86) one obtains the causal propagator for the tachyonic-like field in the k-th term of the series in this model. It is written as

$$G_{\psi^{(k)}}^{(2)}(t, \mathbf{x}; 0, \mathbf{0}) = -\frac{i}{V}$$

$$\sum_{\mathbf{p}^2 \ge k\sigma^2} \sum_{\lambda=2}^3 \frac{\nu_k(\mathbf{p})}{2} \Big(\theta(t) v_{\lambda \mathbf{p}}^{(k)}(t, \mathbf{x}) + \theta(-t) \big(v_{\lambda \mathbf{p}}^{(k)}(t, \mathbf{x}) \big)^* \Big).$$
(87)

One can write the Fourier components of the above equation. We can write the Fourier representation of the twopoint correlation function of the model with the contribution coming from all tachyonic-like fields represented by $G_{\mu(k)}^{(2)}(v_{\lambda}, \mathbf{p})$. We have

$$\bar{G}_{\psi}^{(2)}(v,\mathbf{p}) = \frac{1}{2} \sum_{k=1}^{N} c_{k}^{(1)} \nu_{k}(\mathbf{p}) \left(\frac{1}{v - \nu_{k}(\mathbf{p}) + i\delta} - \frac{1}{v_{\lambda} + \nu_{k}(\mathbf{p}) - i\delta}\right),$$
(88)

where again the infinitesimal term in the denominator indicates in what half-plane of complex frequency the corresponding integrals will converge. We rewrite the above equation in terms of sum over \mathbf{p} , where the volume V appears. We have

$$\bar{G}_{\psi}^{(2)}(\upsilon) = \frac{1}{V} \sum_{k=1}^{N} c_k^{(1)} \sum_{\mathbf{p}^2 \ge k\sigma^2} \frac{\nu_k^2(\mathbf{p})}{\upsilon^2 - \nu_k^2(\mathbf{p}) + i\delta}.$$
 (89)

From the dispersion relation of $\nu_k(\mathbf{p})$ given by Eq. (71) for large k's the contribution of tachyonic-like fields vanishes, due to the breakdown of the linear dispersion relation for large pseudo-momenta. Therefore in the summation we have a finite number of terms. The discrete sum over ${\bf p}$ can be replaced by a continuous integral using that

$$\sum_{\mathbf{p}} \to \frac{V}{(2\pi)^3} \int d^3q. \tag{90}$$

Introducing a Debye quasi-momentum cut-off q_D we write $\bar{G}_{\eta_2}^{(2)}(v)$ as

$$\bar{G}_{\psi}^{(2)}(\upsilon) = \frac{u^2}{2\pi^2} \sum_{k=1}^{N} c_k^{(1)} \int_{k\sigma^2}^{q_D} dq \frac{q^4 - k\sigma^2 q^2}{\upsilon^2 + k\sigma^2 - q^2}.$$
 (91)

It is clear that the same discussion can be performed for the phononic field. Using Eqs. (15) and (69) we can write the Fourier representation for the causal correlation functions. The total contribution of the phonons in the functional series can be written as

$$\bar{G}_{\phi}^{(2)}(v,\mathbf{p}) = \sum_{k=2}^{\infty} c_k^{(1)} \left[\frac{\omega^2(\mathbf{p})}{v^2 - \omega^2(\mathbf{p}) + i\delta} \right]^{k-1}.$$
 (92)

The above expression is quite interesting. Again, it is possible to rewrite the above equation in terms of sums over all the pseudo-momenta **p**. We get

$$\bar{G}_{\phi}^{(2)}(\upsilon) = \frac{1}{V} \sum_{\mathbf{p}} \sum_{k=2}^{\infty} c_k^{(1)} \left(\frac{\omega^2(\mathbf{p})}{\upsilon^2 - \omega^2(\mathbf{p}) + i\delta} \right)^{k-1}.$$
 (93)

VII. SPECIFIC HEAT IN LIQUIDS WITH TACHYONIC-LIKE EXCITATIONS

As we have shown in previous sections, liquids possess a distinct excitation spectrum characterized by gapless longitudinal phonons and k-gapped transverse (tachyonic-like) modes. These unique features should manifest in macroscopic thermodynamic properties, particularly in the specific heat at low temperatures. Here, we derive the temperature dependence of the specific heat in liquids with k-gapped modes and compare our predictions with experimental results.

A. Theoretical prediction of T^2 scaling

The specific heat at constant volume C_V can be generally expressed in terms of the partition function Z(h) as follows:

$$C_V = \beta^2 \sum_{k=1}^{\infty} c_k \frac{\partial^2}{\partial \beta^2} \mathbb{E}[Z^k(h)]$$
(94)

where $\beta = 1/T$ is the inverse temperature, and the expectation value $\mathbb{E}[Z^k(h)]$ is taken over the statistical ensemble. For systems with phononic excitations, the partition function can be related to the functional determinant of

the corresponding operators. After some algebra, we obtain that at low temperatures, the specific heat of liquids with k-gapped excitations scales as:

$$C_V \propto T^2$$
. (95)

This calculation reveals that, in contrast to the familiar T^3 scaling in crystalline solids (Debye law), liquids with tachyonic-like excitations show a T^2 law. Interestingly, this prediction of a quadratic temperature dependence for the low-temperature specific heat can also be formally justified in the topological framework of Ref. [56], by considering the hierarchy of relevant time and frequency scales, and by clarifying the nature of the kgap arising from phase relaxation. At low temperatures, thermally excited modes correspond to Matsubara frequencies $(\omega_n \sim T)$ that lie well below the collision frequency $1/\tau_c$, but crucially, may still be above the Frenkel frequency $\omega_F = 2\pi/t_f$. In this regime, the transverse shear modes, influenced by non-affine displacements and the associated phase relaxation Ω_{\perp} , exhibit a characteristic momentum-gap (k-gap), with a critical propagation frequency $\omega_g \sim \Omega_{\perp}$ [56]. Importantly, this k-gap is not an energy gap, but rather signifies a transition in the nature of transverse modes—from diffusive below ω_q to propagative above it. At very low frequencies ($\omega < \omega_q$), these modes persist as diffusive, non-propagating degrees of freedom, and thus remain thermally accessible even at very low temperatures. This diffusive nature fundamentally modifies the vibrational density of states $q(\omega)$ at low frequencies, changing it from the Debye $q(\omega) \sim \omega^2$ behavior (typical of purely propagative phonons) to a linear dependence $q(\omega) \sim \omega$. It is precisely this linear scaling of the vibrational density of states, induced by the diffusive low-frequency transverse modes below the k-gap, that naturally leads to a specific heat scaling as $C_V \sim T^2$ at low temperatures. Furthermore, the predicted T^2 law may provide a theoretical foundation for understanding the low-temperature specific heat behavior often observed near the glass transition region. Phenomenologically, the specific heat in this regime could be described by the form $C_V = aT + bT^2$, where a and b are temperature independent constants. The linear term is typically attributed to the contribution of two-level systems (TLS), characteristic of the glassy state and arising from local structural rearrangements, as discussed in [120, 121]. Note that since there is no glassy state, and thus no TLS in our model, we find a = 0. The quadratic term, on the other hand, can be associated with collective excitations in the supercooled liquid phase, which in our model are longitudinal phonons and transverse k-gapped modes.

B. Comparison with experimental data

Experimental data from confined benzene studies by [120] align well with our theoretical predictions. Their

adiabatic calorimetry measurements of benzene in MCM-41 silica nanopores (< 2.9 nm) show specific heat decreasing dramatically at low temperatures more rapidly than in typical glasses and without the significant linear term characteristic of TLS. While extracting a pure T^2 dependence remains challenging, this sharp drop in C_p qualitatively supports our predictions. Furthermore, the suppression of the linear TLS contribution under confinement, as observed in [120] and expected theoretically, makes the quadratric term — originating from the collective modes captured by our model — the dominant contribution, strengthening the consistency between our theory and the experiment.

Despite the qualitative agreement, several factors can lead to deviations from the ideal T^2 scaling: (i) Many amorphous materials exhibit an excess contribution to the specific heat at intermediate temperatures, known as the boson peak. This feature corresponds to an excess density of vibrational states above the Debye prediction and can be related to defect scattering [56], suggesting that liquids might show a milder effect related to configurational freezing, like the specific heat (C_p) hump observed in confined benzene [120] and water [121]. (ii) As real liquid samples inevitably fall out of equilibrium at sufficiently low temperatures, some glass-like features may emerge. In particular, TLS effects can introduce a linear term in the specific heat. The experimental data of confined liquids often show suppressed or absent TLS contributions compared to standard glasses, suggesting that equilibrated liquids have fewer frozen-in defects. (iii) Finally, confinement imposes a minimum wavelength (approximately twice the pore diameter), cutting off modes below a certain wavevector. This can further modify the specific heat at very low temperatures, potentially leading to an exponential dependence if T falls below the energy of the lowest allowed mode.

VIII. CONCLUSIONS AND PERSPECTIVES

There are in the literature many attempts to construct a quantum field theory of free and interacting tachyons. For particles with timelike four momenta we get a twosheeted hyperboloid of revolution around the energy axis. We can choose the positive energy sheet, and the energy is always positive definite. Dealing with spacelike four momenta, we have that the (p^0, \mathbf{p}) surface is a singlesheeted hyperboloid of revolution around the p^0 axis. Therefore a proper Lorentz transformation can change the sign of the energy of the tachyon. The main problem in the quantum field theory of tachyons, is that a frame dependence of observed phenomena cannot be avoided. One can ask: is it possible to construct a system in condensed matter where from the quantized acoustic perturbations emerge simultaneously gapless phonon-like excitations and quasi-particles with dispersions relations with gaps in pseudo-momenta space, i.e., tachyonic-like field excitations? Since liquids are predicted to have a gap in momentum space rather than in frequency space, it is an analog model for tachyons. Analog models do not exactly reproduce nature, but we deal with a simplified prototype, that has been used to capture properties of the physical world and of theoretical models which are still without experimental proofs. For instance, this line of research, the so-called analog models in gravity became extremely fertile with theoretical and experimental grounds over the last forty years discussing mainly the Hawking effect, where effective "event" horizons are generated in the laboratory [122, 123].

As we discussed, in spite of considerable efforts, a unifying physical model of thermodynamic properties of liquids is still in construction, due to the complexity of the liquid behavior in different scales, as for example viscoelastic properties. There is an interpolation between the pure elastic solid behavior and the pure dissipative response of a fluid. Maxwell discussed a quite simple model for viscoelastic materials, which is characteristic in liquids. There is a viscous flow on long time scale and elastic behavior for very short time scale. To discuss a solid-like behavior in liquids Frenkel defines τ_f , the liquid relaxation time, i.e., the average time that atoms/molecules spend to traverse the interatomic/intermolecular spacing. For times shorter than τ_f , the behavior of the system is that of a disordered solid with shear elastic waves. This construction led to the theory of gapped momentum states in liquids. In the present work we established in low-temperature liquids the connection between gapped momentum states of elementary excitations and hydrodynamics as an effective field theory. To achieve such connection we use the functional integral formalism of field theory. Here we aim to model the behavior of the elementary excitations of the liquid taking into account processes for frequencies satisfying $\omega \tau_c \geq 1$ i.e., when non-hydrodynamic degrees of freedom are dominant. To model such degrees of freedom we are using an additive delta-correlated in space and time noise field. We use the fact that liquids at small scales or short time-scale must be modeled as dynamical disordered solids.

To describe complex anomalous systems, fractional differential operators and fractional differential equations have been used to explain relaxation behavior of non-Newtonian fluids, and also fluctuation in viscous fluids. Instead of using the non-local fractional Laplacian, an oversimplification is to assume a fluctuating environment and trace out the noise in an extensive quantity, similar to the Gibbs free energy, i.e., the generating functional of connected correlation functions. In the path integral formalism Z(j) is the vacuum persistence functional in the presence of a scalar source, i.e., a functional integral over all classical field histories. We can consider $\mathbb{Q}[...]$ as the "expectation value" of a functional over some specific complex measure. The $\mathbb{Q}[...]$ means a functional integral over all configuration space of the noise field, "averaging" W(j,h), the augmented generating functional of connected correlation functions. After integrating out the noise field, we obtained a new generating functional

describing the emergent non-interacting elementary excitations of the liquid, phononic and quasi-particles with dispersion relations with gaps in pseudo-momenta space, i.e., tachyonic-like excitations.

Our functional approach to emergent phononic and tachyonic-like excitations in liquids can be viewed as complementary to recent symmetry-based interpretations. Notably, Baggioli et al. [56] have developed a unified topological field theory that explains the appearance of the k-gap in liquids through phase relaxation of Goldstone modes. Their work reveals that the fundamental distinction between solids and liquids lies in the conservation (or lack thereof) of a two-form current related to the single-valuedness of the displacement field. This perspective aligns with our findings on the emergence of gapped momentum states due to quantum fluctuations. The combination of our functional approach with their symmetry-based framework provides a more complete understanding of the emergence of tachyoniclike excitations in liquids and their connections to topological properties of the medium.

Our theoretical framework predicts a distinctive lowtemperature thermodynamic signature of k-gapped fluids, i.e., a quadratic temperature dependence of the specific heat $(C_V \propto T^2)$. This scaling emerges naturally from the distributional zeta-function approach when analyzing how tachyonic-like excitations contribute to thermodynamic properties. Unlike crystalline solids with their characteristic Debye T^3 law arising from three acoustic phonon branches, liquids with k-gapped transverse modes effectively possess reduced vibrational density of states at low frequencies. Future experimental investigations of the specific heat of confined and supercooled liquids, similar to existing measurements in benzene [120] and water [121], could provide additional qualitative support for our theoretical predictions, particularly the precipitous decline in specific heat at low temperatures without the characteristic linear term observed in glasses. Although deviations of the T^2 behavior are expected due to excess vibrational modes (manifested as boson peak features) and experimental constraints, the qualitative correspondence would reinforce our comprehensive model of liquids as systems exhibiting emergent phononic and tachyonic-like excitations—effectively establishing a coherent framework linking microscopic dynamics to macroscopic thermodynamic phenomena.

Finally, propagating transverse modes with tachyoniclike dispersion relations have been experimentally detected in viscous liquids [124]. A natural continuation of this work is to discuss the noise field effects in a viscous liquid at finite temperature. In this case we start from the Navier-Stokes equation, with the coefficient of bulk and shear viscosity. Using a linearized equation, an adiabatic assumption and also a linearized equation of continuity one obtains a lossy wave equation given by Eq. (21). Introducing a cut-off in the wave number, one obtains propagating modes. In this case we obtain the usual acoustic phonon branches with different sound velocities. As we discussed, for an elastic medium at finite temperature the effects of anharmonicity $S_{int}(\delta \rho) \neq 0$ are to introduce interaction between the phonons, and in principle can avoid the formation of gapped momentum states. Next one has to introduce not only an additive noise coupled with the quantized fields with transverse and longitudinal modes, but also a multiplicative noise. Even with the interaction between the phonons, one can show that the effective model with additive and multiplicative noise is able to generate elementary excitations of the system with gapped momentum states [125]. This subject is under investigation by the authors.

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