Lax dynamics

Stefano Lepri[1, 2, *]
[1]*Consiglio Nazionale delle Ricerche, Istituto dei Sistemi Complessi, via Madonna del Piano 10, I-50019 Sesto Fiorentino, Italy*
[2]*Istituto Nazionale di Fisica Nucleare, Sezione di Firenze, via G. Sansone 1, I-50019 Sesto Fiorentino, Italy*
(Dated: April 23, 2025)

A novel approach is proposed to characterize the dynamics of perturbed many-body integrable systems. Focusing on the paradigmatic case of the Toda chain under non-integrable Hamiltonian perturbations, this study introduces a method based the time evolution of the Lax eigenvalues λ_α as a proxy of the quasi-particles velocities and of the perturbed Toda actions. A set of exact equations of motion for the λ_α is derived that closely resemble those for eigenenergies of a quantum problem (also known as the Pechukas-Yukawa gas). Numerical simulations suggest that the invariant measure of such dynamics is basically the thermal density of states of the Toda lattice, regardless of the form of the perturbation.

Introduction – The solution of a physical problem usually proceeds by identifying a solvable part and studying the effects of perturbations. For nonlinear systems where the solvable part is described by an integrable classical or quantum Hamiltonian, one can, with varying degrees of mathematical difficulty, separate the independent degrees of freedom (the quasi-particles) and analyze their interactions using, for example, perturbation theory. In many-body problems, a perturbation typically destroys integrability, leaving only a few residual conserved quantities, it is important to assess how and when thermalization, chaotic dynamics, and conventional hydrodynamic behavior occur [1]. Considering that a variety of physical systems as ultracold atoms, one-dimensional magnets, or optical beams are proximate to nonlinear integrable limits [2], such questions are of wide interests in many diverse contexts.

In the classical realm, a paradigmatic example is the celebrated Toda lattice defined by the Hamiltonian [3]

$$H_{\rm Toda} = \sum_{j=1}^{N} \left(\frac{p_j^2}{2} + {\rm e}^{-(q_{j+1}-q_j)}\right), \tag{1}$$

where (q_j, p_j) are position and momentum of the j-th particle (note that the model has no free parameter). The discovery of its full integrability [4, 5] sparkled a vivid reseach activity. However, its thermodynamics received only sporadic consideration [6, 7], and has only recently garnered renewed attention due to the formulation of Generalized Gibbs Ensembles (GGE), which extend the canonical state of standard statistical mechanics to integrable models [8–11]. This represent a great novelty with respect to the very many works dealing with zero-temperature properties of specific solutions e.g. solitons, breathers and nonlinear waves [12], the emphasis being shifted to e.g. dynamical correlations at equilibrium [13, 14].

Beyond this, insights on the effect of integrability-breaking perturbations are relevant for nonequilibrium properties, ranging from the classic thermalization problem, *a lá* Fermi-Pasta-Ulam-Tsingou (FPUT) [15] to heat transport close to quasi-integrable limit [16]. The crucial observation [17] is that, $H_{\rm Toda}$ is the closest integrable approximation of a *whole family* of anharmonic chains with Hamiltonian of the form $H = \sum_j [\frac{1}{2}p_j^2 + \Phi(q_{j+1} - q_j)]$. This viewpoint has been established only recently [18], and implies that for a broad class of inter-particle potentials Φ, Toda's model is a more accurate and insightful approximation than the standard harmonic one. For instance, the slow chaotic motion leading to equipartition is not so much due to the fact a models like FPUT is a discretization of an integrable wave equation, but rather to the fact that the dynamics is (at least at low enough energies) essentially indistinguishable from Toda's on very long time scales [18]. This idea is corroborated by numerical experiment on thermalization [17, 19, 20] and stationary transport [16, 21–24]. In the first case, the metastable state can be seen as a particular GGE state that slowly relaxes to standard thermal equilibrium [19].

In this Letter, a novel point of view of the problem is presented based on the extension of the concept of *Lax eigenvalues*, which are well-known in the theory of integrable models [25], to their perturbed versions. It will be argued that they are insightful quantities and that their time-evolution caused by the perturbation has some generic features of considerable interest.

To start, it is convenient to write the equations of motion for the Toda lattice with the *Flaschka variables* $a_j = {\rm e}^{-r_j/2}, b_j = p_j$, (with $r_j \equiv q_{j+1} - q_j$) as

$$\dot a_j = \tfrac{1}{2} a_j (p_j - p_{j+1}), \quad \dot p_j = a_{j-1}^2 - a_j^2, \tag{2}$$

under periodic boundary conditions, $a_{-1} = a_N, p_{N+1} = p_1$. One then defines the *Lax matrix*, $L = L^T$, $N \geq 3$, and its pair matrix $B = -B^T$ as

$$L_{i,j} = a_{i-1}\delta_{i-1,j} + p_i\delta_{i,j} + a_i\delta_{i+1,j} \tag{3}$$

$$B_{i,j} = \frac{1}{2} a_{i-1}\delta_{i-1,j} - \frac{1}{2} a_i \delta_{i+1,j} \tag{4}$$

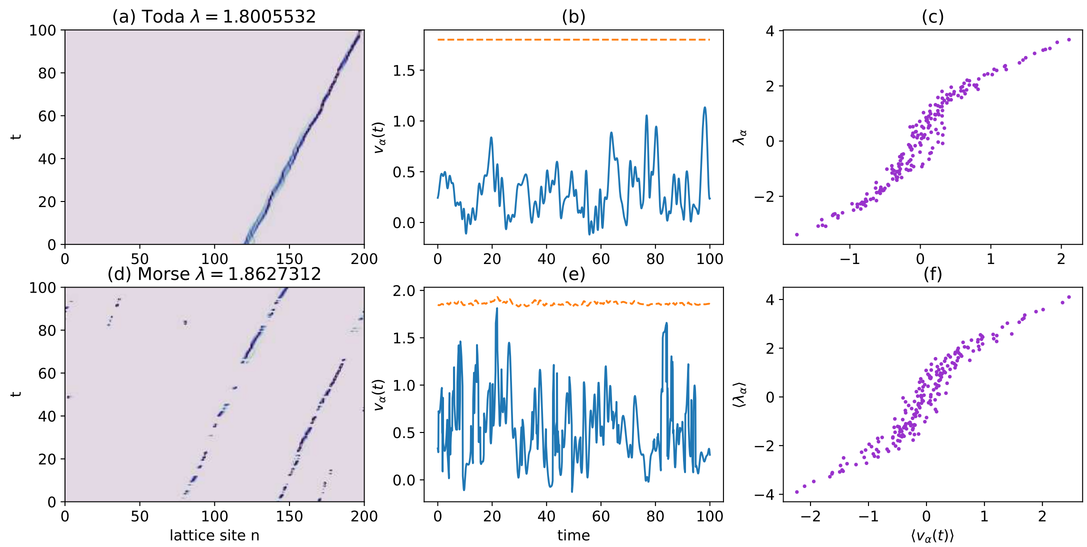

FIG. 1. Simulations of Toda (upper panels) and Morse chains with $\varepsilon = 0.1$ (lower panels); (a,d): Space-time evolution of the square modulus of a Lax eigenvector $|\psi_{\alpha,n}(t)|^2$ $\alpha = 166$ (b,e) quasiparticle velocities computed by Eq.(7) (solid red line) and corresponiding Lax eigenvalue $\lambda_\alpha(t)$; (c,f) plots of $\lambda_\alpha(t)$ versus the time-averaged velocity $\langle v_\alpha \rangle$. For comparison, in the Morse case (f) the average eigenvalue is reported. In both cases, $N = 200$ and initial conditions are sampled from a thermal GGE state of the Toda model with $\beta = 1, \beta P = 1$.

with $L_{1,N} = L_{N,1} = a_N$, $B_{1,N} = -B_{N,1} = a_N$. It is well known [25] that Eqs. (2) can be recasted as $\dot{L} = [B, L] = BL - LB$.

Consider the motion of a perturbed Toda system in the general form [26]

$$\dot{L} = [B, L] + U \tag{5}$$

where $U(a, p)$ is a $N \times N$ symmetric matrix with the same structure as L (i.e. if $L_{i,j} = 0$ then also $U_{i,j} = 0$) whose elements are (nonlinear) functions of the Flashka variables.

Consider the eigenvalue problem

$$L\,|\alpha\rangle = \lambda_\alpha\,|\alpha\rangle\,, \quad \alpha = 1, \ldots, N. \tag{6}$$

where λ_α and $|\alpha\rangle$ are the Lax eigenvalues and eigenvectors, whose components in the lattice basis $|j\rangle$ are $\psi_{\alpha,j} = \langle\alpha|j\rangle$.

Integrability - For Toda ($U = 0$), $L(t)$ is isospectral i.e. λ_α do not change while eigenvectors are time-dependent and satisfy $\dot{|\alpha\rangle} = B\,|\alpha\rangle$. At zero temperature the spectrum is $\lambda_\alpha = 2\cos(2\pi\alpha/N)$ and eigenstates with $|\lambda_\alpha| > 1$ are associated to solitons [27]. The physically relevant conserved charges are given by $\mathrm{tr}(L^n) = \sum_{j=1}^N (L^n)_{j,j} = \sum_{\alpha=1}^N \lambda_\alpha^n$. The first three are the standard ones, namely the sum of the stretches r_j, momenta p_j and local energies $p_j^2 + a_j^2 + a_{j-1}^2$.

A thermodynamic state corresponds to a Generalized Gibbs Ensemble (GGE), with finite energy density fixed by the N independent chemical potentials [10]. In this context, L is a random matrix sampled from each GGE state. The simplest case would be the *thermal* one where the assigned thermodynamic variables are the stretch $\ell = \langle r_j \rangle = -2\langle \log a_j \rangle$, kinetic temperature $1/\beta = \langle p_j^2 \rangle$ and pressure $P = \langle a_j^2 \rangle$ [10].

Lax eigenvalues thus play a major role in the thermodynamics, as averages can be written as integrals over their empirical Density Of States (DOS) $\rho(\lambda) = \lim_{N\to\infty} \sum_{\alpha=1}^N \delta(\lambda - \lambda_\alpha)/N$ that can be computed numerically by sampling the L matrices and direct diagonalization or analytically via the thermodynamic Bethe Ansatz. The thermal DOS $\rho_{th}(\lambda)$ is of particular relevance: in this case the $L_{i,j}$ are independent random variables and the Lax spectrum can be sampled easily. Also, explicit analytical expressions of $\rho_{th}(\lambda)$ are available in some limit cases [10]. Another important property is that the spectral gaps are proportional to Toda actions [28].

Quasiparticles – The quasiparticle concept is insightful to understand the dynamics [10, 29–31]. For integrable systems quasiparticles move ballistically. Upon collisions they retain their velocity but undergo a spatial shift, the case termed *interacting* in Ref. [32]. This results in an *effective velocity*, which for Toda is solution of suitable integral equation [9]. To visualize this concept, one can define the quasiparticle position and velocity x_α, v_α as the center of mass of the Lax eigenvector [30]

$$v_\alpha = \dot{x}_\alpha = \frac{d}{dt}\sum_j \psi_{\alpha,j} q_j \psi_{\alpha,j} \equiv \sum_{ij} \psi_{\alpha,i} V_{i,j} \psi_{\alpha,j} \quad (7)$$

where V is symmetric, tridiagonal with diagonal elements p_j and upper diagonal $\frac{1}{2}a_j \log a_j$ [30]. At variance with λ_α this velocity is not constant due to the spatial shifts.

Conservative perturbations of the Toda chain – Upon multiplying both sides of Eq.(5) by L^{n-1}, and using cyclic and linearity properties of the trace

$$\frac{1}{n}\frac{d}{dt} trL^n = tr(UL^{n-1}). \quad (8)$$

So in general, the conservation laws of the eigenvalues are destroyed except for the case $n = 1$ for which $\sum_\alpha \lambda_\alpha$ is maintained in the class of momentum-conserving perturbations such that $\mathrm{tr}\, U = 0$. In the following, the energy-conserving case will be considered for which the phase-space divergence, $\sum_{ij,j\geq i} \frac{\partial \dot{L}_{ij}}{\partial L_{ij}} \equiv \mathrm{div}\, \dot{L} = \mathrm{div}\, U$, vanishes. Thus, the condition $divU = 0$ ensures that the perturbation is conservative.

Examples – In principle, for any Hamiltonian H one can recast the equation of motion in the form (5). However, for a generic perturbation, U may have a complicated dependence on the $L_{i,j}$: for instance, terms like r_j^p in Φ would yield entries proportional to $\log^{p-1} a_j$ [26]. It is thus useful to examine some simpler cases. The first is the *Morse chain* $\Phi(x) = (e^{-x/2} - \varepsilon)^2$ [33] then (up to a constant)

$$H_{\mathrm{Morse}} = H_{\mathrm{Toda}} + 2\varepsilon \sum_j \mathrm{e}^{-\frac{1}{2}(q_{j+1}-q_j)}, \quad (9)$$

corresponding to a perturbation matrix $U_{i,j} = 2\varepsilon(a_{j-1} - a_j)\,\delta_{ij}$ in Eq. (5). The second example is the Toda model with non-uniform couplings $1 + \varepsilon_j$ among neighbors, namely

$$H_{\mathrm{C}} = H_{\mathrm{Toda}} + \sum_j \varepsilon_j \mathrm{e}^{-(q_{j+1}-q_j)} \quad (10)$$

for which $U_{i,j} = \left(\varepsilon_{j-1}a_{j-1}^2 - \varepsilon_j a_j^2\right)\delta_{ij}$. For both examples, U is diagonal (with $\mathrm{tr}\, U = 0$) and its elements are, respectively, linear and quadratic in the a_j. Thus Eqs.(5) can be integrated directly, which has some computational advantage since only evaluation of algebraic functions is required [34].

Lax dynamics – The main idea is now to look at the time evolution of the λ_α that, under the effect of the perturbation U, are no longer constant [35]. For illustration, let us compare simulations of the Toda and Morse chains. In agreement with intuition, the space-time evolution of a Lax eigenvector $|\psi_{\alpha,n}(t)|^2$ [Fig. 1(a)] looks soliton-like in the integrable case, propagating ballistically with random space shifts, see Fig. 1(b) where the quasi-particle velocity $v_\alpha(t)$ is reported along with the corresponding (constant) λ_α. Plotting λ_α versus the time-averaged velocity $\langle v_\alpha \rangle$ [Fig. 1(c)] confirms an approximate correspondence between the two quantities, indicating that one can be used as a proxy of the other. Also, they become almost exactly proportional for soliton modes located closer to the Lax spectrum band edges.

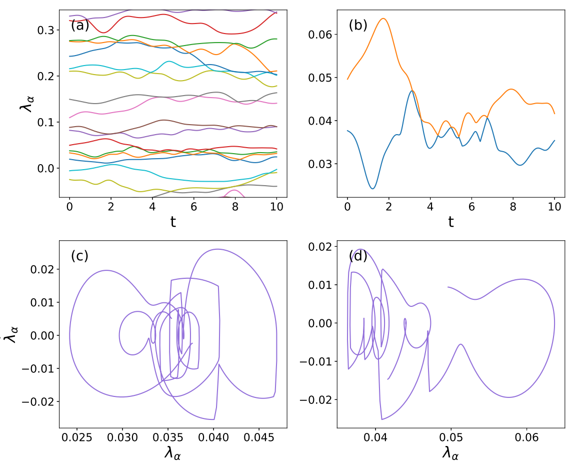

FIG. 2. Lax dynamics for the Morse chain, Eq. (9): time evolution of (a) a subset of eigenvalues $\lambda_\alpha(t)$ and (b) a couple of neighboring ones $\lambda_\alpha(t), \lambda_{\alpha+1}(t)$ illustrating the strong repulsion that yields almost elastic collisions ; (c,d): phase portraits $(\lambda_\alpha, \dot{\lambda}_\alpha)$ for $\alpha = 102, 103$. The chain is initialized with random initial conditions sampled from the Toda thermal GGE state with $\beta = P = 1$, $N = 200$, $\varepsilon = 0.1$.

Remarkably, the above picture remains similar also in presence of the perturbation. The main difference is that the almost-ballistic propagation is interrupted by huge scattering shifts, where the center of mass of the eigenvector very rapidly "jumps" to another site [Fig. 1(d)]. Also, Fig. 1(e,f) confirms that there is a close correspondence between the Lax eigenvalues and the effective velocities [36] .

The eigenvalues trajectories for the Morse chain [Fig. 2(a,b)] manifestly behave as a one-dimensional "gas" of particles. In the simulation, the gas remains confined in a bounded region, with no escape, at least on the considered time-scale. A remarkable feature is that two neighboring eigenvalues undergo almost elastic collisions, in which they approximately exchange their values, as clearly seen in Fig. 2(b). Accordingly, this induces relatively large changes in the velocity v_α which accounts for the large scattering shifts observed in Fig. 1(d). In the phase portraits this yields abrupt changes in $\dot{\lambda}_\alpha$, Fig. 2(c,d).

For an analytical formulation of Lax dynamics, one proceeds by computing $d(L\,|\alpha\rangle)/dt$ from Eq. (6) and

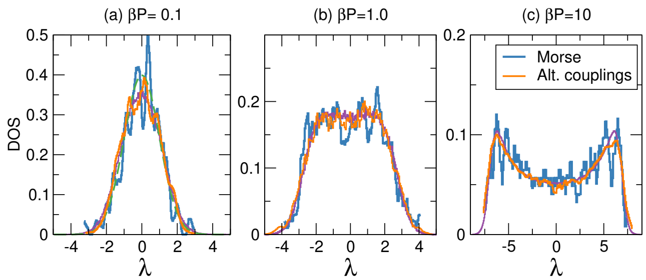

FIG. 3. The DOS $\rho(\lambda)$ obtained from the Lax dynamics of the Morse (blue) and the Toda chain with alternating (staggered) coupling $\varepsilon_j = (-1)^j\varepsilon$ (orange lines), ($N = 200$, $\varepsilon = 0.1$), starting with initial thermal GGE initial conditions with three different βP, $\beta = 1$. Magenta lines are the thermal DOS ρ_{th} for the unperturbed Toda chain, obtained by sampling and diagonalizing the equilibrium Lax matrix. The dashed green line in (a) is the approximate Gaussian DOS expected predicted in the limit of small βP [10].

using Eq. (5), yielding

$$U|\alpha\rangle + (L-\lambda_\alpha)\big(|\dot{\alpha}\rangle - B|\alpha\rangle\big) = \dot{\lambda}_\alpha|\alpha\rangle\,. \tag{11}$$

Upon left-multipying by $\langle\beta|$, using $\langle\alpha|\beta\rangle = \delta_{\alpha,\beta}$, and letting $\langle\alpha|U|\beta\rangle \equiv U_{\alpha\beta}$ one obtains from the diagonal elements that $\dot{\lambda}_\alpha = U_{\alpha\alpha}$ and from the non-diagonal elements the evolution equation for the eigenvectors

$$|\dot{\alpha}\rangle = B|\alpha\rangle + \sum_{\beta\neq\alpha}\frac{U_{\alpha\beta}}{\lambda_\alpha-\lambda_\beta}|\beta\rangle \tag{12}$$

which reduces to the unperturbed Toda evolution for $U = 0$, and is reminiscent of a quantum Hamiltonian with hopping terms induced by the perturbation.

Finally, one computes $\dot{U}_{\alpha\beta}$ and, using Eq.(12),

$$\begin{aligned}
\frac{d\lambda_\alpha}{dt} &= U_{\alpha\alpha} \equiv \pi_\alpha \\
\frac{d\pi_\alpha}{dt} &= F_{\alpha\alpha} + 2\sum_{\beta\neq\alpha}\frac{U^2_{\alpha\beta}}{\lambda_\alpha-\lambda_\beta} \\
\frac{dU_{\alpha\beta}}{dt} &= F_{\alpha\beta} + U_{\alpha\beta}\frac{U_{\beta\beta}-U_{\alpha\alpha}}{\lambda_\alpha-\lambda_\beta} + \\
&+ \sum_{\gamma\neq\alpha,\beta}U_{\alpha\gamma}U_{\gamma\beta}\left[\frac{1}{\lambda_\alpha-\lambda_\gamma}+\frac{1}{\lambda_\beta-\lambda_\gamma}\right] \qquad (\alpha\neq\beta)
\end{aligned} \tag{13}$$

where $F_{\alpha\beta} \equiv \langle\alpha|\left(\dot{U}+[U,B]\right)|\beta\rangle$. Eqs. (13) are exact and hold *whatever the form of* U. Also, using the invariance of the trace with respect to change of basis, if momentum conservation holds for also in presence of perturbation $trU = 0$, the the "momentum", $\sum_\alpha \pi_\alpha$ is conserved too.

If $F_{\alpha\beta}$ would vanish, Eqs. (13) would be *identical* to the so-called Pechukas-Yukawa (PY) equations ruling the evolution of the eigenenergies of a quantum Hamiltonian of the form $\mathcal{H} = \mathcal{H}_0 + t\mathcal{H}_1$, as a function of the fictitious "time" parameter t [37, 38]. Also, Eq. (12) would formally correspond to those for quantum eigenstates [39]. The PY equations are closed and define a $2N + N(N-1)/2$-dimensional dynamical system (U is symmetric) with generalized coordinates $(\lambda_\alpha, \pi_\alpha, U_{\alpha\beta})$. It has the remarkable features to be both Hamiltonian and fully integrable. Indeed, the change of variable $V_{\alpha\beta} = U_{\alpha\beta}(\lambda_\alpha - \lambda_\beta)$ ($\alpha \neq \beta$) transforms the PY equations in a generalized version of the famous Calogero-Moser model [39, 40].

Yet, the presence of the terms $F_{\alpha\beta}$ hinders an exact analysis as in the PY case: indeed, Eqs. (13) are not closed and, in general, are also explicitly time-dependent [41]. Despite those difficulties, some interesting physical consequences can be envisaged.

First, ignoring for the moment the terms $F_{\alpha\alpha}$, the first two of Eqs.(13) describe a one-dimensional Dyson-Coulomb gas coupled through the fluctuating "charges" $U^2_{\alpha\beta}$ provided by the remaining $N(N-1)/2$ degrees of freedom that act as a "heat bath" [42]. Indeed, for a finite number of levels, the DOS is approximatively given by the the invariant measure of the Coulomb gas under an external quadratic potential [38]. On the other hand, for the Toda chain, such measure coincides also with the thermal DOS ρ_{th} [10]. Indeed, based on the above heuristic consideration, one may surmise that the Lax dynamics naturally provides a general thermalization pathway towards the thermal DOS for any perturbation in the class of the above examples. In other words, the fluctuations of the charges provide the chaos (noise) source needed to thermalize the Dyson gas. It may be argued, that the generic mechanism leading to thermalization is provided by level repulsion and quasi-elastic scattering seen in Fig.2.

To support this idea, the data in Fig. 3, show the DOS $\rho(\lambda)$ for models (9) and (10) are basically independent of the choice of U. Moreover, for all the simulations considered here, the equilibrium DOS $\rho(\lambda) \approx \rho_{th}(\lambda)$, within statistical fluctuations. So, the thermal DOS of the unperturbed Toda accurately describes the DOS of the non-integrable models.

As a final remark about integrability, a numerical simulation of the Lax dynamics in simplest case $N = 3$ indicate that the trajectories are compatible with quasiperiodic motion on invariant tori, a hint that some form of integrability may occur also here. This certainly deserve a closer mathematical analysis.

Conclusions – Lax dynamics is a novel and insightful approach to describe the effect of perturbations on a many-body integrable system at finite energy density. Its implementation is computationally straightforward, and it has the potential to be extended to other systems, such as perturbations of integrable discretizations of the nonlinear Schrödinger equation [43], among others. It offers a physically appealing interpretation in terms of quasi-particle collisions as avoided eigenvalue crossings. Furthermore, in the case of weak perturbations, Lax dynamics is expected to evolve on a slower timescale compared to the natural timescale of the Flashka variables. Therefore, it could be an effective approach for studying the slow evolution of Toda actions in the context of thermalization problems [16, 19]. Finally, the analogy with the PY gas is highly suggestive and warrants a more detailed investigation. Equations (13) may allow for an effective, reduced dynamical description of the relevant quantities. The 'universal' evolution described by PY dynamics is one of the arguments used to justify the universal spectral statistics of quantum chaos. Could a similar consideration also apply in the present context? This could be one of the many possible research routes originating from the present work.

Acknowledgements – Support is acknowledged from the PRIN 2022 project *Breakdown of ergodicity in classical and quantum many-body systems* (BECQuMB) Grant No. 20222BHC9Z CUP G53C24000680006 funded by the European Union— NextGenerationEU, M4 C2 1.1. I thank L. Cugliandolo, S.Iubini, A. Politi and S. Ruffo for useful discussions.

* stefano.lepri@isc.cnr.it

[1] A. Bastianello, A. De Luca, and R. Vasseur, Hydrodynamics of weak integrability breaking, Journal of Statistical Mechanics: Theory and Experiment **2021**, 114003 (2021).

[2] B. Doyon, S. Gopalakrishnan, F. Møller, J. Schmiedmayer, and R. Vasseur, Generalized hydrodynamics: a perspective, Physical Review X **15**, 010501 (2025).

[3] M. Toda, *Theory of nonlinear lattices*, Vol. 20 (Springer Science & Business Media, 2012).

[4] M. Hénon, Integrals of the Toda lattice, Phys. Rev. B **9**, 1921 (1974).

[5] H. Flaschka, The Toda lattice. ii. existence of integrals, Physical Review B **9**, 1924 (1974).

[6] N. Theodorakopoulos, Finite-temperature excitations of the classical Toda chain, Physical review letters **53**, 871 (1984).

[7] A. Cuccoli, R. Livi, M. Spicci, V. Tognetti, and R. Vaia, Thermodynamics of the Toda chain, International Journal of Modern Physics B **8**, 2391 (1994).

[8] B. Doyon, Generalized hydrodynamics of the classical Toda system, Journal of Mathematical Physics **60**, 073302 (2019).

[9] H. Spohn, Generalized Gibbs ensembles of the classical Toda chain, Journal of Statistical Physics **180**, 4 (2020).

[10] H. Spohn, Hydrodynamic equations for the Toda lattice, arXiv preprint arXiv:2101.06528 (2021).

[11] M. Baldovin, A. Vulpiani, and G. Gradenigo, Statistical mechanics of an integrable system, Journal of Statistical Physics **183**, 41 (2021).

[12] T. Dauxois and M. Peyrard, *Physics of solitons* (Cambridge University Press, 2006).

[13] A. Kundu and A. Dhar, Equilibrium dynamical correlations in the Toda chain and other integrable models, Physical Review E **94**, 062130 (2016).

[14] G. Mazzuca, T. Grava, T. Kriecherbauer, K. T.-R. McLaughlin, C. B. Mendl, and H. Spohn, Equilibrium spacetime correlations of the Toda lattice on the hydrodynamic scale, Journal of Statistical Physics **190**, 149 (2023).

[15] G. Gallavotti, ed., *The Fermi-Pasta-Ulam Problem: A Status Report*, Lecture Notes in Physics, Vol. 728 (Springer, 2008).

[16] G. Benenti, D. Donadio, S. Lepri, and R. Livi, Non-Fourier heat transport in nanosystems, La Rivista del Nuovo Cimento **46**, 105 (2023).

[17] G. Benettin, H. Christodoulidi, and A. Ponno, The Fermi-Pasta-Ulam problem and its underlying integrable dynamics, Journal of Statistical Physics **152**, 195 (2013).

[18] G. Benettin and A. Ponno, Understanding the FPU state in FPU–like models, Mathematics in Engineering **3**, 1 (2021).

[19] T. Goldfriend and J. Kurchan, Equilibration of quasi-integrable systems, Physical Review E **99**, 022146 (2019).

[20] G. Benettin, G. Orsatti, and A. Ponno, On the role of the integrable Toda model in one-dimensional molecular dynamics, Journal of Statistical Physics **190**, 131 (2023).

[21] S. Chen, J. Wang, G. Casati, and G. Benenti, Nonintegrability and the Fourier heat conduction law, Physical Review E **90**, 032134 (2014).

[22] P. Di Cintio, S. Iubini, S. Lepri, and R. Livi, Transport in perturbed classical integrable systems: The pinned Toda chain, Chaos, Solitons & Fractals **117**, 249 (2018).

[23] A. Dhar, A. Kundu, J. L. Lebowitz, and J. A. Scaramazza, Transport properties of the classical Toda chain: effect of a pinning potential, Journal of Statistical Physics **175**, 1298 (2019).

[24] S. Lepri, R. Livi, and A. Politi, Too close to integrable: Crossover from normal to anomalous heat diffusion, Physical Review Letters **125**, 040604 (2020).

[25] O. Babelon, D. Bernard, and M. Talon, *Introduction to classical integrable systems* (Cambridge University Press, 2003).

[26] D. Bilman and I. Nenciu, On the evolution of scattering data under perturbations of the Toda lattice, Physica D: Nonlinear Phenomena **330**, 1 (2016).

[27] T. Goldfriend, Effective stochastic model for chaos in the Fermi–Pasta–Ulam–Tsingou chain, Journal of Statistical Physics **190**, 70 (2023).

[28] W. Ferguson Jr, H. Flaschka, and D. McLaughlin, Nonlinear normal modes for the Toda chain, Journal of computational physics **45**, 157 (1982).

[29] A. Politi, Heat conduction of the hard point chain at zero pressure, Journal of Statistical Mechanics: Theory and Experiment **2011**, P03028 (2011).

[30] V. B. Bulchandani, X. Cao, and J. E. Moore, Kinetic theory of quantum and classical Toda lattices, Journal of Physics A: Mathematical and Theoretical **52**, 33LT01 (2019).

[31] V. B. Bulchandani, M. Kulkarni, J. E. Moore, and X. Cao, Quasiparticle kinetic theory for Calogero models, Journal of Physics A: Mathematical and Theoreti-

cal **54**, 474001 (2021).

[32] H. Spohn, Interacting and noninteracting integrable systems, Journal of Mathematical Physics **59** (2018).

[33] A. Völkel, A. Cuccoli, M. Spicci, and V. Tognetti, Quantum thermodynamics of one-dimensional systems with Morse-type potentials, Physics Letters A **182**, 60 (1993).

[34] Another class of example studied in the literature is the Toda with non-homogeneous masses m_j [44, 45]. It can be shown that also this case yields a quadratic dependence of U on the a_j, but U is non-diagonal.

[35] To the best of my knowledge, this point of has been considered only very sporadically in the literature, see for example the case of the driven-dissipative Toda chain [46] and Toda with a point-like defect [47].

[36] Strictly speaking, expression (7) holds only for $U = 0$: in presence of the perturbation there is a correction due to additional scattering that we neglect for the sake of simplicity, since we just perform a qualitative comparison here.

[37] P. Pechukas, Distribution of energy eigenvalues in the irregular spectrum, Physical review letters **51**, 943 (1983).

[38] T. Yukawa, New approach to the statistical properties of energy levels, Physical review letters **54**, 1883 (1985).

[39] K. Nakamura and M. Lakshmanan, Complete integrability in a quantum description of chaotic systems, Physical review letters **57**, 1661 (1986).

[40] T. Yukawa, Lax form of the quantum mechanical eigenvalue problem, Physics Letters A **116**, 227 (1986).

[41] Indeed also in the quantum problem, a different (nonlinear) dependence of the Hamiltonian $\mathcal{H}(t)$ on the parameter t may yield additional non-autonomous terms, that can be handled analytically only is specific cases, see e.g. Ref.[48].

[42] X. Yang and J. Burgdörfer, Molecular-dynamics approach to the statistical properties of energy levels, Physical review letters **66**, 982 (1991).

[43] M. Salerno, Quantum deformations of the discrete nonlinear Schrödinger equation, Phys. Rev. A **46**, 6856 (1992).

[44] T. Hatano, Heat conduction in the diatomic Toda lattice revisited, Phys. Rev. E **59**, R1 (1999).

[45] J. Garnier and F. K. Abdullaev, Soliton dynamics in a random Toda chain, Physical Review E **67**, 026609 (2003).

[46] K. Fesser, D. McLaughlin, A. Bishop, and B. Holian, Chaos and nonlinear modes in a perturbed Toda chain, Physical Review A **31**, 2728 (1985).

[47] K. Geist and W. Lauterborn, Chaos upon soliton decay in a perturbed periodic Toda chain, Physica D: Nonlinear Phenomena **23**, 374 (1986).

[48] F. Haake and G. Lenz, Classical hamiltonian dynamics of rescaled quantum levels, Europhysics Letters **13**, 577 (1990).