ON THE CONVERGENCE OF A PERTURBED ONE DIMENSIONAL MANN'S PROCESS

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ABSTRACT. We consider the perturbed Mann's iterative process

(0.1) $x_{n+1} = (1 - \theta_n)x_n + \theta_n f(x_n) + r_n,$

where $f : [0, 1] \to [0, 1]$ is a continuous function, $\{\theta_n\} \in [0, 1]$ is a given sequence, and $\{r_n\}$ is the error term. We establish that if the sequence $\{\theta_n\}$ converges relatively slowly to 0 and the error term r_n becomes enough small at infinity, any sequences $\{x_n\}$ generated by (0.1) converges to a fixed point of the function f. We also study the asymptotic behavior of the trajectories x(t) as $t \to \infty$ of a continuous version of the discrete process (0.1). We investigate the similarities between the asymptotic behaviours of the sequences generated by (0.1) and the trajectories of the corresponding continuous process.

1. INTRODUCTION AND PRESENTATION OF THE MAIN RESULTS

Let $f : [0,1] \to [0,1]$ be a continuous function. The classical Brouwer's fixed point theorem [1]) or a simple application of the intermediate value theorem ensures that the function f has at least one fixed point. In order to determine a numerical approximation of fixed points of f, W. Robert Mann [9] introduced the following iterative process

(1.1)
$$x_{n+1} = \frac{n}{n+1}x_n + \frac{1}{n+1}f(x_n)$$

and proved the following result.

Theorem 1.1. Let $f : [0,1] \to [0,1]$ be a continuous function. If f has a unique fixed point $p \in [0,1]$, then, for any initial data $x_0 \in [0,1]$, the sequence $\{x_n\}$ generated by the process (1.1) converges towards p.

Later, R.L. Franks and R.P. Marzec in a short and a very nice paper [4] proved that the condition on the uniqueness of the fixed points of the function f in the previous theorem is not necessary. Precisely, they proved the following theorem.

Theorem 1.2. If $f : [0,1] \to [0,1]$ is a continuous function then, for any initial data $x_0 \in [0,1]$, the sequence $\{x_n\}$ generated by the process (1.1) converges towards a fixed point of f

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Our first objective in this paper is to investigate the effect of possible small computational errors on the asymptotic behaviour of the iterative process (1.1). We will establish the following result.

Theorem 1.3. Let $f : [0,1] \to [0,1]$ be a continuous function and $\{\theta_n\} \in [0,1]$ a given real sequence. Let $\{x_n\} \in [0,1]$ be a sequence satisfying the perturbed iterative process

(1.2)
$$x_{n+1} = (1 - \theta_n)x_n + \theta_n f(x_n) + r_n,$$

where $\{r_n\}$ is a real sequence. If

(1) $\theta_n \to 0 \text{ as } n \to \infty$, (2) $\sum_{n=0}^{\infty} \theta_n \text{ diverges}$, (3) $\frac{r_n}{\theta_n} \to 0 \text{ as } n \to \infty$, (4) $\sum_{n=0}^{\infty} r_n \text{ converges}$,

then the sequence $\{x_n\}$ converges towards a fixed point of f.

Inspired by the works [8], [6], [7], [10] and [11], we also study the asymptotic behavior of the trajectories x(t) of a continuous version of the discrete dynamical system (1.2).

Theorem 1.4. Let $f : [0,1] \to [0,1]$ and $\theta : [0,\infty) \to [0,\infty)$ be two continuous functions and let $x : [0,\infty) \to [0,1]$ be a continuous differentiable function that satisfies the perturbed differential equation:

(1.3)
$$x'(t) + \theta(t)x(t) = \theta(t)f(x(t)) + r(t), t \ge 0,$$

where $r: [0, \infty) \to \mathbb{R}$ is a continuous function. If

(1) $\lim_{t\to\infty} \frac{r(t)}{\theta(t)} = 0,$ (2) $\int_0^\infty \theta(t) dt \ diverges,$ (3) $\int_0^\infty r(t) dt \ converges,$

then x(t) converges towards a fixed point of f as $t \to \infty$.

The sequel of paper is organized as follows. In the next section, we provide a detail proof of Theorem 1.3. The third section is devoted to the proof of Theorem 1.4. In the fourth section, we study through a numerical experiment the effect of the sequence $\{\theta_n\}$ and the error term $\{r_n\}$ on the rate of convergence of the sequences $\{x_n\}$ generated by the process (1.2). The last section is devoted to a complete and detail proof of Mann's convergence original result (Theorem 1.1).

2. On the convergence of the perturbed discrete dynamical system (1.2)

In this section, we provide a detail proof of Theorem 1.3. The proof is greatly inspired by the original paper [4]. It essentially relies on the classical notion of the omega limit set of a real sequence. We recall here briefly this notion and some of its main properties. **Definition 2.1.** Let $\{x_n\}$ be a real sequence. The omega limit set of the sequence $\{x_n\}$ is the set $\omega(\{x_n\})$ of real numbers z such that $z = \lim_{n\to\infty} x_{n_k}$ for some subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$.

Lemma 2.1. Let $\{x_n\}$ be a real sequence. Then

- (1) $\omega(\{x_n\}) = \bigcap_{n \in \mathbb{N}} \overline{\{x_m : m \ge n\}}^{\mathbb{R}}$ where $\overline{\{x_m : m \ge n\}}^{\mathbb{R}}$ denotes the closure of the set $\{x_m : m \ge n\}$ in \mathbb{R} .
- (2) If $\{x_n\}$ is bounded then $\omega(\{x_n\})$ is a nonempty compact subset of \mathbb{R} .
- (3) If $\{x_n\}$ is bounded then $\{x_n\}$ converges in \mathbb{R} if and only if the set $\omega(\{x_n\})$ is reduced to a single element.
- (4) If $\lim_{n\to\infty} x_{n+1} x_n = 0$ then $\omega(\{x_n\})$ is a connected subset of \mathbb{R} .

The proof of this lemma is classical and easy, it is then left for the readers. Now, we are in position to prove the main result Theorem 1.3.

Proof. Since the sequence $\{x_n\}$ is in [0,1] and satisfies

$$x_{n+1} - x_n = \theta_n [f(x_n) - x_n] + r_n \to 0 \text{ as } n \to \infty,$$

then in view of the previous lemma, there exist two real number $0 \le a \le b \le 1$ such that $\omega(\{x_n\}) = [a, b]$. Let us suppose that a < b. We claim that in this case f(w) = w for any $w \in (a, b)$. Let $w \in (a, b)$. Suppose for the sake of contradiction that f(w) > w. Then thanks to the continuity of f there exist two positive real numbers δ and ε such that $[w - \delta, w + \delta] \subset (a, b)$ and $f(x) \ge x + \varepsilon$ for any $x \in [w - \delta, w + \delta]$. There exists $n_0 \in \mathbb{N}$ such that for any $n \ge n_0$

$$(2.1) |x_{n+1} - x_n| < \delta,$$

(2.2)
$$r_n \geq -\varepsilon \theta_n$$

Moreover, there exist two positive integers $n_2 > n_1 > n_0$ such that $x_{n_1} \in (w + \delta, b)$ and $x_{n_2} \in (a, w - \delta)$. Let *m* be the greatest integer between n_1 and n_2 such that $x_m \ge w - \delta$. Therefore, in view of (2.1), we have $x_{m+1} < w - \delta \le x_m < w$. Now by going back to the dynamical system (1.2) and using the fact that $f(x_m) \ge x_m + \varepsilon$, we get

$$\begin{array}{rcl} x_{m+1} & \geq & x_m + \theta_m \varepsilon + r_m \\ & \geq & x_m & (\mbox{ in view of } (2.2)) \end{array}$$

This contradicts the fact that $x_{m+1} < x_m$. Similarly, we can show that the assumption f(w) < w leads to a contradiction. We therefore conclude f(w) = w for any $w \in (a, b)$ if a < b. We continue working under the assumption a < b. Let $c = \frac{b+a}{2}$ and set $\alpha = \frac{b-a}{8}$. There exists a positive integer m_0 such that

$$|x_{m_0} - c| < \alpha,$$

(2.4)
$$\left|\sum_{n=m_0}^m r_n\right| < \alpha, \forall m \ge m_0.$$

Let $m_1 > m_0$ be the smallest positive integer such that $x_{m_1} \notin [c - 2\alpha, c + 2\alpha]$ (such integer exists since $[c - 2\alpha, c + 2\alpha] \subset (a, b)$). Therefore, for any integer $n \in [m_0, m_1 - 1]$, we have $f(x_n) = x_n$ and

$$x_{n+1} = x_n + r_n$$

Summing up the previous equalities, we obtain

$$x_{m_1} = x_{m_0} + \sum_{n=m_0}^{m_1-1} r_n.$$

Combining this identity with the estimations (2.3) and (2.4), we get

$$|x_{m_1} - c| \le 2\alpha$$

which contradicts the definition of x_{m_1} . We then conclude that a = b which in view of Lemma 2.1 means that the sequence $\{x_n\}$ is converging in \mathbb{R} ; let x_{∞} be its limit. Clearly $x_{\infty} \in [0, 1]$. Let us prove by contradiction that $f(x_{\infty}) = x_{\infty}$. Let us suppose that $f(x_{\infty}) > x_{\infty}$. Thanks to the continuity of the function f, there exists a positive real number $\beta > 0$ and a positive integer p_0 such that, for any $n \ge p_0$,

$$f(x_n) \ge x_n + 2\beta,$$

$$r_n \ge -\beta_n \theta_n.$$

Hence, for any integer $n \ge p_0$,

$$x_{n+1} - x_n = \theta_n (f(x_n) - x_n) + r_n$$

$$\geq 2\beta \theta_n + r_n$$

$$\geq \beta \theta_n.$$

Summing up these inequalities leads to the inequalities

$$\beta \sum_{n=p_0}^{\infty} \theta_n \leq x_{\infty} - x_{p_0} \leq 1$$

that contradict the assumption on the sequence $\{\theta_n\}$. Similarly, we can show that $f(x_{\infty})$ can not be less than x_{∞} . We therefore conclude that $f(x_{\infty}) = x_{\infty}$ which completes the proof. \Box

3. On the convergence of the perturbed continuous dynamical system (1.3)

This section is devoted to the proof of Theorem 1.4. Before starting the proof of this theorem, let us recall the definition and some simple proprieties of the omega limit set of a continuous real valued function defined on the interval $[0, \infty)$. (For more details on the notion of the omega limit in a more general context, we refer the readers to the books [6], [7], and [8]. **Definition 3.1.** let $u : [0, \infty) \to \mathbb{R}$ be a continuous function. The omega limit set associated to the function u (denoted by $\omega(u(t))$) is the set of real numbers z such that $\lim_{n\to\infty} u(t_n) = z$ for some positive sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$.

The following lemma gathers some well known properties of the omega limit set associated to a continuous and bounded real valued function u defined on the interval $[0, \infty)$.

Lemma 3.1. let $u: [0, \infty) \to \mathbb{R}$ be a continuous and bounded function function. Then

- (1) $\omega(u(t)) = \bigcap_{k \in N} \overline{u([k,\infty))}^{\mathbb{R}}$, where $\overline{u([k,\infty))}^{\mathbb{R}}$ denotes the closure of the set $u([k,\infty))$ in \mathbb{R} .
- (2) $\omega(u(t))$ is a nonempty compact interval of \mathbb{R} .
- (3) u(t) converges to some real number u_{∞} as $t \to \infty$ if ad only if $\omega(u(t)) = \{u_{\infty}\}$.

The proof of Theorem 1.4 uses also this simple and technical lemma inspired from the papers and [2] and [5].

Lemma 3.2. let $u : [0, \infty) \to \mathbb{R}$ be a continuous and bounded function. If $\omega(u(t)) = [a, b]$ with a < b then for any real numbers c and d such that a < c < d < b and any $T_0 > 0$ there exist four reals numbers s_1, s_2, τ_1 and τ_2 such that:

- (1) $\tau_i > s_i > T_0, i = 1, 2.$
- (2) $u(t) \in [c, d]$ for any $t \in [s_i, \tau_i], i = 1, 2$.
- (3) $u(s_1) = c$ and $u(\tau_1) = d$.
- (4) $u(s_2) = d$ and $u(\tau_2) = c$.

Proof. Let $T_0 > 0$ and c and d two real number such that a < c < d < b. From the definition of $\omega(u(t))$, there exist two real numbers s_0 and τ_0 such that $\tau_0 > s_0 > T_0$, $u(s_0) < c$ and $u(\tau_0) > d$. Let $s_1 = \sup\{t \in [s_0, \tau_0] : u(t) < c\}$. sing the continuity of u we can easily verify that $s_0 < s_1 < \tau_0$, $u(s_1) = c$ and $u(t) \ge c$ for any $t \in [s_1, \tau_0]$. Let $\tau_1 = \inf\{t \in [s_1, \tau_0] : u(t) > d\}$. Again the continuity of u ensures that $\tau_0 > \tau_1 > s_1$, $u(\tau_1) = d$ and $u(t) \le d$ for every $t \in [s_1, \tau_1]$. This completes the construction of s_1 and τ_1 . The construction of s_2 and τ_2 can be done similarly. In fact, there exist two real numbers s'_0 and τ'_0 such that $\tau_0 > s_0 > T_0$, $u(\tau'_0) < c$ and $u(s'_0) > d$. Now let $s_2 = \sup\{t \in [s'_0, \tau'_0] : u(t) > d\}$ and $\tau_2 = \inf\{t \in [s_2, \tau'_0] : u(t) < c\}$. Again by using the continuity of u we can verify that s_2 and τ_2 satisfy all the required proprieties.

Now we are in position to prove Theorem 1.4.

Proof. From Lemma 3.1, there exist two real numbers $0 \le a \le b \le 1$ such that $\omega(x(t)) = [a, b]$. First, for the sake of a contradiction, we suppose that a < b. Now let w be an arbitrary element of the open interval (a, b). Let us suppose that f(w) > w. Then there exist $\varepsilon, \delta > 0$ such that $[w - \delta, w + \delta] \subset (a, b)$ and $f(z) > z + \varepsilon$ for any $z \in [w - \delta, w + \delta]$. Let $T_0 > 0$ such that $r(t) \ge -\varepsilon \theta(t), \forall t \ge T_0$. According to Lemma 3.2, there exist $\tau_2 > s_2 > T_0$ such that

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 $x(s_2) = w + \delta$, $x(\tau_2) = w - \delta$, and $x(t) \in [w - \delta, w + \delta]$ for any $t \in [s_2, \tau_2]$. Therefore, for every $t \in [s_2, \tau_2]$,

$$\begin{aligned} x'(t) &= \theta(t)(f(x(t)) - x(t)) + r(t) \\ &\geq \varepsilon \theta(t) + r(t) \\ &\geq 0. \end{aligned}$$

Hence by integrating on the interval $[s_2, \tau_2]$, we get the contradiction $-2\delta \ge 0$. Similarly, by sing the times s_1 and τ_1 instead of s_2 and τ_2 , we can verify that the assumption f(w) < w leads also to a contradiction. We then conclude that if a < b then f(w) = w for any $w \in [a, b]$. We continuous working under the assumption a < b. Let $c = \frac{a+b}{2}$ and $\alpha = \frac{b-a}{4}$. Let $T_1 > 0$ be greater enough such that

(3.1)
$$\left| \int_{s}^{\tau} r(t) dt \right| \leq \alpha, \ \forall \tau > s > T_{1}.$$

According Lemma 3.2, there exist $\tau_1 > s_1 > T_1$ such that $x(s_1) = c - \alpha$, $x(\tau_1) = c + \alpha$, and $x(t) \in [c - \alpha, c + \alpha]$ for any $t \in [s_1, \tau_1]$. Therefore, for any $t \in [s_1, \tau_1]$,

$$\begin{aligned} x'(t) &= \theta(t)(f(x(t)) - x(t)) + r(t) \\ &= r(t). \end{aligned}$$

Hence, by integrating the last equality between s_1 and τ_1 we get

$$2\alpha = \int_{s_1}^{\tau_1} r(t)dt,$$

which contradicts (3.1). We therefore conclude that a = b which means thanks to Lemma 3.1, that x(t) converges as $t \to \infty$ to some the real number $x_{\infty} = a = b$. Let s prove It is clear that $x_{\infty} \in [0, 1]$, let us show that it is a fixed point of f. For the sake of absurdity, let us for instance assume that $f(x_{\infty}) < x_{\infty}$. The continuity of x(t) an f and the assumption on r(t) assures the existence of $\varepsilon > 0$ and T_2 such that for any $t \ge T_2$

$$\begin{aligned} x(t) - f(x(t)) &\geq 2\varepsilon, \\ r(t) &\leq \varepsilon \theta(t) \end{aligned}$$

Therefore, for every $t \geq T_2$,

$$\begin{aligned} x'(t) &= \theta(t)(f(x(t)) - x(t)) + r(t) \\ &\leq -\varepsilon \theta(t). \end{aligned}$$

Integrating the last inequality between T_2 and $T > T_2$ and then letting $T \to \infty$, we get the inequality

$$\varepsilon \int_{T_2}^{\infty} \theta(t) dt \le x(T_2) - x_{\infty}$$

that contradict the assumption on θ . Similarly, we can show that $f(x_{\infty})$ can not be greater than x_{∞} . We therefore conclude that $f(x_{\infty}) = x_{\infty}$.

4. A numerical study of the rate of convergence of the processes (1.2) and (1.3)

In this section, we investigate numerically the effect of the speed of the vanishing of the sequence $\{\theta_n\}$ and the amplitude of the error term $\{r_n\}$ on the rate of convergence of the sequence $\{x_n\}$ generated by the perturbed process (1.2) to a fixed point of f. We consider the case where The sequence $\{\theta_n\}$ is defined by $\theta_n = \frac{1}{(n+1)^{\alpha}}$ where $0 < \alpha \leq 1$, and the objective function $f: [0, 1] \to [0, 1]$ is given by:

$$f(x) = \begin{cases} 2(\frac{1}{4} - x), & 0 \le x \le \frac{1}{4}, \\ 4(x - \frac{1}{4}), & \frac{1}{4} \le x \le \frac{1}{2}, \\ 4(\frac{3}{4} - x), & \frac{1}{2} \le x \le \frac{3}{4}, \\ 2(x - \frac{3}{4}), & \frac{3}{4} \le x \le 1, \end{cases}$$

A simple calculation shows that f has exactly three fixed points: $x_{1f} = \frac{1}{6}$, $x_{2f} = \frac{1}{3}$, and $x_{3f} = \frac{3}{5}$. We can now introduce our example of the process (1.2). Let x_0 be an arbitrary element of [0, 1]. The sequence $\{x_n\}$ is generated by the iterative stochastic process

(4.1)
$$x_{n+1} = \Pi((1-\theta_n)x_n + \theta_n f(x_n) + A \frac{M_n}{1+n^2}),$$

where A > 0 is a constant and $\{M_n\}$ is a sequence of i.i.d random variables such that each M_n follows the uniform low U([-1,1]). The function $\Pi : \mathbb{R} \to [0,1]$ is the metric projection on the interval [0,1] which is defined explicitly by:

$$\Pi(x) = \begin{cases} 0, & x \le 0, \\ x, & 0 \le x \le 1, \\ 1, & x \ge 1. \end{cases}$$

Clearly, the sequence $\{x_n\}$ belongs to the interval [0, 1] and satisfies the process (1.2) with an error term $r_n = \prod(y_n + A\frac{M_n}{1+n^2}) - y_n$ where $y_n = (1 - \theta_n)x_n + \theta_n f(x_n)$. Since $y_n \in [0, 1]$,

$$|r_n| \le \frac{A}{1+n^2}.$$

Therefore, according to Theorem 1.3, the sequence $\{x_n\}$ converges to a fixed point x_{∞} . For the two chosen values of the amplitude of the error term A = 0.1 and 0.001 and some chosen values of $\alpha \in [0,1]$ and $\epsilon > 0$, we perform $K_{\max} = 100$ times the process (4.1) with arbitrary initial data x_0 in [0,1] under the stoping criteria $|f(x_n) - x_n| < \epsilon$. Let $N(A, \alpha, \epsilon)$ be the average of number of iterations needed to achieve the stopping criteria. The following two tables give $N(0.1, \alpha, \epsilon)$ and $N(0.001, \alpha, \epsilon)$ for some values of α and ϵ .

We notice that the process (1.2) converges faster if α is close to 1 (i.e., the sequence $\{\alpha_n\}$ converges relatively quickly to zero) and A is small (i.e., the perturbation term $\{r_n\}$ is relatively

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α/ϵ	$\epsilon = 0.1$	$\epsilon=0.01$	$\epsilon=0.001$	$\epsilon = 0.0001$		
$\alpha = 0.1$	4.27	77.07	117.41	133.64		
$\alpha = 0.2$	5.83	24.40	36.46	41.72		
$\alpha = 0.4$	5.05	12.83	17.03	22.50		
$\alpha = 0.6$	4.04	7.65	11.84	20.54		
$\alpha = 0.8$	3.53	6.47	11.20	21.63		
$\alpha = 0.9$	3.65	5.83	10.73	25.07		
$\alpha = 1$	3.40	6.14	11.44	28.89		
TABLE 1. $N(A = 0.1, \alpha, \epsilon)$						

α/ϵ	$\epsilon = 0.1$	$\epsilon=0.01$	$\epsilon=0.001$	$\epsilon=0.0001$	
$\alpha = 0.1$	4.88	76.14	119.15	134.84	
$\alpha = 0.2$	5.14	22.94	35.09	38.39	
$\alpha = 0.4$	4.97	11.52	16.49	17.81	
$\alpha = 0.6$	3.95	7.39	9.49	10.85	
$\alpha = 0.8$	3.46	5.87	6.68	8.63	
$\alpha = 0.9$	3.57	5.72	6.70	9.20	
$\alpha = 1$	3.28	4.89	6.60	8.78	
	TABLE 2. $N(A = 0.001, \alpha, \epsilon)$				

weak). In this case the speed of the convergence of the process (1.2) is slightly better than the rate of the convergence of the classical bisection method (applied for the function f(x) - x) for which $N(\epsilon)$, the number of needed iterations to achieve the precision ϵ , is equal to the entire part of $\frac{-\log(\epsilon)}{\log(2)}$ where $0 < \epsilon < 1$, [3].

5. Annex

In this section, we give a complete and detail proof of original Mann's convergence result (Theorem 1.1).

Proof. The function g(x) = f(x) - x defined on the interval [0, 1] is continuous, satisfies $g(0) \le 0$ and $g(1) \ge 0$, and has p as a unique root, then from the intermediate value theorem, g(x) > 0 on the interval [0, p) and g(x) < 0 on (p, 1]. Now let $\varepsilon > 0$ be an arbitrary but a fixed real number. There exists $\delta > 0$ such that

(5.1)
$$(x \in [0,1] \land |x-p| \ge \varepsilon) \Rightarrow |g(x)| \ge \delta.$$

On the other hand since,

$$|x_{n+1} - x_n| = \frac{1}{n+1} |g(x_n)|$$

 $\leq \frac{1}{n+1},$

there exists a positive integer n_0 such that

$$(5.2) |x_{n+1} - x_n| < \varepsilon, \forall n \ge n_0.$$

Now let us suppose that for any integer $n \ge n_0$, $x_n \notin [p - \varepsilon, p + \varepsilon]$. This implies that either $x_n > p + \varepsilon$ for any $n \ge n_0$ or $x_n for any <math>n \ge n_0$, because otherwise there exists $m \ge n_0$ such that $x_m > p + \varepsilon$ and $x_{m+1} which is impossible in view of (5.2). Let us assume for instance that <math>x_n > p + \varepsilon$ for any $n \ge n_0$. Then, from (5.1) and the fact that g(x) < 0 on (p, 1], we deduce that

$$x_{n+1} - x_n = \frac{1}{n+1}g(x_n) \le -\frac{\delta}{n+1}, \ \forall n \ge n_0.$$

Therefore, for any $n > n_0$, we have the inequality

$$x_n - x_{n_0} \le -\delta \sum_{k=n_0}^{n-1} \frac{1}{k+1}$$

which implies $\lim_{n\to\infty} x_n = -\infty$ contradicting the fact that $\{x_n\}$ is bounded. We therefore conclude that there exists $n_1 \ge n_0$ such that $x_{n_1} \in [p - \varepsilon, p + \varepsilon]$. We will now prove that for any $n \ge n_0$, if $x_n \in [p - \varepsilon, p + \varepsilon]$ then $x_n \in [p - \varepsilon, p + \varepsilon]$. Let $n \ge n_0$ such that $x_n \in [p - \varepsilon, p + \varepsilon]$. We consider the two possible cases:

The first case $x_n \in [0, p + \varepsilon]$. We have $g(x_n) \leq 0$, then $x_{n+1} = x_n + \theta_n g(x_n) \leq x_n$. Combining this inequality with the fact that $|x_{n+1} - x_n| < \varepsilon$, we deduce that $x_{n+1} \in [p - \varepsilon, p + \varepsilon]$.

The second case $x_n \in [p - \varepsilon, 0]$. We have $g(x_n) \ge 0$, then $x_{n+1} \ge x_n$. Hence, by using again the fact $|x_{n+1} - x_n| < \varepsilon$, we also deduce that $x_{n+1} \in [p - \varepsilon, p + \varepsilon]$.

Therefore, by induction we conclude that, for any $n \ge n_1, x_n \in [p - \varepsilon, p + \varepsilon]$. This means that $x_n \to p$ as $n \to \infty$ and therefore completes the proof of the theorem. \Box

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DEPARTMENT OF MATHEMATICS AND STATISTICS, COLLEGE OF SCIENCE, KING FAISAL UNIVERSITY, AL-Ahsa, Kingdom of Saudi Arabia

Email address: rmay@kfu.edu.sa