Gapfull and gapless 1D Topological Superconductivity in Spin-Orbit Coupled Bilayer Graphene

Daniel Skliannyi,* Yuval Oreg, and Ady Stern

Department of Condensed Matter Physics, Weizmann Institute of Science, Rehovot, Israel 76100

(Dated: April 24, 2025)

We propose a way to generate a one-dimensional topological superconductor from a monolayer of a transition metal dichalcogenide coupled to a Bernal-stacked bilayer of graphene under a displacement field. With proper gating, this structure may be tuned to form three parallel pads of superconductors creating two planar Josephson junctions in series, in which normal regions separate the superconductors. Two characteristics of the system which are essential for our discussion are spin orbit coupling induced by the transition metal dichalcogenides and the variation of the Fermi velocities along the Fermi surface. We demonstrate that these two characteristics lead to one-dimensional topological superconductivity occupying large parts in the parameter space defined by the two phase differences across the two junctions and the relative angle between the junctions and the lattice. An angle-shaped device in which this angle varies in space, combined with proper phase tuning, can lead to the formation of domain walls between topological and trivial phases, supporting a zero-energy Majorana mode, within the bulk of carefully designed devices. We derive the spectrum of the Andreev bound states and show that Ising spin-orbit coupling leaves the topological superconductor gapless, and the Rashba spin-orbit coupling opens a gap in its spectrum. Our analysis shows that the transition to a gapped topological state is a result of the band inversion of Andreev states.

Topological superconductors have been a coveted goal in the last two decades, due to their unique physical properties and their potential applications for topological quantum information processing. Of particular interest have been one-dimensional (1D) topological superconductors, due to the localized Majorana zero modes that they carry at their ends. Several ways for engineering such superconductors were proposed, all combining superconductivity, spin-orbit coupling, and breaking of time-reversal symmetry. For most proposed ways superconductivity was induced by the proximity effect, spinorbit coupling was intrinsic to the one-dimensional system, and time-reversal symmetry was broken by coupling a Zeeman field to the electron spin [1–13].

In this work we consider a setup in which spin-orbit coupling is induced into the conducting system by the proximity of a transition metal dichalcogenide (TMD) layer to a Graphene bi-layer; superconductivity is intrinsic to the combined system or achieved by proximity; and time-reversal symmetry is broken by the application of phase differences across Josephson junctions, with or without the application of a Zeeman field. Furthermore, we introduce a new tuning knob, the relative angle β between the Josephson junction and the underlying lattice, and show how it affects the formation of a topological superconductor.

Our setup consists of a structure similar to Ref.[14], comprising four layers: a bottom gate, a layer of transition metal dichalcogenide (TMD, e.g., WSe2), a Bernal stacking bilayer graphene (BBG), and a split top gate (See Fig. 1(A)). In our case, the top gate is divided into three stripes. The angle between the stripes and the underlying Graphene lattice is β .

Tuning the potentials of the bottom and top gates con-

trols the density n and the displacement field D acting on the system. This allows one to adjust the system to achieve a superconducting state (S) or a normal state (N). The creation of two gaps in the top gate leads to the formation of an SNSNS structure that creates two Josephson junctions in series (we will later on name it 2JJ), with the two external superconductors being semiinfinite and the middle superconductor having a width $W_{\rm s}$. We note that the three stripes of the top gate may be replaced by superconducting pads, in which case the superconductivity is induced into the system by proximity rather than being intrinsic.

Once the 2JJ is generated, there are two additional important tuning parameters that control the system's properties - the two-phase differences between the three superconductors, represented by the phases θ and ϕ , where θ is the phase of the order parameter on the central superconductor, and $\pm \phi$ are the phases of the external superconductors.

Earlier works, particularly Ref.[15–18], found that when the electronic band structure leads to two branches with very different Fermi velocities $v_{\rm fs}, v_{\rm fl}$ (s, l stand for small and large) in the direction that crosses the junction, a 1D topological superconductor may be generated by tuning two phase differences across two Josephson junctions, with no need for a Zeeman magnetic field. Having a setup that is free of magnetic fields is desirable due to their adverse effects on superconductivity. A necessary condition for such a topological superconductor is that θ and ϕ form a discrete vortex (DV). When that happens, the formation of a topological superconductor is easiest to understand in the limit where $\xi_s \ll W_{\rm s} \ll \xi_l$, where $W_{\rm s}$ is the width of the middle superconductor and $\xi_s \propto v_{\rm fs}$ and $\xi_l \propto v_{\rm fl}$ are the co-



FIG. 1. (A) A schematic of the system we consider, that consists of Bernal stacked bi-layer of graphene in proximity to TMD (which is not shown), on which patterned gates define superconducting (light blue) and normal (white) regions. The angle β between the superconducting areas and the armchair orientation of the graphene bi-layer determines the velocity difference between the two Fermi branches. With an adjustment of the phase difference between the superconductors that form two Josephson junctions, we can bring the system into a topological state supporting Majorana zero modes. (B) Fermi surface of TMD/BBG heterostructure ($\beta = \pi/2$) in a perpendicular displacement field D = 1 V/nm and $\mu = -12.653$ meV and $\mu = -12.2$ meV. For Fermi energy smaller than -12.652 meV we have three Fermi pockets (FP) per valley and for larger – one Fermi surface (FS) per valley. The $k_{\tilde{x}}$ and $k_{\tilde{y}}$ are the coordinates in momentum space aligned with zig-zag and armchair directions, respectively.

herence lengths for the two velocity branches, which are proportional to v_{fs}, v_{fl} . In that limit, one branch, with coherence length ξ_l , is hardly affected by the middle superconductor and is insensitive to the phase of the middle superconductor, while the other, with coherence length ξ_s , experiences the system as two Josephson junctions in series and is, therefore, sensitive to both phase differences. This difference leads to single gap-closing lines in the ϕ, θ parameter plane, which then leads to topological regions in that plane, see Fig. 2(F).

Two aspects of the BBG/TMD structure contribute unique features to this system: the band structure and the dominance of Ising spin-orbit coupling. We find that the angle β controls the degree of difference between the velocities of the two branches perpendicular to the junction due to the highly anisotropic nature of the Fermi surface. Consequently, β can tune the system between topological and non-topological states. The boundary between such regions contains a Majorana zero mode. Furthermore, we find that the effects of Ising and Rashba spin-orbit couplings on the phase diagram are different.

The structure we consider, BBG [19] in a heterostructure with monolayers of TMD [20, 21], carries two or four Fermi surfaces, one or two in each valley, which are C_3 symmetric, but are anisotropic due to trigonal warping, see Fig. 1(B). It is extensively studied both theoretically [14, 19, 22] and experimentally[20, 21].

Generally, the Hamiltonian of the 2JJ system that we consider is described in terms of valleys $\eta_z = \pm 1$, spins

 $s_z = \pm 1$ and particle-hole $\tau_z = \pm 1$. We use a convention in which the electron spin direction is $\tau_z s_z$. In this convention, *s*-wave pairing is *s*-independent. The Hamiltonian then is,

$$H = \begin{bmatrix} \epsilon(\eta_z \mathbf{k}) - \mu + \alpha_I \eta_z s_z + \alpha_R \mathbf{k} \times \mathbf{s} \end{bmatrix} \tau_z + \mathbf{B} \cdot \mathbf{s} + \Delta_x(y) \tau_x + \Delta_y(y) \tau_y,$$
(1)

where $\epsilon(k_x, k_y)$ is the band energy, α_I is the Ising spinorbit coupling constant, α_R is the Rashba spin-orbit constant, $\mathbf{B} = (B_x, B_y, 0)^T$ is an in-plane magnetic field, \mathbf{k} is measured from the center of the valley, and $\Delta_{x,y}(y)$ is superconducting order parameter that defines the 2JJ. Notice that since the x, y directions are determined by β , the dependence of ϵ on \mathbf{k} is rotated when β is varied, Fig. 1.

Assuming the junction is translation invariant in the x-direction, k_x is a good quantum number and can be considered as a parameter, we can state that the Hamiltonian (1) is of class D, for which states may be classified as either trivial or topological. A transition between trivial and topological states occurs when there is a single gap closing at $k_x = 0$ [23, 24].

We analyze the Hamiltonian (1) in two complementary ways. In the first part, we employ symmetry arguments and approximate solutions to examine the properties of the spectrum of the two Andreev modes that are closest to zero energy. In the second part, we evaluate eigenvalues and study them for the full Hamiltonian analytically, using a symmetry argument to select appropriate couplings. In both parts, we separate between the cases of $\beta = 0$ and $\beta \neq 0$. For each of these two cases we study the effects of Ising coupling, magnetic field, and Rashba coupling on the junction band structure and topology.

As we now show, when the spin-orbit coupling is entirely of the Ising type, the topological region is always gapless due to the gap being closed at $k_x \neq 0$, and becomes gapped only when Rashba spin-orbit coupling is introduced. Furthermore, we find that due to the structure of the Fermi surface, the phase diagram in the ϕ, θ plane strongly depends on β , allowing for Majorana modes to be placed in regions where β is changed.

We first focus on the case $\beta = 0$ and search for solution of linearized BdG equation. The low-energy Andreev mode $E_b(k_x)$ (with b denoting the velocity branch) may be analytically derived in the absence of Rashba coupling and a magnetic field, by linearizing the kinetic term in the BdG equations. In the limit in which the normal regions are wide, such that there are many sub-gap Andreev states for each k_x , the linearized BdG equation yields

$$\cos\left(\frac{E_{p,b}(k_x)}{E_{n,b}^T} - \phi\right) = -\cos(\theta)R_b, \tag{2}$$

where p is an integer that enumerates the mode,

$$R_b = \max\left(\tanh\left(\frac{\Delta}{E_{sc,b}^T}\right) - \Gamma, 0\right),\,$$

b = s, l label the velocity branches and Γ is a phenomenological suppression factor of the Andreev reflection amplitude $R_b \geq 0$. The Thouless energy for the superconducting region is:

$$E_{sc,b}^T(k_x) = \frac{v_{bf}(k_x)}{W_s},$$

where $W_{\rm s}$ is the width of the superconducting region. The energy $E_{n,b}^T(k_x)$ is defined for the normal region in a similar way.

In the limit where the normal regions are narrow $(E_T^n \gg \Delta)$ we find a single set of in-gap modes:

$$\frac{E_b}{\Delta} = \pm \cos \frac{\phi \pm \arccos\left(\frac{2\cos(\theta)R_b + \cos(\phi)(1-R_b)}{(1+R_b)}\right)}{2} \quad (3)$$

Gapless states correspond to $E_b = 0$. As can be seen in the expressions for E_b , when a gapless solution exists for some particular values of θ , ϕ , k_x it may still exist around these values because the changes in θ , ϕ may be compensated by a change in R_b , which is determined by k_x . Gap closings may then disappear or be created only in pairs, either at $k_x = 0$, in which case they correspond to a topological transition, or at $k_x \neq 0$. This stability of the zero energy states may be understood from symmetry considerations, as we now discuss.

Symmetry consideration. For $\beta = 0$ the Hamiltonian (1) anti-commutes with the Particle-hole anti-unitary operator $\hat{P} = K\eta_x s_y \tau_y$, with K being complex conjugation that flips the sign of the momentum. In the absence of Rashba coupling, the Hamiltonian anti-commutes also with a particle-hole operator that is local in k_x , given by $\hat{P}_{kx} = \tilde{K}\eta_x s_y \tau_y$, where \tilde{K} is a complex conjugation that leaves k_x intact. As we now show, when either $\{\hat{P}_{kx}, H\} = 0$ or $[H, s_z] = 0$ the topological region is gapless, and hence Rashba spin-orbit coupling is essential for a gap to open in that region.

When $\{H, P_{k_x}\} = 0$, the spectrum of $H(k_x)$ is symmetric around zero energy. When there are two zero energy states at $k_x = k_0$, close to k_0 the Hamiltonian may be written in the two-dimensional Hilbert space spanned by these two states. All Hermitian operators may then be described in terms of a set of Pauli matrices ρ_i with $i = 0, \ldots, 3$ acting within this subspace, and the operator \hat{P}_{k_x} may be represented in that subspace as the product of complex conjugation and a Pauli matrix ρ_x . The only form of the Hamiltonian allowed by the condition $\{H, K\rho_x\} = 0$ is then $h\rho_z$, with h being a real function of $\delta k_x = k_x - k_0$ and the other system parameters, in particular ϕ, θ and the magnetic field **B**. The two states are degenerate when h = 0. Thus, a small change in



FIG. 2. (A) Fermi velocities (expressed in units of energy $v'_{\rm f} = \hbar v_{\rm f}/a$, where $v_{\rm f}$ is Fermi velocity and a = 2.46 Å. The energy scale 300meV corresponds to $\approx 10^5 \text{m/s}$) in different valleys $(K_+ \text{ and } K_-)$ as a function of angle β $(k_y = 0)$, for $\mu = -12.652$ meV and $\mu = -12.5$ meV. (B) and (C) The spectrum of 2JJ for $\beta = 0$ near the upper part (B) of the phase diagram and the lower part (C) for $\mu = -12.5$, a = 0.1, |d| = 1, **B** = 0 and $\Delta = 0.17$. In panel (B), the conduction and the valence band of the lowest Andreev levels have opposite spins and the topological area is a consequence of band inversion of the Andreev spectrum. The value of s_z of the negative energy band close to E = 0 differs between neighboring topological triangles. Panel (C) shows that when Rashba coupling is added, left and right Fermi pockets can undergo a band inversion earlier than the middle Fermi pocket. (D) A deviation from high symmetry point $\beta = 0$ to $\beta \neq 0$, at which a threshold of Rashba coupling is required for the spectrum to be gapped. Panels (E) and (F) represent the minimal gap $E_M = Q \min_{k_x, b} |E_b^A/\Delta|$ of Andreev state in 2JJ over the phase space without (E) with (F) Rashba SO coupling, where Q = -1 in the topological area and Q = 1 in trivial. White lines correspond to the maximal topological area according to the DV condition. B and B' label topological gapless phases and A and A' are gapped trivial phases related by spin flip.

 ϕ, θ, \mathbf{B} may be compensated by a small change in δk to keep h = 0, rendering the junction's spectrum gapless. This is indeed reflected in the energies from Eq. (2) and from Eq. (3), in which a small change in θ, ϕ may be compensated by a small change in $R^{s/l}$, which depends on k_x . Moreover, this gapless area is the area at which the phases form a discrete vortex. The topological nature of this vortex allows us to define this gapless state as a topological nodal superconductor.

The topological gapless phase and the topological periodic table. The gapless phase we study here possesses D-class symmetries in d = 1. It is a 1D analog to the Weyl semimetal (WS) phase that occurs in 3D. Similar to our discussion, the existence of the WS phase [25] is argued by fixing one momentum component, say k_z , in the 3D Brillouin zone and evaluating the topological invariant of the resulting 2D system as a function of k_z . A change of that invariant requires a gap closure, which forms a Weyl cone. Our case for $\beta = 0$ is similar to that of a WS phase in which the system possesses a crystalline symmetry that makes the d = 2 Hamiltonian invariant to $k_z \to -k_z$, such that the d=2 system of fixed k_z has the same symmetry class of its d = 3 parent phase. Motivated by this observation, we propose the following general principle regarding the periodic table of topological insulators and superconductors: If a topological symmetry class contains a sequence of dimensions $d_1 = d_2 - 1$ characterized both by \mathbb{Z}_2 invariants, it becomes possible, in the presence of an additional non-local symmetry, to realize a topologically gapless phase in dimension d_2 .

Andreev Band inversion. In the absence of \hat{P}_{k_x} breaking terms, the spectrum may be gapped only when the Hilbert space of nearly zero-energy states is larger than two-dimensional, e.g., four-dimensional. This happens in general when zero-energy states at two different values of k_x move toward one another. In particular, it happens when k_0 approaches zero, since for $\beta = 0$ the spectrum is even with respect to k_x . The spectrum may be gapped also by having a term that breaks the relation $\{H, \hat{P}_{k_x}\} = 0$. Examples to this are Rashba spin-orbit coupling and a space-dependent magnetic field with a wave vector that couples the two valleys and leads to a term $\mathbf{B} \cdot \vec{s} \eta_x$. When the gapping happens with two gapless points that merge at $k_0 = 0$, the transition between trivial and topological superconductor is a band inversion of two Andreev bands, see Fig. 2(B).

This general structure may be understood in another way. When the normal regions between the superconductors are small, there are eight sub-gap energy modes. We will focus on the four lowest among them, which are described by the energies in Eq. (3) with a plus sign before the arccos function. The subgap spectrum may then be described in terms of a 4×4 Hamiltonian, which we span by two sets of Pauli matrices λ and s, where s is the same as that appearing in Eq. (1) while λ acts on the particle-hole space and is related to τ by a unitary transformation that is spin-independent. We write the diagonalized 4 × 4 Hamiltonian as $H = \frac{E_A^l + E_A^s}{2} \lambda_z + \frac{E_A^l - E_A^s}{2} \delta_z$. This Hamiltonian anti-commutes with the particle-hole operator, which may be chosen here to be $K\lambda_u s_u$. At the transition, then, the two states of zero energy differ both by their λ_z and s_z values, i.e., they have the same eigenvalue of $\lambda_z s_z$. For an operator to couple these two states and open a gap, it must therefore flip both λ_z and s_z . However, for such an operator to flip both of these numbers and still anti-commute with the particle-hole operator, it must be odd under $k_x \to -k_x$.

These observations allow us to analyze the effect of various perturbations on the low energy Andreev modes.



FIG. 3. (A) $\beta - \alpha_R$ qualitative phase diagram of a 2JJ. The phases of the superconductors are fixed in the middle of a topological region. The diagram shows two gapped regions (Trivial and Topological) and a gapless region. The gapless region connects between gapped trivial and topological superconductors. The angle β_t represents the alignment angle of the device with respect to the tangential direction of the Fermi pockets that separates the trivial and the topological states. (B) A possible configuration for a 2JJ device oriented on the lattice of BBG in such a way that, for some value of θ and ϕ , it acquires a trivial region in one half of the device and topological region at another, with Majorana zero modes between them. The $\beta_1 = 0$ and β_2 are corresponding angles, and color crosses are phase-controlling fluxes. (C) Phase diagram for the case of four velocity branches, for $\mu = -15$ meV and $\beta = 0$. The colored regions represent gapped topological phases. The diagram shows that the topological region can expand when the system has two bands per valley instead of one. (D) Phase diagram for $\beta_1 = 0$ (the shaded blue region is topological) and $\beta_2 = \pi/24$ (the shaded gold region is topological). The chemical potential is $\mu = -12$ meV. In a knee geometry with angle $\pi/24$, when both parts are topological (the blue and gold regions overlap), there is no MZM at the knee. When only one of them is topological, the knee hosts an MZM.

The energy modes in Eq. (3) are calculated by linearizing the BdG equations, thus setting normal reflections to zero, and by setting the magnetic field and the Rashba spin-orbit coupling to zero as well. Normal reflection does not flip the spin, and will therefore introduce terms of the form $V_x \lambda_x + V_y \lambda_y$. These terms would not couple the two states close to zero energy, and would therefore not open a gap. An in-plane Zeeman field will introduce terms of the form $b_x s_x + b_y s_y$. Again, these terms will not open a gap, for the same reason. Finally, Rashba spinorbit coupling, being odd with respect to $k_x \rightarrow -k_x$, will be able to couple two states of the same $\lambda_z s_z$, and will therefore open a gap that will have band inversion.

We now turn to the case $\beta \neq 0$. In the case of $\beta \neq 0$ the Hamiltonian does not anti-commute with \hat{P}_{k_r} . However,

in the absence of Rashba coupling and Zeeman field, it commutes with s_z . For every β and for any k_x Eq. (3) still holds, but the dispersion is not symmetric with respect to $k_x \rightarrow -k_x$. Thus, crossing points of the modes with two opposite values of s_z do not necessarily occur at zero energy. Close to these points the effective Hamiltonian may be written as $E_0 + \alpha \delta k_x s_z$, where $\delta k_x = k_x \mp k_0$, and there are then four crossing points of the zero energy line, again making the topological region gapless. Furthermore, in this case, infinitesimal Rashba coupling removes the degeneracy at E_0 , but does not gap the spectrum at zero energy, necessitating a threshold value of the Rashba coupling to introduce a gapped topological region. A schematic plot of the resulting phase diagram in the β - α_R plane is given in Fig. 3(A).

When $\beta \neq 0$ the zero dimensional system defined when k_x is fixed has no symmetry, and thus belongs to class A, which follows a topological classification of \mathbb{Z} . Again, transitions between different values of the topological invariant as a function of k_x result in gap closures and gapless 1D superconductivity. Since for class A the spectrum is not symmetric around zero, single gap crossings are allowed. More generally, then, we see that if in the sequence of dimension $d_1 = d_2 - 1$ the d_1 dimension corresponds to a symmetry class with \mathbb{Z}_2 invariant and d_2 corresponds to a symmetry class with \mathbb{Z}_2 invariant it becomes possible to realize a topological gapless phase in d_2 dimension.

The β -dependence of the phase diagram may be used in geometries in which β varies along the junction. For example, Fig. 3(B) shows a knee-like 2JJ, where β varies on the two sides of the knee by an obtuse angle, and where each end of the junction has a phase controlling loop, to control θ and ϕ . The variation of β gives rise to different phase diagrams for each orientation, see Fig 3(D). This geometry creates a spatial mismatch of 2JJ topological properties and, for some particular θ and ϕ , gives rise to a topological domain wall that supports a Majorana zero mode at the knee itself.

Before concluding, we employ the band structure calculations described in Appendix A.1 to compute the phase diagram as a function of ϕ , θ , and the chemical potential μ . As shown in Fig. 3(C), the gapped topological region is highly sensitive to the chemical potential. When the chemical potential is such that two bands are being filled in each valley (see the inset to Fig. 3(C) for the band structure in one valley) the topological region expands. In the absence of normal reflection each of the bands forms a phase diagram of its own as a function of θ, ϕ . In regions where both bands form topological superconductors, normal reflection may turn both to be trivial, in view of the Z_2 nature of the classification of Class D.

To summarize, we studied a 2JJ geometry made of a two-dimensional structure of Bernal bi-layer of graphene tunnel coupled to a monolayer of a TMD material. We found that when the spin-orbit coupling induced by the TMD in the graphene bilayer is purely of the Ising type, a region of topological superconductivity exists in the phase diagram, but the topological region is gapless. Spin-rotating couplings, such as Rashba coupling, is crucial for the existence of a nonzero gap in the topological region. We also showed that the relative direction of the junction with respect to the lattice determines the fraction of the phase diagram that is occupied by a topological state.

ACKNOWLEDGMENT

We thank Aleksandr Osin, Vishal Bhardwaj, Omri Lesser, Yuli Nazarov, Mikhail Feigel'man, Boris Altshuler and Alexei Kitaev for their helpful discussions. D.S. acknowledges the Ljubljana Summer School on Quantum Physics, which was held at the University of Ljubljana, the Jozef Stefan Institute, and the CENN Nanocenter. This work was supported by the Israeli Science Foundation, the Israeli Ministry of Science Technology and Space, the Minerva Stiftung, the European Union's Horizon 2020 research and innovation programme (Grant Agreement LEGOTOP No. 788715), the DFG (CRC/Transregio 183, EI 519/7-1), the Israel Science Foundation ISF (Grant No 1914/24) and ISF Quantum Science and Technology (2074/19).

* daniil.skliannyi@weizmann.ac.il

- Y. Oreg, G. Refael, and F. von Oppen, Phys. Rev. Lett. 105, 177002 (2010).
- [2] R. M. Lutchyn, J. D. Sau, and S. Das Sarma, Phys. Rev. Lett. 105, 077001 (2010).
- [3] A. Romito, J. Alicea, G. Refael, and F. von Oppen, Phys. Rev. B 85, 020502 (2012).
- [4] H. Pan and S. Das Sarma, Phys. Rev. Res. 2, 013377 (2020).
- [5] H. Pan and S. Das Sarma, Phys. Rev. B 103, 224505 (2021).
- [6] S. D. Escribano, A. Levy Yeyati, Y. Oreg, and E. Prada, Phys. Rev. B 100, 045301 (2019).
- [7] S. Vaitiekėnas, G. W. Winkler, B. van Heck, T. Karzig, M.-T. Deng, K. Flensberg, L. I. Glazman, C. Nayak, P. Krogstrup, R. M. Lutchyn, and C. M. Marcus, Science **367**, eaav3392 (2020), https://www.science.org/doi/pdf/10.1126/science.aav3392.
- [8] E. Sela, A. Altland, and A. Rosch, Phys. Rev. B 84, 085114 (2011).
- [9] E. M. Stoudenmire, J. Alicea, O. A. Starykh, and M. P. Fisher, Phys. Rev. B 84, 014503 (2011).
- [10] Y. Oreg, E. Sela, and A. Stern, Phys. Rev. B 89, 115402 (2014).
- [11] M. Aghaee, A. Alcaraz Ramirez, Z. Alam, R. Ali, M. Andrzejczuk, A. Antipov, M. Astafev, A. Barzegar, B. Bauer, J. Becker, U. K. Bhaskar, A. Bocharov, S. Boddapati, D. Bohn, J. Bommer, L. Bourdet, A. Bous-

quet, S. Boutin, L. Casparis, B. J. Chapman, S. Chatoor, A. W. Christensen, C. Chua, P. Codd, W. Cole, P. Cooper, F. Corsetti, A. Cui, P. Dalpasso, J. P. Dehollain, G. de Lange, M. de Moor, A. Ekefjärd, T. El Dandachi, J. C. Estrada Saldaña, S. Fallahi, L. Galletti, G. Gardner, D. Govender, F. Griggio, R. Grigoryan, S. Grijalva, S. Gronin, J. Gukelberger, M. Hamdast, F. Hamze, E. B. Hansen, S. Heedt, Z. Heidarnia, J. Herranz Zamorano, S. Ho, L. Holgaard, J. Hornibrook, J. Indrapiromkul, H. Ingerslev, L. Ivancevic, T. Jensen, J. Jhoja, J. Jones, K. V. Kalashnikov, R. Kallaher, R. Kalra, F. Karimi, T. Karzig, E. King, M. E. Kloster, C. Knapp, D. Kocon, J. V. Koski, P. Kostamo, M. Kumar, T. Laeven, T. Larsen, J. Lee, K. Lee, G. Leum, K. Li, T. Lindemann, M. Looij, J. Love, M. Lucas, R. Lutchyn, M. H. Madsen, N. Madulid, A. Malmros, M. Manfra, D. Mantri, S. B. Markussen, E. Martinez, M. Mattila, R. McNeil, A. B. Mei, R. V. Mishmash, G. Mohandas, C. Mollgaard, T. Morgan, G. Moussa, C. Nayak, J. H. Nielsen, J. M. Nielsen, W. H. P. Nielsen, B. Nijholt, M. Nystrom, E. O'Farrell, T. Ohki, K. Otani, B. Paquelet Wütz, S. Pauka, K. Petersson, L. Petit, D. Pikulin, G. Prawiroatmodjo, F. Preiss, E. Puchol Morejon, M. Rajpalke, C. Ranta, K. Rasmussen, D. Razmadze, O. Reentila, D. J. Reilly, Y. Ren, K. Reneris, R. Rouse, I. Sadovskyy, L. Sainiemi, I. Sanlorenzo, E. Schmidgall, C. Sfiligoj, M. B. Shah, K. Simoes, S. Singh, S. Sinha, T. Soerensen, P. Sohr, T. Stankevic, L. Stek, E. Stuppard, H. Suominen, J. Suter, S. Teicher, N. Thiyagarajah, R. Tholapi, M. Thomas, E. Toomey, J. Tracy, M. Turley, S. Upadhyay, I. Urban, K. Van Hoogdalem, D. J. Van Woerkom, D. V. Viazmitinov, D. Vogel, J. Watson, A. Webster, J. Weston, G. W. Winkler, D. Xu, C. K. Yang, E. Yucelen, R. Zeisel, G. Zheng, J. Zilke, and M. A. Quantum, Nature 638, 651 (2025).

- [12] F. Zatelli, D. van Driel, D. Xu, G. Wang, C.-X. Liu, A. Bordin, B. Roovers, G. P. Mazur, N. van Loo, J. C. Wolff, A. M. Bozkurt, G. Badawy, S. Gazibegovic, E. P. A. M. Bakkers, M. Wimmer, L. P. Kouwenhoven, and T. Dvir, Nature Communications 15, 7933 (2024).
- [13] K. Flensberg, F. von Oppen, and A. Stern, Nature Reviews Materials 6, 944 (2021).
- [14] Y.-M. Xie, E. Lantagne-Hurtubise, A. F. Young, S. Nadj-Perge, and J. Alicea, Phys. Rev. Lett. 131, 146601 (2023).
- [15] O. Lesser, Y. Oreg, and A. Stern, Phys. Rev. B 106, L241405 (2022).
- [16] O. Lesser and Y. Oreg, Journal of Physics D: Applied Physics 55, 164001 (2022).
- [17] O. Lesser, K. Flensberg, F. von Oppen, and Y. Oreg, Phys. Rev. B 103, L121116 (2021).
- [18] O. Lesser, A. Saydjari, M. Wesson, A. Yacoby, and Y. Oreg, Proceedings of the National Academy of Sciences 118, e2107377118 (2021), https://www.pnas.org/doi/pdf/10.1073/pnas.2107377118.
- [19] E. McCann and M. Koshino, Reports on Progress in Physics 76, 056503 (2013).
- [20] Y. Zhang, R. Polski, A. Thomson, É. Lantagne-Hurtubise, C. Lewandowski, H. Zhou, K. Watanabe, T. Taniguchi, J. Alicea, and S. Nadj-Perge, Nature 613, 268 (2023).

- [21] L. Holleis, C. L. Patterson, Y. Zhang, Y. Vituri, H. M. Yoo, H. Zhou, T. Taniguchi, K. Watanabe, E. Berg, S. Nadj-Perge, and A. F. Young, Nematicity and orbital depairing in superconducting bernal bilayer graphene with strong spin orbit coupling (2024), arXiv:2303.00742 [cond-mat.supr-con].
- [22] J. Y. Khoo, A. F. Morpurgo, and L. Levitov, Nano Letters 17, 7003 (2017).
- [23] F. Pientka, A. Keselman, E. Berg, A. Yacoby, A. Stern, and B. I. Halperin, Phys. Rev. X 7, 021032 (2017).
- [24] M. Hell, M. Leijnse, and K. Flensberg, Phys. Rev. Lett. 118, 107701 (2017).
- [25] N. P. Armitage, E. J. Mele, and A. Vishwanath, Rev. Mod. Phys. **90**, 015001 (2018).
- [26] A. Y. Kitaev, Physics-Uspekhi 44, 131 (2001).
- [27] C. L. Kane and E. J. Mele, Phys. Rev. Lett. 95, 226801 (2005).
- [28] L. Fu and C. L. Kane, Phys. Rev. Lett. 100, 096407 (2008).
- [29] J. Jung and A. H. MacDonald, Phys. Rev. B 89, 035405 (2014).
- [30] A. L. Chernyshev, D. Galanakis, P. Phillips, A. V. Rozhkov, and A.-M. S. Tremblay, Phys. Rev. B 70, 235111 (2004).
- [31] M. Gmitra and J. Fabian, Phys. Rev. B 92, 155403 (2015).
- [32] Y. Li and M. Koshino, Phys. Rev. B 99, 075438 (2019).
- [33] T. Naimer, K. Zollner, M. Gmitra, and J. Fabian, Phys. Rev. B 104, 195156 (2021).
- [34] M. Gmitra and J. Fabian, Phys. Rev. Lett. 119, 146401 (2017).
- [35] A. David, P. Rakyta, A. Kormányos, and G. Burkard, Phys. Rev. B 100, 085412 (2019).
- [36] A. Banerjee, O. Lesser, M. A. Rahman, H.-R. Wang, M.-R. Li, A. Kringhøj, A. M. Whiticar, A. C. C. Drachmann, C. Thomas, T. Wang, M. J. Manfra, E. Berg, Y. Oreg, A. Stern, and C. M. Marcus, Phys. Rev. B 107, 245304 (2023).
- [37] S. Manzeli, D. Ovchinnikov, D. Pasquier, O. V. Yazyev, and A. Kis, Nature Reviews Materials 2, 17033 (2017).
- [38] C. W. J. Beenakker, Phys. Rev. Lett. 67, 3836 (1991).
- [39] P. Karnatak, Z. Mingazheva, K. Watanabe, T. Taniguchi, H. Berger, L. Forró, and C. Schönenberger, Nano Letters 23, 2454 (2023).
- [40] X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. 83, 1057 (2011).
- [41] J. Alicea, Reports on Progress in Physics 75, 076501 (2012).
- [42] M. Leijnse and K. Flensberg, Semiconductor Science and Technology 27, 124003 (2012).
- [43] A. Kitaev, Annals of Physics 321, 2 (2006), january Special Issue.
- [44]V. Mourik, K. Zuo, S. M. Frolov. S. **R**. Plissard, E. Ρ. Α. М. Bakkers, and L. Ρ. Kouwenhoven. Science **336**. 1003(2012),https://www.science.org/doi/pdf/10.1126/science.1222360.
- [45] A. Das, Y. Ronen, Y. Most, Y. Oreg, M. Heiblum, and H. Shtrikman, Nature Physics 8, 887 (2012).
- [46] S. M. Albrecht, A. P. Higginbotham, M. Madsen, F. Kuemmeth, T. S. Jespersen, J. Nygård, P. Krogstrup, and C. M. Marcus, Nature 531, 206 (2016).
- [47] M. T. Deng, S. Vaitiekėnas, E. B. Hansen, J. Danon, M. Leijnse, K. Flensberg, J. Nygård, P. Krogstrup, and C. M. Marcus, Science **354**, 1557 (2016),

https://www.science.org/doi/pdf/10.1126/science.aaf3961.

- [48] A. Grivnin, E. Bor, M. Heiblum, Y. Oreg, and H. Shtrikman, Nature Communications 10, 1940 (2019).
- [49] S. Vaitiekėnas, Y. Liu, P. Krogstrup, and C. M. Marcus, Nature Physics 17, 43 (2021).
- [50] A. Vuik, B. Nijholt, A. R. Akhmerov, and M. Wimmer, SciPost Phys. 7, 061 (2019).
- [51] R. Hess, H. F. Legg, D. Loss, and J. Klinovaja, Phys. Rev. B 104, 075405 (2021).
- [52] D. Sabonis, O. Erlandsson, A. Kringhøj, B. van Heck, T. W. Larsen, I. Petkovic, P. Krogstrup, K. D. Petersson, and C. M. Marcus, Phys. Rev. Lett. **125**, 156804 (2020).
- [53] V. Shmidt, P. Müller, and A. Ustinov, The Physics of Superconductors (Springer, 1997).
- [54] T. Laeven, B. Nijholt, M. Wimmer, and A. R. Akhmerov, Phys. Rev. Lett. **125**, 086802 (2020).
- [55] P. P. Paudel, T. Cole, B. D. Woods, and T. D. Stanescu, Phys. Rev. B 104, 155428 (2021).

- [56] J.-D. Pillet, V. Benzoni, J. Griesmar, J.-L. Smirr, and Çağlar Ö. Girit, SciPost Phys. Core 2, 009 (2020).
- [57] C. W. J. Beenakker, Rev. Mod. Phys. 87, 1037 (2015).
- [58] G. E. Blonder, M. Tinkham, and T. M. Klapwijk, Phys. Rev. B 25, 4515 (1982).
- [59] A. Andreev, JETP **22**, 655 (1965).
- [60] M. Masseroni, M. Gull, A. Panigrahi, N. Jacobsen, F. Fischer, C. Tong, J. D. Gerber, M. Niese, T. Taniguchi, K. Watanabe, L. Levitov, T. Ihn, K. Ensslin, and H. Duprez, Nature Communications 15, 9251 (2024).
- [61] B. van Heck, S. Mi, and A. R. Akhmerov, Phys. Rev. B 90, 155450 (2014).
- [62] R.-X. Zhang and C.-X. Liu, Topological invariants for three dimensional dirac semimetals and four dimensional topological rotational insulators (2016), arXiv:1605.04451 [cond-mat.mes-hall].
- [63] X.-L. Qi, T. L. Hughes, and S.-C. Zhang, Phys. Rev. B 78, 195424 (2008).

APPENDIXES

A.1 Explicit derivation of perturbative effective Hamiltonian.

BBG tights binding model

We begin with the tight-binding Hamiltonian for bilayer graphene (BBG) under a perpendicular displacement field D:

$$H_{\rm BBG} = \begin{pmatrix} u/2 & v_0 k_{\eta}^{\dagger} & -v_4 k_{\eta}^{\dagger} & -v_3 k_{\eta} \\ v_0 k_{\eta} & f_{\rm BBG} + u/2 & \gamma_1 & -v_4 k_{\eta}^{\dagger} \\ -v_4 k_{\eta} & \gamma_1 & f_{\rm BBG} - u/2 & v_0 k_{\eta}^{\dagger} \\ -v_3 k_{\eta}^{\dagger} & -v_4 k_{\eta} & v_0 k_{\eta} & -u/2 \end{pmatrix}.$$
(4)

Our basis is defined as $\psi_{\eta}(k) = (\psi_{\eta,A_1}(k), \psi_{\eta,B_1}(k), \psi_{\eta,A_2}(k), \psi_{\eta,B_2}(k))$, where $\eta = \pm 1$ corresponds to the valleys $\pm K = (\pm \frac{4\pi}{3a}, 0)$. The term $k_{\eta} = a(\eta k_x + ik_y)$ represents the momentum dependence of the BBG Hamiltonian, with a = 0.24 nm being the graphene lattice constant. The indices A_i/B_i denote sublattices, while i = 1, 2 labels the two layers.

The parameters $v_j \equiv \frac{\sqrt{3}}{2} \gamma_j$, along with γ_1 , f_{BBG} , and u, define the band structure:

- γ_i various hopping amplitudes;
- $f_{\rm BBG}$ on-site potential difference arising from stacking;
- $u = -d_{\perp}D/\epsilon_{\text{BBG}}$ interlayer potential difference induced by the displacement field, where $d_{\perp} = 0.33$ nm is the interlayer separation and $\epsilon_{\text{BBG}} \approx 4.3$ is the relative permittivity of BBG.

From [29], we adopt the *ab initio* calculated parameters: the intralayer nearest-neighbor hopping is $\gamma_0 = 2.61$ eV, while the interlayer hopping amplitudes are $\gamma_1 = 361$ meV, $\gamma_3 = 283$ meV, and $\gamma_4 = 138$ meV. The on-site potential difference is $f_{\text{BBG}} = 15$ meV.

This Hamiltonian differs from that used in [22], where a simplified version was considered, specifically setting $v_3 = v_4 = 0$.

Adding of transition metal dichalcogenides top layer

We can treat the addition of the TMD top layer like a perturbation that affects the top layer of BBG; namely, we take into account only virtual transitions between the top monolayer graphene (MLG) and TMD material. Notice

8

that the chemical potential is always in the gap of the TMD. In this case, we can project the Hamiltonian on the BBG bands. Transitions take place only from the BBG top layer to the TMD and lead to the following spin-dependent Hamiltonian:

$$H_{\rm SOC}(k) = \mathcal{P}_1 \delta H_{\rm MLG} \mathcal{P}_1, \tag{5}$$

$$\delta H_{\rm MLG} = \frac{f_{\rm TMD}}{2} \sigma_z s_0 + \frac{\lambda_I}{2} \eta \sigma_0 s_z + \frac{\lambda_R}{2} \left(\eta \sigma_x s_y - \sigma_y s_x \right) \tag{6}$$

– proximity contribution of WSe₂/MLG Hamiltonian. Where in MLG Hamiltonian σ and s is Pauli matrix that act on sublattice (A, B) and spin degrees of freedom respectively, f_{TMD} – internal displacement field that arise due to stacking, λ_I is rising spin-orbit coupling constant and λ_R – Rashba coupling constant. It was described in work [31] and added spin-orbit coupling potential that arises from the coupling between BBG and TMD material after stacking.

Full effective Hamiltonian, in this case, will be:

$$H_{\rm eff} = H_{\rm BBG} \otimes s_0 + \mathcal{P}_1 \delta H_{\rm MLG} \mathcal{P}_1, \tag{7}$$

In addition to all of the above, one can note that the spin-orbit coupling constants in the Hamiltonian depend on the rotation angle of the TMD sheet relative to the BBG. The values of these parameters are obtained in paper [32] in the case of MLG and can be extrapolated to the case considered in this article.

Brillouin-Wigner perturbation theory

It is well-known, that for generic momentum near K points the strongest hopping in BBG occurs between the B1 and A2 sites – γ_1 , hence we will treat:

$$H_{\gamma_1} = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & \gamma_1 & 0\\ 0 & \gamma_1 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes s_0, \tag{8}$$

like initial Hamiltonian and all other terms:

as a perturbation.

Initial Hamiltonian and perturbation in new basis, where H_{γ_1} is diagonal, have the next form:

$$H' = \begin{pmatrix} \frac{1}{2}(2w_z + (u + f_{\text{TMD}})s_0) & -k_\eta v_3 s_0 & \frac{k_\eta^{\dagger} v_{+} s_0 + a_{R,\eta}}{\sqrt{2}} & \frac{k_\eta^{\dagger} v_{-} s_0 + a_{R,\eta}}{\sqrt{2}} \\ -k_\eta^{\dagger} v_3 s_0 & -\frac{u}{2} s_0 & -\frac{k_\eta v_{+}}{\sqrt{2}} s_0 & \frac{k_\eta v_{-}}{\sqrt{2}} s_0 \\ \frac{k_\eta v_{+} s_0 + a_{R,\eta}^{\dagger}}{\sqrt{2}} & -\frac{k_\eta^{\dagger} v_{+}}{\sqrt{2}} s_0 & (f_{\text{BBG}} - \frac{f_{\text{TMD}}}{4}) s_0 + \frac{w_z}{2} & \frac{1}{2}(w_z + (u - \frac{f_{\text{TMD}}}{2}) s_0) \\ \frac{k_\eta v_{-} s_0 + a_{R,\eta}^{\dagger}}{\sqrt{2}} & \frac{k_\eta^{\dagger} v_{-}}{\sqrt{2}} s_0 & \frac{1}{2}(w_z + (u - \frac{f_{\text{TMD}}}{2}) s_0) & (f_{\text{BBG}} - \frac{f_{\text{TMD}}}{4}) s_0 + \frac{w_z}{2} \end{pmatrix},$$
(11)

where we denote $a_{R,\eta} = \frac{\lambda_R}{2} (\eta s_y + i s_x)$, $v_{\pm} = v_0 \pm v_4$ and $w_z = \frac{1}{2} \lambda_I s_z \eta$. Let us notice that we can label eigenstates by a quantum number m = -1, 0, 1 in a next way:

$$H_{\gamma_1} \left| m \right\rangle = m \gamma_1 \left| m \right\rangle. \tag{12}$$

Our focus will be on the low energy subspace denoting m = 0. One can find that we can express our perturbation through ladder operators $T_i |m\rangle \propto |m+i\rangle$ and $T_i = T_{-i}^{\dagger}$:

$$H' = T_{-2} + T_{-1} + T_0 + T_1 + T_2 + V, (13)$$

where:

$$T_{0} = \begin{pmatrix} \frac{1}{2}(2w_{z} + (u + f_{\text{TMD}})s_{0}) & 0 & 0 & 0 \\ 0 & -\frac{1}{2}us_{0} & 0 & 0 \\ 0 & 0 & (f_{\text{BBG}} - \frac{f_{\text{TMD}}}{4})s_{0} + \frac{w_{z}}{2} & 0 \\ 0 & 0 & 0 & (f_{\text{BBG}} - \frac{f_{\text{TMD}}}{4})s_{0} + \frac{w_{z}}{2} \end{pmatrix},$$
(14)

$$T_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & k_\eta^{\dagger} v_+ s_0 + a_{R,\eta} & 0\\ 0 & 0 & -k_\eta v_+ s_0 & 0\\ 0 & 0 & 0 & 0\\ k_\eta v_- s_0 + a_{R,\eta}^{\dagger} & k_\eta^{\dagger} v_- s_0 & 0 & 0 \end{pmatrix},$$
(15)

For H' we have the next stationary Schrödinger equation:

$$(E - H_{\gamma_1} - T_{-2} - T_{-1} - T_0 - T_1 - T_2 - V) |\psi\rangle = 0.$$
(18)

From it, according to Brilluen-Wigner perturbation theory, we will receive all possible "excitation" of the lowest energy level wave function and the effective Hamiltonian that describes it.

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Let's define projection to high energy states $Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and derive the iterative equation for wave-function

by direct action of projection operator to Schrödinger equation:

$$Q\left(E - H_{\gamma_1} - T_{-2} - T_{-1} - T_0 - T_1 - T_2 - V\right) |\psi\rangle = 0,$$
(19)

We know that E, T_0, V and H_{γ_1} don't change our quantum number, hence $[Q, E - T_0 - V] = 0$. Substitution of this equality leads to the equation:

$$(E - T_0 - V - H_{\gamma_1}) Q |\psi\rangle = Q (T_{-2} + T_{-1} + T_1 + T_2) |\psi\rangle, \qquad (20)$$

$$Q |\psi\rangle = \frac{1}{(E - T_0 - V - H_{\gamma_1})} Q (T_{-2} + T_{-1} + T_1 + T_2) |\psi\rangle, \qquad (21)$$

where last equality is well defined in terms of inverse operator due to the fact that we act in a high-energy state, according to projection operator, so the absolute value of the lowest eigenvalue $|\lambda|$ of $E - T_0 - V - H_{\gamma_1}$ stay in vicinity of γ_1 , namely $|\lambda| \sim \gamma_1$. We want to find low-energy effective Hamiltonian, which corresponds to low-energy wave functions:

$$\left|\psi_{0}\right\rangle = (1-Q)\left|\psi\right\rangle. \tag{22}$$

Inserting it into Eq. (21) we receive:

$$\begin{aligned} |\psi\rangle &= |\psi_0\rangle + \frac{1}{(E - T_0 - V - H_{\gamma_1})} Q \left(T_{-2} + T_{-1} + T_1 + T_2\right) |\psi\rangle ,\\ &= |\psi_0\rangle + \hat{A} |\psi\rangle . \end{aligned}$$
(23)

So let's divide our perturbed wave function labelled by order of $\frac{1}{\gamma_1}$, namely $|\psi^{(i)}\rangle = \dots + O(\frac{1}{\gamma^{i+1}})$:

$$\left|\psi^{(0)}\right\rangle = \left|\psi^{(0)}_{0}\right\rangle,\tag{24}$$

$$\left|\psi^{(1)}\right\rangle = \left|\psi_{0}^{(1)}\right\rangle - \left(\frac{1}{H_{\gamma_{1}}}Q\right)\left(T_{1} + T_{-1}\right)\left|\psi_{0}^{(1)}\right\rangle.$$
(25)

The Wave function is a 4-dimensional vector on spin and zero energy (m = 0) spaces and we will treat it as a 2-dimensional that is block-diagonal on spin space. Consequently Effective Hamiltonian in this case will be:

$$\mathcal{H}_{\text{eff}}^{i} = \begin{pmatrix} \left\langle \psi_{1}^{(i)} \middle| H \middle| \psi_{1}^{(i)} \right\rangle & \left\langle \psi_{1}^{(i)} \middle| H \middle| \psi_{2}^{(i)} \right\rangle \\ \left\langle \psi_{2}^{(i)} \middle| H \middle| \psi_{1}^{(i)} \right\rangle & \left\langle \psi_{2}^{(i)} \middle| H \middle| \psi_{2}^{(i)} \right\rangle \end{pmatrix}.$$
(26)

In the zeros order effective Hamiltonian has a next form:

$$\mathcal{H}_{\text{eff}}^{0} = \begin{pmatrix} \frac{1}{2} (2w_{z} + (f_{\text{TMD}} + u)s_{0}) & -k_{\eta}v_{3}s_{0} \\ -k_{\eta}^{\dagger}v_{3}s_{0} & -\frac{1}{2}us_{0} \end{pmatrix}$$
(27)

and in first:

$$\mathcal{H}_{\text{eff}}^{1} = \mathcal{H}_{\text{eff}}^{0} + \frac{1}{2\gamma_{1}} \left(\begin{array}{c} (v_{+} - v_{-})(|k_{\eta}|^{2}(v_{+} + v_{-})s_{0} + k_{\eta}a_{R,\eta} + k_{\eta}^{\dagger}a_{R,\eta}^{\dagger}) & -(v_{+} + v_{-})\left(\frac{(v_{+}^{2} + v_{-}^{2})}{(v_{+} + v_{-})}(k_{\eta}^{\dagger})^{2}s_{0} + k_{\eta}a_{R,\eta}\right) \\ -(v_{+} + v_{-})\left(\frac{(v_{+}^{2} + v_{-}^{2})}{(v_{+} + v_{-})}(k_{\eta})^{2}s_{0} + k_{\eta}a_{R,\eta}^{\dagger}\right) & (v_{+} - v_{-})|k_{\eta}|^{2}\left(v_{+} + v_{-}\right)s_{0} \end{array} \right) \right).$$

$$(28)$$

One interesting phenomenon can be extracted from this Hamiltonian – the decay of renormalized Rashba Spin-Orbit coupling in low energy subspace according to the properties of BBG. Also, that claim was independently received in the supplementary materials of the work [60].

A.2 Explicit construction of BdG Hamiltonian

We construct the BdG Hamiltonian using the material Hamiltonian in the standard way, accounting for a subtle aspect arising due to the presence of two valleys, $\eta = \pm 1$, in a material with a hexagonal lattice. Specifically, we derive Eq. (28) the effective low-energy Hamiltonian $\mathcal{H}_{\text{eff},\eta}$, which have zero-momentum points at the K and K' points, respectively. To formulate the BdG model, we must return to the real zero points after diagonalizing this Hamiltonian. This can be achieved using the translation operator $U_{\eta} = \exp(i\eta Kx)$, leading to the relation:

$$\sum_{\eta} \int d\vec{k} \vec{\psi}^{\dagger}_{\eta}(\vec{k}) \mathcal{H}_{\mathbf{v},\eta} \vec{\psi}_{\eta}(-\vec{k}) = \sum_{\eta} \int d\vec{k} \vec{\psi}^{\dagger}_{\eta}(\eta K \hat{x} + \vec{k}) \left(U_{\eta} \mathcal{H}_{\mathbf{v},\eta} U^{\dagger}_{\eta} \right) \vec{\psi}_{\eta}(\eta K \hat{x} - \vec{k}), \tag{29}$$

where \vec{k} is the 2D momentum vector, $\psi_{\eta}(\vec{k})$ is the annihilation operator in the η -valley, $\mathcal{H}_{v,\eta}$ is the projection of effective Hamiltonian on to valence band, and the operator U_{η} acts as:

$$U_{\eta}\vec{\psi}_{\eta}(\vec{k}) = \vec{\psi}_{\eta}(\eta K \hat{x} + \vec{k}). \tag{30}$$

Where we emphasise that operators are 2-dimensional cause we have two spins at each valley.

To construct the BdG Hamiltonian, we define the Nambu-Gor'kov spinor basis as:

$$\Psi(\vec{k}) = \begin{pmatrix} \vec{\psi}_{\eta=1}(K\hat{x} - \vec{k}) \\ \vec{\psi}_{\eta=-1}(-K\hat{x} - \vec{k}) \\ \vec{\psi}_{\eta=1}^{\dagger}(K\hat{x} + \vec{k}) \\ -\vec{\psi}_{\eta=-1}^{\dagger}(-K\hat{x} + \vec{k}) \end{pmatrix},$$
(31)

and write the resulting BdG Hamiltonian as:

$$\mathcal{H}_{\rm BdG} = \int d\vec{k} \Psi^{\dagger}(\vec{k}) \begin{pmatrix} U_{+} \mathcal{H}_{\rm v,+}(\vec{k}) U_{+}^{\dagger} & 0 & U_{-} \hat{\Delta} U_{+} & 0 \\ 0 & U_{-} \mathcal{H}_{\rm v,-}(\vec{k}) U_{-}^{\dagger} & 0 & U_{+} \hat{\Delta} U_{-} \\ U_{-}^{\dagger} \hat{\Delta}^{*} U_{+}^{\dagger} & 0 & -U_{+}^{\dagger} \mathcal{H}_{\rm v,+}^{\dagger}(-\vec{k}) U_{+} & 0 \\ 0 & U_{+}^{\dagger} \hat{\Delta}^{*} U_{-}^{\dagger} & 0 & -U_{-}^{\dagger} \mathcal{H}_{\rm v,-}^{\dagger}(-\vec{k}) U_{-} \end{pmatrix} \Psi(\vec{k}).$$
(32)

where $\hat{\Delta} = \Delta s_0$ in spin space

Simplifying further, we note the property $U_{\eta}^{\dagger} = U_{-\eta}$, which allows us to rewrite the Hamiltonian as:

$$\mathcal{H}_{\rm BdG} = \int d\vec{k} \Psi^{\dagger} \begin{pmatrix} U_{+} & 0 & 0 & 0 \\ 0 & U_{-} & 0 & 0 \\ 0 & 0 & U_{+}^{\dagger} & 0 \\ 0 & 0 & 0 & U_{-}^{\dagger} \end{pmatrix} \begin{pmatrix} \mathcal{H}_{\rm v,+}(\vec{k}) & 0 & \hat{\Delta} & 0 \\ 0 & \mathcal{H}_{\rm v,-}(\vec{k}) & 0 & \hat{\Delta} \\ \hat{\Delta}^{*} & 0 & -\mathcal{H}_{\rm v,+}^{\dagger}(-\vec{k}) & 0 \\ 0 & \hat{\Delta}^{*} & 0 & -\mathcal{H}_{\rm v,+}^{\dagger}(-\vec{k}) \end{pmatrix} \begin{pmatrix} U_{+}^{\dagger} & 0 & 0 & 0 \\ 0 & U_{-}^{\dagger} & 0 & 0 \\ 0 & 0 & U_{+} & 0 \\ 0 & 0 & 0 & U_{-} \end{pmatrix} \Psi.$$

$$(33)$$

Thus, we demonstrate that the BdG Hamiltonian in the k = 0 basis can be translated directly to one centred around the K and K' points.

A.3 Spectrum of spin-orbit coupled 2JJ

The sub-gap excitation spectrum can be directly derived in 2JJ systems made from spin-orbit coupled van der Waals materials under the zero normal scattering limit. This simplification is feasible because, unlike 2D electron gases in semiconductors, these materials exhibit dominant Ising spin-orbit coupling for small momenta, significantly simplifying the calculations.

Let's start with the expansion of our Hamiltonian near the Fermi energy:

$$H = -iv_{\mathrm{f},b}(k_x)\partial_y \tag{34}$$

where $v_{f,b}$ is momentum dependent Fermi velocity on branch b = s, l. Thus, we can regularly write BdG Hamiltonian:

$$H = \begin{pmatrix} -iv_{\mathrm{f},\mathrm{b}}(k_x)\partial_y & \Delta(y) \\ \Delta^*(y) & iv_{\mathrm{f},b}(k_x)\partial_y \end{pmatrix}$$
(35)

where

$$\Delta(y) = \Delta e^{-i\phi}\Theta(W_{\rm s}/2 + W_{\rm n} - y) + \Delta e^{i\theta}\left(\Theta(y + W_{\rm s}/2) - \Theta(y - W_{\rm s}/2)\right) + \Delta e^{i\phi}\Theta(y - W_{\rm s}/2 - W_{\rm n})$$
(36)

is the superconducting order parameter.

Matching of the boundary conditions through the junction gives us the following equation for energy:

$$c_{s} \begin{pmatrix} 1 \\ \exp(-i(\phi + \nu_{b})) \end{pmatrix} = \begin{pmatrix} \exp(ik_{e,b}W_{n}) & 0 \\ 0 & \exp(ik_{h,b}W_{n}) \end{pmatrix} \\ \times \begin{pmatrix} \exp\left(\frac{i\kappa_{-,b}W_{s}}{2}\right) & \exp\left(\frac{i\kappa_{+,b}W_{s}}{2}\right) \\ \exp(i(\nu_{b} - \theta))\exp\left(\frac{i\kappa_{-,b}W_{s}}{2}\right) & \exp\left(-i(\theta + \nu_{b})\right)\exp\left(\frac{i\kappa_{+,b}W_{s}}{2}\right) \end{pmatrix} \\ \times \begin{pmatrix} \exp\left(-\frac{i\kappa_{-,b}W_{s}}{2}\right) & \exp\left(-\frac{i\kappa_{+,b}W_{s}}{2}\right) \\ \exp(i(\nu_{b} - \theta))\exp\left(-\frac{i\kappa_{-,b}W_{s}}{2}\right) & \exp\left(-i(\theta + \nu_{b})\right)\exp\left(-\frac{i\kappa_{+,b}W_{s}}{2}\right) \end{pmatrix} \end{pmatrix}^{-1} \\ \times \begin{pmatrix} \exp(ik_{e,b}W_{n}) & 0 \\ 0 & \exp(ik_{h,b}W_{n}) \end{pmatrix} \begin{pmatrix} 1 \\ \exp(i(\phi + \nu_{b})) \end{pmatrix}, \end{cases}$$
(37)

where $\nu_b = \arccos E/\Delta$, $k_{e/h,b} = \pm \frac{E}{v_{\mathrm{f},b}(k_x)}$ and $\kappa_{\pm} = \pm i \frac{\sqrt{\Delta^2 - E^2}}{v_{\mathrm{f},b}(k_x)}$. After some simple math transformations, one can obtain the final equation:

$$\sin(\nu_{b} + \phi - (k_{e,b} - k_{h,b}) 2W_{n}) \sin(\nu_{b}) \left(-i \cot\left(\frac{1}{2}W_{s}(\kappa_{-,b} - \kappa_{+,b})\right) \right) - \cos(\nu_{b} + \phi - (k_{e,b} - k_{h,b}) 2W_{n}) \cos(\nu_{b}) = -\cos(\theta)$$
(38)

Solution for $\beta \neq 0$

The corresponding solution for the arbitrary angle of the pads can be obtained by direct rotation of our coordinate system:

$$\begin{cases} k_{\tilde{x}} \to \cos\left(\beta\right) k_x \pm \sin\left(\beta\right) \left(-i\partial_y\right) \\ -i\partial_{\tilde{y}} \to \mp \sin\left(\beta\right) k_x + \cos\left(\beta\right) \left(-i\partial_y\right) \end{cases}$$
(39)

where \pm refers to electrons and holes respectively, it results in the gap closings of the form depicted in Fig. 2. From such plots, we can understand that we will have no gap for any relative angle (for large β gap closings will occur in other Fermi pockets). Thus one can conclude that we must have two "directions" on Pauli matrices for non-trivial topological superconductivity.

Also, it can be observed that outside the high symmetry point, $\beta = 0$, exists minimal energy that we will need to pay to open a gap in the topological region. This takes place because the branch touching points arise not in zero energy but shifted to non-zero positive and negative energy with respect to particle-hole symmetry (see Fig. 2(D)).

Г