Dimensional Uplift in Conformal Field Theories

Ferdinando Gliozzi

Dipartimento di Fisica dell'Università di Torino Regge Center for Algebra, Geometry and Theoretical Physics, via Pietro Giuria 1, I-10125 Torino, Italy

Abstract

The n-point functions of any Conformal Field Theory (CFT) in ddimensions can always be interpreted as spatial restrictions of corresponding functions in a higher-dimensional CFT with dimension d' > d. In particular, when a four-point function in d dimensions has a known conformal block expansion, this expansion can be easily extended to d' = d + 2 due to a remarkable identity among conformal blocks, discovered by Kaviraj, Rychkov, and Trevisani (KRT) as a consequence of Parisi-Sourlas supersymmetry and confirmed to hold in any CFT with d > 1.

In this note, we provide an elementary proof of this identity using simple algebraic properties of the Casimir operators. Additionally, we construct five differential operators, Λ_i , which promote a conformal block in d dimensions to five conformal blocks in d + 2 dimensions. These operators can be normalized such that $\sum_i \Lambda_i = 1$, from which the KRT identity immediately follows. Similar, simpler identities have been proposed, all of which can be reformulated in the same way.

1 Introduction

Dimensional reduction is a property emerging in some disordered systems. In the random-field Ising model it was observed that the correlation functions reduced to those of the pure Ising model, without the random source, in two fewer dimensions [1, 2]. This was understood by Parisi and Sourlas [3] to be due to a hidden supersymmetric formulation. This dimensional reduction holds near the upper critical dimension of six, but fails in sufficiently low dimensions [4–12]. This failure is now believed to result from operators that are irrelevant near six dimensions becoming relevant around five dimensions [6, 10, 13].

Another notable example is the connection, again discovered by Parisi and Sourlas [14], between randomly branched polymers or lattice animals in d dimensions [15–19] and the Yang-Lee edge singularity [20] in two fewer dimensions. In this case, dimensional reduction works not only near the upper critical dimension of eight but continues down to two dimensions, as rigorously demonstrated by Brydges and Imbrie [21, 22]. Even in these instances, the correlation functions on branched polymers reduce to those of the Yang-Lee edge singularity.

In the present note we explore such a connection across the space dimensions from the point of view of conformal field theory (CFT). As a result, we give a simple algebraic proof of a surprising exact relation discovered by Kaviraj, Rychkov and Trevisani (KRT) as a direct consequence of the Parisi-Sourlas (PS) supersymmetry and confirmed to hold in any CFT with d > 1 [10]; this relation expresses any (d - 2)-dimensional conformal block as a linear combination of five conformal blocks in dimension d. We in a sense reverse such a relation by constructing, only utilizing some commutation properties of the Casimir operators, five differential operators, Λ_i , transforming an arbitrary (d - 2)-dimensional conformal block $g_{\Delta,\ell}^{(d-2)}$ (describing the contribution to a four-point function of a primary of scaling dimension Δ and spin ℓ) into five different conformal blocks in dimension d,

$$\Lambda_i g_{\Delta,\ell}^{(d-2)} \propto g_{\Delta_i,\ell_i}^{(d)}, \qquad (1)$$

where the conformal blocks in the *rhs* are precisely those contributing to the KRT identity. Equation (1) can be utilized to build new conformal blocks in higher dimensions once their form in lower dimensions is known. The Λ_i 's

can be normalized so that

$$\sum_{i=1}^{5} \Lambda_i = 1 \quad , \tag{2}$$

therefore $\sum_i \Lambda_i g_{\Delta,\ell}^{(d-2)}$ yields at once the KRT identity. In such a form it is clear that this identity holds for any real d > 1 under the only assumption that the conformal blocks are eigenfunctions of the Casimir operators.

Б

The KRT identity allows one to reconstruct the spectrum of primary operators contributing to a four-point function of a CFT in dimension d once this spectrum is known in a (d-2)-dimensional CFT with the same external scalars [23].¹

2 Notation

In a generic CFT in d dimensions the 4pt function of arbitrary scalars \mathcal{O}_i of scaling dimension Δ_i can be parametrized as [24, 25]

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = K(x_i)\,g(u,v)\,,\tag{3}$$

where $K(x_i)$ is a kinematic factor given by

$$K(x_i) = \frac{1}{|x_{12}|^{\Delta_1 + \Delta_2} |x_{34}|^{\Delta_3 + \Delta_4}} \left(\frac{x_{14}^2}{x_{24}^2}\right)^a \left(\frac{x_{14}^2}{x_{13}^2}\right)^b, \qquad (4)$$

with $a = \frac{\Delta_2 - \Delta_1}{2}$ and $b = \frac{\Delta_3 - \Delta_4}{2}$; g(u, v) is a theory-dependent function of the two cross-ratios $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$ and $v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$, which are linearly independent for d > 1.

It is important to emphasize that although g(u, v) depends on the scaling dimensions of the external scalars, it is completely independent of the spacetime dimension d:

$$\partial_d g(u, v) = 0 \tag{5}$$

In principle, we can choose d to be any real number, provided that $d \ge d$, where d represents the dimension of the linear space spanned by the four points x_i (hence, $1 < d \le 3$). Under this condition, we obtain the expansion

$$g(u,v) = \sum_{\Delta,\ell} c_{\Delta,\ell} g_{\Delta,\ell}^{(d)}(u,v), \qquad (6)$$

¹In [23] this result is obtained by assuming the PS supersymmetry.

where the coefficients $c_{\Delta,\ell}$ are the operator product expansion (OPE) coefficients.

The freedom in choosing d has intriguing physical consequences. For instance, if g(u, v) corresponds to the four-point function of a random branched polymer system in dimension d, then, due to dimensional reduction, its expansion in (d-2)-dimensional conformal blocks provides insight into the spectrum of primary operators associated with the Yang-Lee edge singularity. The dimensional shift by 2 in this case is well understood in terms of Parisi-Sourlas supersymmetry.

A natural question then arises: can this reduction by 2, or the reverse approach, the dimensional uplift, be explained purely through conformal invariance? A simple example helps to clarify the issue. Consider the four-point function $\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$, where ϕ is a scalar generalized free field (GFF) with an arbitrary scaling dimension δ . The corresponding function g(u, v) is given by

$$g(u,v) = 1 + u^{\delta} + \left(\frac{u}{v}\right)^{\delta}.$$
(7)

The OPE coefficients and the spectrum of contributing primaries have been exactly evaluated in various ways [26–28]. Since δ is an arbitrary real number, generalized free CFTs are not necessarily local CFTs. For locality, the theory must have a non-vanishing coupling to the conserved energy-momentum tensor $T_{\mu\nu}$, which is a primary operator with scaling dimension d and spin 2. The spectrum of primaries contributing to the expansion of g(u, v) is

$$\{\Delta, \ell\} = \{2\delta + 2m + 2n, \ell = 2n\}, \quad (m, n = 0, 1, 2, \dots).$$
(8)

Keeping δ fixed, if the equation $2\delta + 2m + 2 = d$ has a solution, then clearly there exist infinitely many solutions for dimensions d' = d + 2k with $k = 1, 2, \ldots$.

This behaviour seems to be a general feature of local conformal field theories. Actually, as already pointed out in [10], the KRT identity implies that if a scalar primary of scaling dimension δ has a non-vanishing coupling to the energy-momentum tensor in dimension d, then it describes a *local* CFT in any space dimension d' = d + 2k, $(k \in \mathbb{N})$, and this property holds for any real d > 1.

3 The proof

The quadratic Casimir operator $C_2[d]$ can be written as [24]

$$C_2[d] = D_z + D_{\bar{z}} + (d-2)\frac{z\,\bar{z}}{z-\bar{z}}\left((1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}}\right)\,,\tag{9}$$

with

$$D_z = z^2 (1-z)\partial_z^2 - (1+a+b)z^2 \partial_z + ab z .$$
 (10)

The two variables z, \bar{z} are related to the cross-ratios as $u = z \bar{z}$ and $v = (1-z)(1-\bar{z})$.

We aim to demonstrate a special relationship between the quadratic Casimir oprators in d and d-2 dimensions. Specifically, we iteratively construct five differential operators Y_i (i = 1, ..., 5) where $Y_1 = 1$, Y_2 is first-order in z and \overline{z} , Y_3 and Y_4 are second-order, and Y_5 is third-order, such that

$$C_2[d] Y_i = Y_i C_2[d-2] + Y_j \alpha_i^j , \qquad (11)$$

where the α_i^j 's are simple linear functions of the Casimir in dimension d-2. If we attempt to replace in the *rhs* $C_2[d-2]$ with $C_2[d']$ with $d' \neq d-2$, the linearity in the Y_i is immediately lost, and the algebraic structure of the equation no longer closes.

Basic linear algebra manipulations of this equation allow us to define five operators Λ_i , which promote any conformal block in d-2 dimensions to a suitable conformal block in d dimensions - leading to the KRT identity.

We begin with $Y_1 = 1$ and write

$$C_2[d] Y_1 = Y_1 C_2[d-2] + Y_2 , \qquad (12)$$

where $Y_2 = \frac{2z \bar{z}}{z-\bar{z}} \left((1-z) \partial_z - (1-\bar{z}) \partial_{\bar{z}} \right)$. We proceed with

$$C_2[d] Y_2 = Y_2 \left(C_2[d-2] - d + 2 \right) + 2 Y_1 C_2[d-2] + 2 Y_3 - 2 Y_4 , \qquad (13)$$

where $Y_3 = \frac{z \bar{z}}{z - \bar{z}} (D_z - D_{\bar{z}})$ and $Y_4 = \frac{z + \bar{z}}{z \bar{z}} Y_3$. Now we have

$$C_2[d] Y_3 = Y_3 \left(C_2[d-2] - d + 2 \right) ; \tag{14}$$

this remarkable relation, first discovered by Dolan and Osborn [24] and recently reobtained by Trevisani [23] as a consequence of the PS supersymmetry, tells us that $Y_3 g_{\Delta,\ell}^{(d-2)}$ is proportional to $g_{\Delta+1,\ell-1}^{(d)}$. Next we obtain

$$C_2[d] Y_4 = Y_4 \left(C_2[d-2] + 2 \right) + Y_3 \left(2(a+b) - d \right) + 2 Y_5 , \qquad (15)$$

where $Y_5 = ((z-1)\partial_z + (\bar{z}-1)\partial_{\bar{z}})Y_3$.

Finally we find

$$C_{2}[d] Y_{5} = Y_{5} (C_{2}[d-2] - 3d + 8) + Y_{4} (C_{2}[d-2] - (d-2)^{2}) - Y_{1} C_{4}[d-2] + Y_{3} (2ab - 2(d-3)(a+b) + (d-2)^{2} - C_{2}[d-2]) ,$$
(16)

where C_4 is the quartic Casimir operator defined by Dolan and Osborn in [24] :

$$C_4[d] = \left(\frac{z\bar{z}}{z-\bar{z}}\right)^{d-2} \left(D_z - D_{\bar{z}}\right) \left(\frac{z\bar{z}}{z-\bar{z}}\right)^{-d+1} Y_3 .$$
(17)

Clearly, when we apply the $C_2[d] Y_i$'s to an arbitrary conformal block in d-2 dimensions, these differential operators span a five-dimensional space where the action of $C_2[d]$ is represented by a 5 × 5 matrix:

$$C_{2}[d]\mathbf{Y} = \begin{pmatrix} c_{2} & 1 & 0 & 0 & 0\\ 2c_{2} & c_{2} - d + 2 & 2 & -2 & 0\\ 0 & 0 & c_{2} - d + 2 & 0 & 0\\ 0 & 0 & 2(a+b) - d & c_{2} + 2 & 2\\ -c_{4} & 0 & \kappa & c_{2} - (d-2)^{2} & c_{2} - 3d + 8 \end{pmatrix} \mathbf{Y}$$
(18)

where Y is the column-vector formed by the Y_i 's, κ is the coefficient of Y_3 in Eq.(16) and c_2, c_4 are the eigenvalues of the quadratic and quartic Casimir operators in d-2 dimensions:

$$c_2 = \ell(\ell + d - 4)/2 + \Delta(\Delta - d + 2)/2 ; \qquad (19)$$

$$c_4 = -\ell(\ell + d - 4)(d - \Delta - 3)(\Delta - 1) \quad . \tag{20}$$

One can check at once that the five eigenvectors V_i , (i = 1, 2, ..., 5) of the transposed of such a matrix define five differential operators $\Lambda_i = V_i \cdot Y$ which transform $g_{\Delta,\ell}^{(d-2)}$ into a suitable conformal block in d dimensions.

$$\Lambda_i g_{\Delta,\ell}^{(d-2)} \equiv \Lambda_{\alpha\beta} g_{\Delta,\ell}^{(d-2)} \propto g_{\Delta+\alpha,\ell-\beta}^{(d)}, \, \alpha,\beta \in \{0,1,2\}, \, \alpha+\beta \text{ even}$$
(21)

Precisely we obtain

$$\begin{split} \Lambda_{00} &= n_{00} \left[Y_{5} + \left(3d - 8 + \ell - \Delta \right) Y_{4} + \left(a + b + \frac{2ab}{d - 2 + \ell - \Delta} - \frac{3d - 8 + \ell - \Delta}{2} \right) Y_{3} + \\ &+ \frac{(d - 4 + \ell)(3 - d + \Delta)}{2} Y_{2} + \frac{(d - 4 + \ell)(3 - d + \Delta)(d + \ell - \Delta + 2)}{2} \right], \\ \Lambda_{20} &= n_{20} \left[Y_{5} + \frac{2d - 6 + \ell - \Delta}{2} Y_{4} + \left(3 + a + b + \frac{2ab}{\ell + \Delta} - \frac{\Delta + \ell}{2} \right) Y_{3} - \\ &- \frac{(\Delta - 1)(d - 4 + \ell)}{2} Y_{2} - \frac{(d - 4 + \ell)(\Delta - 1)(\Delta + \ell)}{2} \right], \\ \Lambda_{11} &= n_{11} Y_{3}, \\ \Lambda_{12} &= n_{02} \left[Y_{5} + \left(d - 2 - \frac{\ell + \Delta}{2} \right) Y_{4} + \left(a + b - \frac{2ab}{\ell + \Delta - 2} + \frac{4 - 2d + \ell + \Delta}{2} + \ell + \Delta - 2 \right) Y_{3} + \\ &+ \frac{\ell (d - 3 - \Delta)}{2} Y_{2} + \frac{\ell (d - 3 - \Delta)(\Delta + \ell - 2)}{2} \right], \\ \Lambda_{22} &= n_{22} \left[Y_{5} + \frac{d - 2 - \ell + \Delta}{2} Y_{4} + \frac{2(a + b + 1) + \ell - \Delta - d - \frac{4ab}{d - 4 + \ell - \Delta}}{2} Y_{3} + \\ &+ \frac{\ell (\Delta - 1)}{2} Y_{2} - \frac{\ell (\Delta - 1)(d - 4 + \ell - \Delta)}{2} \right]. \end{split}$$

The normalization coefficients $n_{\alpha\beta}$ can be chosen in such a way that $\sum_i \Lambda_i = 1$, which implies of course five linear equations for the $n_{\alpha\beta}$'s. Thier solution yields

$$n_{00} = \frac{-2}{(d-4+2\ell)(d-2\Delta)(d-3+\ell-\Delta)},$$

$$n_{20} = \frac{-2}{(d-4+2\ell)(d-2-2\Delta)(\ell+\Delta-1)},$$

$$n_{11} = \frac{-8ab}{(d-4+\ell-\Delta)(d-2+\ell-\Delta)(\ell+\Delta-2)(\ell+\Delta)},$$

$$n_{02} = \frac{2}{(d-4+2\ell)(d-2-2\Delta)(d-3+\ell-\Delta)},$$

$$n_{22} = \frac{2}{(d-4+2\ell)(d-2-2\Delta)(\ell+\Delta-1)}.$$
(23)

Taking advantage of this normalization we finally obtain the KRT identy typy $_{5}$

$$g_{\Delta,\ell}^{(d-2)} \equiv \sum_{i=1}^{5} \Lambda_i \, g_{\Delta,\ell}^{(d-2)} = \sum_{\alpha,\beta \in \{0,1,2\},\,\alpha+\beta \text{ even}} k_{\alpha\beta} \, g_{\Delta+\alpha,\ell-\beta}^{(d)} \,. \tag{24}$$

The $k_{\alpha\beta}$'s depend of course on the normalization of the conformal blocks. In accordance with [10] we have

$$k_{20} = -\frac{(\Delta - 1)\Delta(\Delta - \Delta_{12} + \ell)(\Delta + \Delta_{12} + \ell)(\Delta - \Delta_{34} + \ell)(\Delta + \Delta_{34} + \ell)}{4(d - 2\Delta - 4)(d - 2\Delta - 2)(\Delta + \ell - 1)(\Delta + \ell)^2(\Delta + \ell + 1)},$$

$$k_{11} = -\frac{(\Delta - 1)\Delta_{12}\Delta_{34}\ell}{(\Delta + \ell - 2)(\Delta + \ell)(d - \Delta + \ell - 4)(d - \Delta + \ell - 2)},$$

$$k_{02} = -\frac{(\ell - 1)\ell}{(d + 2\ell - 6)(d + 2\ell - 4)}, \quad k_{00} = 1,$$

$$k_{22} = \frac{(\Delta - 1)\Delta(\ell - 1)\ell(d - \Delta - \Delta_{12} + \ell - 4)(d - \Delta + \Delta_{12} + \ell - 4)(d - \Delta - \Delta_{34} + \ell - 4)(d - \Delta + \Delta_{34} + \ell - 4)}{4(d - 2\Delta - 4)(d - 2\Delta - 2)(d + 2\ell - 6)(d + 2\ell - 4)(d - \Delta + \ell - 5)(d - \Delta + \ell - 4)^2(d - \Delta + \ell - 3)}.$$
(25)

Similar, simpler dimensional uplift identities for two-point and three-point functions of CFT with boundary [29, 30] and for two-point functions in real projective space [31] have been found. All of them can be reformulated in terms of differential operators transforming a conformal block in dimension d-2 into a conformal block in dimension d. For instance, in the case of CFT with boundaries, the boundary conformal blocks $f_{bdy}^{(d)}[\Delta, \xi]$ are eigenfunctions of the following quadratic Casimir

$$\mathcal{C}[d] = -\xi(\xi+1)\partial_{\xi}^{2} - d(\xi+\frac{1}{2})\partial_{\xi} , \ \mathcal{C}[d] f_{bdy}^{(d)}[\Delta,\xi] = \Delta (d-1-\Delta) f_{bdy}^{(d)}[\Delta,\xi]$$
(26)

This operator acts linearly on $\{y_1 = 1, y_2 = -(1+2\xi)\partial_{\xi}\}$, namely

$$C[d] y_1 = y_1 C[d-2] + y_2$$
 (27)

$$\mathcal{C}[d] y_2 = y_2 \left(\mathcal{C}[d-2] + 2(d-3) \right) - 4y_1 \mathcal{C}[d-2].$$
(28)

By applying the same approach as before we build two differential operators, λ_0 and λ_2 ,

$$\lambda_0 = \frac{y_2 + 2(\Delta - d + 3)}{2(2\Delta - d + 3)}, \quad \lambda_2 = \frac{-y_2 + 2\Delta}{2(2\Delta - d + 3)}, \quad \lambda_0 + \lambda_2 = 1 \quad .$$
(29)

such that 2

$$\lambda_0 f_{bdy}^{((d-2)}[\Delta,\xi] = f_{bdy}^{(d)}[\Delta,\xi] \quad , \ \lambda_2 f_{bdy}^{(d-2)}[\Delta,\xi] = k f_{bdy}^{(d)}[\Delta+2,\xi] \quad , \tag{30}$$

where k is a normalization-dependent coefficient. If for $\xi \to \infty f_{bdy}^{(d)}[\Delta, \xi] \simeq 1/\xi^{\Delta}$, we have $k = \frac{\Delta(\Delta+1)}{4(d-2\Delta-5)(d-2\Delta-3)}$.

²Since it has been noticed [32] that the three-dimensional boundary conformal blocks are expressible as elementary algebraic functions, the mere existence of λ_0 and λ_2 allows us to conclude that this property holds for any odd d.

4 Conclusions

We provided a straightforward algebraic proof of a remarkable identity discovered some time ago by Kaviraj, Rychkov, and Trevisani (KRT). This identity expresses a conformal block contributing to the four-point function in (d-2) dimensions as a linear combination of five conformal blocks in d dimensions. The key mechanism involved the iterative construction of five differential operators that transform linearly under the action of the quadratic Casimir. This, in turn, enabled us to define five differential operators, denoted by Λ_i , which transform any conformal block in (d-2) dimensions into five conformal blocks in d dimensions. These operators can be normalized such that $\sum_i \Lambda_i = 1$, from which the KRT identity follows immediately. Similar, simpler identities have been proposed in recent literature, all of which can be reformulated using the present approach. As an example, we explicitly worked out the case of boundary conformal blocks contributing to the two-point functions of boundary conformal field theories.

References

- G. Grinstein and A. Luther, "Application of the renormalization group to phase transitions in disordered systems," Phys. Rev. B 13 (1976), 1329-1343 doi:10.1103/PhysRevB.13.1329
- [2] A. Aharony, Y. Imry and S. K. Ma, "Lowering of Dimensionality in Phase Transitions with Random Fields," Phys. Rev. Lett. 37 (1976), 1364-1367 doi:10.1103/PhysRevLett.37.1364
- [3] G. Parisi and N. Sourlas, "Random Magnetic Fields, Supersymmetry and Negative Dimensions," Phys. Rev. Lett. 43 (1979), 744 doi:10.1103/PhysRevLett.43.744
- [4] J. "Lower Critical Ζ. Imbrie, Dimension of the Random Field Ising Model," Phys. Rev. Lett. 53(1984),1747 doi:10.1103/PhysRevLett.53.1747
- [5] E. Brezin and C. De Dominicis, "New phenomena in the random field Ising model," EPL 44 (1998), 13-19 doi:10.1209/epl/i1998-00428-0 [arXiv:cond-mat/9804266 [cond-mat]].

- [6] M. Tissier and G. Tarjus, "Nonperturbative Functional Renormalization Group for Random Field Models. IV: Supersymmetry and its spontaneous breaking," Phys. Rev. B 85 (2012), 104203 doi:10.1103/PhysRevB.85.104203 [arXiv:1110.5500 [cond-mat.statmech]].
- [7] M. Baczyk, G. Tarjus, M. Tissier and I. Balog, "Fixed points and their stability in the functional renormalization group of random field models," J. Stat. Mech. **1406** (2014), P06010 doi:10.1088/1742-5468/2014/06/P06010 [arXiv:1312.6375 [cond-mat.dis-nn]].
- [8] John L. Cardy, "Nonperturbative aspects of supersymmetry in statistical mechanics," Physica D: Nonlinear Phenomena 15 (1985), 123-128, ISSN 0167-2789, https://doi.org/10.1016/0167-2789(85)90154-X.
- [9] N. G. Fytas, V. Martín-Mayor, G. Parisi, Picco М. and Ν. Sourlas, Phys. Rev. Lett. 122(2019)no.24, 240603 doi:10.1103/PhysRevLett.122.240603 [arXiv:1901.08473 [cond-mat.statmech]].
- [10] A. Kaviraj, S. Rychkov and E. Trevisani, "Random Field Ising Model and Parisi-Sourlas supersymmetry. Part I. Supersymmetric CFT," JHEP 04 (2020), 090 doi:10.1007/JHEP04(2020)090 [arXiv:1912.01617 [hep-th]].
- [11] A. Kaviraj, S. Rychkov and E. Trevisani, "Random field Ising model and Parisi-Sourlas supersymmetry. Part II. Renormalization group," JHEP 03 (2021), 219 doi:10.1007/JHEP03(2021)219 [arXiv:2009.10087 [condmat.stat-mech]].
- [12] A. Kaviraj, S. Rychkov and E. Trevisani, "Parisi-Sourlas Supersymmetry in Random Field Models," Phys. Rev. Lett. **129** (2022) no.4, 045701 doi:10.1103/PhysRevLett.129.045701 [arXiv:2112.06942 [cond-mat.statmech]].
- [13] K. J. Wiese, "Theory and experiments for disordered elastic manifolds, depinning, avalanches, and sandpiles," Rept. Prog. Phys. 85 (2022) no.8, 086502 doi:10.1088/1361-6633/ac4648 [arXiv:2102.01215 [cond-mat.disnn]].

- [14] G. Parisi and N. Sourlas, "Critical Behavior of Branched Polymers and the Lee-Yang Edge Singularity," Phys. Rev. Lett. 46 (1981), 871 doi:10.1103/PhysRevLett.46.871
- [15] J. L. Cardy, "Exact scaling functions for selfavoiding loops and branched polymers," J. Phys. A 34 (2001), L665-L672 doi:10.1088/0305-4470/34/47/101 [arXiv:cond-mat/0107223 [cond-mat]].
- [16] J. L. Cardy, "Lecture on branched polymers and dimensional reduction", (2003), cond-mat/0302495.
- [17] H. -P. Hsu, W. Nadler and P. Grassberger, "Statistics of lattice animals", Comput. Phys. Commun. 169 (2005), 114–116.
- [18] S. Luther and S. Mertens, "Counting lattice animals in high dimensions", J. Stat. Mech. 2011 (2011), P09026.
- [19] A. Kaviraj and E. Trevisani, "Random field ϕ^3 model and Parisi-Sourlas supersymmetry," JHEP **08** (2022), 290 doi:10.1007/JHEP08(2022)290 [arXiv:2203.12629 [hep-th]].
- [20] J. L. Cardy, "The Yang-Lee Edge Singularity and Related Problems," [arXiv:2305.13288 [cond-mat.stat-mech]].
- [21] D.C. Brydges and J.Z. Imbrie, "Branched polymers and dimensional reduction", Ann. Math. 158 (2003), 1019–1039.
- [22] D. C. Brydges and J. Z. Imbrie, "Dimensional Reduction Formulas for Branched Polymer Correlation Functions", J. Stat. Phys. **110** (2003) 503-518.
- "The Parisi-Sourlas uplift |23| E. Trevisani, and infinitely many CFTs," SciPost solvable 4dPhys. $\mathbf{18}$ (2025)no.2, 056 doi:10.21468/SciPostPhys.18.2.056 [arXiv:2405.00771 [hep-th]].
- [24] F. A. Dolan and H. Osborn, "Conformal Partial Waves: Further Mathematical Results," [arXiv:1108.6194 [hep-th]].
- [25] M. Hogervorst, H. Osborn and S. Rychkov, "Diagonal Limit for Conformal Blocks in *d* Dimensions," JHEP 08 (2013), 014 doi:10.1007/JHEP08(2013)014 [arXiv:1305.1321 [hep-th]].

- [26] A. L. Fitzpatrick and J. Kaplan, "Unitarity and the Holographic S-Matrix," JHEP 10 (2012), 032 doi:10.1007/JHEP10(2012)032 [arXiv:1112.4845 [hep-th]].
- [27] F. Gliozzi, A. Guerrieri, A. C. Petkou and C. Wen, "Generalized Wilson-Fisher Critical Points from the Conformal Operator Product Expansion," Phys. Rev. Lett. **118** (2017) no.6, 061601 doi:10.1103/PhysRevLett.118.061601 [arXiv:1611.10344 [hep-th]].
- [28] F. Gliozzi, A. L. Guerrieri, A. C. Petkou and C. Wen, "The analytic structure of conformal blocks and the generalized Wilson-Fisher fixed points," JHEP 04 (2017), 056 doi:10.1007/JHEP04(2017)056 [arXiv:1702.03938 [hep-th]].
- [29] X. Zhou, "How to Succeed at Witten Diagram Recursions without Really Trying," JHEP 08 (2020), 077 doi:10.1007/JHEP08(2020)077 [arXiv:2005.03031 [hep-th]].
- [30] J. Chen and X. Zhou, "Aspects of higher-point functions in $BCFT_d$," JHEP **09** (2023), 204 doi:10.1007/JHEP09(2023)204 [arXiv:2304.11799 [hep-th]].
- [31] S. Giombi, H. Khanchandani and X. Zhou, "Aspects of CFTs on Real Projective Space," J. Phys. A 54 (2021) no.2, 024003 doi:10.1088/1751-8121/abcf59 [arXiv:2009.03290 [hep-th]].
- [32] F. Gliozzi, P. Liendo, M. Meineri and A. Rago, "Boundary and Interface CFTs from the Conformal Bootstrap," JHEP 05 (2015), 036 [erratum: JHEP 12 (2021), 093] doi:10.1007/JHEP05(2015)036 [arXiv:1502.07217 [hep-th]].