A RANK-2 VECTOR BUNDLE ON $\mathbb{P}^2 \times \mathbb{P}^2$ AND PROJECTIVE GEOMETRY OF NONCLASSICAL ENRIQUES SURFACES IN CHARACTERISTIC 2

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ABSTRACT. We construct a rank-2 indecomposable vector bundle on $\mathbb{P}^2 \times \mathbb{P}^2$ in characteristic 2 that does not come from a bundle on \mathbb{P}^2 by factor projection nor from a bundle on \mathbb{P}^m by central projection. We show that the zero-sets of a suitable twist of *E* form a family of nonclassical smooth Enriques surfaces of bidegree (4, 4) whose general member is 'singular' in the sense that Frobenius acts isomorphically on H^1 , and there is a divisor consisting of smooth supersingular surfaces (Frobenius acts as zero). Every nonclassical Enriques surface of bidegree (4, 4) that is bilinearly normal arises as a zero-set in this way.

A well-known result of Serre asserts that a codimension 2 locally complete intersection in projective space is the zero scheme of the section of a rank 2 vector bundle if and only if its canonical line bundle is the restriction of a line bundle on the ambient space. Serre's result is also valid for certain other ambient spaces. When the canonical bundle is trivial, the condition is automatically satisfied. Among algebraic surfaces, there are three wellknown classes of minimal surfaces with a trivial canonical bundle, and two of them have models in \mathbb{P}^4 with corresponding bundles:

- (i) Abelian surface can be embedded into \mathbb{P}^4 by an ample (1,5)-polarization, and the corresponding bundle has first been described by Horrocks and Mumford in 1973.
- (ii) K3 surfaces with an ample divisor of degree 6 can be embedded in P⁴ as a complete intersection of a quadric and a cubic hypersurface, hence the corresponding vector bundle splits.
- (iii) In characteristic 2 only, there exist non-classical (singular and supersingular) Enriques surfaces with a trivial canonical bundle; however, the self-intersection formula [8, App. A] shows that they cannot be embedded into P⁴.

In light of this one can naturally ask whether non-classical Enriques surfaces can be embedded in other 'standard' homogeneous rational 4-folds and give rise to new rank-2 cector bundles..

In this paper we study in detail a particular family of rank-2 vector bundles E on $\mathbb{P}^2 \times \mathbb{P}^2$ in char. 2, which is a special case of a series of families of rank-*n* bundles on $\mathbb{P}^n \times \mathbb{P}^n$ in all positive characteristics constructed in [14]. Alongside E we study some of the associated zero-sets. The main results may be summarized as follows.

(i) Denote by *L*, *h* and Q_L, Q_h respective hyperplane line bundles and quotient bundles on two copies of \mathbb{P}^2 . Then the family of bundles *E*, constructed in [14] via

Key words and phrases. vector bundle, projective space, monad, Enriques surface.

arxiv.org.

Date: 2025-04-24 00:02:56Z.

²⁰¹⁰ Mathematics Subject Classification. 14n05, 14j60.

an elementary modification, coincides up to a suitable twist with the family of cohomology bundles of a monad on $\mathbb{P}^2 \times \mathbb{P}^2$ of the following type

$$\mathcal{O} \to Q_L \otimes Q_h \to \mathcal{O}(L+h).$$

- (ii) The zero set *Y* of a general section of a suitable twist of *E* is a smooth nonclassical Enriques surface in characteristic 2 with trivial canonical bundle and irregularity 1. The general smooth zero set is *singular* in the sense of Bombieri-Mumford (Frobenius acts isomorphically on $H^1(\mathcal{O}_{\gamma})$) and there is a nontrivial codimension-1 subfamily of supersingular smooth zero sets (Frobenius is zero on $H^1(\mathcal{O}_Y)$).
- (iii) The same twist has special sections with zero-set that is a normal-crossing surface of the form $Y_1 \cup_W Y_2$ where Y_1 is isomorphic to \mathbb{P}^2 , Y_2 is an elliptic ruled surface and W is a smooth supersingular elliptic curve
- (iv) Any nonclassical (K = 0) Enriques surface in $\mathbb{P}^2 \times \mathbb{P}^2$ that is 'bilinearly normal', i.e. does not lift to a higher dimensional $\mathbb{P}^r \times \mathbb{P}^s$, occurs as one of the zero-sets Y above.

The paper is organized as follows. In Part 1 we construct and study a bundle via a monad as above, and ultimately prove that it is the special case n = 2, p = q = 2, k = 1 of the bundle constructed in [14]. In part 2 we show that the zero scheme of a general section is a smooth nonclassical Enriques surface Y and give a criterion for detecting whether Y is singular or supersingular (both kinds exist). In §8 we construct the aforementioned reducible zero-set. In §9 we briefly discuss the relations between the moduli spaces of the bundles and the surfaces. Finally in §10 we review embeddings of Enriques surfaces into $\mathbb{P}^2 \times \mathbb{P}^2$ and show that any non-classical Enriques surface in $\mathbb{P}^2 \times \mathbb{P}^2$ with cycle class $4L^2 + 5Lh + 4h^2$ is the zero scheme of one of the bundles in our family.

PRELIMINARIES

0.1. **Projective planes.** Consider a copy of \mathbb{P}^2 with hyperplane class *L* and let $V_L = H^0(\mathcal{O}(L))^*$ so we have an exact sequence

$$0 \to \mathcal{O}(-L) \to V_L \to Q_L \to 0$$

where $Q_L = T_{\mathbb{P}^2}(-L) \cong \Omega^1_{\mathbb{P}^2}(2L)$ is the universal quotient bundle. Any nonzero section of Q_L vanishes at a unique point $p \in \mathbb{P}^2$ and yields an exact sequence

$$0 \to \mathcal{O} \to Q_L \to \mathcal{I}_v(L) \to 0.$$

Note that $\wedge^2 Q_L = \mathcal{O}(L)$ whence an isomorphism

$$egin{aligned} Q_L &\simeq \operatorname{Hom}(Q_L, \mathcal{O}(L)), \ v &\mapsto v^t \end{aligned}$$

and a skew-symmetric pairing

$$egin{aligned} Q_L imes Q_L o \mathcal{O}(L), \ (v,w) \mapsto \langle v,w
angle. \end{aligned}$$

Given $\phi \in H^0(Q_L)$ vanishing at p and $\psi \in H^0(Q_L) = H^0(\text{Hom}(Q_L, \mathcal{O}(L)))$, if ψ vanishes at $q \neq p$, then the composite $\langle v, w \rangle = \psi \circ \phi \in H^0(\mathcal{O}(L))$ vanishes on the line spanned by *p*, *q*, while if q = p, i.e. $\psi = \phi^t$, then the composite $\psi \circ \phi = 0$. For any nonzero section ϕ vanishing at *p*, the map $\psi \mapsto \psi \circ \phi$ yields a surjection $H^0(Q_L) \to H^0(\text{Hom}(Q_L, \mathcal{I}_p(L))) = H^0(\mathcal{I}_p(L))$ with kernel generated by ϕ^t .

Note under the above isomorphism the composition map

$$H^0(Q_L) \times H^0(\operatorname{Hom}(Q_L, \mathcal{O}(L)) \to H^0(\mathcal{O}(L)))$$

is compatible up to scalar multiples with wedge product

$$\wedge^2 V \to V^*$$

and in particular we have $\phi^t \circ \phi = 0$. For $v \in V$ we denote the 2-dimensional subspace $v \wedge V \subset V^*$ by U_v .

0.2. **Products and maps.** Now consider two copies of \mathbb{P}^2 , denoted \mathbb{P}^2_L , \mathbb{P}^2_h with respective line generators *L*, *h* and respective tangent bundles T_L , T_h , and quotient bundles $Q_L = T_L(-L)$, $Q_h = T_h(-h)$ as above. Setting $V_L = H^0(\mathcal{O}(L))^*$, $V_h = H^0(\mathcal{O}(h))^*$, we then have

$$H^0(Q_L \otimes Q_h) = V_L \otimes V_h$$

and this 9-dimensional vector space is endowed with an increasing filtration (by cones)

$$R_1 \subset R_2 \subset R_3 = V_L \otimes V_h$$

where R_i denotes the elements expressible as a sum of *i* decomposable elements. The component R_i can be identified as the locus of 3×3 matrices of rank *i* or less. Note that R_1 has dimension 3 + 3 - 1 = 5 while R_2 has dimension 8. A section in R_2 vanishes at a unique point $(p, p') \in \mathbb{P}^2_L \times \mathbb{P}^2_h$, while any section in $R_3 \setminus R_1$ is nowhere vanishing.

Now consider a rank-1 element $\phi = v \otimes w \in V_L \otimes V_h$. It is easy to check that the map $\phi^t : V_L \otimes V_h \to V_L^* \otimes V_h^* = \mathcal{O}(L+h)$ has kernel $v \otimes V_h + V_L \otimes w$ (a 5-dimensional subspace) and image the 4-dimensional subspace $U_v \otimes U_w z$, which coincides with the set of blinear forms vanishing on $Z_{v,w} := \langle v \rangle \times \mathbb{P}_h^2 \cup \mathbb{P}_L^2 \times \langle w \rangle$.

Next, for a rank-2 element $\phi_1 + \phi_2 = v_1 \otimes w_1 + v_2 \otimes w_2$, we have that

$$(v_1 \otimes V_h + V_L \otimes w_1) \cap (v_2 \otimes V_h + V_L \otimes w_2) = \mathfrak{k}v_1 \otimes w_2 + \mathfrak{k}v_2 \otimes w_1$$

is 2 dimensional while

$$U_{v_1} \otimes U_{w_1} \cap U_{v_2} \otimes U_{w_2} = \mathfrak{t} \langle v_1, v_2 \rangle \otimes \langle w_1, w_2 \rangle$$

is 1-dimensional, hence $U_{v_1} \otimes U_{w_1} + U_{v_2} \otimes U_{w_2}$ is 7-dimensional. Thus the image of $\phi_1^t + \phi_2^t$ is 7-dimensional, and consists exactly of the forms vanishing on the 2 points $Z_{v_1,w_2} \cap Z_{v_2,w_1} = (\langle v_1 \rangle, \langle w_2 \rangle) \cup (\langle v_2 \rangle, \langle w_1 \rangle)$.

Similarly, for a rank-3 element $\phi = \phi_1 + \phi_2 + \phi_3$, the image of ϕ^t is at least 8-dimensional, and since we know $\phi^t \circ \phi = 0$, it follows that ker(ϕ^t) =< ϕ > and moreover the image of ϕ^t is base-point free.

0.3. **Principal parts and smooth zero-sets.** For a bundle *E* on a smooth variety *X*, we denote by P(E) its bundle of 1-st order principal parts or jets, which fits in an exact sequence

$$0 \to \Omega_X \otimes E \to P(E) \to E \to 0.$$

This sequence admits a canonical additive splitting (and in particular induces a surjection $H^0(P(E)) \rightarrow H^0(E)$), where a section $\mathcal{O} \rightarrow E$ maps to the following section of P(E), called its canonical lift:

$$\mathcal{O}_X \to P(\mathcal{O}_X) = \mathcal{O}_X \oplus \Omega_X \to P(E)$$

where the left map is $f \mapsto (f, df)$.

In positive characteristic, a globally generated bundle need not have a smooth general zero-set (cf. [9], Prop. 1.4). However the following criterion in terms of principal parts holds:

Lemma 1. Let *E* be a vector bundle over a smooth variety *X* and $V \subset H^0(E)$ a finite-dimensional subspace whose canonical lift generates P(E). Then the zero-set of a general element of *V* is smooth.

Proof. Let us say that a section of P(E) is degenerate at a point $p \in X$ if the image in E vanishes at p and the induced element of $\Omega \otimes E$ at p is a degenerate tensor (corresponds to a matrix of non-maximal rank). If P(E) is generated by V, the set of sections in V degenerate at a given point $p \in X$ has codimension at least 1 + dim(X), hence the set of sections degenerate somewhere is a proper subset. Therefore the zero set of the image in E of a nowhere degenerate section is smooth.

Remark 2. The Lemma implies the usual Bertini Theorem that the general hyperplane section of a smooth projective variety *X* is smooth. This is because

$$P(\mathcal{O}_{\mathbb{P}^n}(1)) = H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}.$$

This sheaf is globally generated, hence so is its quotient $P_X(\mathcal{O}_X(1))$.

0.4. **Supersingular plane cubics.** Recall that an elliptic curve *C* over a field of positive characteristic is called *supersingular*, if the action of Frobenius on $H^1\mathcal{O}_C$ is zero. Supersingularity of elliptic curves can be detected from the equation in a plane embedding as a cubic curve. In the following we need an analogous result for arbitrary subschemes defined by a cubic equation. The proof from [8, IV 4.21] applies to this more general situation without modification and yields the next result.

Proposition 3. Let $X \subset \mathbb{P}^2 = Proj(k[x, y, z])$ be the subvariety defined by a homogeneous cubic equation f(x, y, z) = 0 in characteristic 2. Then the morphism $F^* \colon H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$ induced by Frobenius is 0 if and only if the coefficient of xyz in f is 0.

Note that degree 3 polynomials in characteristic 2 with vanishing *xyz*-term are the image of $(F^*H^0\mathcal{O}_{\mathbb{P}^2}(1)) \otimes H^0\mathcal{O}_{\mathbb{P}^2}(1)$, hence can be described without reference to a particular choice of coordinates.

Example 4. We can group the subvarieties defined by cubic equations as follows:

$F^* = 0$	$F^* = id_X$
supersingular elliptic curve	ordinary elliptic curve
cuspidal rational cubic	nodal rational cubic
conic and a line tangent to it	conic and a line intersecting transversely
three lines intersecting in a point	three lines with three points of intersection
a double line and another line	-
a triple line	

Part 1. The bundle

Here we work in char. 2 and undertake a detailed study of our bundle *E* on $\mathbb{P}^2 \times \mathbb{P}^2$. Notably, we will develop an equivalent construction of *E* as the cohomology of a very simple monad.

1. CONSTRUCTION VIA A MONAD

 \mathbb{P}_L^2 denotes a copy of \mathbb{P}^2 with *L* a line, $V_L = H^0(\mathcal{O}(L))^*$ and $Q_L = V_L \otimes \mathcal{O}/\mathcal{O}(-L)$ the universal quotient bundle. Ditto to \mathbb{P}_h^2 , *h*, *V_h*, *Q_h*.

1.1. Construction. We use notation as of §0.1. Set

$$V_{L,h} = V_L \otimes V_h = H^0(\operatorname{Hom}(Q_L \otimes Q_h, \mathcal{O}(L+h))),$$

a 9-dimensional vector space. We have a symmetric pairing

$$V_{L,h} \times V_{L,h} \to H^0(\mathcal{O}(L+h)) \otimes \det(V_{L,h}),$$

$$(v_1 \otimes w_1, v_2 \otimes w_2) \mapsto \langle v_1, v_2 \rangle \otimes \langle w_1, w_2 \rangle,$$

which corresponds up to scalars to

$$(\phi, \psi) \mapsto \psi \circ \phi.$$

This pairing can be viewed as a 9-dimensional system of symmetric bilinear forms whose corresponding system of quadratic forms is just the 9-dimensional system of quadrics on \mathbb{P}^8 cutting out the Segre image $\mathbb{P}^2_L \times \mathbb{P}^2_h$ (though this will not be important in the sequel). Note that

$$\langle v \otimes w, v \otimes w \rangle = 0$$

Therefore in char. 2, we have

$$\langle \phi, \phi^t \rangle = 0 \ \forall \phi \in V_{L,h}.$$

Moreover for general ϕ (specifically of rank 3), ϕ is fibrewise injective while ϕ^t is fibrewise surjective. Hence such a ϕ defines a monad

(1)
$$\mathcal{O} \xrightarrow{\phi} Q_L \otimes Q_h \xrightarrow{\phi^t} \mathcal{O}(L+h).$$

We call such a monad *symmetric*. More generally any pair (ϕ , ψ) of rank 3 such that (ϕ , ψ) = 0 defines a monad

(2)
$$\mathcal{O} \xrightarrow{\phi} Q_L \otimes Q_h \xrightarrow{\psi} \mathcal{O}(L+h)$$

Then

$$E_0 = \ker(\psi) / \operatorname{im}(\phi)$$

is a rank-2 vector bundle on $\mathbb{P}^2_L \times \mathbb{P}^2_h$. As we show in §2, ker $(H^0(\phi^t)) = \operatorname{im}(H^0(\phi))$, and it follows that

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(3)
$$h^0(E_0) = 0.$$

and likewise

(4)
$$h^1(E_0) = 1.$$

A straightforward calculation shows that E_0 has Chern class

(5)
$$c(E_0) = 1 + L + h + 2L^2 + 2h^2 + Lh.$$

The display of the monad (2) is as follows:

1.2. A vanishing result. Now I claim that for E_0 given by the monad (2), we have $H^1(E_0(h)) = 0$. More complete cohomological results are given in the next section. Note that the claimed vanishing is equivalent to $H^0(E_0(h)) = 3$ Via the above monad, the claim is equivalent to surjectivity of the map induced by ψ

$$H^0(Q_L \otimes T_h) \to H^0(\mathcal{O}(L+2h)),$$

We will prove the surjectivity under the assumption that ψ corresponds to a rank-2 tensor

$$v_{p_1} \otimes v_{q_1} + v_{p_2} \otimes v_{q_2}, p_1 \neq p_2 \in \mathbb{P}^2_L, q_1 \neq q_2 \in P^2_h.$$

This will imply the result for a rank-3 tensor as well, which is the case we need. Then for $q \in P_L^2, q \in P_h^2, g \in H^0(\mathcal{O}(h))$, we have

$$\psi(v_p \otimes gv_q) = \langle v_{p_1}, v_p \rangle \otimes \langle v_{q_1} \otimes gv_q \rangle + \langle v_{p_2}, v_p \rangle \otimes \langle v_{q_2} \otimes gv_q \rangle$$

Now taking $p = p_2$ (resp. $p = p_1$) the images generate $\langle p_1, p_2 \rangle \otimes H^0(\mathcal{O}(2h - q_1))$ (resp. $\langle p_1, p_2 \rangle \otimes H^0(\mathcal{O}(2h - q_2))$). and together these generate $\langle p_1, p_2 \rangle \otimes H^0(\mathcal{O}(2h))$. Similarly, taking $q = Q_1, q = q_2$ the images generate $H^0(\mathcal{O}(L)) \otimes H^0(\mathcal{O}(2h - \langle q_1, q_2 \rangle))$.

2. COHOMOLOGY VIA MONAD

This part begins our analysis of the bundle *E*. Key results are:

- (i) determination of the cohomology groups $h^i E(aL + bh)$ for $a, b \ge -5$ (Theorem 6)
- (ii) there is a unique jumping line of *E* in each fibre $x \times \mathbb{P}^2$, and the union of these lines forms a divisor $A \in |L + h|$ (Corollary 11)

Remark 5. If the bundle *E* is defined by $\phi = \sum v_i \otimes w_i$, then *A* has the equation $\sum v_i w_i = 0$.

In this section we set O(aL + bh) = O(a, b) and let *E* denote $E_0(L + h)$ where E_0 is given by the monad (1). We shall determine the cohomology of E(a, b) in the range $a \ge -5$, $b \ge -5$. For $-5 \le a, b \le 3$ the result can be seen in the tables below. There is no table for $h^4E(a, b)$, because these groups are 0 in the whole range, and blanks correspond to zeros.

Theorem 6. (*i*) The higher cohomology groups of E(a,b) vanish for $a \ge -1, b \ge 0$ and for $a \ge 0, b \ge -1$, hence

$$h^{0}E(a,b) = \chi E(a,b) = \frac{a'b'(a'b'-1)}{2} - a'^{2} - b'^{2} + 1$$
 (where $a' = a + 3, b' = b + 3$)

as long as $a, b \ge -1$ *and* $(a, b) \ne (-1, -1)$ *.*

(ii) The dimension of other cohomology groups is as follows.

 $h^0 E(a,b) h^1 E(a,b)$

					b								1				b				
		-5	-4	-3	-2	-1	0	1	2	3			-5	-4	-3	-2	-1	0	1	2	3
	-5											-5									
	-4											-4					1	3	6	10	15
	-3											-3					3	8	15	24	35
	-2											-2				1	3	6	10	15	21
	-1						3	9	17	27	l	ı -1		1	3	3	1				
а	0					3	19	42	72	109		0		3	8	6					
	1					9	42	89	150	225		1		6	15	10					
	2					17	72	150	251	375		2		10	24	15					
	3					27	109	225	375	559		3		15	35	21					
	4					39	153	314	522	777		4		21	48	28					

h^2E	(a,	b)
	··· /	- /

 $h^3E(a,b)$



Proof. Our general strategy is to apply the Leray spectral sequence with respect to the projection to one of the factors. It turns out that only one of the direct image sheaves is nonzero for each twist. The cohomology of E(a, b) for $-5 \le a \le -1$ can be read off from the results in Proposition 10 below.

Since $h^1 E(0, -1) = h^2 E(0, -2) = h^3 E(0, -3) = h^4 E(0, -4) = 0$, we conclude that $h^1 E(0, l) = 0$ for any $l \ge 0$, and inductively that $h^i E(a, b) = 0$ for any $i > 0, a, b \ge 0$.

We now proceed to outline the details of the calculation. First we recall the following well-known result.

Lemma 7. Let *E* be a vector bundle on \mathbb{P}^2 , and assume that $h^1E(-1) = 1$ and $h^2E(-2) = 0$. Then $h^1E = 0$.

Proof. Let *M* be an arbitrary line. *E* splits on *M* as $\oplus \mathcal{O}_M(a_i)$, and the assumptions imply that $h^1(E(-1) \otimes \mathcal{O}_M) \leq 1$. Thus $a_i \geq -1$ for all *i* and $h^1(E \otimes \mathcal{O}_M) = 0$.

Now assume that $h^1 E > 0$. Since $h^1(E \otimes \mathcal{O}_M) = 0$, the multiplication by the linear form corresponding to M in $H^1 E(-1) \rightarrow H^1 E$ must be surjective, hence $h^1 E = 1$, and the multiplication map is even bijective. On the other hand the multiplication map

$$H^1E(-1) \times H^0\mathcal{O}_{\mathbb{P}^2}(1) \to H^1E$$

cannot be injective, as the source has dimension 3. Therefore there must be a linear form corresponding to a line *M* so that the induced multiplication is zero.

We have arrivd at a contradiction, and therefore we must have $h^1 E = 0$.

Proposition 8. Let E_x be the restriction of E to a fiber $\mathbb{P}^2_L \times \{x\}$.

1. The cohomology $h^i E_x(l)$ in the range $-5 \le l \le -1$ is as follows:

2. The cokernel of the map $H^0E_x(-1)\otimes \mathcal{O}_{\mathbb{P}^2} \to E_x(-1)$ is isomorphic to $\mathcal{O}_M(-1)$ for a line M.

3. $E_x(-1)$ splits generically as $\mathcal{O} \oplus \mathcal{O}(1)$. *M* is its only jumping line, and $E_x(-1) \otimes \mathcal{O}_M \cong \mathcal{O}_M(2) \oplus \mathcal{O}_M(-1)$.

Proof. 1. Recall that the bundle *E* is given by the monad

$$0 \to \mathcal{O}(1,1) \to Q_L(L) \otimes Q_h(h) \to \mathcal{O}(2,2) \to 0$$

Hence the restriction of *E* to a fiber $\mathbb{P}^2 \times \{x\}$ is given by the restricted monad

$$0
ightarrow \mathcal{O}(1)
ightarrow 2Q(1)
ightarrow \mathcal{O}(2)
ightarrow 0$$

We note that $c_1(E_x) = 3$, $c_2(E_x) = 4$ and $\chi(E_x(l)) = (l+4)(l+2) - 1$.

The cohomology for $E_x(-3)$ and $E_x(-2)$ can be read off from the display of the monad via a diagram chase.

Lemma 7 implies that $h^{1}E(-1) = 0$, hence $h^{0}E_{x}(-1) = \chi E_{x}(-1) = 2$.

The cohomology for $E_x(-4)$ and $E_x(-5)$ follows by Serre duality.

2. The cokernel of the map $H^0E_x(-1) \otimes \mathcal{O}_{\mathbb{P}^2} \to E_x(-1)$ is a sheaf of projective dimension 1 supported on a divisor of degree $c_1E_x(-1) = 1$, hence it is locally free on a line M. Its Hilbert polynomial is $\chi E_x(l-1) - \chi(2\mathcal{O}(l)) = l$, hence it has rank 1 and is isomorphic to $\mathcal{O}_M(-1)$.

3. Restricting the sequence $0 \to 2\mathcal{O} \to E_x(-1) \to \mathcal{O}_M(-1)$ to M, one finds an exact sequence $0 \to \mathcal{O}_M(2) \to E_x(-1) \otimes \mathcal{O}_M \to \mathcal{O}_M(-1)$ which necessarily splits.

On any other line the restriction of $E_x(-1)$ is globally generated, hence the splitting type must be $\mathcal{O} \oplus \mathcal{O}(1)$.

Remark 9. Hulek [10] investigated stable vector bundles \mathcal{F} of rank 2 with odd first Chern class in characteristic 0. He normalizes them so that $c_1 = -1$ and studies "jumping lines of the second kind", i.e., lines M with the property that $H^0(M^2, \mathcal{F} \otimes \mathcal{O}_{M^2}) \neq 0$, where M^2 is the double structure on M induced by its embedding in \mathbb{P}^2 .

Hulek defines a sheaf supported on a divisor $C(\mathcal{F})$ of degree $2c_2(\mathcal{F}) - 2$ in $(\mathbb{P}^2)^{\vee}$ and proves that this sheaf characterizes \mathcal{F} .

He further constructs moduli spaces $M(-1, c_2)$.

Regarding our case $c_2 = 2$, he shows that:

- (i) $C(\mathcal{F})$ consists of two distinct lines. Their point of intersection corresponds to the unique jumping line of \mathcal{F} .
- (ii) All bundles with $c_2 = 2$ are projectively equivalent under automorphisms of \mathbb{P}^2 .
- (iii) The moduli space M(-1,2) is isomorphic to the quotient of $(\mathbb{P}^2)^{\vee} \times (\mathbb{P}^2)^{\vee} \Delta$ by the natural symmetry action that exchanges the two factors.

These results do not extend to our example in characteristic 2: It is not hard to see (using the alternative construction of *E* from the following section) that all lines are jumping lines of the second kind for E_x , hence the support of $C(E_x)$ is all of $(\mathbb{P}^2)^{\vee}$.

Proposition 10. The direct image sheaves $R^i pr_{2,*}E(l,0)$ in the range $-5 \le l \le -1$ are as follows:

where $R = pr_{2,*}E(-1, -1)$ is a rank 2 vector bundle with Chern class $c(R) = 1 + 3h^2$, Euler characteristic $\chi R(l) = l^2 + 3l - 1$ and the following cohomology $h^i R(l)$ in the range $-4 \le l \le 1$



Proof. The calculation of the direct images follows the same approach as for the fibers in the previous Proposition: The direct images for l = -3 and l = -2 can be derived from the display of the monad via a diagram chase. For l = -1 we use from Proposition 8 that the higher direct images vanish. The columns for l = -5 and l = -4 follow by relative duality for the projection.

The direct image of the monad for E(-1, -1) produces the monad

$$0 \rightarrow \mathcal{O} \rightarrow 3Q \rightarrow 3\mathcal{O}(1) \rightarrow 0$$

for *R*, from which we can compute its Chern class. The cohomology for l = -1 and l = 0 can be determined most easily from the display of the dual monad.

Lemma 7 implies that $h^1R(1) = 0$, hence $h^0R(1) = \chi R(1) = 3$, and the other columns follow by Serre duality.

Corollary 11. *There is a short exact sequence*

$$0 \to pr_2^* R \to E(-L-h) \to \mathcal{O}_{\mathcal{A}}(-L+2h) \to 0.$$

where A is a divisor in |L + h|.

Proof. The cokernel of the map

$$pr_2^*pr_{2,*}(E(-1,-1))\otimes \mathcal{O}_{\mathbb{P}^2} \to E(-1,-1)$$

is a sheaf of projective dimension 1 supported on a divisor in the class

$$c_1E(-1,-1) - c_1(pr_2^*pr_{2,*}E(-1,-1)) = L + h,$$

hence it is locally free on a divisor A in |L + h|.

Our analysis of the restriction to the fibres implies that the cokernel has rank 1 and is isomorphic to $\mathcal{O}_{\mathcal{A}}(-L+ah)$ for some integer *a*.

Finally, a short calculation yields a = 2.

3. FROM MONAD TO ELEMENTARY MODIFICATION

In this section, we consider a bundle *E* defined via a self-dual monad

$$0 \to \mathcal{O}(L+h) \xrightarrow{\phi} Q_L \otimes Q_h \xrightarrow{\phi^*} \mathcal{O}(2L+2h) \to 0.$$

We show (Theorem 13) that E(-2L - h) can be represented as an elementary modification, namely as the kernel of the composition

$$pr_2^*F^*Q \to \pi_2^*F^*Q \stackrel{F^*\psi}{\to} \mathcal{O}_{\mathcal{A}}(2L)$$

where $\mathcal{A} \cong \mathbb{P}(Q)$ is a smooth divisor in |L + h|, $\pi_2 \colon \mathcal{A} \to \mathbb{P}^2$ the restriction of the canonical projection pr_2 , and ψ is the canonical map $\pi_2^*Q \to \mathcal{O}_{\mathcal{A}}(L)$ of sheaves on \mathcal{A} .

This representation has several important applications:

- (i) a second monad (Corollary 14),
- (ii) the symmetry of \mathcal{A} (Corollary 15),
- (iii) and the splitting of *E* on \mathcal{A} (Corollary 16).

These are all specializations of properties of the general construction in [14, §2].

Later in §4 we will show show that the representation via the elementary modification induces the monad that we started with.

We assume E_0 is the cohomology of a monad (1).

Recall the exact sequence

(7)
$$0 \to p_2^*(R) \to E_0 \to \mathcal{O}_{\mathcal{A}}(-L+2h) \to 0$$

from corollary 11.

Proposition 12. *A* is isomorphic to the \mathbb{P}^1 -bundle $\mathbb{P}(Q)$ in its standard embedding given by the global sections $3\mathcal{O} \to Q$.

Proof. We know from our investigation of the cohomology of *E* that A is a divisor in |L + h|, fibered in lines over the second factor of \mathbb{P}^2 . These lines represent the jumping lines of the vector bundle *E* in the fibers $\mathbb{P}^2 \times \{x\}$.

Applying $pr_{2,*}$ to the sequence $0 \to \mathcal{O}(-h) \to \mathcal{O}(L) \to \mathcal{O}_{\mathcal{A}}(L) \to 0$ we find a sequence $0 \to \mathcal{O}(-1) \to 3\mathcal{O} \to pr_{2,*}\mathcal{O}_{\mathcal{A}}(L) \to 0$ on \mathbb{P}^2 where the last term is a rank 2 locally free sheaf on \mathbb{P}^2 . However, there is only one sheaf with such a resolution, the twisted tangent bundle Q, hence $\mathcal{A} \cong \mathbb{P}(Q)$, and $\mathcal{O}_{\mathcal{A}}(L)$ is isomorphic to the twisting sheaf $\mathcal{O}_{\mathbb{P}(Q)}(1)$. \Box

Theorem 13. E(-2L - h) is isomorphic to the kernel of the composition

 $pr_2^*F^*Q \to \pi_2^*F^*Q \stackrel{F^*\psi}{\to} \mathcal{O}_{\mathcal{A}}(2L)$

where ψ is the canonical map $\pi_2^* Q \to \mathcal{O}_{\mathcal{A}}(L)$ of sheaves on \mathcal{A} .

Proof. Recall from §2 that we have a short exact sequence

$$0 \to pr_2^*R \to E(-L-h) \to \mathcal{O}_{\mathcal{A}}(-L+2h) \to 0$$

where $R = pr_{2,*}E(-L - h)$.

Its twisted dual (by $\mathcal{O}(h)$) is a sequence

(8)
$$0 \to E(-2L-h) \to (pr_2^*R)(h) \to \mathcal{O}_{\mathcal{A}}(2L) \to 0$$

Our plan is to identify the two sheaves on the right and the map between them.

Restricting (8) to \mathcal{A} , we obtain $0 \to \mathcal{O}_{\mathcal{A}}(-2L+2h) \to \pi_2^*R(1) \to \mathcal{O}_{\mathcal{A}}(2L) \to 0$, hence applying $\pi_{2,*}$ yields an exact sequence

(9)
$$0 \to R(1) \to S^2 Q \to \mathcal{O}_{\mathbb{P}^2}(1) \to 0.$$

The morphism $S^2Q \rightarrow O(1)$ corresponds to a section of the twisted divided power $(D^2Q)(-1)$. By Example [14, Example 1], this section is essentially unique, and given by the inclusion of the (twisted) exterior power.

As a consequence, the sequence (9) agrees with the sequence in Lemma [14, Lemma 1], and $R(1) \cong F^*Q$, hence $pr_2^*(R(1)) \cong pr_2^*F^*Q$.

Finally, consider the diagram

$$0 \longrightarrow \mathcal{O}_{\mathcal{A}}(-2L+2h) \longrightarrow F^*\pi_2^*Q \longrightarrow \mathcal{O}_{\mathcal{A}}(2L) \longrightarrow 0$$

$$\downarrow F^*\psi$$

$$F^*\mathcal{O}_{\mathcal{A}}(L)$$

where ψ is the Frobenius pullback of the canonical map $\pi_2^* Q_h \to \mathcal{O}_A(L)$.

Since the dashed diagonal map corresponds to a section of $\mathcal{O}_{\mathcal{A}}(4L-2h)$, hence must be 0, $F^*\psi$ factors through the map $F^*\pi_2^*Q \to \mathcal{O}_{\mathcal{A}}(2L)$, and for degree reasons the resulting map $\mathcal{O}_{\mathcal{A}}(2L) \to F^*\mathcal{O}_{\mathcal{A}}(L)$ is an isomorphism.

We conclude that the map $\pi_2^* R(1) \to \mathcal{O}_{\mathcal{A}}(2L)$ in the restriction of (8) to \mathcal{A} is isomorphic to the Frobenius pullback $\pi_2^* F^* Q \to F^* \mathcal{O}_{\mathcal{A}}(L)$, as required to finish the proof.

For the next result, we choose coordinates so that $\mathbb{P}^2 \times \mathbb{P}^2 = \operatorname{Proj}[a, b, c, x, y, z]$, and the equation of \mathcal{A} is given by f = ax + by + cz.

Corollary 14. E(-2L-h) is isomorphic to the homology of the monad

$$0 \to \mathcal{O}(-2h) \xrightarrow{\phi_2} \mathcal{O}^{\oplus 3} \oplus \mathcal{O}(L-h) \xrightarrow{\phi_1} \mathcal{O}(2L) \to 0$$

where the maps are given by $\phi_2 = \begin{pmatrix} x^2 & y^2 & z^2 & f \end{pmatrix}^t$ and $\phi_1 = \begin{pmatrix} a^2 & b^2 & c^2 & f \end{pmatrix}$

Proof. By Theorem 13, E(-2L - h) can be represented as the kernel of the right column in the following diagram:

The mapping cone of the two rows, with the right column excluded, provides the required complex.

As a consequence of the above monad, we get a symmetry result (for another proof see [14], §1):

Corollary 15. Assume *E* is given by a symmetric monad (1) . E(-L-2h) is isomorphic to the kernel of the composition

$$pr_1^*F^*Q \to \pi_1^*F^*Q \stackrel{F^*\psi'}{\to} \mathcal{O}_{\mathcal{A}}(2h)$$

where ψ' is the canonical map $\pi_1^* Q \to \mathcal{O}_{\mathcal{A}}(h)$.

In particular, \mathcal{A} is also the locus of the jumping lines of the restriction of E to the fibers $\{x\} \times \mathbb{P}^2$.

Proof. The dual of the monad in Corollary 14 is a monad for E(-L-2h). Unwinding it we obtain the diagram

$$\begin{array}{cccc} 0 & \longrightarrow & \mathcal{O}(-2L) & \xrightarrow{\begin{pmatrix} a^2 \\ b^2 \\ c^2 \end{pmatrix}} & \mathcal{O}^{\oplus 3} & \longrightarrow & pr_1^*F^*Q & \longrightarrow & 0 \\ & & & \downarrow^{(x^2 \ y^2 \ z^2)} & \downarrow & \\ 0 & \longrightarrow & \mathcal{O}(h-L) & \xrightarrow{f} & \mathcal{O}(2h) & \longrightarrow & \mathcal{O}_{\mathcal{A}}(2h) & \longrightarrow & 0 \end{array}$$

where E(-L - 2h) is the kernel of the right column. The left column establishes the first statement of the corollary.

The second statement follows by restricting the right column to the fibers of pr_2 .

Corollary 16. The restriction of *E* to *A* splits as $\mathcal{O}_{\mathcal{A}}(3L) \oplus \mathcal{O}_{\mathcal{A}}(3h)$, and the subbundle $\mathcal{O}_{\mathcal{A}}(3L)$ (resp. $\mathcal{O}_{\mathcal{A}}(3h)$) is uniquely determined as the image of the map $(pr_1^*F^*Q)(L) \to E$ (resp. $(pr_2^*F^*Q)(h) \to E$). In particular, the canonical map

$$(pr_1^*F^*Q)(h) \oplus (pr_2^*F^*Q)(L) \to E$$

is surjective.

Proof. We restrict the exact sequence from Corollary 11

$$0 \to (pr_2^*F^*Q)(-h) \to E(-L-h) \to \mathcal{O}_{\mathcal{A}}(-L+2h) \to 0.$$

to \mathcal{A} and obtain

$$0 \to \mathcal{O}_{\mathcal{A}}(-2L+h) \to (\pi_2^* F^* Q)(-h) \to E \otimes \mathcal{O}_{\mathcal{A}}(-L-h) \to \mathcal{O}_{\mathcal{A}}(-L+2h) \to 0.$$

Twisting by $\mathcal{O}(L+h)$ we find an epimorphism $E \otimes \mathcal{O}_{\mathcal{A}} \to \mathcal{O}_{\mathcal{A}}(3h)$ with kernel $\mathcal{O}_{\mathcal{A}}(3L)$ which is the image of $(\pi_2^* F^* Q)(L)$.

By symmetry (Corollary 15) we obtain another epimorphism $E \otimes \mathcal{O}_A \to \mathcal{O}_A(3L)$ with kernel $\mathcal{O}_A(3h)$ which is the image of $(\pi_1^*F^*Q)(h)$.

The injection $\mathcal{O}_{\mathcal{A}}(3h) \to E \otimes \mathcal{O}_{\mathcal{A}}$ now provides the required splitting of the surjection $E \otimes \mathcal{O}_{\mathcal{A}} \to \mathcal{O}_{\mathcal{A}}(3h)$.

The remaining statements of the corollary follow immediately.

4. FROM ELEMENTARY MODIFICATION TO MONAD

4.1. **Cohomology.** Here we assume E_0 is given by a (dual) elementary modification

(10)
$$0 \to (p_2^*(F^*Q_h))(-h) \to E_0 \to \mathcal{O}_{\mathcal{A}}(-L+2h) \to 0$$

where \mathcal{A} is a divisor of class L + h which is a \mathbb{P}^1 -bundle over \mathbb{P}^2_h of the form $\mathbb{P}(Q_h)$.

Twisting (10) by L + h and using $H^1(R(h)) = 0$, we recover the fact that E is globally generated, though E(-L) and E(-h) are not. Also this vanishing plus the vanishing of $H^1(\mathcal{O}_A(3h))$ (obvious) imply again that $H^1(E) = 0$, hence $h^0(E) = 19$. The same sequence also implies easily that

(11)
$$H^{i}(E(aL+bh)) = 0, \forall i \geq 1, a, b \geq 0.$$

Note that it follows from the results of the previous section that the fibres of \mathcal{A} over the two factors are precisely the 'jumping rulings' of E: i.e. that $E_{l\times x} \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)$ when $\ell \times x$ is not a ruling of \mathcal{A} over \mathbb{P}^2_h and $\mathcal{O}(3) \oplus \mathcal{O}$ otherwise; and ditto with fibres interchanged.

4.2. The monad. Here the characteristic is 2 and $R = (F^*Q_h)(-h)$. Finally we claim that the exact sequence (10) implies that *E* is the cohomology of a monad (1), extending the fibrewise result in the previous section. To see this we use the projection pr_2 and the corresponding fibrewise Beilinson monad, which shows that *E* is the abutment of a spectral sequence with E_1 term

$$E_1^{p,q} = R^q p_{2*}(E((-3+p)L)) \otimes \wedge^{2-p}(Q_L(-L)).$$

Now the vanishing of the higher direct images for $q \neq 1$ follows from the corresponding fibrewise result, and the evaluation of the images for q = 1 is done similarly, using that $\mathcal{A} = \mathbb{P}(Q_h)$.

Part 2. Zero sets: Enriques surfaces

The next four sections investigate the zero sets of sections of *E*. Key results are:

- (i) The general zero set is nonsingular (Corollary 21)
- (ii) the quantitative invariants of smooth zero sets, specifying them as nonclassical Enriques surfaces (Theorem 22)
- (iii) an analysis of certain reducible zero sets (§ 8)
- (iv) a criterion to distinguish F-singular and supersingular (smooth) zero sets (Theorem 28)

Section 9 discusses moduli of *E* and relates them to moduli of its zero sets, non-classical Enriques surfaces.

The final section 10 shows how to recover the monad from an embedded nonclassical Enriques surface of bidegree (4, 4).

5. GEOMETRY OF ZERO SETS

5.1. **Minimal zero sets.** By the results of §2, E(-L) and E(-h) are minimal twists with nonzero sections. They satisfy $h^0 = 3$, $h^1 = 0$ and every zero-scheme is two-dimensional. The analysis of their zero-schemes is the same in both cases, and we will discuss below only E(-L).

If $s \in H^0E(-L)$ has zero scheme *Y*, then *Y* has class

$$[Y] = 2L^2 + 2Lh + 4h^2, \quad \omega_Y = \mathcal{O}_Y(-2L)$$

Proposition 17. Let $s \in H^0E(-L)$ be a section with zero scheme Y. Then $Y = Y_1 \cup Y_2$ where:

- (1) $Y_1 = 2Z_1 \subset \mathcal{A}$ is a double structure on $Z_1 = \pi_2^*(L_1)$ for some line $L \subset \mathbb{P}^2_L$.
- (2) $Y_2 = \mathbb{P}^2 \times N_2(p)$ is a multiplicity-4 structure where $N_2(p)$ is the infinitesimal neighborhood of a point $p \in \mathbb{P}^2$ of length 4 with ideal (x^2, y^2) (where p is defined by the vanishing of two local coordinates x, y).

Proof. First of all, the map $F^*Q \to E(-L)$ induces an isomorphism on sections, and moreover, we have $E(-L) \otimes \mathcal{O}_{\mathcal{A}} \cong \mathcal{O}(2L) \oplus \mathcal{O}(3h - L)$. Hence the sections of F^*Q restrict on \mathcal{A} to the squares in $H^0\mathcal{O}_{\mathcal{A}}(2L)$. This means that Y must intersect \mathcal{A} in a surface of the form $2Z_1$ where $Z = pr_2^{-1}(L_1)$ for some line $L \subset \mathbb{P}^2_L$. Note that Z_1 is isomorphic to $\mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O})$, a rational ruled surface of type F_1 , and moreover

$$Y = 2Z_1 + Y_2$$

where $|Y_2| = 4h^2 = 4[\mathbb{P}_L^2 \times \{x\}]$. In fact, in view of the exact sequence

$$0 \to F^*Q_h \to E(-L) \to \mathcal{O}_{\mathcal{A}}(3h-L),$$

 Y_2 is also a zero-set of F^*Q , and we conclude that Y_2 is a multiplicity-4 structure of the form $\mathbb{P}^2_L \times N_2(p)$ where $N_2(p)$ is an infinitesimal neighborhood of a point $p \in \mathbb{P}^2_h$ with ideal of the form (x^2, y^2) .

Next an easy consequence of the minimality of E(-L), E(-h):

Corollary 18. *The map of sections*

$$\psi \colon \left(H^0 E(-L) \otimes H^0 \mathcal{O}(L) \right) \oplus \left(H^0 E(-h) \otimes H^0 \mathcal{O}(h) \right) \to H^0 E$$

is injective, and its image is 18-dimensional.

Proof. In principle the statement can be read off from the computer calculations of Corollary 45 in the appendix, but we would like to give an independent proof.

Since $H^0E(-L-h) = 0$, the restriction of sections $H^0E \to H^0(E \otimes \mathcal{O}_A)$ is injective, and it suffices to validate the statement on A.

By Corollary 16, sections of $E \otimes \mathcal{O}_{\mathcal{A}}$ correspond to sections of $\mathcal{O}_{\mathbb{P}^2}(3L) \oplus \mathcal{O}_{\mathbb{P}^2}(3h)$. Proposition 17 implies that the sections of E(-L) correspond to the squares of linear polynomials in $H^0\mathcal{O}(2L)$, hence the sections in $H^0E(-L) \otimes H^0\mathcal{O}(L)$ map to cubic polynomials in $H^0\mathcal{O}(3L)$ without an *abc*-term. Similarly, sections of $H^0E(-h) \otimes H^0\mathcal{O}(h)$ map to cubic polynomials in $H^0\mathcal{O}(3h)$ without an *xyz*-term.

Injectivity of ψ is now clear, and the dimension of the image can be read off from the known dimensions of the terms on the left.

Remark 19. We have inclusions

$$\operatorname{im}(\psi) \subset H^0(E) \subset H^0(E \otimes \mathcal{O}_{\mathcal{A}}) \cong H^0\mathcal{O}(3L) \oplus H^0\mathcal{O}(3h)$$

of codimension 1 vector spaces. Let $(s_L, s_h) \in H^0\mathcal{O}(3L) \oplus H^0\mathcal{O}(3h)$ be a section.

The space on the left includes all sections s_L , s_h where the corresponding polynomials have no terms *abc*, *xyz*.

We will see later (see Corollary 45) that a section $(s_L, s_h) \in H^0E$ but not in the image of ψ must have non-vanishing *abc*- and *xyz*-terms.

5.2. **Sections of** *E***.** We start our investigation of the sections by considering a subset of them, namely those in the image of

$$\psi \colon \left(H^0 E(-L) \otimes H^0 \mathcal{O}(L) \right) \oplus \left(H^0 E(-h) \otimes H^0 \mathcal{O}(h) \right) \to H^0 E$$

Theorem 20. The general section in the image of ψ has a smooth zero scheme.

Proof. This is a local question, and we will treat the cases $x \in A$ and $x \notin A$ separately.

Case 1 ($x \in A$): On A, E splits as $\mathcal{O}_A(3L) \oplus \mathcal{O}_A(3h)$, and the sections in the image of ψ correspond to those in the direct sum of $F^*H^0\mathcal{O}(L) \otimes H^0\mathcal{O}(L)$ and $F^*H^0\mathcal{O}(h) \otimes H^0\mathcal{O}(h)$.

Regarding the first summand, we know that the linear system corresponding to $F^*H^0\mathcal{O}(L)$ is basepointfree, while $H^0\mathcal{O}(L)$ is very ample, hence by [11, 3.5] the general section in $F^*H^0\mathcal{O}(L) \otimes H^0\mathcal{O}(L)$ is nonsingular on \mathbb{P}^2 . A similar argument holds for the second summand, and taken together they imply that the general section of *E* is nonsingular on \mathcal{A} .

Case 2 ($x \notin A$): We claim that the stalks of P(E) outside A are generated by pull-back of sections of R_L (=_{def} $pr_{1,*}E_0$) and R_h (=_{def} $pr_{2,*}E_0$) via

$$pr_1^*P(R_L) \oplus pr_2^*P(R_h) \to P(pr_1^*R_L) \oplus P(pr_2^*R_h) \to P(E).$$

If this holds, then we can apply Lemma 1 to conclude that the general section of *E* will be nonsingular outside A.

Regarding the claim, recall the following well-known facts:

- (1) The injections $pr_1^*R_L \to E$ (resp. $pr_2^*R_h \to E$) are bijective outside \mathcal{A} .
- (2) There is a canonical short exact sequence

$$0 \to E \otimes \Omega^1_{\mathbb{P}^2 \times \mathbb{P}^2} \to P(E) \to E \to 0$$

that is split as a sequence of sheaves of abelian groups.

(3) the cotangent bundle of $\mathbb{P}^2 \times \mathbb{P}^2$ splits as $\Omega^1_{\mathbb{P}^2 \times \mathbb{P}^2} \cong pr_1^* \Omega^1_{\mathbb{P}^2} \oplus pr_2^* \Omega^1_{\mathbb{P}^2}$. Taken together, they imply that

$$P(E) \cong (E \otimes pr_1^* \Omega_{\mathbb{P}^2}^1) \oplus (E \otimes pr_2^* \Omega_{\mathbb{P}^2}^1) \oplus E$$

as sheaves of abelian groups, and this decomposition for P(E) holds outside A also, if E is replaced by $pr_1^*R_L$ or by $pr_2^*R_h$ on the right side.

The bundle F^*Q is globally generated on \mathbb{P}^2 , and $\mathcal{O}_{\mathbb{P}^2}(1)$ is very ample. Therefore the canonical lifts (see Lemma 3) of the sections of $R_L = F^*Q \otimes \mathcal{O}(1)$ generate the stalks of the bundle $P(R_L) \cong (R_L \otimes \Omega^1_{\mathbb{P}^2}) \oplus R_L$ (split as sheaves of abelian groups) of principal parts of R_L on \mathbb{P}^2 .

Hence the pullback of sections under the map

$$pr_1^*P(R_L) \rightarrow P(pr_1^*R_L) \rightarrow pr_1^*(R_L \otimes \Omega_{\mathbb{P}^2}^1) \oplus pr_1^*(R_L)$$

generates the stalks of the sheaf of the right outside A, and the same holds for the analogous pullback of R_h from the second factor.

Taken together, these establish our claim.

Corollary 21. *The zero scheme of a general section of E is smooth.*

Proof. Nonsingularity of a section is a generic property. We proved in Theorem 20 that there are sections with a smooth zero scheme in the image of ψ . Hence the same holds for the general section, and the general section is not in the image of ψ .

6. ENRIQUES SURFACES

6.1. Enriques surface basics. General reference for Enriques surfaces: [6].

Within the Enriques-Castelnuovo classification of complex algebraic surfaces, Enriques surfaces are part of the surfaces of Kodaira dimension 0 (equivalently, with trivial or torsion canonical bundle), alongside abelian, K3 and hyperelliptic (sometimes also called bielliptic) surfaces. They are characterized by $q = p_g = 0$ and $K^{\otimes 2} = O$.

The fundamental group of a complex Enriques surface is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and the universal double cover is a K3 surface. (see [2, VIII 15]).

When Bombieri and Mumford extended the classification of algebraic surfaces to characteristic p > 0, they discovered new phenomena in characteristic 2. They redefined Enriques surfaces as surfaces X with numerically trivial canonical bundle and second Betti number $B_2 = 10$. They found that Enriques surfaces in characteristic 2 may be divided in three classes as follows:

- (i) *classical* Enriques surfaces satisfy $h^1 \mathcal{O}_X = 0$, hence $K_X \not\cong \mathcal{O}_X$, but $K_X^{\otimes 2} \cong \mathcal{O}_X$,
- (ii) *F-singular* Enriques surfaces (or μ_2 -surfaces) with $h^1 \mathcal{O}_X = 1$, $K_X \cong \mathcal{O}_X$, and Frobenius acts bijectively on $H^1 \mathcal{O}_X$,

(iii) supersingular Enriques surfaces (or α_2 -surfaces) with $h^1 \mathcal{O}_X = 1$, $K_X \cong \mathcal{O}_X$, and Frobenius on $H^1 \mathcal{O}_X$ is zero.

(Their original terminology for F-singular was 'singular'; we have chosen F-singular to avoid confusion).

Enriques surfaces in positive characteristic continue to have a special double cover. For classical Enriques surfaces in characteristic $\neq 2$ and for F-singular Enriques surfaces in characteristic 2 the double cover is an unramified map from a K3 surface, while for classical and supersingular surfaces in characteristic 2 the map is purely inseparable, and the covering surface (which is either K3 or rational) has singularities.

To measure positivity of line bundles, one uses

$$\Phi(D) = \min\{|D \cdot f| : f^2 = 0 \text{ in } \operatorname{Num}(S)\}.$$

A big and nef primitive divisor D such that $D^2 = 4$, $\Phi(D) = 2$ is called a *Cossec-Verra polarization*. Every Enriques surface possesses a Cossec-Verra polarization [6, 3.4.2]. However, this polarization may not be ample; in fact, there exist "extra-special" Enriques surfaces in characteristic 2 with no ample Cossec-Verra polarization [12, 3.4]. Their non-ample locus is a union of smooth rational curves with self-intersection -2. There is a birational contraction $\pi: S \to X$ such that D induces an ample divisor on the Gorenstein surface X. The singularities of X consist of a finite number of rational double points.

6.2. Enriques zero-sets. Enriques surfaces occur as zero-sets of sections of our bundle E:

Theorem 22. The zero scheme of a general section $s \in H^0E$ is a nonsingular surface S with the following properties:

- 1. $\chi O_S = 1$ and $K_S = O_S$, *i.e.*, *S* is a non-classical Enriques surface.
- The cycle class of S in P² × P² is 4L² + 5Lh + 4h²; i.e., the line bundles L_S = O_S(L) and h_S = O(h) have the following intersections: L²_S = h²_S = 4 and L_S ⋅ h_S = 5.
 L_S and h_S are Cossec-Verra polarizations, but may not be ample (for an example, see 27
- 3. L_S and h_S are Cossec-Verra polarizations, but may not be ample (for an example, see 27 below).
- 4. The embedding of S in \mathbb{P}^8 under the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ is a linear projection of the image of S in \mathbb{P}^9 under the complete linear system $|L_S + h_S|$.
- 5. The homogeneous ideal of S is generated by 3 polynomials each of bidegrees (2,3) and (3,2), and one additional generator of bidegree (3,3).

Proof. Nonsingularity of the zero scheme of the general section was proved in Corollary 21. K_S is the restriction of $c_1 E \otimes \omega_{\mathbb{P}^2 \times \mathbb{P}^2}^{-1} = \mathcal{O}$, and $\chi \mathcal{O}_S = \chi \mathcal{O} - (\chi E(-3L - 3h) - \chi \mathcal{O}(-3L - 3h)) = 1$. According to the extension of the Enriques-Kodaira classification of surfaces by Bombieri and Mumford [3], *S* is an Enriques surface with trivial canonical bundle.

The cycle class of *S* agrees with c_2E , and the intersection of classes on *S* agree with the intersections on $\mathbb{P}^2 \times \mathbb{P}^2$.

 L_S and h_S are basepointfree linear systems of degree 4, hence by [6, 2.4.11,2.4.14], we must have $\Phi(L_S) = \Phi(h_S) = 2$, i.e., L_S and h_S are Cossec-Verra polarizations.

As $h^0 \mathcal{O}_S(L+h) = 10$ and $h^0 \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(L+h) = 9$, the embedding of *S* inside the Segre image of $\mathbb{P}^2 \times \mathbb{P}^2$ is a linear projection.

6.3. **linear systems on Enriques surfaces.** The Neron-Severi group of an Enriques surface has rank 10. The intersection form on $\text{Num}(S) = \text{NS}(S)/(K_S) \simeq \mathbb{Z}^{10}$ is unimodular, even and has signature (1,9), hence is isomorphic to $H \oplus (-E_8)$.

There exists a specific root basis $\alpha_0, \ldots, \alpha_9$ on the lattice Num(*S*) of an Enriques surface which at the same time is a basis of Num(*S*) [6, section 1.5]. The elements of the corresponding dual basis vectors $\omega_0, \ldots, \omega_9$ are called the *fundamental weights* of the Enriques lattice.

We can identify candidates for the classes of L_S and h_S inside Num(S) as follows:

The fundamental weights ω_1, ω_2 generate a primitive subgroup with intersection matrix $\begin{pmatrix} 4 & 9 \\ 9 & 18 \end{pmatrix}$ [6]. The vector $2\omega_1 - \omega_2$ is a root of Num(*S*), and the associated reflection maps ω_1 to $\omega_1 + (\omega_1, 2\omega_1 - \omega_2)(2\omega_1 - \omega_2) = \omega_2 - \omega_1$.

If the classes ω_1 and $\omega_2 - \omega_1$ are both nef (e.g., if *S* does not contain any (-2)-curves), then they are Cossec-Verra polarizations, and they combine to yield a basepointfree morphism $S \to \mathbb{P}^2 \times \mathbb{P}^2$.

7. DETECTING SUPERSINGULARITY

Our next task will be to identify F-singular resp. supersingular Enriques surfaces among the zero schemes of the sections of *E*. Our plan will be to identify suitable elliptic curves on *S* and use the following criterion:

Suppose $C \subset S$ is a curve such that the restriction $\mathcal{O}_S \to \mathcal{O}_C$ induces an isomorphism $H^1\mathcal{O}_S \to H^1\mathcal{O}_C$. Then S is supersingular if and only if C is supersingular.

Proposition 23. Let $s \in H^0E$ be a section vanishing on a nonsingular Enriques surface S. Then $S \cap A$ is a complete intersection $D_L \cap D_h \cap A$ for uniquely determined divisors $D_L \in |3L|$, $D_h \in |3h|$.

Proof. Since $E \otimes \mathcal{O}_A \cong \mathcal{O}_A(3L) \oplus \mathcal{O}_A(3h)$ of E on \mathcal{A} (Corollary 15), there exist uniquely determined divisors $D_{L\mathcal{A}} \in |3L_{\mathcal{A}}|$, $D_{h,\mathcal{A}} \in |3h_{\mathcal{A}}|$ such that $S \cap \mathcal{A} = (D_{L,\mathcal{A}} \cap D_{h,\mathcal{A}}) \cap \mathcal{A}$. Furthermore, \mathcal{A} is a divisor in |L + h| on $\mathbb{P}^2 \times \mathbb{P}^2$, and therefore the divisors in the linear systems $|3L_{\mathcal{A}}|$ and $|3h_{\mathcal{A}}|$ on \mathcal{A} are restrictions of uniquely determined effective divisors D_L , D_h from the corresponding linear systems on $\mathbb{P}^2 \times \mathbb{P}^2$.

In the following we write D_L , D_h for both the divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ and the cubic polynomial defining it. This should lead no confusion.

Proposition 24. Let $C_L \subset S$ be the residual of $S \cap A$ in $S \cap D_L$.

- 1. C_L is an effective divisor in $|2L h|_S$.
- 2. We have $(2L h)^2 = 0$, $h^0 \mathcal{O}_S(2L h) = 1$ and $h^1 \mathcal{O}_S(2L h) = h^2 \mathcal{O}_S(2L h) = 0$. Hence C_L is a half-fiber (in the sense of [6, 2.2.9])
- 3. C_L is connected, $h^1 \mathcal{O}_{C_L} = 1$, and the map $H^1 \mathcal{O}_S \to H^1 \mathcal{O}_{C_L}$ is bijective.
- 4. We have $p_{L*}C_L \in |3L| \subset \mathbb{P}^2_L$

Proof. We calculate $C_L = 3L - (L+h) = 2L - h$ for the divisor class of C_L . The second statement follows from the intersection pairing on *S* and the cohomology of *E*. The third follows from the first (and Serre duality) upon considering the cohomology of $0 \rightarrow O_S(-2L+h) \rightarrow O_S \rightarrow O_{C_L} \rightarrow 0$.

The cycle class of C_L in $\mathbb{P}^2 \times \mathbb{P}^2$ is $(2L - h) \cdot (4L^2 + 5Lh + 4h^2) = 6L^2h + 3Lh^2$, hence $C_L \cdot L = 3$.

Surjectivity of $H^0\mathcal{O}_S(L) \to H^0\mathcal{O}_C(L)$ follows from the vanishing of $H^1\mathcal{O}_S(-L+h)$. \Box

Proposition 25. Notation as in Proposition 24. Let

$$C_L \xrightarrow{f'} D'_L \xrightarrow{g} D_L$$

be the Stein factorization of the projection $C_L \rightarrow D_L$ *. Then the following hold:*

- 1. f' induces an isomorphism between $H^1\mathcal{O}_{C_L}$ and $H^1\mathcal{O}_{D'_L}$.
- 2. D'_L and D_L have the same number of irreducible components with multiplicities, and g is generically bijective for each irreducible component of D'_L and its image.

Proof. We need to investigate the structure of the divisor C_L .

(1) The components: Every effective divisor class *C* with $C^2 \ge 0$ can be expressed as

$$C \sim C' + \sum m_i R_i, \qquad (m_i \ge 0)$$

where C' is nef and the R_i are (-2)-curves [6, 2.3.3]. Since C_L is unique in its divisor class by Proposition 24, the above decomposition is a decomposition of divisors, and not only of divisor classes.

The potential structures for numerically effective half-fibers C' (note that C' is automatically an indecomposable divisor of canonical type in the sense of Mumford) have been classified [6, 2.2.5]. If C' is irreducible, then (being of arithmetic genus 1) it is an elliptic curve, or a nodal or cuspidal rational curve. If C' is reducible, then its components are nonsingular rational curves of self-intersection -2.

(2) Vertical and horizontal components: We can divide the components of *C*_{*L*} into horizontal and vertical (fibral) components. The images of the former are 1-dimensional, while the latter are mapped to a point.

Vertical components are (-2)-curves on *S* contracted by the Cossec-Verra polarization L_S . There is a Gorenstein surface *X* with a finite number of rational double points, and a contraction $\pi: S \to X$ such that L_S induces an ample line bundle on *X*; in particular we note $R^1\pi_*\mathcal{O}_S = 0$ [6, 2.4.16].

The cycle class of C_L in $\mathbb{P}^2 \times \mathbb{P}^2$ is $(2L - h) \cdot (4L^2 + 5Lh + 4h^2) = 6L^2h + 3Lh^2$, hence the intersection of C_L with a general divisor of *h* consists of 3 points (counted with multiplicity), and C_L has at most 3 horizontal components.

(3) Stein factorization of the projection: Now consider the Stein factorization

$$C_L \xrightarrow{f'} D'_L \xrightarrow{g} D_L$$

where f' has connected fibers and g is finite. The components of D'_L are the images of the horizontal components of C_L . We have $f'_* \mathcal{O}_{C_L} = \mathcal{O}_{D'_L}$ by construction, and we claim that $R^1 f'_* \mathcal{O}_{C_L} = 0$.

To this end, consider the restriction to C_L of the contraction $S \to X$. Since π contracts all the vertical fibers that f' contracts, we have $\pi|_{C_L} = \pi' \circ f'$ for some morphism π' . If $R^1 f'_* \mathcal{O}_{C_L}$ were not 0, then it would be supported on a finite set of

points, and $\pi'_* R^1 f'_* \mathcal{O}_{C_L}$ would also be non-zero. But the latter agrees (by Leray) with $R^1 \pi_* \mathcal{O}_{C_L}$, which is a quotient of $R^1 \pi_* \mathcal{O}_S = 0$. Therefore we must have $R^1 f'_* \mathcal{O}_{C_L} = 0$.

We can thus conclude that $H^1\mathcal{O}_{C_L} = H^1f'_*\mathcal{O}_{C_L}$, i.e., f' induces an isomorphism between $H^1\mathcal{O}_{D'_t}$ and $H^1\mathcal{O}_{C_L}$.

(4) Let $E
ightharpoondown D_L$ be an irreducible component of degree $d \le 3$. The 1-cycle $E \cap D_h \cap \mathcal{A}$ has the class $dL \cdot 3h \cdot (L+h) = 3d(L^2h + Lh^2)$ and is contained in $S \cap \mathcal{A}$. The residual of this cycle in $S \cap E$ has the class $(5dL^2h + 4dLh^2) - 3d(L^2h + Lh^2) = 2dL^2h + dLh^2$, hence intersects a general divisor in |L| in d points, counted with multiplicity. This implies that C_L , and hence D'_L , contains exactly one horizontal irreducible component mapping to $E \subset D_L$.

We have now reached to the following situation: There is a scheme D'_L and a finite map $D'_L \rightarrow D_L$ that is generically bijective for each irreducible component, with the same multiplicities. Each component of D'_L is the isomorphic image of a subscheme of *X*.

In particular, the map *g* from (3) is locally an isomorphism over each nonsingular point of D_L .

There is a finite list of isomorphism classes for D_L (see Example 4) and we will inspect them individually.

Proposition 26. Notation as in Propositions 24, 25. Then

- 1. D_L is reduced.
- 2. If D_L is irreducible, then $g: D'_L \to D_L$ is an isomorphism.
- 3. If D_L has several irreducible components, then $g: D'_L \to D_L$ is also an isomorphism.

Proof. 1. We argue by contradiction, and assume that D_L has a non-reduced component. By Proposition 25 this means that D'_L consists of a double line together with another line intersecting it in one point, or D'_L is a multiplicity 3 structure on a line.

First assume that D'_L consists of a double structure E_1 on a line E, and a single line E'. The ideal sheaf of E on E_1 has square 0, hence is isomorphic to the line bundle $\mathcal{O}(a)$ on Ewhere $a = -E^2 = 2$ (intersection taken on S). This implies that $h^0 \mathcal{O}_{E_1} \ge h^0 \mathcal{O}_E(2) = 3$.

Now consider the Mayer-Vietoris sequence

$$0 \to H^0 \mathcal{O}_{D'_t} \to H^0 \mathcal{O}_{E_1} \oplus H^0 \mathcal{O}_{E'} \to H^0 \mathcal{O}_{E_1 \cap E'}:$$

As $h^0 \mathcal{O}_{E_1 \cap E'} \leq 2$, we conclude that $h^0 \mathcal{O}_{D'_L} \geq 4 - 2 = 2$. However, by construction we have $h^0 \mathcal{O}_{D'_L} = h^0 \mathcal{O}_{C_L} = 1$, hence this contradiction implies that the assumed double structure cannot exist.

A similar argument rules out the case where D_L is a triple structure on a line.

2. If D_L is as smooth elliptic curve, then the projection $D'_L \to D_L$ is automatically an isomorphism. If D_L is a singular rational curve, then D'_L must be irreducible, rational with $h^1 \mathcal{O}_{D'_L} = 1$. The last condition implies that D'_L is singular, and it is immediate that the singularity must be of the same type as the singularity of D_L .

3. We already know that D'_L and D_L have the same number of components, and these are smooth rational curves. Furthermore, any intersection of two components in D'_L has to lie above an intersection of the corresponding components in D_L .

There are 4 different cases to consider: a line and a conic intersecting in two points or touching in one point, three lines intersecting in one point or in three separate points.

In each case, if an intersection of two components of D_L were not mirrored in D'_L , then we would have $h^1 \mathcal{O}_{D'_L} = 0$ in contradiction to the results of the previous proposition. \Box

Example 27. Consider the Enriques surface defined by (notation as in Theorem 43)

 $w = (b+c \quad b+c \quad x+z \quad a+c \quad z \quad y \quad 1).$

The unique divisor $C_L \in |2L - h|$ has three components: two vertical lines over the points $a = b, b = \alpha_i c$ (where α_1, α_2 are the solutions of the quadratic equation $X^2 + X + 1 = 0$), and an ordinary elliptic curve whose projection is $D_L = a^2b + b^3 + a^2c + abc + b^2c + ac^2 + c^3$.

The unique divisor $C_h \in |2h - L|$ has four components: three vertical lines over the points y = z, x = 0 and x = y + z, $z = \alpha_i y$ (i = 1, 2), and an ordinary elliptic curve whose projection is $D_h = y^3 + x^2 z + xyz + xz^2 + z^3$.

In this example neither Cossec-Verra polarization $\mathcal{O}_S(L)$ nor $\mathcal{O}_S(h)$ is ample, as it contracts at least 2 resp. 3 rational curves.

Finally we are ready for the main theorem of this section.

Theorem 28. Let $s \in H^0E$ be a section vanishing on a nonsingular Enriques surface *S* with trivial canonical bundle. The following conditions are equivalent:

- (i) *S* is supersingular.
- (ii) *s* is in the image of the canonical map

$$\psi$$
: $\left(H^0E(-L)\otimes H^0\mathcal{O}(L)\right)\oplus \left(H^0E(-h)\otimes H^0\mathcal{O}(h)\right)\to H^0E.$

- (iii) The restriction $s|_{\mathcal{A}} \in H^0(E|_{\mathcal{A}}) = H^0(\mathcal{O}_{\mathbb{P}^2_L}(3) \oplus \mathcal{O}_{\mathbb{P}^2_h}(3))$ corresponds to a pair of cubic polynomials D_L , D_h defining possibly degenerate supersingular cubic curves (as explained in 4).
- (iv) The homogeneous bigraded ideal of *S* is not generated by its elements in bidegrees (3,2) and (2,3); *i.e.* there is an additional generator in bidegree (3,3).
- *Proof.* The equivalence of (ii) and (iii) is immediate from the discussion in Corollary 45. (ii) and (iv) are equivalent: Consider the diagram

$$\begin{array}{c} V_L \oplus V_h \\ & \downarrow \\ 0 \longrightarrow \mathcal{O} \longrightarrow E \longrightarrow \mathcal{I}_S(3,3) \longrightarrow 0 \end{array}$$

where $V_L = H^0 E(-L) \otimes L$ and $V_h = H^0 E(-h) \otimes h$. We know that $h^0(V_L \oplus V_h) = 18$ and $h^0 E = 19$.

The section $s \in H^0E$ is in the image of $H^0(V_L \oplus V_h)$ if and only if the morphism $H^0(V_L \oplus V_h) \to H^0\mathcal{I}_S(3,3)$ is not surjective.

(i) and (iii) are equivalent: Recall that we identified in Proposition 24 two (possibly reducible) half-fibers on *S* that are supersingular if and only if *S* is. Furthermore these

divisors are contained in the intersection of *S* with the hypersurfaces of $\mathbb{P}^2 \times \mathbb{P}^2$ cut out by the pullbacks of the polynomials D_L resp. D_h .

The equivalence follows if we can show that the projection $C_L \rightarrow D_L$ (or the projection $C_h \rightarrow D_h$) induces an isomorphism on $H^1\mathcal{O}$.

But this is exactly what we established in Proposition 26.

Remark 29. Theorem 28 probably extends to Enriques surfaces with rational double points. In this case, *L* and *h* restrict to Cossec-Verra polarizations but |L + h| is not ample.

There is a quasi-projective dense open subscheme $U \subset \mathbb{P}H^0E = \mathbb{P}^{18}$ corresponding to sections of *E* whose zero scheme is a nonsingular Enriques surface. Theorem 28 showed that the closed points corresponding to supersingular Enriques surfaces form a linear subvariety of codimension 1.

We now want to show a slightly stronger result:

Consider the universal family $\pi: X_U \to U$, which is a subscheme of $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^{18}$. By Grauert's theorem, $R^1 \pi_* \mathcal{O}_{X_U}$ is a line bundle on U, and the relative Frobenius morphism on X_U defines via

$$(R^1\pi_*)(F^*): R^1\pi_*\mathcal{O}_{X_{II}} \to R^1\pi_*\mathcal{O}_{X_{II}}$$

a subscheme of *U*, which corresponds to the subfamily of supersingular Enriques surfaces in our family. It is defined by a global function on *U*.

Corollary 30. The subscheme of supersingular Enriques surfaces is reduced and nonsingular of codimension 1.

Proof. Corollary 45 in Appendix 1 provides an explicit parametrization of the family of all sections, and we can use it to compute the action of Frobenius on the direct image sheaf.

Step 1: Every section $s \in H^0E$ defines by restriction to \mathcal{A} two sections $p \in H^0\mathcal{O}_{\mathbb{P}^2}(3)$, $q \in H^0\mathcal{O}_{\mathbb{P}^2}(3)$, and we claim that the subscheme of supersingular Enriques surfaces agrees with the subscheme of supersingular cubic curves defined by p.

To see this, we go back to our previous analysis.

In Proposition 24 we constructed an effective divisor $C \subset X$ that represents an elliptic half-fiber on each surface. The construction generalizes to the family over U and induces a divisor $C_U \subset X_U$ such that the morphism

$$R^1\pi_*\mathcal{O}_{X_{U}}\to R^1\pi_*\mathcal{O}_{C_{U}}$$

is bijective (as the map $H^1\mathcal{O}_{X_u} \to H^1\mathcal{O}_{C_u}$ is bijective for each $u \in U$) and commutes with the relative Frobenius.

Furthermore, by Proposition 23) there exists an effective divisor $D_U \in H^0(pr_1^*\mathcal{O}_{\mathbb{P}^2}(3))$ such that the morphism

$$R^1\pi_*\mathcal{O}_{C_{II}} \to R^1\pi_*\mathcal{O}_{D_{II}}$$

is bijective and commutes with relative Frobenius (we proved this for the fibers in Propositions 25 and 26).

Now the divisor D_U corresponds to the section p mentioned in the beginning of step 1.

Step 2: In order to complete the proof, we need to investigate the relative Frobenius morphism

$$(R^1\pi_*)(F^*)\colon R^1\pi_*\mathcal{O}_{D_U}\to R^1\pi_*\mathcal{O}_{D_U}$$
²³

for the family of cubic curves $D_U \in H^0(pr_1^*\mathcal{O}_{\mathbb{P}^2}(3))$ over U, defined by p.

According to Corollary 45 we can write $p = f_1 \cdot a^2 + f_2 \cdot b^2 + f_3 \cdot c^2 + h \cdot abc$ where $\mathbb{P}^2 = \operatorname{Proj}(k[a, b, c])$, and $f_i = \lambda_{i,a}a + \lambda_{i,b}b + \lambda_{i,c}c$ (for i = 1, 2, 3), $h = \lambda_{abc}$ for suitably indexed coordinate functions (λ_*) on $\mathbb{P}H^0E$.

We now apply the discussion from the proof of [8, 4.21]: Identifying $H^1\mathcal{O}_{D_U}$ with $H^0\mathcal{O}_{\mathbb{P}^2_U}(3)$, represented by the one-dimensional vector space with natural basis $(abc)^{-1}$ of $H^0\mathcal{O}_{\mathbb{P}^2_U}(3)$, we obtain that the image of $H^1\mathcal{O}_{D_U}$ under the relative Frobenius is $(abc)^{-1}$ times the coefficient of *abc* in *p*, hence agrees with with the coordinate function λ_{abc} .

Therefore we can conclude that the subscheme of *U* corresponding to supersingular Enriques surfaces is defined by the vanishing of a coordinate function on $\mathbb{P}H^0E$, hence is reduced and nonsingular.

8. A REDUCIBLE MODEL

Here we discuss a reducible limit of the general zero-set *Y*.

Proposition 31. A special zero-set of E has the form

$$(s)_0 = Y_1 \cup Y_2$$

where $Y_1 \simeq \mathbb{P}^2$, Y_2 is an elliptic ruled surface and $Y_1 \cap Y_2$ is a smooth cubic in Y_1 and an unramified bisection of Y_2 .

Proof. Consider the modification sequence

$$0 \to (p_2^*F^*Q_h)(L) \to E \to \mathcal{O}_{\mathcal{A}}(3h) \to 0.$$

Note that A induces a duality

(12)
$$\mathbb{P}_L^2 \simeq \mathbb{P}_h^{2*}, \mathbb{P}_h^2 \simeq \mathbb{P}_L^{2*}.$$

For a section *s* of *E* coming from a general section $s_1 \in H^0((p_2^*F^*Q_h)(L))$, the zero set takes the form $(s)_0 = Y_1 \cup Y_2.$

where

$$Y_1 = (s_1)_0, Y_2 \in |\mathcal{O}_{\mathcal{A}}(3L)|,$$

$$Y_1] = c_2(F^*Q_h(L)) = 4h^2 + 2hL + L^2, [Y_2] = [L+h).3L = 3L^2 + 3Lh.$$

Now s_1 has the form $u_0v_0^2 + u_1v_1^2 + u_2v_2^2$ where u. are homogeneous coordinates on \mathbb{P}_L^2 , and the v. are a basis for $H^0(Q_h)$ vanishing respectively at p_0, p_1, p_2 that are the vertices of the coordinates triangle. Clearly Y_1 is smooth off the p_i . At p_0 , we may assume that, with respect to a suitable local basis for Q_h, v_0^2 has the form (x^2, y^2) while $v_1^2 = (1, 0), v_2^2 = (0, 1)$, and the local equations for Y_1 are $x^2u_0 + u_1 = 0, y^2u_0^2 + u_2 = 0$. Hence Y_1 is smooth at p_0 . Thus Y_1 is smooth everywhere.

Note that because \mathcal{A} is smooth over \mathbb{P}^2_L , this argument also proves that $W = Y_1 \cap \mathcal{A}$ is smooth.

Note that Y_1 is anticanonical, i.e. $K_{Y_1} = \mathcal{O}_{Y_1}(-L-h)$ and because has class $[Y_1] = 4h^2 + 2hL + L^2$, Y_1 maps birationally to \mathbb{P}_h^2 and under the Segre embedding $\mathcal{Y}_1 \subset \mathbb{P}_L^2 \times \mathbb{P}_h^2 \to \mathbb{P}^8$, which is also an anticanonical map, Y_1 has degree $9 = (-K_{Y_1})^2 = h^0(K_{Y_1}) - 1$. It follows

that Y_1 must project isomorphically to \mathbb{P}_h^2 and its image in \mathbb{P}^8 is a projection of its full anticanonical image in \mathbb{P}^9 .

As for Y_2 , it is a ruled surface of the form $\mathbb{P}(Q_L|_Z)$ where $Z \subset \mathbb{P}_L^2$ is the cubic with equation, in the above notation, $u_0^t x_0^2 + u_1^t x_1^2 + u_2^t x_2^2$ where x_i are the coordinates dual to v_i by the duality (12), hence Z is smooth. Y_2 may be obtained from $Z \times \mathbb{P}^1$ by elementary transformations (blowing up a point and blowing down the proper transform of its fibre) at 3 collinear points.

As for the intersection $W = Y_1 \cap A$, it is a zero-set of $F^*Q_h(L)|_A$ which fits in an exact sequence

$$0 \to \mathcal{O}_{\mathcal{A}}(2h-L) \to F^*Q_h(L)|_{\mathcal{A}} \to \mathcal{O}_{\mathcal{A}}(3L) \to 0$$

hence W projects to Z in \mathbb{P}^2_L . Moreover

$$[W] = [Y_1].(L+h) = 6h^2L + 3L^2h$$

hence W.L = 6, hence W maps with degree 2 to Z. On the other hand W.h = 3 sp W maps isomorphically to a cubic in \mathbb{P}_{h}^{2} . This also implies that the map $W \to Z$ is unramified.

Note that the fact that $\omega_{Y_1 \cup_W Y_2} = \omega_{Y_1}(W) \cup \omega_{Y_2}(W)$ is the trivial bundle a priori forces W to be a cubic in Y_1 and a bisection in Y_2 .

The Mayer-Vietoris sequence

$$0 o \mathcal{O}_{Y_0} o \mathcal{O}_{Y_1} \oplus \mathcal{O}_{Y_2} o \mathcal{O}_W o 0$$

yields an injection $H^1(\mathcal{O}_{Y_0}) \to H^1(\mathcal{O}_{Y_2})$ (induced by restriction). Since $Y_2 \to Z$ is a \mathbb{P}^1 bundle, we have an isomorphism of 1-dimensional vector spaces $H^1(\mathcal{O}_Z) \simeq H^1(\mathcal{O}_{Y_2})$ induced by pullback. Since $H^1(\mathcal{O}_{Y_0}) \neq 0$, we conclude $H^1(\mathcal{O}_{Y_0}) \simeq H^1(\mathcal{O}_{Y_2}) \simeq H^1(\mathcal{O}_Z)$. Both these isomorphisms are compatible with the action of Frobenius. As the equation of *Z* has no $x_0x_1x_2$ term, *Z* is supersingular by [8], IV.4.21. Hence we conclude

Corollary 32. *Notations as above,* Y₀ *is supersingular.*

Note that in the family of all pairs (E, s) so that the zero-set $(s)_0 = Y_1 \cup Y_2$ as above has codimension 10 in the family of all pairs (E, s) where $(s)_0$ is a smooth or normal- crossing surface. On the other hand, the locus where $(s)_0$ is supersingular is a divisor. Corollary 32 implies that this divisor is nontrivial. A priori, a general surface in this divisor may have finitely many double points. However the results of §7, specifically Theorem 28, proven by other means, shows that a general supersingular zero-set is in fact smooth.

9. Moduli

As we saw earlier 5, there is a bijection between the set of bundles *E* and the set of smooth divisors in |L + h|. This bijection is in fact an isomorphism of moduli spaces:

Theorem 33. (*i*) Let $U \subset |L+h|$ be the subset parametrizing smooth divisors and let $A_U \subset U \times \mathbb{P}^2 \times \mathbb{P}^2$ be the universal divisor. Then the kernel \mathcal{E} of the natural surjection

$$F^*Q_h \to \mathcal{O}_{\mathcal{A}_U}(2L)$$

is a vector bundle whose restriction E_u on $u \times \mathbb{P}^2 \times \mathbb{P}^2$ for each $u \in U$ is indecomposable and coincides with one of the bundles E studied above and every such bundle occurs in this way; moreover for each u, \mathcal{E} is the universal deformation of E_u .

(ii) There is an isomorphism between the space V of pairs (E, s) where $s \in H^0(E)$ has smooth zero-set and the space V' of pairs $(S, \Phi_L \times \Phi_h)$ where S is a smooth nonclassical Enriques surface and $\Phi_L \times \Phi_h : S \to \mathbb{P}^2 \times \mathbb{P}^2$ is an embedding whose image S' has $S'.L^2 = S'.h^2 = 4$, S'.L.h = 5.

Proof. By construction each bundle *E* as above corresponds to a smooth divisor A, hence to a point $u \in U$ and conversely. The divisor A depends canonically on *E* as the degeneracy divisor of the map

$$p_h^* p_{h*} E \to E.$$

Conversely given a smooth divisor $A \in |L + h|$, it determines a bundle *E* as a suitable twist of the kernel of the canonical map

$$F^*Q_h \to \mathcal{O}_{\mathcal{A}}(2L).$$

Moreover the correspondence $E \leftrightarrow \mathcal{A}$ clearly extends to deformations over any local scheme *S*: given a bundle E_S on $S \times \mathbb{P}^2 \times \mathbb{P}^2$, we get a divisor \mathcal{A}_S on $S \times \mathbb{P}^2 \times \mathbb{P}^2$ as the degeneracy locus of $p_2^* p_{h*} E_S \to E_S$ and conversely given a divisor \mathcal{A}_S we get E_S as the kernel of $F^*Q_h \to \mathcal{O}_{\mathcal{A}_S}(2L)$. Therefore the deformation space of *E* can be identified with *U* and in particular it is smooth and 8-dimensional.

Now because $h^0(E) = 19$, $h^1(E) = 0$ for each $E = E_u$, $\mathbb{P}(\pi_{U*}(\mathcal{E}))$ is smooth and relatively 18-dimensional over U. Let $V \subset \mathbb{P}(\pi_{U*}(\mathcal{E}))$ be the open subset of sections with smooth zero-sets. The V is the deformation space of pairs (E, t_s) , where $s \in H^0(E)$ has smooth (Enriques) zero-set, and V is smooth and 26 dimensional.

On the other hand the deformation space V' of pairs (Y, Φ) where Y is an Enriques surface (10 moduli) and $\Phi : Y \to \mathbb{P}^2_L \times \mathbb{P}^2_h$ is an embedding with $L^2 = h^2 = 4$, L.h = 5(uniquely determined by (Y, L, h) up to an automorphism of $\mathbb{P}^2_L \times \mathbb{P}^2_h$) is smooth of relative dimension 16 over the 10-dimensional moduli space M of Enriques surfaces. There are mutually inverse maps

$$V \xrightarrow{\alpha} V' \xrightarrow{\beta} V$$

where α is the zero-set map and β is given by the Serre construction. Hence α , β are isomorphisms. We have established an isomorphism between deformation spaces of (E, s) and of (Y, Φ) . This implies in particular that a general zero set Y is general in the moduli of Enriques surfaces, and that for a general Enriques surface in char. 2 there exist divisors L, h with $L^2 = h^2 = 4, L.h = 5$ such that $\Phi_L \times \Phi_h : Y \to \mathbb{P}^2 \times \mathbb{P}^2$ is an embedding.

10. FROM SURFACE TO BUNDLE

The purpose of this section is to prove Theorem 40 below which asserts that every non-classical Enriques surface of bidegree (4,4) in $\mathbb{P}^2 \times \mathbb{P}^2$ occurs as a zero-set one of the 'monadic' bundles constructed above.

Note that the Chow group of 2-cycles on $\mathbb{P}^2 \times \mathbb{P}^2$ has a basis that consists of L^2 , *Lh* and h^2 .

Also, note at the outset that given a nef and big line bundle *L* on a nonclassical Enriques surface *S* (i.e. $K_S = 0$), we have $h^1(L) = 0$ [6, 2.1.7], hence $h^0(L) = \chi(L) = L^2/2 + 1$.

We assume ¹ in the following that $S \subset \mathbb{P}^2 \times \mathbb{P}^2$ is 'bilinearly normal', i.e. it is not a 'linear projection' from a higher-dimensional $\mathbb{P}^r \times \mathbb{P}^s$. This is equivalent to $\mathcal{O}(L)$, $\mathcal{O}(h)$ restricting to Cossec-Verra polarizations on *S* (i.e., $L_S^2 = h_S^2 = 4$, and $\Phi(L_S) = \Phi(h_S) = 2$).

Setting $\lambda = L_S \cdot h_S$, the self-intersection formula for *S* (see [8, App. A 4.1.3]) now shows that $\lambda^2 - 9\lambda + 20 = 0$, hence $\lambda = 4$ or $\lambda = 5$.

If $\lambda = 4$, then we have $(L_S - h_S)^2 = 0$, and the Hodge index theorem implies that $L_S - h_S$ is numerically trivial, thus necessarily *S* is a classical Enriques surface and $h_S = L_S \otimes K_S$.

Conversely, given a Cossec-Verra polarization *L* on a classical Enriques surface *S*, there is a birational contraction $f: S \to X$ to a Gorenstein surface *X* such that $|L| \times |L \otimes K_X|$ defines an embedding of *X* into $\mathbb{P}^2 \times \mathbb{P}^2$ [6, 3.4.7].

Since $K_s = h \otimes L^{-1}$, the canonical bundle is restriction of the global line bundle $\mathcal{O}(-1,1)$ on $\mathbb{P}^2 \times \mathbb{P}^2$. According to Serre's theorem [13], there exists a rank 2 vector bundle on $\mathbb{P}^2 \times \mathbb{P}^2$ with a section *s* whose zero scheme is the given surface. These bundles have been constructed by Casnati and Ekedahl [4, 6.5] who showed that they are (up to twist) pullbacks of a bundle from one of the factors \mathbb{P}^2 , so they are not 'new'. The corresponding bundles on \mathbb{P}^2 are in turn pullbacks of the tangent bundle by quadratic maps; they have a minimal graded resolution $0 \to \mathcal{O}(-2)^{\oplus 3} \to \mathcal{O} \to \mathcal{F} \to 0$, and their moduli were described by Barth [1]. Both [4] and [1] are written in char. 0 but it seems likely that they largely extend, with some exceptions, to char. *p*.

Now assume $\lambda = 5$, i.e., we have an embedding $S \to \mathbb{P}^2 \times \mathbb{P}^2$ of an Enriques surface such that the class of the surface *S* is $4h^2 + 5hL + 4L^2$.

We do not know if there exist classical Enriques surfaces with such an embedding.

If *S* is non-classical, then the canonical bundle is trivial, and there exists a rank 2 vector bundle *E* on $\mathbb{P}^2 \times \mathbb{P}^2$ with a section *s* whose vanishing scheme is *S*.

In the remainder of this section we want to show that the bundle *E* belongs to the family of bundles that we constructed in §1 as the cohomology of a monad.

Lemma 34. *S uniquely determines the bundle E.*

Proof. The Serre construction [13], says that given *S*, the choice of bundle *E* corresponds to that of a line bundle $L \in \text{Pic}(\mathbb{P}^2 \times \mathbb{P}^2) = \mathbb{Z} \oplus \mathbb{Z}$ that restricts on *S* to K_S . Since $L_S \cdot h_S = 5$, the restrictions L_S resp. h_S of the bundles $\mathcal{O}_{\mathbb{P}^2}(1,0)$ and $\mathcal{O}_{\mathbb{P}^2}(0,1)$ are linearly independent in Pic(S), so *L* is uniquely determined, hence so is *E*, up to isomorphism.

Remark 35. Let *L* be an ample Cossec-Verra polarization on a classical Enriques surface (any characteristic). $|L \times (L + K_S)|$ determines an embedding $S \rightarrow \mathbb{P}^2 \times \mathbb{P}^2$ [6, 3.4].

The canonical sheaf of *S* is 2-torsion, hence we have

$$K_S \cong h_S - L_S \cong L_S - h_S,$$

and *S* can be represented in two different ways as the zero section of a rank 2 bundle. As mentioned above, up to twist these bundles arise by pullback from one of the factors of \mathbb{P}^2 .

Our strategy will be to reconstruct *E* from its (higher) direct images under pr_2 via the relative Beilinson spectral sequence. To calculate the direct images, we need to make use of the two auxiliary exact sequences $0 \rightarrow \mathcal{O} \rightarrow E \rightarrow \mathcal{I}_S(3,3) \rightarrow 0$ and $0 \rightarrow \mathcal{I}_S \rightarrow \mathcal{O} \rightarrow \mathcal{O}$

¹Recall the analogy with \mathbb{P}^4 : All smooth surfaces in \mathbb{P}^4 are linearly normal except the Veronese surface.

 $\mathcal{O}_S \rightarrow 0$, and work backwards, starting with the determination of the higher direct images of \mathcal{O}_S .

As mentioned earlier, the line bundles $L_S = O_S(1,0)$ and $h_S = O_S(0,1)$ are Cossec-Verra polarizations, i.e., they are basepointfree and have self-intersection 4. This implies that $h^0L_S = h^0h_S = 3$ and the higher cohomology groups of L_S and h_S vanish.

Proposition 36. $pr_{2,*}\mathcal{O}_S \cong \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3) \oplus \Omega^1_{\mathbb{P}^2}$.

Proof. There is a birational contraction of smooth rational (-2)-curves $\pi \colon S \to X$ such that π_*L_S is ample on X, and the induced morphism $X \to \mathbb{P}^2$ is finite [6, 2.4.16].

Therefore the higher direct images of \mathcal{O}_S under pr_2 are zero, and we have isomorphisms $H^i pr_{2,*} \mathcal{O}_S(0,l) \cong H^i \mathcal{O}_S(0,l)$ for any i, l.

Considering the composition $S \to X \to \mathbb{P}^2$, we obtain a short exact sequence

(13)
$$0 \to \mathcal{O}_{\mathbb{P}^2} \to pr_{2,*}\mathcal{O}_S \to \mathcal{E} \to 0,$$

with a locally free sheaf \mathcal{E} of rank 3 on \mathbb{P}^2 .

We now use the sequence (13) to determine the cohomology groups $h^i \mathcal{E}(l)$. Since the higher cohomology groups of a big and nef line bundle on an Enriques surface vanish [6, 2.1.16], we find (where empty slots mean 0)

The Beilinson spectral sequence for $\mathcal{E}(1)$ shows that there is an exact sequence $0 \to \Omega^1(1) \to \mathcal{E}(1) \to \mathcal{E}' \to 0$ where \mathcal{E}' fits into $0 \to \mathcal{E}' \to \mathcal{O}(-1)^{\oplus 3} \to \Omega^1(1) \to 0$. The second sequence implies that $\mathcal{E}' \cong \mathcal{O}(-2)$, and therefore the extension class of the first sequence is zero. We obtain $\mathcal{E}(1) \cong \Omega^1(1) \oplus \mathcal{O}(-2)$.

The extension class of the sequence (13) vanishes as well, hence $pr_{2,*}\mathcal{O}_S \cong \mathcal{E} \oplus \mathcal{O}$. \Box

Proposition 37. $pr_{2,*}\mathcal{O}_S(1,0) \cong \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^2}(-1).$

Proof. As explained in the beginning of the proof of the previous proposition, the fibers of pr_2 are finite with the potential exception of a finite number of smooth rational curves.

 $\mathcal{O}_{S}(1,0)$ has positive degree on these curves, and therefore we obtain (as in the previous proof) that the higher direct images of $\mathcal{O}_{S}(1,0)$ under pr_{2} vanish, and there are isomorphisms $H^{i}pr_{2,*}\mathcal{O}_{S}(1,l) \cong H^{i}\mathcal{O}_{S}(1,l)$ for any i, l.

We now determine the cohomology groups $h^i \mathcal{O}_S(1, l)$ for $-2 \le l \le 0$:

l = 0: Higher cohomology groups of a big and nef line bundle vanish.

l = -1: The line bundle $\mathcal{O}_S(1, -1)$ is non-trivial and has intersection 0 with the ample divisor L + h, hence it has no sections, and its inverse also has no sections, thus (by duality) $h^2 = 0$. The vanishing of h^1 follows from Riemann-Roch on *S*.

l = -2: $\mathcal{O}_S(1, -2)$ has negative intersection with L + h, hence it has no sections. If $h^1 > 0$, then $\mathcal{O}_S(-1, 2)$ would be an elliptic pencil and thus have a half-fiber [6, 2.2.9], i.e., the divisor class would be be divisible by 2. But this contradicts the primitivity of this divisor class. Hence $h^1 = 0$ and $h^2 = 1$.

Finally, the Beilinson spectral sequence provides an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \to pr_{2,*}\mathcal{O}_S(1,0) \to \mathcal{O}_{\mathbb{P}^2}(-1) \to 0$$

which necessarily splits.

Corollary 38. The direct image sheaves $R^i pr_{2,*} \mathcal{I}_S(l,0)$ in the range $0 \le l \le 1$ are as follows:

$$\begin{array}{c|c} 2\\i & 1\\0\\ \hline \\ 0\\ 1\\ \hline \\ l\\ \end{array} \mathcal{O}_{\mathbb{P}^2}(-3)\oplus\Omega^1_{\mathbb{P}^2} \quad \mathcal{O}_{\mathbb{P}^2}(-1)\\ \hline \\ 0\\ \hline \\ 1\\ \hline \\ l\\ \end{array}$$

Proof. Using the previous two propositions, the table follows from the exact sequence $0 \rightarrow \mathcal{I}_S \rightarrow \mathcal{O} \rightarrow \mathcal{O}_S \rightarrow 0$ by twisting (for l = 1) and taking direct images under pr_2 . For l = 0 we obtain

Since $H^0 \mathcal{O}_{\mathbb{P}^2} \to H^0 \mathcal{O}_S$ is bijective, we obtain the column for l = 0.

For l = 1 we obtain

Since $H^0\mathcal{O}_{\mathbb{P}^2}(1,0) \to H^0\mathcal{O}_S(1,0)$ is bijective, we obtain the column for l = 1.

Proposition 39. The direct image sheaves $R^i pr_{2,*}E(l,-3)$ in the range $-4 \le l \le -2$ are as follows:

Proof. We propose to twist the exact sequence $0 \rightarrow \mathcal{O} \rightarrow E \rightarrow \mathcal{I}_S(3,3) \rightarrow 0$ and apply direct images under pr_2 .

Twisting by $\mathcal{O}(-2, -3)$ and taking direct images, we find that

$$R^{i}pr_{2,*}E(-2,-3) \cong R^{i}pr_{2,*}\mathcal{I}(1,0)$$

for any *i*, and the column for l = -2 follows from Corollary 38.

Twisting by $\mathcal{O}(-3, -3)$ and taking direct images, we find that $pr_{2,*}E(-3, -3) = pr_{2,*}\mathcal{I}_S = 0$, and further that there is an exact sequence of sheaves on \mathbb{P}^2

Relative duality implies for the last term that

$$(R^2 pr_{2,*}E(-3,-3))^{\vee} \cong pr_{2,*}(E(-3,-3)) \otimes \mathcal{O}_{\mathbb{P}^2}(3) = 0,$$

hence $R^1 pr_{2,*}E(-3, -3)$ is the kernel of an epimorphism $\mathcal{O}(-3) \oplus \Omega^1 \to \mathcal{O}(-3)$. Since there is no surjection $\Omega^1 \to \mathcal{O}(-3)$, the induced map $\mathcal{O}(-3) \to \mathcal{O}(-3)$ must be nontrivial, hence an isomorphism, and we conclude that $R^1 pr_{2,*}E(-3, -3) \cong \Omega^1$.

This establishes the column for l = -3.

Finally, relative duality implies that

$$(R^i pr_{2,*}E(-4,-3))^{\vee} \cong (R^{2-i} pr_{2,*}E(-2,-3)) \otimes \mathcal{O}_{\mathbb{P}^2}(3),$$

hence the entries in the column for l = -2 imply the entries for l = -4.

We can now state and prove the main result of this section:

Theorem 40. Let *S* be a smooth nonclassical Enriques surface in $\mathbb{P}^2 \times \mathbb{P}^2$ such that $L^2 \cdot S = h^2 \cdot S = 4$. Then *S* is a zero set of a rank-2 vector bundle *E* that is the homology sheaf of a monad of the form

$$0 o \mathcal{O}(L+h) o Q_L(L) \otimes Q_h(h) o \mathcal{O}(2L+2h) o 0.$$

Proof. Using the table from the previous proposition, the relative Beilinson spectral sequence implies that E(-2, -3) is isomorphic to the middle homology of a complex with the following terms

$$0 \to \mathcal{O}(-1, -2) \to pr_1^*\Omega^1(1) \otimes pr_2^*\Omega^1 \to \mathcal{O}(0, -1) \to 0.$$

The Corollary follows by twisting with $\mathcal{O}(2,3)$ and noting that $Q_L = pr_1^*\Omega^1(2)$, $Q_h = pr_2^*\Omega^1(2)$, $\mathcal{O}(L) = \mathcal{O}(1,0)$ and $\mathcal{O}(h) = \mathcal{O}(0,1)$.

11. APPENDIX 1: THE GRADED MODULE OF SECTIONS OF E

Here we present a resolution of the module of sections of *E* (Theorem 42) and generators for the ideal sheaves of the zero schemes (Theorem 43). Some results were obtained with the aid of the computer algebra package *Macaulay2* [7]. The supporting code can be found in the next appendix.

Proposition 41. The bigraded module $\bigoplus_{m,n} H^0 E(m,n)$ has 7 minimal generators: three each in bidegrees (-1,0), (0,-1) and one in bidegree (0,0).

Proof. The computation of the cohomology in section 5 exhibits three generators each in bidegrees (-1, 0) and (0, -1). Furthermore, the homomorphisms $H^0E(-L) \otimes H^0\mathcal{O}(L) \rightarrow H^0E$ (resp. $H^0E(-h) \otimes H^0\mathcal{O}(h) \rightarrow H^0E$) are both injective, and the restrictions of these sections to \mathcal{A} lie by Corollary 16 in different subbundles of $E \otimes \mathcal{O}_{\mathcal{A}}$, hence the images intersect only in 0. Therefore their sum generates an 18-dimensional subspace of H^0E . Since $h^0E = 19$, one additional generator is needed in bidegree (0,0). These together generate the sections of E(m,n) in all twists $m, n \geq 0$.

11.1. The minimal bigraded resolution for *E*.

Theorem 42. The vector bundle *E* has a minimal bigraded resolution with the following terms:

Proof. Proposition 41 showed that $\bigoplus_{m,n} H^0 E(m, n)$ is generated by $H^0 E(0, -1)$, $H^0 E(-1, 0)$ and one additional section of $H^0 E$.

The computer algebra package *Macaulay2* [7] can create a bigraded module and compute its minimal bigraded resolution. A priori we only know that the sheafification of the bigraded module will agree with *E*. However, inspection of the generators of this module show that their numbers and degrees agree with those of $\bigoplus_{m,n} H^0E(m,n)$, hence the resolution captures all sections of all twists of *E*.

11.2. generators of the ideal sheaf of a section of *E*. In the following, we let k[a, b, c, x, y, z] be the bihomogeneous coordinate ring of $\mathbb{P}^2 \times \mathbb{P}^2$.

Theorem 43. Let $s \in H^0E(m, n)$ be a non-zero section. Then there exist functions $f_1, f_2, f_3 \in H^0\mathcal{O}(m+1,n)$, $g_1, g_2, g_3 \in H^0\mathcal{O}(m, n+1)$ and $h \in H^0\mathcal{O}(m, n)$ such that the homogeneous ideal of the zero scheme Z of s is generated by the 7 entries of the row vector

$$w = \begin{pmatrix} f_1 & f_2 & g_1 & f_3 & g_2 & g_3 & h \end{pmatrix} \cdot B$$

where B is the following 7×7 -matrix:

$$\begin{bmatrix} b^2y^2 + c^2z^2 & axy^2 + by^3 + cy^2z & a^2y^2 & 0\\ b^2x^2 & ax^3 + bx^2y + cx^2z & a^2x^2 + c^2z^2 & axz^2 + byz^2 + cz^3\\ ab^2x + b^3y + b^2cz & a^2x^2 + b^2y^2 & a^3x + a^2by + a^2cz & a^2z^2\\ c^2x^2 & 0 & c^2y^2 & axy^2 + by^3 + cy^2z\\ 0 & c^2x^2 & ac^2x + bc^2y + c^3z & b^2y^2 + c^2z^2\\ ac^2x + bc^2y + c^3z & c^2y^2 & 0 & a^2y^2\\ b^2cxy + bc^2xz & acx^2y + bcxy^2 & a^2cxy + ac^2yz & aby^2z + acyz^2 \end{bmatrix}$$

$axz^2 + byz^2 + cz^3$	$a^{2}z^{2}$	$aby^2z + acyz^2$
0	$b^{2}z^{2}$	$abx^2z + bcxz^2$
$b^{2}z^{2}$	0	$a^2bxz + ab^2yz$
$ax^3 + bx^2y + cx^2z$	$a^2x^2 + b^2y^2$	$acx^2y + bcxy^2$
$b^2 x^2$	$ab^2x + b^3y + b^2cz$	$b^2 cxy + bc^2 xz$
$a^2x^2 + c^2z^2$	$a^3x + a^2by + a^2cz$	$a^2 c x y + a c^2 y z$
$abx^2z + bcxz^2$	$a^2bxz + ab^2yz$	0

Conversely, the entries of $(f_1 \ f_2 \ g_1 \ f_3 \ g_2 \ g_3 \ h) \cdot B$ generate the zero scheme of a section of E(m, n) for any functions f_1, \ldots, g_3, h as specified above.

The matrix *B* is presented in the form provided by *Macaulay2*. The entries of *w* are not ordered by bidegree.

Proof. (compare [5, 5.1.11]) Let *s* be a section of E(a, b), and consider the following diagram



where $E_1 \rightarrow E_0 \rightarrow E \rightarrow 0$ is the presentation of *E* from Theorem 42 above.

The dashed diagonal arrow determines bihomogeneous polynomials P_1 , P_2 , $P_3 \in H^0\mathcal{O}(m + 2, n + 3)$, Q_1 , Q_2 , $Q_3 \in H^0\mathcal{O}(m + 3, n + 2)$ and $R \in H^0(\mathcal{O}(m + 3, n + 3))$.

Now write $v^t = (P_1 \ P_2 \ P_3 \ Q_1 \ Q_2 \ Q_3 \ R)$ and let *A* be the 7 × 14-matrix specifying the map $E_1 \rightarrow E_0$. Since v^t corresponds to a section of E(m, n), we conclude that $v^t \cdot A = 0$.

The latter is equivalent to $A^t \cdot v = 0$, $v \in \ker A^t$. Since all rings and modules are noetherian, ker A^t is finitely generated, and a set of generators can be calculated by *Macaulay2*, forming the columns of a matrix B^t .

Hence we can write $v = B^t \cdot w^t$, where $w = (f_1 \ f_2 \ g_1 \ f_3 \ g_2 \ g_3 \ h)$. Inspection of *B* shows that the entries of *w* are homogeneous polynomials with the stated bidegrees.

Conversely, given any vector *w* with entries as above, we have $(w \cdot B) \cdot A = w \cdot (B \cdot A) = w \cdot 0 = 0$, hence $w \cdot B$ determines a section of E(m, n).

Remark 44. We know that the sheaf map

$$F^*Q_L(h) \oplus F^*Q_h(L) \to E$$

is surjective. Therefore the ideal sheaf of the zero scheme of any section can be generated by the first 6 entries of *w* alone. In Theorem 28 we showed that sometimes even the homogeneous ideal is generated by only these 6 polynomials.

Corollary 45. Let *s* be a section of E(m, n) corresponding to the row vector

$$w = \begin{pmatrix} f_1 & f_2 & g_1 & f_3 & g_2 & g_3 & h \end{pmatrix}$$

where $f_1, f_2, f_3 \in H^0\mathcal{O}(m+1, n), g_1, g_2, g_3 \in H^0\mathcal{O}(m, n+1)$ and $h \in H^0\mathcal{O}(m, n)$.

The restriction of s to A *agrees with the section* (p,q) *of* $\mathcal{O}_A(m+3,n) \oplus \mathcal{O}_A(m,n+3)$ *where*

$$p = f_1 \cdot a^2 + f_2 \cdot b^2 + f_3 \cdot c^2 + h \cdot abc$$

$$q = g_1 \cdot z^2 + g_2 \cdot x^2 + g_3 \cdot y^2 + h \cdot xyz$$

Proof. p, *q* depend k[a, b, c, x, y, z]-linearly on the f_i , g_i , *h* so that it suffices to check the equations on the basis vectors, e.g., by *Macaulay2*.

APPENDIX 2: MACAULAY2 CODE

This is the Macaulay2 code used for the calculations in Theorems 42, 43 and Example 27:

```
R = ZZ/2[a,b,c,x,y,z, Degrees => \{\{1,0\},\{1,0\},\{1,0\},\{0,1\},\{0,1\},\{0,1\}\}\}
m1 = R^{\{\{0,3\},\{1,2\},\{1,2\},\{1,2\}\}}
phi1 = map( m1, R<sup>{{-1,2}}</sup>, matrix{{a*x+b*y+c*z},{a^2},{b^2},{c^2}})
phi2 = map( R<sup>{</sup>{1,4}}, m1, matrix{{a*x+b*y+c*z,x^2,y^2,z^2}})
E = trim( (ker phi2) / (image phi1) )
E1 = resolution E
B = transpose generators ker transpose presentation E
A = ideal(a*x+b*y+c*z)
w = matrix\{\{b+c, b+c, x+z, a+c, z, y, 1\}\}
S = ideal(w * B)
sat = i -> saturate(saturate(i, ideal(a,b,c)), ideal(x,y,z))
sing = i -> sat( minors( codim i, jacobian(i)) + i)
codim S
                                                         -- should be 2
SingS = sing S
                                                       -- should be ideal 1
AS = sat(A + S)
for i from 0 to (numgens AS)-1 do if degree AS_i = \{3, 0\} then p1 = ideal(AS_i)
for i from 0 to (numgens AS)-1 do if degree AS_i = \{0,3\} then p_i^2 = ideal(AS_i)
C1 = sat(p1 + S) : AS
C2 = sat(p2 + S) : AS
minimalPrimes C1
minimalPrimes C2
```

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