Continuity Conditions for Piecewise Quadratic Functions on Simplicial Conic Partitions are Equivalent

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***Abstract*—Analysis of continuous-time piecewise linear (PWL) systems based on piecewise quadratic (PWQ) Lyapunov functions typically requires continuity of these functions over a partition of the state space. Several conditions for guaranteeing continuity of PWQ functions over state space partitions can be found in the literature. In this technical note, we show that these continuity conditions are equivalent over so-called simplicial conic partitions. A key element in our proof is a technical lemma, which, in addition to being of independent interest, plays a crucial role in demonstrating the equivalence of these conditions. As a consequence, the choice of which condition to impose can be based solely on practical considerations such as specific application or numerical aspects, without introducing additional conservatism in the analysis.**

***Index Terms*—Linear matrix inequalities (LMIs), Lyapunov methods, Piecewise linear (PWL) systems, Piecewise quadratic (PWQ) functions**

I. Introduction

Piecewise linear (PWL) systems represent a particular class of switched systems characterised by a partition of the state space into regions where the system dynamics can be described by linear models [1], [2]. PWL models have become useful within a wide range of applications, including nonsmooth mechanical systems, electrical circuits [3], hybrid control [4]–[7], model predictive control [8], nonlinear system approximation [9], dynamic optimisation in operations research and economics [10], and neural networks [11], to name but a few.

In this technical note, we consider piecewise quadratic (PWQ) functions, which are particularly useful for analysing e.g. stability of continuous-time PWL systems whose state space partition consists of *convex polyhedral cones*. In the literature, this class of systems is also known as *conewise linear systems* [12], [13]. Formally, a conewise linear system is described as

$$\dot{x} = A_i x, \text{ if } Cx \in \mathcal{S}_i, \tag{1}$$

where $x \in \mathbb{R}^n$ is the vector of states, $A_i \in \mathbb{R}^{n\times n}$, $i \in \mathcal{N} := \{1, 2, \dots, N\}$, and $C \in \mathbb{R}^{m\times n}$, $m \leq n$, are known system matrices, and $\mathcal{S}_i \subseteq \mathbb{R}^m$ are convex polyhedral cones. The collection of polyhedral cones $\mathcal{S}_i$, $i \in \mathcal{N}$, forms a partition of (a subset of) the state space.

Stability of conewise linear systems in (1) is often assessed using PWQ functions of the form

$$V(x) = V_i(x) = x^\top P_i x, \text{ when } Cx \in \mathcal{S}_i, \tag{2}$$

where $P_i = P_i^\top$. The reason for considering the class of PWQ functions for stability analysis of conewise linear systems, comes from the fact that they naturally generalise analysis of linear systems based on quadratic functions, and from their success in reducing conservatism in the analysis, see, e.g., [4], [6], [7], [13]–[17]. At the same time, their specific mathematical structure facilitates the analysis to be cast into linear matrix inequalities (LMIs), which can be solved systematically using numerical programs. Typical conditions for functions of the form (2) to provide a certificate for stability of the conewise linear system in (1) are formulated in terms of i) positive definiteness of V, i.e., $V(0) = 0$ and $V(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$, ii) negative definiteness of a suitable generalised time-derivative of V, and iii) (locally Lipschitz) continuity of V over adjacent cones in the partition, that is, $V_i(x) = V_j(x)$ for all $x \in \mathbb{R}^n$ with $Cx \in \mathcal{S}_i \cap \mathcal{S}_j$, $i, j \in \mathcal{N}$. For PWQ functions as in (2), locally Lipschitz continuity is guaranteed by continuity,

Condition i) and ii) are fairly standard and can be guaranteed by searching for matrices P_i that satisfy typical constraints of the form $P_i \succ 0$ and $A_i^\top P_i + P_i A_i \prec 0$, possibly appended with S-procedure relaxation terms [18] or formulated as a *cone-copositive* problem [13]. The arising LMIs can be effectively handled by numerical solvers [19], [20]. For guaranteeing continuity of the PWQ function over partitions, as stated in condition iii) above, several methods exist in the literature. These methods are either based on posing explicit equality constraints on the matrix P_i [4], [6], or on directly incorporating the continuity condition in the parametrisation of the matrix P_i [13], [14], [16]. Both approaches have advantages and disadvantages. For example, equality constraints can be applied to generic partitions but are difficult to solve numerically. The latter results from the fact that solvers work with finite precision and, therefore, return a solution that typically violates the equality constraints [21]. Solutions that deal with this numerical inaccuracy have been proposed in specific scenarios [6], [22], but the problem remains unsolved in general. On the other hand, using a specific matrix parametrisation removes the need for equality constraints, but the parametrisation may be difficult to con-

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struct. Despite these apparent differences, it turns out that for certain partitions based on *simplicial* cones, these approaches are *equivalent*. This equivalence result is not completely surprising – hints at this fact are found scattered across the literature, but has not been proven explicitly and rigorously before. In this technical note, we provide an overview of the various methods for guaranteeing continuity available in the literature, and show their equivalence explicitly. The value of this result lies in demonstrating that none of these approaches introduces additional conservatism in the analysis. Hence, choosing which approach is most suitable can be solely based on practical arguments such as implementation and numerical aspects. In addition to the equivalence result, we present a technical lemma inspired by *the non-strict projection lemma* [23]. Although of independent interest, this lemma will be instrumental in proving the aforementioned equivalence.

The remainder of this technical note is organised as follows. Preliminaries on cones and state space partitions are provided in Section II. A key technical lemma is presented in Section III. Different continuity conditions for PWQ functions are discussed in Section IV, and a proof of their equivalence is given in Section V. An illustrative example is presented in Section VI. Conclusions and suggestions for future work are provided in Section VII.

II. Preliminaries

To make the discussions in this paper precise, in this section, we introduce some mathematical notation used throughout the paper and review some definitions for cones and partitions.

A. Notation

The set of nonnegative real numbers is denoted $\mathbb{R}_{\geq 0} := \{x \in \mathbb{R} \mid x \geq 0\}$. The set of vectors in $\mathbb{R}^n$ (matrices in $\mathbb{R}^{m\times n}$) whose elements are nonnegative real numbers is denoted $\mathbb{R}^n_{\geq 0}$ ($\mathbb{R}^{m\times n}_{\geq 0}$). The set of symmetric matrices in $\mathbb{R}^{n\times n}$ is denoted $\mathbb{S}^n := \{A \in \mathbb{R}^{n\times n} \mid A = A^\top\}$. A positive (semi) definite matrix is denoted $P \succ 0$ ($P \succeq 0$). Similarly, a negative (semi) definite matrix is denoted $P \prec 0$ ($P \preceq 0$). The transpose of a matrix inverse $\left(A^{-1}\right)^\top$ is compactly written as $A^{-\top}$. The symbol $\star$ is used to complete a symmetric matrix. Given a matrix $A \in \mathbb{R}^{m\times n}$, its image is denoted $\operatorname{im} A := \{Av \mid v \in \mathbb{R}^n\}$, its kernel is denoted $\ker A := \{v \in \mathbb{R}^n \mid Av = 0\}$, its (Moore-Penrose) pseudoinverse is denoted A^+, and $A_\perp$ denotes any matrix whose columns form a basis of $\ker A$, and thus, $AA_\perp = 0$. The interior of a set $\mathcal{S}$ is denoted $\operatorname{int}(\mathcal{S})$.

B. Cones and partitions

Given a set of K vectors $z_k \in \mathbb{R}^m$, $k \in \{1, 2, \ldots, K\}$, its *positive hull* (or *conical hull*) is the set of vectors $z \in \mathbb{R}^m$ such that $z = \sum_{k=1}^K \lambda_k z_k$, with $\lambda_k \geq 0$. A set $\mathcal{C} \subseteq \mathbb{R}^m$ is a *polyhedral cone*, if it is the positive hull of a finite set of vectors. A *face* of a polyhedral cone $\mathcal{C}$, is any set of the form $\mathcal{F} = \mathcal{C} \cap \{z \in \mathbb{R}^m \mid c^\top z = c_0\}$, $c_0 \in \mathbb{R}$, that also satisfies $c^\top z \leq c_0$ for all $z \in \mathcal{C}$. A set $\mathcal{S} \subseteq \mathbb{R}^m$ is a *simplicial cone*, if it is the positive hull of m linearly independent vectors.[1]

[1]The standard definition of a simplicial cone considers the positive hull of (any number of) linearly independent vectors, see, e.g., [24, Definition 1.39].

Polyhedral cones with nonempty interior can always be partitioned into a finite number of simplicial cones [24, Lemma 1.40]. For that reason, without loss of generality, in the remainder of this paper, we solely focus our attention to simplicial cones. Hence, the dynamics of the conewise linear system in (1) is considered to be defined over simplical cones, that is, the cones $\mathcal{S}_i$, $i \in \mathcal{N}$ in (1) are considered to be simplicial. Given a simplicial cone $\mathcal{S} \subseteq \mathbb{R}^m$ there exists a nonsingular matrix $R \in \mathbb{R}^{m\times m}$, such that $\mathcal{S} = \{R\lambda \mid \lambda \in \mathbb{R}^m_{\geq 0}\}$. The matrix R is called an *extremal ray matrix* of the simplicial cone $\mathcal{S}$. The fact that R is nonsingular follows from our definition of a simplicial cone. The columns of R define the so-called *extremal rays* of the simplicial cone and are uniquely defined up to a positive multiple. The set of extremal rays of a simplicial cone $\mathcal{S}$ is denoted $\mathcal{R}_\mathcal{S}$.

Given a set $\mathcal{Z} \subseteq \mathbb{R}^m$ and a finite positive integer N, a *simplicial conic partition* of $\mathcal{Z}$ is a family $\{\mathcal{S}_h\}_{h=1}^N$ of simplicial cones satisfying $\mathcal{Z} = \bigcup_{h=1}^p \mathcal{S}_h$, with $\operatorname{int}(\mathcal{S}_i) \neq \emptyset$ for all $i \in \mathcal{N}$ and $\operatorname{int}(\mathcal{S}_i) \cap \operatorname{int}(\mathcal{S}_j) = \emptyset$ for $i, j \in \mathcal{N}, i \neq j$.

We define the extremal ray matrices of a given simplicial conic partition $\{\mathcal{S}_h\}_{h=1}^p$, as a matrix $\bar{R} \in \mathbb{R}^{m\times r}$ with $r \geq m$ whose columns are all *distinct* extremal rays r_j of the simplicial conic partition $\{\mathcal{S}_h\}_{h=1}^p$. For each simplicial cone $\mathcal{S}_i \in \{\mathcal{S}_h\}_{h=1}^p$ define a so-called extraction matrix $E_i \in \mathbb{R}^{r\times m}$ having its j-th row equal to zero for all $r_j \notin \mathcal{S}_i$, and the remaining rows equal to the rows of the m-dimensional identity matrix. Then, the extremal ray matrix of $\mathcal{S}_i$ is given by $R_i = \bar{R}E_i$, see also [13], [14].

The following assumption is made for simplicial conic partitions throughout this paper.

Assumption 1. *The boundary shared by any two cones of a simplicial conic partition, i.e.,* $\mathcal{S}_i \cap \mathcal{S}_j$*, is a face of both.*

A direct result of Assumption 1, is that the extremal rays of the boundary $\mathcal{S}_i \cap \mathcal{S}_j$, are equal to the extremal rays shared by the two cones, i.e., $\mathcal{R}_{\mathcal{S}_i\cap\mathcal{S}_j} = \mathcal{R}_{\mathcal{S}_i} \cap \mathcal{R}_{\mathcal{S}_j}$. Let the matrix Z_{ij} be a matrix whose columns are equal to the shared extremal rays, i.e., equal to the elements in $\mathcal{R}_{\mathcal{S}_i\cap\mathcal{S}_j}$. Let $H_{ij} := ((Z_{ij}^\top)_\perp)^\top$, such that $Z_{ij} = (H_{ij})_\perp$, and thus, $H_{ij}Z_{ij} = 0$. By definition, the matrix Z_{ij} is tall and has full column rank, whereas the matrix H_{ij} is wide and has full row rank. With the previous definitions, and due to Assumption 1, the boundary shared by two simplicial cones, $\mathcal{S}_i \cap \mathcal{S}_j$, satisfies

$$\mathcal{S}_i \cap \mathcal{S}_j = \{Z_{ij}v \mid v \geq 0\} \subseteq \operatorname{im} Z_{ij} = \ker H_{ij}. \tag{3}$$

III. Technical Lemma

In this section, we present a technical lemma in the spirit of the non-strict projection lemma in [23]. Although this lemma is of independent interest, it will be particularly useful in proving equivalence of the continuity conditions presented in Section IV.

Lemma 1. *Let* $\boldsymbol{U} \in \mathbb{R}^{m\times n}$*,* $\boldsymbol{V} \in \mathbb{R}^{p\times n}$ *and* $\boldsymbol{Q} \in \mathbb{S}^n$*. Consider the following statements:*

(L1.1) There exists a matrix $\boldsymbol{X} \in \mathbb{R}^{m\times p}$ *such that*

$$\boldsymbol{Q} + \boldsymbol{U}^\top \boldsymbol{X} \boldsymbol{V} + \boldsymbol{V}^\top \boldsymbol{X}^\top \boldsymbol{U} = 0;$$

(L1.2) $x^\top \boldsymbol{Q} x = 0$ *for all* $x \in \ker \boldsymbol{U} \cup \ker \boldsymbol{V}$;
(L1.3) $\ker \boldsymbol{U} \cap \ker \boldsymbol{V} \subset \ker \boldsymbol{Q}$.

Then, (L1.1) holds if and only if (L1.2) and (L1.3) hold.

Proof. **Necessity:** Suppose that (L1.1) holds. Then, using the fact that either $\boldsymbol{U}x = 0$ or $\boldsymbol{V}x = 0$ when $x \in \ker \boldsymbol{U} \cup \ker \boldsymbol{V}$, it follows that (L1.2) holds. Due to (L1.1), it holds, for any $x \in \mathbb{R}^n$, that

$$\left(\boldsymbol{Q} + \boldsymbol{U}^\top \boldsymbol{X} \boldsymbol{V} + \boldsymbol{V}^\top \boldsymbol{X}^\top \boldsymbol{U}\right) x = 0. \tag{4}$$

Let $x \in \ker \boldsymbol{U} \cap \ker \boldsymbol{V}$. Then, it holds that

$$\left(\boldsymbol{Q} + \boldsymbol{U}^\top \boldsymbol{X} \boldsymbol{V} + \boldsymbol{V}^\top \boldsymbol{X}^\top \boldsymbol{U}\right) x = \boldsymbol{Q} x \overset{(4)}{=} 0, \tag{5}$$

i.e., (L1.3) holds.

Sufficiency: Suppose (L1.2) and (L1.3) hold. Let $\boldsymbol{T} \in \mathbb{R}^{n \times n}$ be a nonsingular matrix, whose columns in the partition $\boldsymbol{T} = \begin{bmatrix} \boldsymbol{T}_1 & \boldsymbol{T}_2 & \boldsymbol{T}_3 & \boldsymbol{T}_4 \end{bmatrix}$ are chosen to satisfy

$$\operatorname{im} \begin{bmatrix} \boldsymbol{T}_1 & \boldsymbol{T}_3 \end{bmatrix} = \ker \boldsymbol{U}, \tag{6}$$

$$\operatorname{im} \begin{bmatrix} \boldsymbol{T}_2 & \boldsymbol{T}_3 \end{bmatrix} = \ker \boldsymbol{V}, \tag{7}$$

$$\operatorname{im} \boldsymbol{T}_3 = \ker \boldsymbol{U} \cap \ker \boldsymbol{V}. \tag{8}$$

Clearly, (L1.1) is equivalent to the existence of $\boldsymbol{X} \in \mathbb{R}^{m \times p}$ such that

$$\boldsymbol{Y} := \boldsymbol{T}^\top \left(\boldsymbol{Q} + \boldsymbol{U}^\top \boldsymbol{X} \boldsymbol{V} + \boldsymbol{V}^\top \boldsymbol{X}^\top \boldsymbol{U}\right) \boldsymbol{T} = 0. \tag{9}$$

We partition $\boldsymbol{W} := \boldsymbol{T}^\top \boldsymbol{Q} \boldsymbol{T}$ in accordance with $\boldsymbol{T}$ to obtain

$$\boldsymbol{W} = \begin{bmatrix} \boldsymbol{W}_{11} & \boldsymbol{W}_{12} & \boldsymbol{W}_{13} & \boldsymbol{W}_{14} \\ \boldsymbol{W}_{12}^\top & \boldsymbol{W}_{22} & \boldsymbol{W}_{23} & \boldsymbol{W}_{24} \\ \boldsymbol{W}_{13}^\top & \boldsymbol{W}_{23}^\top & \boldsymbol{W}_{33} & \boldsymbol{W}_{34} \\ \boldsymbol{W}_{14}^\top & \boldsymbol{W}_{24}^\top & \boldsymbol{W}_{34}^\top & \boldsymbol{W}_{44} \end{bmatrix}. \tag{10}$$

Using (6), (7) and (8), we write the term $(\boldsymbol{U}\boldsymbol{T})^\top \boldsymbol{X} (\boldsymbol{V}\boldsymbol{T})$ in (9) as

$$\begin{bmatrix} \boldsymbol{U}\boldsymbol{T}_2 & \boldsymbol{U}\boldsymbol{T}_4 \end{bmatrix}^\top \boldsymbol{X} \begin{bmatrix} \boldsymbol{V}\boldsymbol{T}_1 & \boldsymbol{V}\boldsymbol{T}_4 \end{bmatrix} =: \begin{bmatrix} \boldsymbol{K} & \boldsymbol{L} \\ \boldsymbol{M} & \boldsymbol{N} \end{bmatrix}, \tag{11}$$

where, due to (6) and (7), $\begin{bmatrix} \boldsymbol{U}\boldsymbol{T}_2 & \boldsymbol{U}\boldsymbol{T}_4 \end{bmatrix}$ and $\begin{bmatrix} \boldsymbol{V}\boldsymbol{T}_1 & \boldsymbol{V}\boldsymbol{T}_4 \end{bmatrix}$ have full column rank. Hence, using (10) and (11), (9) reads as

$$\begin{aligned} \boldsymbol{Y} &= \left[\begin{array}{c|c} \boldsymbol{Y}_1 & \boldsymbol{Y}_2 \\ \hline \boldsymbol{Y}_2^\top & \boldsymbol{Y}_3 \end{array}\right] \\ &= \left[\begin{array}{ccc|c} \boldsymbol{W}_{11} & \boldsymbol{W}_{12} + \boldsymbol{K}^\top & \boldsymbol{W}_{13} & \boldsymbol{W}_{14} + \boldsymbol{M}^\top \\ \boldsymbol{W}_{12}^\top + \boldsymbol{K} & \boldsymbol{W}_{22} & \boldsymbol{W}_{23} & \boldsymbol{W}_{24} + \boldsymbol{L} \\ \boldsymbol{W}_{13}^\top & \boldsymbol{W}_{23}^\top & \boldsymbol{W}_{33} & \boldsymbol{W}_{34} \\ \hline \boldsymbol{W}_{14}^\top + \boldsymbol{M} & \boldsymbol{W}_{24}^\top + \boldsymbol{L}^\top & \boldsymbol{W}_{34}^\top & \boldsymbol{W}_{44} + \boldsymbol{N} + \boldsymbol{N}^\top \end{array}\right] \\ &= 0. \end{aligned} \tag{12}$$

It follows from (L1.2) that

$$\begin{bmatrix} \boldsymbol{W}_{11} & \boldsymbol{W}_{13} \\ \boldsymbol{W}_{13}^\top & \boldsymbol{W}_{33} \end{bmatrix} = 0 \text{ and } \begin{bmatrix} \boldsymbol{W}_{22} & \boldsymbol{W}_{23} \\ \boldsymbol{W}_{23}^\top & \boldsymbol{W}_{33} \end{bmatrix} = 0. \tag{13}$$

Clearly, to ensure that $\boldsymbol{Y}_1 = 0$, we should construct $\boldsymbol{X}$ such that $\boldsymbol{K} = -\boldsymbol{W}_{12}^\top$. Similarly, we will aim to construct $\boldsymbol{X}$ such that $\boldsymbol{L} = -\boldsymbol{W}_{24}$, $\boldsymbol{M} = -\boldsymbol{W}_{14}^\top$ and $\boldsymbol{N} = -\frac{1}{2}\boldsymbol{W}_{44}$. Note that, due to $\begin{bmatrix} \boldsymbol{U}\boldsymbol{T}_2 & \boldsymbol{U}\boldsymbol{T}_4 \end{bmatrix}$ and $\begin{bmatrix} \boldsymbol{V}\boldsymbol{T}_1 & \boldsymbol{V}\boldsymbol{T}_4 \end{bmatrix}$ having full column rank, we can construct such $\boldsymbol{X}$ by taking

$$\begin{aligned} \boldsymbol{X} &= \begin{bmatrix} (\boldsymbol{U}\boldsymbol{T}_2)^\top \\ (\boldsymbol{U}\boldsymbol{T}_4)^\top \end{bmatrix}^+ \begin{bmatrix} \boldsymbol{K} & \boldsymbol{L} \\ \boldsymbol{M} & \boldsymbol{N} \end{bmatrix} \begin{bmatrix} \boldsymbol{V}\boldsymbol{T}_1 & \boldsymbol{V}\boldsymbol{T}_4 \end{bmatrix}^+ \\ &= \begin{bmatrix} (\boldsymbol{U}\boldsymbol{T}_2)^\top \\ (\boldsymbol{U}\boldsymbol{T}_4)^\top \end{bmatrix}^+ \begin{bmatrix} -\boldsymbol{W}_{12}^\top & -\boldsymbol{W}_{24} \\ -\boldsymbol{W}_{14}^\top & -\frac{1}{2}\boldsymbol{W}_{44} \end{bmatrix} \begin{bmatrix} \boldsymbol{V}\boldsymbol{T}_1 & \boldsymbol{V}\boldsymbol{T}_4 \end{bmatrix}^+. \end{aligned} \tag{14}$$

Note that all entries now equal zero except for $\boldsymbol{W}_{34}$. Hence, it remains to show that $\boldsymbol{W}_{34} = 0$. It follows from (L1.3) that $\boldsymbol{Q}\boldsymbol{T}_3 = 0$ and, thus, $\boldsymbol{W}_{34}^\top = \boldsymbol{T}_4^\top \boldsymbol{Q} \boldsymbol{T}_3 = 0$. □

Lemma 1 is closely related to the non-strict projection lemma [23], but it deals with equalities instead of (matrix) inequalities. Interestingly, it turns out that, as in the non-strict projection lemma, an additional coupling condition (L1.3) is needed to achieve the equivalence in Lemma 1.

Next, we introduce two useful corollaries of Lemma 1.

Corollary 1. *Let* $\boldsymbol{U} \in \mathbb{R}^{m \times n}$ *and let* $\boldsymbol{Q} \in \mathbb{S}^n$. *Consider the following statements:*

(C1.1) There exists a matrix $\boldsymbol{X} \in \mathbb{R}^{m \times n}$ *such that*

$$\boldsymbol{Q} + \boldsymbol{U}^\top \boldsymbol{X} + \boldsymbol{X}^\top \boldsymbol{U} = 0;$$

(C1.2) $x^\top \boldsymbol{Q} x = 0$ *for all* $x \in \ker \boldsymbol{U}$.

Then, (C1.1) holds if and only if (C1.2) holds.

Corollary 1 follows from Lemma 1 with $\boldsymbol{V} := I_n$. To see this, note that $\ker \boldsymbol{V} = \{0\}$, and thus, (L1.3) trivially holds.

Corollary 2. *Let* $\boldsymbol{U} \in \mathbb{R}^{m \times n}$ *and let* $\boldsymbol{Q} \in \mathbb{S}^n$. *Consider the following statements:*

(C2.1) There exists a symmetric matrix $\boldsymbol{X} \in \mathbb{S}^m$ *such that*

$$\boldsymbol{Q} + \boldsymbol{U}^\top \boldsymbol{X} \boldsymbol{U} = 0;$$

(C2.2) $\ker \boldsymbol{U} \subset \ker \boldsymbol{Q}$.

Then, (C2.1) holds if and only if (C2.2) holds.

Corollary 2 follows from Lemma 1 with $\boldsymbol{V} := \frac{1}{2}\boldsymbol{U}$, in which case $\ker \boldsymbol{U} = \ker \boldsymbol{V}$. Thus, (L1.3) simplifies to (C2.2), which immediately implies (L1.2). Corollary 2 is closely related to the non-strict finsler's lemma [25], but it deals with equalities instead of (matrix) inequalities.

IV. Continuity Conditions

In this section, we formalise the equivalence of different conditions that can be found in the literature for guaranteeing continuity of a PWQ function over state space partitions. In particular, we consider PWQ functions of the form as in (2), i.e.,

$$V(x) = x^\top P_i x, \text{ when } Cx \in \mathcal{S}_i, \tag{15}$$

where $P_i \in \mathbb{S}^n$, $i \in \mathcal{N}$, $C \in \mathbb{R}^{m \times n}$ has full row rank, and $\mathcal{S}_i$ are simplicial cones. We want to guarantee continuity of these PWQ functions and thus, locally Lipschitz continuity, in order for it to be useful in stability analysis.

Before stating the main theorem, it is emphasised that definitions from Section II are used, e.g., for the matrices Z_{ij}, H_{ij}, E_i, and R_i.

Theorem 1. *Let $\mathcal{N} := \{1, 2, \dots, N\}$. Consider a simplicial conic partition $\{\mathcal{S}_i\}_{i=1}^N$ of a set $\mathcal{Z} \subseteq \mathbb{R}^m$, and a set $\{P_i\}_{i=1}^N$ of symmetric matrices $P_i \in \mathbb{S}^n$, $i \in \mathcal{N}$. Then, the following statements are equivalent:*

(T1.1) The matrices $P_1, P_2, \dots, P_N$ satisfy, for all $i, j \in \mathcal{N}$

$$x^\top (P_i - P_j)x = 0, \text{ for all } Cx \in \mathcal{S}_i \cap \mathcal{S}_j, \quad (16)$$

and thus, the function $V(x)$ as in (15) is continuous.

(T1.2) The matrices $P_1, P_2, \dots, P_N$ satisfy, for all $i, j \in \mathcal{N}$

$$x^\top (P_i - P_j)x = 0, \text{ for all } Cx \in \operatorname{im} Z_{ij}. \quad (17)$$

(T1.3) Let

$$W_{ij} = T^{-1} \begin{bmatrix} Z_{ij} & 0 \\ 0 & I \end{bmatrix}, \text{ with } T := \begin{bmatrix} C \\ C_\perp^\top \end{bmatrix}.$$

For all $i, j \in \mathcal{N}$, it holds that

$$W_{ij}^\top (P_i - P_j) W_{ij} = 0. \quad (18)$$

(T1.4) Let

$$F_i = \begin{bmatrix} E_i R_i^{-1} C \\ V \end{bmatrix},$$

where V is any matrix that satisfies $\operatorname{im} V^\top \supseteq \operatorname{im} C_\perp$. There exists a symmetric matrix Φ, such that, for all $i \in \mathcal{N}$

$$P_i = F_i^\top \Phi F_i. \quad (19)$$

(T1.5) There exist matrices Γ_{ij}, for all $i, j \in \mathcal{N}$, such that

$$P_i - P_j + (H_{ij}C)^\top \Gamma_{ij} + \Gamma_{ij}^\top (H_{ij}C) = 0. \quad (20)$$

In the next section, we will give an explicit proof of the equivalence in Theorem 1. However, before continuing with the proof, we provide a few comments and discussions on the various elements of Theorem 1:

1) Item (T1.1) expresses necessary and sufficient conditions for continuity of a PWQ function as in (15), over generic state space partitions. However, we will only show its equivalence with the other conditions, (T1.2)-(T1.5), over simplicial conic partitions. Hence, in general, equivalence may not be guaranteed. Conditions (16) and (17) require checking an infinite number of equalities, that is, one for each $x \in \mathbb{R}^n$. On the other hand, (T1.3)–(T1.5) express continuity conditions in terms of computationally tractable conditions on the matrices P_i directly.
2) An example illustrating the difference between the sets $\mathcal{S}_i \cap \mathcal{S}_j$ and $\operatorname{im} Z_{ij}$, used in (T1.1) and (T1.2), is shown in Fig. 1. The set $\operatorname{im} Z_{ij}$ is the minimal linear subspace of $\mathbb{R}^m$ that contains $\mathcal{S}_i \cap \mathcal{S}_j$ (minimal in the sense that its dimension is equal to the dimension of $\mathcal{S}_i \cap \mathcal{S}_j$, or equivalently, that $\operatorname{im} Z_{ij}$ is equal to the intersection of all possible linear subspaces of $\mathbb{R}^m$ that contain $\mathcal{S}_i \cap \mathcal{S}_j$). We can say that $\operatorname{im} Z_{ij}$ is the so-called *linear hull* of the boundary region $\mathcal{S}_i \cap \mathcal{S}_j$. Surprisingly, the equivalence between (T1.1) and (T1.2) means that, for PWQ functions, continuity on the boundary, $\mathcal{S}_i \cap \mathcal{S}_j$, is equivalent to continuity on the whole (generalised) plane (of some dimension) containing $\mathcal{S}_i \cap \mathcal{S}_j$.
3) Continuity conditions of the form presented in (T1.3) are used in, e.g., [4], [6]. The equality constraint in (18) is simple to formulate, but generally difficult to solve numerically. The reason for this, is that solvers work with finite precision and, as a result, often return solutions that slightly violate the equality constraints (see [21, Section 4.5.2]). Note that the matrix $T = \begin{bmatrix} C^\top & C_\perp \end{bmatrix}^\top$ is nonsingular, as we assume, without loss of generality, that C has full row rank.
4) The parametrisation of P_i in (T1.4) was first proposed in [14], and has been used successfully, e.g. in [13], [16]. This parametrisation removes the need for explicit equality constraints, which may provide a significant advantage from a computational point of view [17]. For the matrix V, the most obvious choice is $V = C_\perp^\top$. This choice minimises the number of parameters in the matrix Φ, which can be numerically beneficial. Another simple choice is $V = I$, which avoids the need to compute the matrix $C_\perp$, but leads to more parameters in Φ.

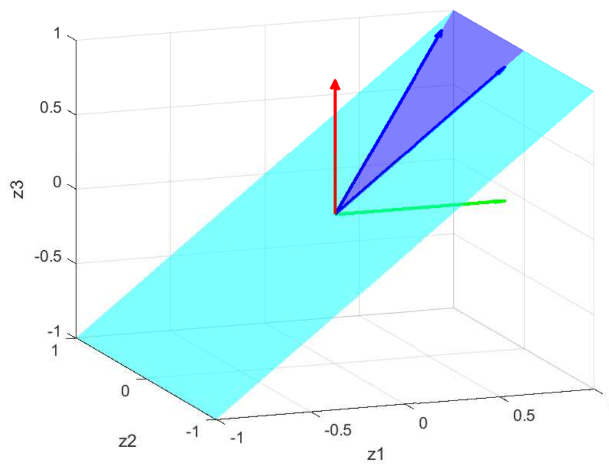

Fig. 1: Example illustrating the difference between the regions considered in the first two items of Theorem 1. The figure includes the distinct extremal rays of two example cones, $\mathcal{S}_1$ and $\mathcal{S}_2$ (in red and green), their shared extremal rays (in blue), their boundary region, $\mathcal{S}_1 \cap \mathcal{S}_2$ (also in blue), and the extended boundary region, $\operatorname{im} Z_{ij}$ (in cyan).

V. Proof of Theorem 1

In this section, we present the proof of Theorem 1. The proof is carried out in the following order: $(T1.1) \iff (T1.2)$, $(T1.2) \iff (T1.3)$, $(T1.2) \iff (T1.5)$, $(T1.1) \implies (T1.4)$, $(T1.4) \implies (T1.1)$.

$(T1.1) \iff (T1.2)$. Since $\mathcal{S}_i \cap \mathcal{S}_j \subseteq \operatorname{im} Z_{ij}$, the necessity, $(T1.1) \impliedby (T1.2)$, is trivial. Hence, the focus is on the sufficiency, $(T1.1) \implies (T1.2)$. Suppose that $x^\top (P_i - P_j)x = 0$ for all $Cx \in \mathcal{S}_i \cap \mathcal{S}_j$. Consider the state transformation $\bar{x} = Tx$, with $T = \begin{bmatrix} C^\top & C_\perp \end{bmatrix}^\top$, such that

$\bar{x} = \begin{bmatrix} z^\top & \hat{x}^\top \end{bmatrix}^\top$, with $z = Cx \in \mathbb{R}^m$ and $\hat{x} = C_\perp^\top x \in \mathbb{R}^{n-m}$. Now, partition the matrix $\bar{P}_i := T^{-\top} P_i T^{-1}$ as

$$\bar{P}_i = \bar{P}_i^\top = \begin{bmatrix} \bar{P}_i^{11} & \star \\ \bar{P}_i^{21} & \bar{P}_i^{22} \end{bmatrix}, \tag{21}$$

according to $(z, \hat{x})$, such that

$$\begin{aligned} 0 &= x^\top (P_i - P_j) x = \bar{x}^\top \left(\bar{P}_i - \bar{P}_j \right) \bar{x} \\ &= \begin{bmatrix} z \\ \hat{x} \end{bmatrix}^\top \left(\begin{bmatrix} \bar{P}_i^{11} & \star \\ \bar{P}_i^{21} & \bar{P}_i^{22} \end{bmatrix} - \begin{bmatrix} \bar{P}_j^{11} & \star \\ \bar{P}_j^{21} & \bar{P}_j^{22} \end{bmatrix} \right) \begin{bmatrix} z \\ \hat{x} \end{bmatrix} \\ &= z^\top (\bar{P}_i^{11} - \bar{P}_j^{11}) z + 2\hat{x}^\top (\bar{P}_i^{21} - \bar{P}_j^{21}) z \\ &\quad + \hat{x}^\top (\bar{P}_i^{22} - \bar{P}_j^{22}) \hat{x}, \end{aligned} \tag{22}$$

for all $z \in \mathcal{S}_i \cap \mathcal{S}_j$ and all $\hat{x} \in \mathbb{R}^{n-m}$. Since $0 \in \mathcal{S}_i \cap \mathcal{S}_j$, (22) may be evaluated separately for $z = 0$ and $\hat{x} = 0$. As such, one finds that

$$z^\top (\bar{P}_i^{11} - \bar{P}_j^{11}) z = 0, \text{ for all } z \in \mathcal{S}_i \cap \mathcal{S}_j, \tag{23a}$$

$$\hat{x}^\top (\bar{P}_i^{21} - \bar{P}_j^{21}) z = 0, \text{ for all } z \in \mathcal{S}_i \cap \mathcal{S}_j,\ \hat{x} \in \mathbb{R}^{n-m}, \tag{23b}$$

$$\hat{x}^\top (\bar{P}_i^{22} - \bar{P}_j^{22}) \hat{x} = 0, \text{ for all } \hat{x} \in \mathbb{R}^{n-m}. \tag{23c}$$

Firstly, (23c) implies, due to symmetry of $\bar{P}_i^{22} - \bar{P}_j^{22}$, that

$$\bar{P}_i^{22} - \bar{P}_j^{22} = 0. \tag{24}$$

Secondly, since $\mathcal{R}_{\mathcal{S}_i \cap \mathcal{S}_j} \subseteq \mathcal{S}_i \cap \mathcal{S}_j$, it follows from (23b) that $\hat{x}^\top (\bar{P}_i^{21} - \bar{P}_j^{21}) r = 0$ for every $r \in \mathcal{R}_{\mathcal{S}_i \cap \mathcal{S}_j}$ and all $\hat{x} \in \mathbb{R}^{n-m}$. As such,

$$(\bar{P}_i^{21} - \bar{P}_j^{21}) r = 0, \text{ for every } r \in \mathcal{R}_{\mathcal{S}_i \cap \mathcal{S}_j}. \tag{25}$$

Thirdly, it follows from (23a) that

$$r^\top (\bar{P}_i^{11} - \bar{P}_j^{11}) r = 0, \text{ for every } r \in \mathcal{R}_{\mathcal{S}_i \cap \mathcal{S}_j}. \tag{26}$$

Note that

$$r_m + r_n \subseteq \mathcal{S}_i \cap \mathcal{S}_j, \text{ for every } r_m, r_n \in \mathcal{R}_{\mathcal{S}_i \cap \mathcal{S}_j}, \tag{27}$$

that is, the sum of two shared extremal rays is contained in the boundary region $\mathcal{S}_i \cap \mathcal{S}_j$. As such, by substituting $z = r_m + r_n$ into (23a), it follows that, for every $r_m, r_n \in \mathcal{R}_{\mathcal{S}_i \cap \mathcal{S}_j}$

$$\begin{aligned} 0 &\overset{(23a)}{=} z^\top (\bar{P}_i^{11} - \bar{P}_j^{11}) z \\ &= (r_m + r_n)^\top (\bar{P}_i^{11} - \bar{P}_j^{11}) (r_m + r_n) \\ &= r_m^\top (\bar{P}_i^{11} - \bar{P}_j^{11}) r_m + r_n^\top (\bar{P}_i^{11} - \bar{P}_j^{11}) r_n \\ &\quad + 2 r_m^\top (\bar{P}_i^{11} - \bar{P}_j^{11}) r_n \\ &\overset{(26)}{=} 2 r_m^\top (\bar{P}_i^{11} - \bar{P}_j^{11}) r_n. \end{aligned} \tag{28}$$

Note that (26) is a special case of (28). Finally, to prove (T1.2), consider $z \in \operatorname{im} Z_{ij} \supseteq \mathcal{S}_i \cap \mathcal{S}_j$. By construction, the columns of Z_{ij} are equal to the elements of $\mathcal{R}_{\mathcal{S}_i \cap \mathcal{S}_j}$ (see Section II). Thus, $z \in \operatorname{im} Z_{ij}$ if and only if there exist numbers $v_m \in \mathbb{R}$ such that $z = \sum_m v_m r_m$, where the sum is taken over all $r_m \in \mathcal{R}_{\mathcal{S}_i \cap \mathcal{S}_j}$. Now, substitute $z = \sum_m v_m r_m$ into (22), such that, for all $z \in \operatorname{im} Z_{ij}$, or equivalently, for all $v_m \in \mathbb{R}$ and all $\hat{x} \in \mathbb{R}^{n-m}$, one finds that

$$\begin{aligned} x^\top (P_i - P_j) x &= z^\top (\bar{P}_i^{11} - \bar{P}_j^{11}) z + 2\hat{x}^\top (\bar{P}_i^{21} - \bar{P}_j^{21}) z \\ &\quad + \hat{x}^\top (\bar{P}_i^{22} - \bar{P}_j^{22}) \hat{x} \\ &= \left(\sum_m v_m r_m \right)^\top (\bar{P}_i^{11} - \bar{P}_j^{11}) \left(\sum_n v_n r_n \right) \\ &\quad + 2\hat{x}^\top (\bar{P}_i^{21} - \bar{P}_j^{21}) \sum_m v_m r_m \\ &\quad + \hat{x}^\top (\bar{P}_i^{22} - \bar{P}_j^{22}) \hat{x} \\ &= \sum_m \sum_n v_m v_n r_m^\top (\bar{P}_i^{11} - \bar{P}_j^{11}) r_n \\ &\quad + 2\hat{x}^\top \sum_m v_m (\bar{P}_i^{21} - \bar{P}_j^{21}) r_m \\ &\quad + \hat{x}^\top (\bar{P}_i^{22} - \bar{P}_j^{22}) \hat{x} \\ &= 0, \end{aligned} \tag{29}$$

where the sums are taken over all $r_m, r_n \in \mathcal{R}_{\mathcal{S}_i \cap \mathcal{S}_j}$, and where the last equality follows from (24), (25), and (28). Since, by definition, $z = Cx$, (T1.2) follows.

$(T1.2) \iff (T1.3)$. Let W_{ij} be given as in (T1.3). Consider again the state transformation $\bar{x} = Tx$, with $T = \begin{bmatrix} C^\top & C_\perp \end{bmatrix}^\top$, such that $\bar{x} = \begin{bmatrix} z^\top & \hat{x}^\top \end{bmatrix}^\top$, with $z = Cx \in \mathbb{R}^m$ and $\hat{x} = C_\perp^\top x \in \mathbb{R}^{n-m}$. Moreover, $Cx \in \operatorname{im} Z_{ij}$ if and only if there exists a real vector v such that $Cx = z = Z_{ij} v$. As such,

$$\begin{aligned} x &= T^{-1} \bar{x} = \begin{bmatrix} C \\ C_\perp^\top \end{bmatrix}^{-1} \begin{bmatrix} z \\ \hat{x} \end{bmatrix} \\ &= \begin{bmatrix} C \\ C_\perp^\top \end{bmatrix}^{-1} \begin{bmatrix} Z_{ij} v \\ \hat{x} \end{bmatrix} = \begin{bmatrix} C \\ C_\perp^\top \end{bmatrix}^{-1} \begin{bmatrix} Z_{ij} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} v \\ \hat{x} \end{bmatrix} = W_{ij} w, \end{aligned} \tag{30}$$

for some $w = \begin{bmatrix} v^\top & \hat{x}^\top \end{bmatrix}^\top$ if and only if $Cx \in \operatorname{im} Z_{ij}$. By substituting $x = W_{ij} w$ into (17), we obtain the equivalent condition

$$w^\top W_{ij}^\top (P_i - P_j) W_{ij} w = 0, \text{ for all real } w, \tag{31}$$

which, due to symmetry of $W_{ij}^\top (P_i - P_j) W_{ij}$, is equivalent to (18).

$(T1.2) \iff (T1.5)$. For each pair $(i, j) \in \mathcal{N} \times \mathcal{N}$, consider Corollary 1 with the substitutions

$$\boldsymbol{U} := H_{ij} C, \tag{32a}$$

$$\boldsymbol{Q} := P_i - P_j. \tag{32b}$$

With the above substitutions, (C1.1) in Corollary 1 reads as follows: There exists a matrix $\boldsymbol{X} \in \mathbb{R}^{m \times n}$ such that

$$P_i - P_j + (H_{ij} C)^\top \boldsymbol{X} + \boldsymbol{X}^\top (H_{ij} C) = 0, \tag{33}$$

i.e., exactly as (T1.5) of Theorem 1. On the other hand, (C1.2) in Corollary 1 reads

$$x^\top (P_i - P_j) x = 0, \text{ for all } x \in \ker H_{ij} C, \tag{34}$$

which, because

$$x \in \ker H_{ij} C \iff Cx \in \ker H_{ij} = \operatorname{im} Z_{ij},$$

is equivalent to (T1.2). Due to Corollary 1, (33) is equivalent to (34), and thereby, $(T1.2) \iff (T1.5)$.

$(T1.1) \implies (T1.4)$. Suppose that $x^\top(P_i - P_j)x = 0$, for all $Cx \in \mathcal{S}_i \cap \mathcal{S}_j$. As in the proof of $(T1.1) \implies (T1.2)$, it follows that (24), (25), and (28) hold. Hence, for every $r_m, r_n \in \mathcal{R}_{\mathcal{S}_i \cap \mathcal{S}_j}$

$$r_m^\top \bar{P}_i^{11} r_n = r_m^\top \bar{P}_j^{11} r_n =: \phi_{mn}, \tag{35a}$$
$$\bar{P}_i^{21} r_m = \bar{P}_j^{21} r_m =: \phi_m, \tag{35b}$$
$$\bar{P}_i^{22} = \bar{P}_j^{22} =: \Phi^{22}, \tag{35c}$$

where $\phi_{mn} \in \mathbb{R}$, $\phi_m \in \mathbb{R}^{n-m}$, and $\Phi^{22} \in \mathbb{S}^{n-m}$. Recall that $\mathcal{R}_{\mathcal{S}_i \cap \mathcal{S}_j}$ denotes the set of extremal rays of $\mathcal{S}_i \cap \mathcal{S}_j$.

From (35a), and on the basis of [16, Lemma 1], there exists a symmetric matrix $\Phi^{11} := \{\phi_{pq}\} \in \mathbb{S}^r$ for all $p, q \in \{1, 2, \ldots, r\}$. Clearly, Φ^{11} can always be constructed, by collecting the elements in (35a) and giving arbitrary values to the remaining elements (see [16, Remark 4]). On a per region basis, using the extremal ray matrices R_i of $\mathcal{S}_i$, one can write

$$R_i^\top \bar{P}_i^{11} R_i = E_i^\top \Phi^{11} E_i, \tag{36}$$

which follows from the construction of the extraction matrices E_i (see Section II). Since R_i is invertible, one finds

$$\bar{P}_i^{11} = R_i^{-\top} E_i^\top \Phi^{11} E_i R_i^{-1}. \tag{37}$$

In a similar manner as before, collecting the elements in (35b) in a matrix $\Phi^{21} = \{\phi_p\} \in \mathbb{R}^{(n-m)\times r}$ for all $p \in \{1, 2, \ldots, r\}$, results in

$$\bar{P}_i^{21} R_i = \Phi^{21} E_i, \tag{38}$$

such that, by invertibility of R_i, one finds

$$\bar{P}_i^{21} = \Phi^{21} E_i R_i^{-1}. \tag{39}$$

Using (35c), (37), and (39), the partitioned matrix in (21) is equivalently written as

$$\begin{aligned}\bar{P}_i &= \begin{bmatrix} R_i^{-\top} E_i^\top \Phi^{11} E_i R_i^{-1} & \star \\ \Phi^{21} E_i R_i^{-1} & \Phi^{22} \end{bmatrix} \\ &= \begin{bmatrix} E_i R_i^{-1} & 0 \\ 0 & I \end{bmatrix}^\top \begin{bmatrix} \Phi^{11} & \star \\ \Phi^{21} & \Phi^{22} \end{bmatrix} \begin{bmatrix} E_i R_i^{-1} & 0 \\ 0 & I \end{bmatrix}.\end{aligned} \tag{40}$$

Then, using $P_i = T^\top \bar{P}_i T$ with $T = \begin{bmatrix} C^\top & C_\perp \end{bmatrix}^\top$, one finds

$$\begin{aligned}P_i &= \begin{bmatrix} C \\ C_\perp^\top \end{bmatrix}^\top \begin{bmatrix} E_i R_i^{-1} & 0 \\ 0 & I \end{bmatrix}^\top \begin{bmatrix} \Phi^{11} & \star \\ \Phi^{21} & \Phi^{22} \end{bmatrix} \begin{bmatrix} E_i R_i^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} C \\ C_\perp^\top \end{bmatrix} \\ &= \begin{bmatrix} E_i R_i^{-1} C \\ C_\perp^\top \end{bmatrix}^\top \begin{bmatrix} \Phi^{11} & \star \\ \Phi^{21} & \Phi^{22} \end{bmatrix} \begin{bmatrix} E_i R_i^{-1} C \\ C_\perp^\top \end{bmatrix}.\end{aligned} \tag{41}$$

By assumption, the matrix V satisfies $\ker V \subseteq \ker C_\perp^\top$ (or equivalently, $\operatorname{im} V^\top \supseteq \operatorname{im} C_\perp$). It follows that there exists a matrix X such that

$$XV = C_\perp^\top. \tag{42}$$

Then, continuing from (41) with $C_\perp = (XV)^\top$, one finds

$$\begin{aligned}P_i &= \begin{bmatrix} E_i R_i^{-1} C \\ XV \end{bmatrix}^\top \begin{bmatrix} \Phi^{11} & \star \\ \Phi^{21} & \Phi^{22} \end{bmatrix} \begin{bmatrix} E_i R_i^{-1} C \\ XV \end{bmatrix} \\ &= \begin{bmatrix} E_i R_i^{-1} C \\ V \end{bmatrix}^\top \begin{bmatrix} \Phi^{11} & \star \\ X^\top \Phi^{21} & X^\top \Phi^{22} X \end{bmatrix} \begin{bmatrix} E_i R_i^{-1} C \\ V \end{bmatrix} \\ &= F_i^\top \Phi F_i,\end{aligned} \tag{43}$$

where $F_i = \begin{bmatrix} (E_i R_i^{-1} C)^\top & V^\top \end{bmatrix}^\top$.

Hence, (T1.1) implies that there exists a symmetric matrix Φ such that $P_i = F_i^\top \Phi F_i$ for all $i \in \mathcal{N}$.

$(T1.4) \implies (T1.1)$. Let $F_i = \begin{bmatrix} (E_i R_i^{-1} C)^\top & V^\top \end{bmatrix}^\top$, where V satisfies $\operatorname{im} V^\top \supseteq \operatorname{im} C_\perp$. Suppose that there exists a symmetric matrix Φ, such that, $P_i = F_i^\top \Phi F_i$ for all $i \in \mathcal{N}$. Recall from Section II that the extremal ray matrix of each simplicial cone is constructed as $R_i = \bar{R} E_i$, where $\bar{R} \in \mathbb{R}^{m\times r}$ contains all distinct extremal rays of the simplicial conic partition $\{S_i\}_{i=1}^N$, and $E_i \in \mathbb{R}^{r\times m}$ are selection matrices.

Clearly, $Cx \in \mathcal{S}_i \cap \mathcal{S}_j$ if and only if there exist vectors $\lambda_i, \lambda_j \geq 0$ such that $Cx = R_i\lambda_i = R_j\lambda_j$. Furthermore, $E_i\lambda_i = E_j\lambda_j$ if and only if $Cx \in \mathcal{S}_i \cap \mathcal{S}_j$, due to Assumption 1. Hence, for all $Cx \in \mathcal{S}_i \cap \mathcal{S}_j$

$$\begin{aligned}(E_i R_i^{-1} C - E_j R_j^{-1} C)x &= (E_i R_i^{-1} R_i \lambda_i - E_j R_j^{-1} R_j \lambda_j) \\ &= (E_i\lambda_i - E_j\lambda_j) = 0.\end{aligned} \tag{44}$$

As such,

$$\begin{aligned}(F_i - F_j)x &= \left(\begin{bmatrix} E_i R_i^{-1} C \\ V \end{bmatrix} - \begin{bmatrix} E_j R_j^{-1} C \\ V \end{bmatrix} \right) x \\ &= \begin{bmatrix} (E_i R_i^{-1} C - E_j R_j^{-1} C)x \\ 0 \end{bmatrix} \\ &\overset{(44)}{=} 0, \text{ for all } Cx \in \mathcal{S}_i \cap \mathcal{S}_j.\end{aligned} \tag{45}$$

From (45), it follows that $F_i x = F_j x$ for all $Cx \in \mathcal{S}_i \cap \mathcal{S}_j$. Hence,

$$\begin{aligned}x^\top(P_i - P_j)x &= x^\top(F_i^\top \Phi F_i - F_j^\top \Phi F_j)x \\ &= x^\top F_i^\top \Phi F_i x - x^\top F_j^\top \Phi F_j x \\ &\overset{(45)}{=} 0, \text{ for all } Cx \in \mathcal{S}_i \cap \mathcal{S}_j,\end{aligned} \tag{46}$$

i.e., (T1.1) is satisfied.

As we have shown $(T1.1) \iff (T1.2)$, $(T1.2) \iff (T1.3)$, $(T1.2) \iff (T1.5)$, and $(T1.1) \iff (T1.4)$, the proof is complete. □

VI. Illustrative Example

Consider the PWQ function $V : \mathcal{Z} \to \mathbb{R}_{\geq 0}$ given by

$$V(x) = \begin{cases} x^\top P_1 x, & \text{when } R_1^{-1}x \geq 0, \\ x^\top P_2 x, & \text{when } R_2^{-1}x \geq 0, \end{cases} \tag{47}$$

where $\mathcal{Z} = \{x \in \mathbb{R}^2 \mid x_2 \geq 0\}$ and

$$P_1 = I = R_1, \; P_2 = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, \; R_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{48}$$

A contour plot of the function V in (47) are shown in Fig. 2. Here, because $C = I$, the matrix $C_\perp$ is omitted. Now, the

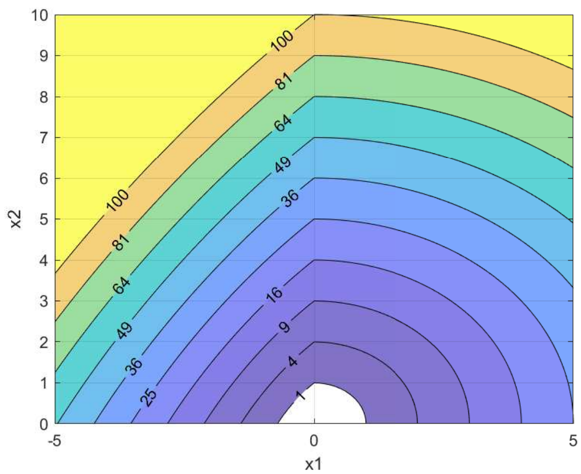

Fig. 2: Contour plot of the PWQ function (47) analysed in the example in Section VI.

different continuity conditions in Theorem 1 will be analysed explicitly.

$(T1.1)$: The boundary region between $\mathcal{S}_1$ and $\mathcal{S}_2$ is given by $\mathcal{S}_1 \cap \mathcal{S}_2 = \{x \in \mathbb{R}^2 \mid x_1 = 0, x_2 \geq 0\}$. At the boundary, we have $V_1(0, x_2) = V_2(0, x_2) = x_2^2$. Hence, (T1.1) of Theorem 1 is satisfied and $V(x)$ is (locally Lipschitz) continuous. The continuity of $V(x)$ is also observed in Fig. 2.

$(T1.2)$: The so-called image representation of the boundary region is given by $\operatorname{im} Z_{12} = \{x \in \mathbb{R}^2 \mid x_1 = 0\}$ (compared to $\mathcal{S}_1 \cap \mathcal{S}_2$, this set also includes $x_2 < 0$). Since, $(-x)^\top P_1(-x) = x^\top P_1 x$ holds trivially, it follows that (T1.2) holds.

$(T1.3)$: With $T = C = I$, everything except the top-left block of W_{ij} in (18), is omitted. Hence, the condition is reduced to checking whether $Z_{ij}^\top (P_i - P_j) Z_{ij} = 0$ holds. By direct computation, one gets

$$Z_{12}^\top (P_1 - P_2) Z_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^\top \begin{bmatrix} 1-2 & 0+1 \\ 0+1 & 1-1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0, \quad (49)$$

where $Z_{12} = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top$ corresponds to the only boundary region. Hence, $(T1.3)$ is satisfied, as we expected (due to the equivalence).

$(T1.4)$: As $C = I$, it is sufficient to consider $\Phi^{11} \in \mathbb{R}^{3\times 3}$, such that $P_i = (E_i R_i^{-1})^\top \Phi^{11} E_i R_i^{-1}$. If the extremal rays are ordered as

$$\bar{R} = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (50)$$

the selection matrices, E_i, are defined as

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (51)$$

and the so-called continuity matrices, $E_i R_i^{-1}$ (see [2]), become

$$E_1 R_1^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_2 R_2^{-1} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (52)$$

We find that

$$\Phi^{11} = \begin{bmatrix} 1 & c & 0 \\ c & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad (53)$$

satisfies $P_i = (E_i R_i^{-1})^\top \Phi^{11} E_i R_i^{-1}$, for any $c \in \mathbb{R}$. To match the structure of Φ given in Theorem 1, we extend Φ^{11} by zeros, such that

$$\Phi = \begin{bmatrix} \Phi^{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad (54)$$

and with this choice, (19) is satisfied.

$(T1.5)$: The matrix H_{12} is constructed as $H_{12} = H_{21} = \begin{bmatrix} 1 & 0 \end{bmatrix}$. Now, with $\Gamma_{12} := \begin{bmatrix} -1/2 & 1 \end{bmatrix}$, we get

$$H_{12}^\top \Gamma_{12} + \Gamma_{12}^\top H_{12} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = P_1 - P_2, \quad (55)$$

and $\Gamma_{21} = -\Gamma_{12}$ follows. Hence, (20) is satisfied.

VII. Conclusion

Continuity conditions known in the literature for PWQ functions on simplicial conic partitions are shown to be equivalent. This result is particularly useful in the context of stability analysis of PWL systems using PWQ Lyapunov functions. Furthermore, the results indicate that the choice of approach for guaranteeing continuity solely can be based on practical considerations, without introducing additional conservatism in the analysis. In addition, a technical lemma useful for showing equivalence of the continuity conditions is presented. This lemma is of independent interest and has significant potential for applications beyond those explored in this technical note.

Future research directions include investigating whether the assumptions on the simplicial conic partitions introduce conservatism in the analysis or not. In particular, investigate whether any simplicial conic partition can be finitely refined to satisfy the assumptions in this paper.

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