arXiv:2504.15891v1 [physics.flu-dyn] 22 Apr 2025

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Optimal body force for heat transfer in turbulent vertical heated pipe flow

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As buoyancy can help drive a flow, the vertical heated-pipe arrangement is widely used in thermal engineering applications. However, buoyancy suppresses and can even laminarise turbulence in the flow, thereby seriously damaging the heat transfer, measured by the Nusselt number Nu. As buoyancy, measured by the parameter C, is increased, three flow regimes are possible: shear-driven turbulence, laminarised flow, and convective turbulence. In Chu *et al.* (2024) we employed a variational optimisation method to investigate how the buoyancy changes the structure of the minimal flow perturbation that triggers turbulence. Here, we extend the method to find an optimal body force of limited magnitude A_0 that maximises heat transfer, and examine how time-dependence of the flow affects the optimisation in each of the three flow regimes. Optimisations are performed at $Re = 3000$, and the force is found to laminarise convective turbulence, or make it only weakly chaotic for C up to 8. Consistent with previous computations that assume steady flow, the optimal force induces streamwise-independent rolls, but at larger A_0 the force triggers time-dependent turbulent flow. Transition from the laminar streamwise-independent state to turbulent flow can either enhance Nu or reduce Nu. For highly chaotic flows, either shear turbulence at $C = 1$ or convective turbulence at $C = 16, 32$, optimisations place rolls closer to the wall than calculations with the steady flow assumption. At any given A_0, however, the enhanced Nu is only weakly dependent on the number of induced rolls.

Key words:

1. Introduction

Vertical heated pipe flow is widely used in engineering applications, e.g. in geothermal energy capture, nuclear reactor cooling systems and fossil-fuel power plants, to transfer heat from one device to another. The important difference from iso-thermal pipe flow is that buoyancy, caused by the expansion of the fluid near the heated wall, can partially or even fully drive the flow, referred to as mixed or natural convection. Mixed convection has been widely researched, due to the great effects of buoyancy on the dynamics of the flow,

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as well as heat-transfer performance. Buoyancy plays a different role in downward versus upward flow. In the latter, buoyancy acts in the opposite direction to the flow and always enhances heat transfer. In an upward flow, instead, buoyancy first deteriorates the heat transfer (Ackerman 1970), then recovers only when buoyancy is strong enough. Three typical regimes for the heat-transfer characteristics are so classified in an upward heated flow, namely shear turbulence, laminarised flow, and convective turbulence (Parlatan *et al.* 1996; Yoo 2013; Zhang *et al.* 2020).

Extensive research has been conducted to understand the mechanism of heat transfer deterioration in a heated upward flow. Hall & Jackson (1969) proposed that the reduced shear stress in the buffer layer caused by buoyancy leads to a reduction of turbulence production, suppressing turbulence and even laminarising the flow, consequently deteriorating the heat transfer. More recently, He *et al.* (2016) successfully reproduced the laminarisation phenomenon by modelling the buoyancy with a radially dependent body force added to the isothermal flow. They noticed that the body force causes little difference to the key characteristics of turbulence, and proposed that laminarisation is caused by the reduction of the 'apparent Reynolds number', which is calculated based only on the pressure force of the flow (i.e. excluding the contribution of the body force). A similar laminarisation phenomenon is also found in isothermal pipe flow, where it has attracted much attention due to its implications for drag reduction (Hof *et al.* 2010; He *et al.* 2016; Kühnen *et al.* 2018; Marensi *et al.* 2019). Kühnen *et al.* (2018) examined the phenomenon of laminarisation from the perspective of the self-sustaining process of shear turbulence (Hamilton *et al.* 1995) and suggested that the decay of turbulence is caused by the reduction of transient growth. Marensi *et al.* (2019) investigated this phenomenon using nonlinear stability analysis (Pringle & Kerswell 2010), and found that nonlinear stability is enhanced in the presence of a body force that flattens the velocity profile. Recently, Marensi *et al.* (2021) systematically studied the flow regimes in a vertical heated pipe flow and found evidence that heat transfer deterioration and laminarisation are caused by weakened rolls.

Enhanced heat transfer means more effective and efficient energy conversion or cooling. There have been many interesting investigations aimed at improving the heat transfer of fluid systems. Strategies can generally be classified as active, passive and compound remedies (Webb & Bergies 1983; Liu & Sakr 2013; Kumar & Kim 2015; Suri *et al.* 2018). Active methods (Ohadi *et al.* 1991; Wang *et al.* 2020; Yuan *et al.* 2023) require an external power input to improve heat transfer. For example, Ohadi *et al.* (1991) studied the effect of corona discharge on forced-convection heat transfer in a tube. Wang *et al.* (2020); Yuan *et al.* (2023) proposed a method of vibrating the boundary layer to enhance the heat transfer. Passive methods include curving or twisting flow geometry, adding extended surfaces and so on. Compound methods (Gau & Lee 1992; Naphon *et al.* 2017; Kareem & Gao 2018) adopt both active and passive techniques. These strategies have significantly improved heat transfer in many systems.

Many techniques for heat transfer enhancement have been developed empirically. Here we seek a strategy that is 'optimal' with respect to a constrained magnitude of an applied body force. In principle, maximisation of the heat transfer can be solved by variational methods. However, the heat-transfer, measured by the Nusselt number *Nu*, is a local field variable, depending only on the gradient of the temperature evaluated at the wall, which leads to awkward delta functions in a variational approach. A quantity is required for variational formulations that is spatially global, but which measures the heat transfer at the wall. Guo *et al.* (1998, 2007) defined the *entransy* quantity to describe the heat-transfer ability of a system, using an analogy between heat conduction and electrical conduction. The entransy dissipation rate was then derived and applied to measure the irreversibility of the heat transfer process. Maximum and minimum entransy dissipation principles were proposed for the optimisation

of heat transfer for fixed boundary temperature and fixed heat flux respectively. The theory has been successfully used to optimise heat transfer in various thermal systems, e.g. in heat exchangers (Guo & Xu 2012; Guo *et al.* 2010) and heat exchanger networks (Chen *et al.* 2009).

For pipe flow, Meng *et al.* (2005) have sought a steady velocity field that maximises heat transfer. Although the Navier-Stokes equation was not prescribed as a constraint, it was shown that the velocity field must satisfy a similar equation, subject to a particular force called the synergy force, which produces a velocity field that tends to align with the temperature gradient (Guo 2001). Jia *et al.* (2014) did a similar optimisation but set power consumption as a constraint condition, they also found that longitudinal swirl flow with multi-vortex structure can enhance heat transfer greatly, and the number of vortexes of the optimal velocity field increases with a larger power consumption. Wang *et al.* (2015) proposed a new criterion, exergy destruction minimisation, to optimise the heat transfer, and a similar optimal velocity field was found. Such heat transfer optimisations in pipe flow have motivated several heat-transfer enhancement designs, e.g. the alternating elliptical axis tube (Meng *et al.* 2005), discrete double-inclined ribs tubes (Li *et al.* 2009) and many other interesting attempts (Liu & Sakr 2013; Sheikholeslami *et al.* 2015). However, the above calculations have assumed a steady laminar flow, while it is common to find that the flow is turbulent. There have been efforts to construct variational equations based on the Reynolds Averaged Navier Stokes (RANS) turbulence description (Chen *et al.* 2007), but this approach does not capture the detailed dynamical characteristics of the flow under heating conditions, the self-sustaining mechanisms of the flow and transitions between the flow regimes, that we wish to retain and optimise here. Motoki *et al.* (2018) also adopted a variational method to find the optimal steady velocity field for plane Couette flow with the largest Nusselt number. The 'scalar dissipation' was set as the objective function, which coincides with the entransy dissipation. They found the optimal flow state is composed of streamwise-independent rolls at $Re \sim 10^1$, but there appear smaller-scale hierarchical quasi-streamwise vortex tubes near the walls in addition to the large-scale rolls at $Re \geqslant 10^2$. Although Motoki *et al.* (2018) performed calculations up to $Re = 10^4$, their analysis assumes a time-independent velocity field.

Optimisations need to be extended to time-dependent flows, such as turbulence. This requires a new framework that includes the dynamical effects of the flow on the mean heat transfer. The fully nonlinear variational method has been used in isothermal pipe flow by Pringle & Kerswell (2010) to find initial flow perturbations that grow maximally. The smallest perturbation which triggers transition is called the 'minimal seed'. For a review of wider applications, see Kerswell (2018). In the context of pipe flow, this framework has been successfully employed to find the minimal seed under various conditions affecting the flow (Pringle & Kerswell 2010; Pringle *et al.* 2012; Marensi *et al.* 2019) and extended to induce transition 'the other way', i.e. to construct an optimal 'baffle' that destabilises turbulence to cause transition to a laminar state (Marensi *et al.* 2020; Ding *et al.* 2020). The minimal seed for transition in heated pipe flow has been calculated using the model of §2 (Chu *et al.* 2024). Here, we extend this new nonlinear variational framework to maximise the heat transfer for vertical heated pipe flow. While previous optimisations in this geometry have identified optimal stationary velocity fields that maximise the heat transfer, here we introduce and seek to optimise a time-independent body force that modifies the time-dependent flow. Although hard to accurately reproduce body forces in an engineering application, this is a step towards guiding such an approach. It should also be noted that the present study mainly focuses on improving heat transfer and understanding the physical mechanism, without considering the changes in pumping power.

The plan of the paper is as follows. In §2, we present our Direct Numerical Simulation (DNS) model of vertical heated pipe flow and the variational equations for optimisation. In §3,

we first show behaviour for preliminary optimisations, including the features of optimal force, and the effects of target time. Optimisations are then performed in the laminarisatin regime, shear turbulence regime and convective turbulence regime. Finally, the paper concludes with a summary in §4.

2. Formulation

2.1. *The heated pipe flow model*

We begin with the formulation of Chu *et al.* (2025), which models a vertical heated pipe with a background streamwise temperature gradient. A particular feature of the model is that the gradient is allowed to vary in time, reflecting the flow-dependent nature of heat transfer, e.g. the flow can transition from the shear-driven turbulent state to the convective state causing a significant drop in the heat flux.

Using cylindrical coordinates $\boldsymbol{x} = (r, \phi, z)$ for a pipe of radius R, the total temperature is decomposed as

$$T_{tot}(\boldsymbol{x}, t) = T_w(z, t) + T(\boldsymbol{x}, t) - 2T_b \qquad (2.1)$$

where the wall temperature is given by $T_w(z, t) = a_{tot}(t)z + b$, with some constant reference temperature b, and $T(\boldsymbol{x}, t)$ carries the temperature fluctuations. Let $\langle \cdot \rangle$ denote the spatial average over the domain. Temperature fluctuations in the model have a fixed positive spatial average $T_b = \langle T \rangle$, maintained by adjustments in the temperature gradient $a_{tot}(t)$. The factor $-2T_b$ has been inserted in (2.1) so that the fluctuations are positive and largest at the heated wall, where $T|_{r=R} = 2T_b$.

Nondimensionalisation using the temperature scale $2T_b$, the length scale R, and velocity scale $2U_b$, where U_b is the mean flow speed, and assuming the Boussinesq approximation, leads to the dimensionless governing equations

$$\frac{\partial T}{\partial t} + (\boldsymbol{u}_{tot} \cdot \boldsymbol{\nabla})T = \frac{1}{Re\,Pr}\nabla^2 T - \boldsymbol{u}_{tot} \cdot \hat{\boldsymbol{z}}\, a_{tot}(t)\,, \qquad (2.2)$$

$$\frac{\partial \boldsymbol{u}_{tot}}{\partial t} + (\boldsymbol{u}_{tot} \cdot \boldsymbol{\nabla})\boldsymbol{u}_{tot} = -\boldsymbol{\nabla} p_{tot} + \frac{1}{Re}\nabla^2 \boldsymbol{u}_{tot} + \frac{4}{Re}(1 + \beta'(t) + CT)\hat{\boldsymbol{z}}\,, \qquad (2.3)$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}_{tot} = 0\,, \qquad (2.4)$$

where $\boldsymbol{u}_{tot}(\boldsymbol{x}, t)$ is the velocity field. The dimensionless boundary condition for the temperature is then $T = 1$ and no-slip is applied to the velocity $\boldsymbol{u}_{tot} = \boldsymbol{0}$ at $r = 1$. Axial periodicity over a distance L is assumed for the temperature fluctuations and velocity. The dimensionless fixed bulk temperature and flow rate are respectively $\langle T \rangle = 1/2$ and $\langle \boldsymbol{u}_{tot} \cdot \hat{\boldsymbol{z}} \rangle = 1/2$, and these two conditions determine values for $a_{tot}(t)$ and the excess pressure fraction $\beta'(t)$ via the spatial averages of the respective governing equation. The Reynolds and Prandtl numbers are $Re = 2U_bR/\nu$ and $Pr = \nu/\kappa$, where ν and κ are the kinematic viscosity and thermal diffusivity respectively. The third dimensionless parameter

$$C = \frac{Gr}{16\,Re}\,, \qquad (2.5)$$

measures the buoyancy force relative to the force that drives laminar isothermal shear flow, where $G_r = \gamma g(T|_{r=R} - T_b)(2R)^3/\nu^2$ is the Grashof number, γ is the coefficient of volume expansion, and g is gravitational acceleration. The observed quantity that measures the heat flux is the Nusselt number

$$Nu = \frac{2R\,q_w}{\lambda\,(T|_{r=R} - T_b)}\,, \qquad (2.6)$$

where λ is the thermal conductivity of the fluid, $q_w = \lambda\,\overline{(\partial T/\partial r)|_{r=R}}$ is the wall heat flux per

unit area, and the overline denotes the time average. We sometimes plot the instantaneous $Nu(t)$, where the time average is dropped.

For this configuration, the laminar solution does not have a simple analytic form, and must be computed numerically. We consider perturbations from the laminar state, subscripted with 0, and decompose variables as $\boldsymbol{u}_{tot} = \boldsymbol{u}_0 + \boldsymbol{u}$, $p_{tot} = p_0 + p$, $1 + \beta'(t) = 1 + \beta_0 + \beta(t)$, $T = \Theta_0 + \Theta$, $a_{tot}(t) = a_0 + a(t)$. The perturbations then satisfy

$$\frac{\partial \Theta}{\partial t} + u_0 \frac{\partial \Theta}{\partial z} + u_r \frac{d\Theta_0}{dr} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\Theta = \frac{1}{RePr}\nabla^2\Theta - u_z a_0 - (u_0 + u_z)a(t), \qquad (2.7)$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + u_0 \frac{\partial \boldsymbol{u}}{\partial z} + u_r \frac{du_0}{dr}\hat{\boldsymbol{z}} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla}p + \frac{1}{Re}\nabla^2\boldsymbol{u} + \frac{4}{Re}(C\Theta + \beta(t))\hat{\boldsymbol{z}}, \qquad (2.8)$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \qquad (2.9)$$

where $\boldsymbol{u} = (u_r, u_\phi, u_z)$. Further details on the numerical model can be found in Chu *et al.* (2024) and Chu *et al.* (2025).

2.2. *Variational optimisation of a body force for heat transfer*

In the following, we suppose that a body force $\boldsymbol{F}(\boldsymbol{x})$ is appended to the right-hand sides of the Navier–Stokes equations (2.3) and (2.8), then seek to optimise the form of $\boldsymbol{F}(\boldsymbol{x})$ such that it maximises Nu, subject to a constraint on the magnitude of $\boldsymbol{F}$. Following Guo *et al.* (2007), we use the *entransy dissipation*, $\frac{1}{2}\lambda(\boldsymbol{\nabla}T)^2$, as a proxy for the heat transfer. Here, the time-averaged quantity

$$J = \frac{1}{\mathcal{T}}\int_0^\mathcal{T} \frac{1}{2}\langle(\nabla T)^2\rangle \, dt. \qquad (2.10)$$

is used as our objective function. The way in which the entransy dissipation is used depends upon the thermal boundary condition. As the fixed temperature boundary condition is applied, heat flux is maximised when the entransy dissipation J maximised. (For the fixed heat flux boundary condition, minimal J corresponds to minimal thermal resistance within the body, and hence maximum heat flux.)

A Lagrangian is constructed as follows:

$$\begin{aligned}
\mathcal{L} = \frac{1}{N}\sum_{i=1}^N J_i - \lambda_0(\langle\frac{1}{2}(\boldsymbol{F})^2\rangle - A_0) - \sum_{i=1}^N \int_0^\mathcal{T}\langle \boldsymbol{v}_i \cdot (\text{NS}(\boldsymbol{u}_i))\rangle dt \\
- \sum_{i=1}^N \int_0^\mathcal{T}\langle \Pi_i(\boldsymbol{\nabla} \cdot \boldsymbol{u}_i)\rangle dt - \sum_{i=1}^N \int_0^\mathcal{T}\langle \pi_i(\text{Tem}(\Theta_i))\rangle dt \qquad (2.11) \\
- \sum_{i=1}^N \int_0^\mathcal{T} \Gamma_i\langle(\boldsymbol{u}_i \cdot \hat{\boldsymbol{z}})\rangle dt - \sum_{i=1}^N \int_0^\mathcal{T} Q_i\langle(\Theta_i)\rangle dt.
\end{aligned}$$

As the initial condition for the velocity field could be turbulent, to improve the robustness of the results we apply the optimisation to N initial velocity fields. The variables λ_0, Π_i, $\pi_i(\boldsymbol{x}, t)$, $\Gamma_i(t)$, $Q_i(t)$ and $\boldsymbol{v}_i(\boldsymbol{x}, t) = (v_{r,i}, v_{\phi,i}, v_{z,i})$ are Lagrange multipliers. The first term, the ensemble average of all time-averaged entransy dissipation, is the objective function to be maximised. The second term fixes the amplitude of the body force. Next, the velocity perturbation $\boldsymbol{u}$ is constrained to satisfy the Navier–Stokes equation $\text{NS}(\boldsymbol{u})$ and the continuity equation, and the temperature perturbation satisfies the temperature equation $\text{Tem}(\Theta)$, each over the period from $t = 0$ to $t = \mathcal{T}$. The last two terms ensure that the velocity satisfies the fixed mass flux and fixed bulk temperature conditions.

Taking variations of $\mathcal{L}$ with respect to each variable and setting them equal to zero, we

obtain the following set of Euler–Lagrange equations. The adjoint Navier–Stokes, temperature equation and continuity equations are

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{u}_i} = \frac{\partial \boldsymbol{v}_i}{\partial t} + u_0 \frac{\partial \boldsymbol{v}_i}{\partial z} - v_{z,i} u_0' \hat{\boldsymbol{r}} + \boldsymbol{\nabla} \times (\boldsymbol{v}_i \times \boldsymbol{u}_i) - \boldsymbol{v}_i \times \boldsymbol{\nabla} \times \boldsymbol{u}_i + \boldsymbol{\nabla}\Pi_i +$$

$$\frac{1}{Re}\boldsymbol{\nabla}^2 \boldsymbol{v}_i - \pi_i \Theta_0' \hat{\boldsymbol{r}} - \pi_i \boldsymbol{\nabla}\Theta_i - \pi_i(a(t) + a_0(t))\hat{\boldsymbol{z}} - \Gamma_i \hat{\boldsymbol{z}} = 0. \quad (2.12)$$

$$\frac{\partial \mathcal{L}}{\partial \Theta_i} = \frac{\partial \pi_i}{\partial t} + u_0 \frac{\partial \pi_i}{\partial z} + \frac{4}{Re} v_{z,i} C + \boldsymbol{u}_i \cdot \boldsymbol{\nabla}\pi_i + \frac{1}{RePr}\boldsymbol{\nabla}^2 \pi_i - Q_i - \frac{1}{\mathcal{T}}\boldsymbol{\nabla}^2 T_i = 0, \quad (2.13)$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{v}_i = 0. \quad (2.14)$$

The compatibility condition (terminal conditions for backward integration of (2.12) and (2.13)) is given by

$$\frac{\delta \mathcal{L}}{\delta \boldsymbol{u}_i(\boldsymbol{x}, \mathcal{T})} = -\boldsymbol{v}_i(\boldsymbol{x}, \mathcal{T}) = 0, \quad (2.15)$$

$$\frac{\delta \mathcal{L}}{\delta \Theta_i(\boldsymbol{x}, \mathcal{T})} = -\pi_i(\boldsymbol{x}, \mathcal{T}) = 0 \quad (2.16)$$

and the optimality condition is

$$\frac{\delta \mathcal{L}}{\delta \boldsymbol{F}} = -\lambda_0 \boldsymbol{F} + \frac{1}{N} \sum_{i=1}^{N} \int_0^{\mathcal{T}} \langle \boldsymbol{v}_i \rangle \mathrm{d}t = 0. \quad (2.17)$$

For an arbitrary initial $\boldsymbol{F}$ and set of initial conditions $\boldsymbol{u}_i$, the force $\boldsymbol{F}$ is incrementally updated to produce a maximum in $\mathcal{L}$ where where $\delta \mathcal{L}/\delta \boldsymbol{F}$ should vanish. An iterative algorithm similar to that in Pringle *et al.* (2012) is applied. The update for $\boldsymbol{F}$ at $(j+1)$th iteration is

$$\boldsymbol{F}^{(j+1)} = \boldsymbol{F}^{(j)} - \epsilon_0 \frac{\delta \mathcal{L}}{\delta \boldsymbol{F}^{(j)}}. \quad (2.18)$$

where ϵ_0 is a small value, controlled using a procedure described in Pringle *et al.* (2012). λ_0 is adjusted to set $\langle [\boldsymbol{F}(\boldsymbol{x})^{(j+1)}]^2 \rangle = 2 A_0$.

2.3. *Numerical methods*

Calculations are carried out using the open-source code Openpipeflow (Willis 2017). Variables are discretised in the domain $\{r, \phi, z\} = [0,1] \times [0, 2\pi] \times [0, 2\pi/\alpha]$, where $\alpha = 2\pi/L$, using Fourier decomposition in the azimuthal and streamwise direction and finite difference in the radial direction, e.g.

$$\boldsymbol{u}(r_s, \phi, z) = \sum_{k<|K|} \sum_{m<|M|} \boldsymbol{u}_{skm} \mathrm{e}^{\mathrm{i}(\alpha k z + m\phi)}, \qquad s = 1, ..., S \quad (2.19)$$

where the radial points r_s are clustered towards the wall. Temporal discretisation is via a second-order predictor-corrector scheme, with Euler predictor for the nonlinear terms and Crank-Nicolson corrector. To keep the nonlinear optimisations manageable, a Reynolds number $Re = 3000$ and $Pr = 0.7$ are adopted with a domain of length $L = 5D$. We use mesh resolution of $S = 64, M = 48, K = 42$, and the size of the time step is $\Delta t = 0.01$. This resolution is sufficient to maintain a drop-off in the amplitude of the coefficients by three to four orders magnitude, which experience has shown to be sufficient for accurate simulation of shear-driven turbulence. For the C considered here, the convective state is less computationally demanding to simulate.

Figure 1: (*a*) Instantaneous Nusselt number over time as the iteration proceeds, where $Nu_{F=0}$ refers to the value of the unforced case. (*b*) Dimensionless time-averaged entransy dissipation, (time-averaged) Nu and $Nu(\mathcal{T})$ versus iteration, normalised by their corresponding values at the zeroth iteration. (*c*) The residual $\langle(\delta\mathcal{L}/\delta\boldsymbol{F})^2\rangle$ versus iteration. (*d*) Magnitude of the components of body force $\frac{1}{2}\langle F_r^2\rangle$, $\frac{1}{2}\langle F_\phi^2\rangle$, $\frac{1}{2}\langle F_z^2\rangle$ versus iteration. The optimisation is run at $A_0 = 5\times10^{-7}, \mathcal{T} = 400, C = 3, Re = 3000$.

3. Results

We first show preliminary optimisation in §3.1. Then, we optimise the body force to maximise the heat transfer in three typical flow regimes of vertical heated pipe flow, i.e. the laminarisation regime (§3.2), shear turbulence regime (§3.3) and convective turbulence regime (§3.4). (Further details on the parameter regimes for this model can be found in figure 3(*a*) of Chu *et al.* (2024).)

3.1. *Preliminary optimisation*

For the laminar case, the state is unique and we require only one initial velocity field, $N = 1$. We start with the unforced laminarised flow at $Re = 3000$, $C = 3$, and take random fields for the initial force (such as a turbulent velocity field). Results from the preliminary optimisation with $A_0 = 5\times10^{-7}, \mathcal{T} = 400$ are shown in figure 1. The instantaneous Nusselt number, $Nu(t)$, normalised by the mean for the unforced flow, $Nu_{F=0}$, is shown in figure 1(*a*) for each iteration. The final value increases by more than 80% over the unforced laminar case. Figure 1(*b*) shows the objective function J (2.10), the (time-averaged) Nusselt number (2.6) and the final value of the instantaneous Nusselt number $Nu(\mathcal{T})$, versus iteration, normalised by values for the unforced case. Changes in these quantities show good agreement, indicating that the global 'entransy dissipation' quantity effectively captures the local (boundary) heat transfer behaviour measured by the Nusselt number. Figure 1(*c*) shows the residual of the

Figure 2: Contours of the optimal body force optimised with target time horizons (a)
$\mathcal{T} = 50$, (b) $\mathcal{T} = 100$, (c) $\mathcal{T} = 200$, (d) $\mathcal{T} = 600$ at $C = 3$, $A_0 = 5e − 7$, $Re = 3000$, $N = 1$.
Contours are coloured by the streamwise body force, while arrows represent the
cross-stream components of the body force. The first colorbar is for (a, b), the second one
is for (c, d). The largest arrow has magnitude 2.38×10^{-4} in (a), 2.36×10^{-4} in (b),
2.93×10^{-4} in (c) and 3.21×10^{-4} in (d).

calculation, which drops quickly in first 50 iterations then more gradually. (Spikes are related to the method that seeks to increase ϵ_0 as much a possible, which affects the magnitude of the residual via λ_0.) Usually, the optimisation is stopped when the change in the Nusselt number drops below 10^{-5}. Figure 1(d) tracks the amplitude of the three components of the body force versus iteration. The amplitude of the streamwise component drops significantly, while the amplitude of the cross-stream components increases. This suggests that the cross-stream components of the body force play a dominant role in enhancing heat transfer, whereas the contribution of the streamwise component is nearly negligible.

The time horizon $\mathcal{T}$ should be long enough such that Nu is optimised for the steady response to the force, and figure 1(a), suggests that $\mathcal{T}$ should be greater than 200. Indeed, the form of the optimal is found to change when increasing $\mathcal{T}$ from 50 to 100, and again to 200, but increasing further to 600, the optimal is essentially the same up to a rotation, as shown in figure 2. Interestingly, the optimal body force optimised for a short target time has perfect rotational symmetry. As $\mathcal{T}$ increases, the azimuthal wave number m decreases. This is consistent with smaller-scale vortices growing more rapidly (Schmid 2007), thereby increasing the heat transfer within a shorter time. In the longer target time, however, the larger-scale mode is more effectively amplified for the given magnitude of force. The optimal force in the form of rolls is consistent with the results of Meng $et\ al.$ (2005), although their force corresponds to the sum of our force and the buoyancy term itself, discussed in §4.

It is found that the distribution of the body force is almost uniform in the streamwise direction. To simplify the form of the body force and to accelerate convergence, we constrain the body force to be streamwise-independent in the optimisations of the following sections by zeroing its streamwise-dependent Fourier modes.

3.2. *Optimisation in the laminarisation regime*

3.2.1. *The optimal force for the laminar state*

Having examined properties of the parameters necessary for optimisation, in this section we consider optimisation in the laminarised regime at $C = 3$, $Re = 3000$ in more detail. In particular, we examine the dependence of the rotational symmetry on A_0 and the presence of local optimals (dependent on the initial guess for the force).

For the laminar initial condition, as we have assumed streamwise independence for the force, a small streamwise-dependent perturbation must be added to the initial velocity so that transition to turbulence may be triggered if the resulting two-dimensional flow is unstable. We add a perturbation of magnitude $E_0 = \frac{1}{2}\langle u^2 \rangle \approx 10^{-7}$ and set a longer $\mathcal{T} = 600$ to allow the occurrence of transition. Figure 3 shows the instantaneous Nusselt number, $Nu(t)$, for

Figure 3: Time evolution of instantaneous Nusselt number normalised by unforced value
for different force amplitudes, starting from a laminar initial condition at
$C = 3, Re = 3000$. The vertical dashed line indicates the optimisation target time $\mathcal{T} = 600$.

Figure 4: Contours of optimal body force at
$(a) A_0 = 10^{-8}, (b) A_0 = 10^{-7}, (c) A_0 = 5 \times 10^{-7}, (c) A_0 = 6 \times 10^{-7}$, starting from a
laminar initial condition at $C = 3, \mathcal{T} = 600, Re = 3000$. The arrows represent the
cross-stream components of body force. The largest arrow has magnitude 4.85×10^{-5} in
(a), 1.87×10^{-4} in (b), 4.05×10^{-4} in (c) and 4.26×10^{-4} in (d). As the axial
component is at least an order of magnitude smaller, it is not shown.

several A_0. Optimisation improves the heat transfer substantially: for $A_0 = 10^{-7}$ heat transfer
is almost 50% greater than that of the unforced flow, and for $A_0 = 5 \times 10^{-7}$ is almost doubled.
When the force amplitude is $A_0 = 6 \times 10^{-7}$, $Nu(t)$ experiences a sudden increase near the end
of the optimisation target time and fluctuates thereafter, indicating the onset of turbulence.
(At $A_0 = 5 \times 10^{-7}$, transition is observed very late, at around $t = 1000$, and interestingly, the
transition does not lead to a larger Nu. This phenomenon will be discussed later.)

The typical amplitude-dependent form of the forces obtained from optimisations are given
in figure 4(a-d). (As the axial component is at least an order of magnitude smaller, it is not
shown.) At small A_0, the body force has a single pair of rolls. At increased force amplitude,
figure 4(b) illustrates how the vortex structure gradually approaches the wall, reducing the
spatial scale in both the radial and spanwise directions. This is actually found to be a local
optimal for this A_0, as two pairs of rolls may be squeezed in to increase Nu a little further. At
larger A_0, more rolls are seen in figure 4(c-d). For the largest A_0, turbulence is triggered within
$\mathcal{T}$, and the optimisation struggles to converge to a well-organised optimal force. However, a
preference for roll structures of larger m is clear. The form and increase in wavenumber is
consistent with the calculations of Meng *et al.* (2005); Jia *et al.* (2014); Wang *et al.* (2015)
for steady flow.

We have observed that optimal body forces occur with rotational symmetry of different
azimuthal wave numbers m in figures 2 and figure 4. We let O_2 denote an optimal with

Figure 5: (a) The Nusselt number of final state $Nu(\mathcal{T})$ versus iteration for different initial forces with $A_0 = 10^{-7}, C = 3, Re = 3000$. The legend indicates the initial force, and the resulting local optimal forces are labelled on the curves. (b) The time-averaged Nusselt numbers at $C = 3, Re = 3000$ versus O_{Fm} for different force amplitudes, indicated in the legend. The global optimal is highlighted with a dashed circle. For dashed lines, the forced flow state remains laminar. For solid lines, the forced state is turbulent and the values are time-averages.

2-fold rotational symmetry. This optimal will have non-zero Fourier coefficients for $m = 0, 2, 4, 6, 8, ...$, but note that this does not exclude a force with only non-zero modes $m = 0, 4, 8, ...$, which corresponds to an optimal O_4 with 4-fold rotational symmetry. To examine the influence of the azimuthal periodicity, and to simplify the optimisation further, we consider optimal forces restricted to azimuthal Fourier modes of wavenumbers 0 and m only, and denote them O_{Fm}. Note that rotational symmetry is imposed only on the force, and not on the velocity field.

Optimisations have been computed for O_{Fm}, then used as starting forces for optimisations in the full space O_1. In this way, we examine the dependence on m to determine which rotational symmetry is the global optimal. The Nusselt numbers of the final states, $Nu(\mathcal{T})$, as a function of iteration are shown in figure 5(a) starting from the O_{Fm} forces. Three optimisations starting from random initial forces are also shown. Several observations can be made. Firstly, there are multiple local optimals O_m, of which O_2 (figure 4(c)) is the global optimal for this A_0. Secondly, the optimal of type O_m does not produce much greater Nu than the optimal O_{Fm}, of the reduced Fourier space, used as the starting force. Thirdly, if the starting force is quite perturbed, such as for the random initial forces, it is most likely to end up at the global optimal. Similarly, the optimisation starting from O_{F5} (dark orange line) appears to pass close to O_1, but ends at the global optimal O_2.

As we have observed that O_m does not produce much greater Nu than O_{Fm}, for small A_0 at least, we directly compare Nu of the flow forced by O_{Fm} for several m to determine the global optimal. Figure 5(b) shows the Nusselt number, calculated using averages over 5000 time units for each simulation. For the dashed lines, the final state is still laminar and good convergence is easily achieved. For the solid lines at larger A_0, the final state is turbulent, which renders convergence difficult. For the latter case, and for convenience in this section, the force has been calculated using an artificially stabilised two-dimensional flow, by putting $K = 1$ in (2.19) (equivalent to adding no three-dimensional perturbation, $E_0 = 0$). The optimal force leads to an artificially stabilised optimal velocity field, as for the optimal steady velocity fields reported by Meng et al. (2005); Jia et al. (2014); Wang et al. (2015) . The force is then applied, here resulting in a fully three-dimensional time-dependent turbulent

Figure 6: Enhanced Nu of forced flows at $C = 3$, $Re = 3000$. (*a*) Instantaneous $Nu(t)$ for flows forced by O_{F1}. (*b*) Time-averaged Nu for flows forced by O_{Fm} in the turbulent state (solid lines) and $Nu(\mathcal{T})$ for the artificially stabilised streamwise-independent state (dashed lines).

simulation, from which Nu is calculated. (In principle, the artificial stabilisation of the flow during optimisation may render the force no longer optimal. This is examined further for each initial flow regime, and is found to be a good approach for intermediate C. We will show this for the convective case in §3.4.) As A_0 increases, the rotational symmetry m of the (global) optimal increases, from $m = 1$ at $A_0 = 10^{-8}$, to $m = 2$ at $A_0 = 10^{-7}$, and is $m \approx 5$ at substantially larger $A_0 = 10^{-5}$. However, it should be noted that Nu is not strongly dependent on m.

3.2.2. *The path to transition and effect on heat transfer*

Figure 6(*a*) shows the instantaneous Nusselt number $Nu(t)$ for flows forced by O_{F1} at several force amplitudes. At small A_0, the flow is reshaped into a two-dimensional forced laminar state. With an increase of force amplitude to $A_0 = 6 \times 10^{-7}$, the flow does not transition to turbulence directly — instead, the two-dimensional state quickly forms, then transitions later to a travelling wave solution. Figure 7 shows the two-dimensional reshaped laminar solution and a travelling wave solution found in the flow forced by O_{F1} at $A_0 = 6 \times 10^{-7}$ (at $t \approx 500$ and $t \approx 1500$ respectively). In isothermal flow, forces have been used to find travelling wave solutions, via homotopy (Wedin & Kerswell 2004), but were only found by this method for higher rotational symmetry $m \geqslant 2$. The travelling wave solution has larger Nu compared with the two-dimensional reshaped laminar solution. With further increase of force amplitude, the flow transitions from the two-dimensional state to a (mildly) chaotic three-dimensional state, and finally also converges to a travelling wave state at a later time (not shown here). At $A_0 = 10^{-5}$, the flow directly transitions to a strong chaotic three-dimensional state, along with a greatly increased Nu. However, this is not always the case, and in fact figure 6(*b*) shows that at larger A_0, the transition from two-dimensional flow (stabilised by setting $K = 1$) to the chaotic three-dimensional state tends to lead to a decrease in Nu.

It is interesting that the more chaotic state does not necessarily lead to an improvement in heat transfer. The instantaneous Nusselt number $Nu(t)$ and the roll energy $E_{roll}(t) = E(u_r) + E(u_\phi)$ are shown in figure 8 for the flow forced by O_{F4} at two amplitudes. At $A_0 = 10^{-6}$ heat transfer increases after transition, but at $A_0 = 10^{-5}$ it is reduced after transition. In both cases, the energy of rolls increases, but in the latter case by not as much. Stronger rolls are typically associated with enhanced heat transfer, but this shows that higher roll energy alone does not necessarily correspond to more efficient heat transfer.

Figure 7: The isosurface of streamwise velocity at (*a*) $t = 500$ and (*b*) $t = 1500$ when the flow is forced by O_{F1} with $A_0 = 6 \times 10^{-7}$. Red/yellow are 20% of the min/max streamwise velocity.

Figure 8: Time evolution of (*a*) instantaneous $Nu(t)$ and (*b*) instantaneous energy of rolls $E_{roll} = E(u_r) + E(u_\phi)$, $E(u_r) = \frac{1}{2}\langle u_r^2 \rangle$, $E(u_\phi) = \frac{1}{2}\langle u_\phi^2 \rangle$, when the flow is forced by O_{F4} at $C = 3$, $Re = 3000$ with different forcing amplitudes. Solid lines are for the forced turbulent state, and dashed lines are for the artificially-stabilised streamwise-independent state.

Figure 9 shows the contours of the radial temperature gradient $\partial T/\partial r$ evaluated at the boundary, for the forced laminar and forced turbulent states, normalised by the mean radial temperature gradient of the unforced flow. When the flow is forced by O_{F4} with $A_0 = 10^{-6}$, the laminar state (figure 9(a)) exhibits distinct regions of strong and weak heat transfer. After the transition (figure 9(b)), these regions are not fixed, and the regions of higher heat flux widen and intensify a little. At $A_0 = 10^{-5}$ (figure 9(c-d)), the regions of higher heat flux also widen after the transition, but are less organised and weaker than for the steady flow, despite the slightly increased roll energy. This suggests that the streamwise vortices in the forced turbulent states are not as efficient as those in the forced laminar states. The rolls are unsteady in forced turbulent states and move further from the wall intermittently due to the waving of low-speed streaks, leading to a nonuniform heat transfer distribution. Heat transfer is enhanced on the sides where the rolls are positioned close to the wall, but weakens as the rolls move further away from the wall.

Overall, the transition to the forced turbulent state involves two main competing effects on heat transfer: the first is the enhancement of rolls, which increases heat transfer by facilitating better mixing, the second is the unsteady rolls, which can reduce heat transfer efficiency as the rolls are not consistently positioned near the wall, where they are most effective at transferring heat. At $A_0 = 10^{-6}$, the enhancement of rolls wins and heat transfer is enhanced, while for

Figure 9: Contours of the radial temperature gradient $\frac{\partial T}{\partial r}$ at the boundary at $C = 3$, $Re = 3000$, normalised by the unforced laminar radial temperature gradient. The flow is forced by O_{F4} with (a,b) $A_0 = 10^{-6}$ and (c,d) $A_0 = 10^{-5}$, the flow states are taken in (a,c) from the forced laminar state ($t = 1000$) and (b,d) forced turbulent state ($t = 1000$).

$A_0 = 10^{-5}$ the rolls are not much stronger than in forced laminar state and so the reduction of heat transfer due to unsteady rolls prevails. The case O_{F1} is an exception in figure 6(b), where the transition to a chaotic state leads to an increase in Nu for both amplitudes. In this case, only one pair of rolls is inefficient and the unsteadiness can lead to the creation of additional rolls.

3.3. *Optimisation in the shear turbulence regime*

Returning to smaller C, the shear-turbulence case is the most challenging, as the flow state remains highly chaotic. As turbulence is already effective in enhancing heat transfer relative to the laminarised case, it is not obvious that optimisation should be able to improve heat transfer substantially.

We focus on the case $C = 1$ at $Re = 3000$. For the highly chaotic flow, it is difficult to apply the method with a large target time (Pringle & Kerswell 2010; Pringle *et al.* 2012; Marensi *et al.* 2019), and convergence was found to fail for $\mathcal{T}$ even as low as 100. However, reasonably good convergence was found for $\mathcal{T} = 50$. Although this is not sufficient time to capture the statistics of the end state, it is sufficient time for a response to be observed, so that it is reasonable to examine whether the heat transfer has been pushed in the right direction.

Figure 10(a) shows the instantaneous Nusselt number $Nu(t)$ for flows forced by the optimised force in the full space O_1 from a random initial force, with $A_0 = 10^{-6}$. The optimisation, although with a short target time, still increases the Nusselt number significantly. Surprisingly, table 1 shows that the (subsequent time-averaged) $Nu/Nu_{F=0}$ does not change significantly with more initial conditions N, despite that the short $\mathcal{T}$ might suggest greater dependence on the initial condition, nor does Nu vary significantly with rotational symmetry. The structures of the optimised forces are shown in figure 11. Although Nu varies very little with N, for O_1, the structure of the optimal force does change – going from $N = 1$ to 9 initial conditions, the force develops towards a structure more like the O_3 optimal, but larger N would probably improve convergence. Optimising for O_3 itself, the structure clearly becomes more regular as N is increased. For O_{F3}, O_5 (third and fourth columns) and O_{F5} (not shown), the structure of the force does not change for larger N, so that only one initial velocity field is sufficient. This is reasonable for larger m, as the angular section $[0, 2\pi/m]$ of the force is determined by m angular subsections of the rotationally unconstrained velocity field.

Although we have computed optimals only for a short $\mathcal{T}$, our results strongly suggest that inducing rolls remain optimal even for flow already in the shear-turbulence regime. As before, it is interesting to examine whether or not there is a strong dependence on the

Figure 10: The time evolution of instantaneous Nusselt number of the flow forced by different optimal forces in full space O_1 with $A_0 = 10^{-6}$ and time horizon $\mathcal{T} = 50$ (marked by the vertical dashed line). The legend indicates the number of initial conditions used in the optimisation.

N	O_1	O_3	O_5	O_{F3}	O_{F5}
1	1.22	1.16	1.25	1.18	1.27
3	1.22	1.18	1.27	1.18	1.26
9	1.20	1.18	1.27	1.17	1.25

Table 1: The time-averaged Nusselt number $Nu/Nu_{F=0}$ when the flow is forced by different optimal forces. $N = 1, 3, 9$ means the optimal force is optimised from N initial conditions.

rotational symmetry m. Figure 12(a) shows the Nusselt number for flows forced by O_{Fm} (solid) and O_m (dashed) for several m, using $N = 1$ for O_{Fm} and $N = 3$ for O_m. For small $m = 1, 2$, the force O_m appears to produce larger Nu than O_{Fm}, but this is because the optimisation for O_1 actually found a structure closer to that for O_3, previously seen in figure 11(cf. (i) and (j)), and similarly, the O_2 optimal is structurally more like O_4, seen in figure 13 (cf. (a) and (c)). Constraining the number of rolls strictly, by using the single Fourier mode, the rolls of O_{F3} are a little stretched (figure 13(f)) relative to those of O_3 (figure 13(b)), but for $m > 3$ there is essentially no visible difference between O_m and O_{Fm}. For $m \geqslant 3$, Nu is relatively insensitive to the wavenumber (figure 12(a)).

An interesting observation is the occurrence of laminarisation of the shear turbulence when forced by O_{F1} at low amplitudes $A_0 = 10^{-7}$. A very similar observation was reported by Willis *et al.* (2010), where there the roll-force was beneficial for drag reduction in isothermal flow. Here, as the turbulent flow enhances heat transfer, the laminarisation can lead to a reduction in the heat transfer, i.e. $Nu/Nu_{F=0} < 1$, see figure 12(b).

Optimisation for a steady laminar flow can be imposed by setting $K = 1$ in (2.19), as applied in §3.2. Figure 12(b) compares optimisation with the short $\mathcal{T}$ (solid lines) with steady laminar optimisation (dashed lines) with $\mathcal{T} = 600$. Particularly for larger A_0, it should be noted that including the time dependence of the flow in the optimisation does improve the resulting Nu over the steady assumption, despite the short $\mathcal{T}$. There is an exception for the $m = 1, 2$ case at the largest A_0, however, where the short-time optimal results in an unusual

Figure 11: The optimal force O_m and O_{Fm} optimised for an increasing number of initial conditions at $A_0 = 10^{-6}$, $C = 1$. The largest arrow has magnitude 6.95×10^{-4} in (a), 7.52×10^{-4} in (b), 5.76×10^{-4} in (c), 5.38×10^{-4} in (d), 8.14×10^{-4} in (e), 6.60×10^{-4} in (f), 5.76×10^{-4} in (g), 5.46×10^{-4} in (h), 8.45×10^{-4} in (i), 7.12×10^{-4} in (j), 5.89×10^{-4} in (k), 5.55×10^{-4} in (m).

force structure, see, e.g., figure 13(e). For the structure of the optimal forces, when optimised for the steady two-dimensionalised state (figure 13$(i\text{-}m)$) these forces are expected to be close to those of previous calculations (Meng *et al.* 2005; Jia *et al.* 2014; Wang *et al.* 2015). When optimised for the time-dependent shear-turbulent flow (figure 13$(e\text{-}h)$) the rolls are notably closer to the wall. Such a difference may be linked to the flattened turbulent mean velocity profile in the shear turbulence state, which leads to a more localised lift-up process towards the near-wall region.

3.4. *Optimisation in the convective turbulence regime*

Here, optimisation was first considered for a weakly convective turbulent state at $C = 4$. As the velocity fields in the convective state is time-dependent, optimisations were initially performed using several initial velocity fields ($N > 1$) at this C and random initial forces. However, even for small $A_0 = 10^{-7}$ it was found that the flow is rapidly laminarised by the force, so that, like at $C = 3$ for the laminarisation regime, $N = 1$ is sufficient. Also like the laminarised case, the optimal Nusselt number shows substantial improvement compared to the unforced case, 50% at $A_0 = 10^{-7}$, and the structure of the optimal, although calculated for O_1, looks very similar to that of figure 13(i), close to O_{F2} symmetry. Optimisations for O_{Fm} for other m exhibits similar behaviour to those for the laminarised case at $C = 3$, this time due to the laminarisation by the new force, therefore, further detailed results at $C = 4$ are omitted.

Figure 12: Time-averaged Nusselt number for flows at $C = 1$, $Re = 3000$ subject to optimal forces in the full and reduced rotational symmetries. (a) Comparison between O_{Fm} (solid lines) and O_m (dash lines). (b) Comparison between O_{Fm} optimised for shear turbulence (solid line) and a steady two-dimensional laminar state (dashed line).

Figure 13: Optimal forces at $A_0 = 10^{-6}$, $C = 1$, comparing calculations including time-dependence with stabilised two-dimensional calculations. The largest arrow has magnitude 6.01×10^{-4} in (a), 6.60×10^{-4} in (b), 5.50×10^{-4} in (c), 5.46×10^{-4} in (d), 5.15×10^{-4} in (e), 5.76×10^{-4} in (f), 5.40×10^{-4} in (g), 5.45×10^{-4} in (h), 6.34×10^{-4} in (i), 5.55×10^{-4} in (j), 5.99×10^{-4} in (k), 6.34×10^{-4} in (m). $N = 1$ initial velocity condition is used for O_{Fm} and $N = 3$ initial velocity conditions are used for O_m.

Figure 14: Instantaneous Nusselt number for O_1 optimisations at several C, conditions with $A_0 = 10^{-7}$. The vertical dashed line indicates the optimisation target time $\mathcal{T} = 600$.

We therefore optimise heat transfer at larger C to examine how far this laminarisation phenomenon occurs. Time evolutions for $Nu(t)$ optimised at $C = 4-8, \mathcal{T} = 600, A_0 = 10^{-7}$, for O_1 are shown in figure 14. With increased buoyancy force, laminarisation by the force still occurs at $C = 5, 6$, but disappears at $C = 7, 8$. Although the turbulence does not decay, the Nusselt number at $C = 7$ only fluctuates slightly. At $C = 8$, the amplitude of fluctuations in $Nu(t)$ are similar to those of the unforced flow, but with a higher mean value.

Due to the laminarisation, the optimisations at $C = 5-7$ were found to be well converged, but as turbulence remained stronger for $C = 8$, convergence was poor. Following the success of the short-time optimisation in the shear turbulence regime, we performed a short-$\mathcal{T}$ optimisation at $C = 8$. In this case, an intermediate value of $\mathcal{T} = 200$ was sufficient to achieve good convergence. Based on the observations for optimisations in the shear turbulence regime, we consider optimisation in the reduced space O_{Fm}. Figure 15(a) shows $Nu(t)$ for O_{F4} as an example. The target time $\mathcal{T}$ is not sufficient to capture the statistics of the endstate, but is sufficient to capture the initial response to the force. With an increase in force amplitude, the Nusselt number gradually increases. Figure 15(b) compares O_{Fm} optimised for the unsteady convective turbulence versus optimisation for the artificially stabilised steady two-dimensional laminar state (setting $K = 1$ as before). Unlike for shear-turbulence, this time they show little difference, and the optimal forces are close in structure, similar to figure 13(i-m). For the convective flow case, as long as it does not become too chaotic, including time dependence in the optimisation does not improve the Nusselt number further.

Towards more chaotic convective turbulence, a further optimisation was carried out at $C = 16$. Due to the stronger chaos, the target time was again reduced to $\mathcal{T} = 100$. Comparison between O_{Fm} optimised with the short $\mathcal{T}$ (solid line) and a steady two-dimensional laminar state (dash line) is shown in figure 16(a). At $A_0 = 10^{-7}$ and $A_0 = 10^{-6}$, the Nu of the chaotic forced flows do not show a difference resulting from the way the force was optimised, but greater improvement for the short-$\mathcal{T}$ optimisation starts to be seen at $A_0 = 10^{-5}$. The roll structures of the force optimised with the short $\mathcal{T}$ at all amplitudes are found to be located closer to the wall than for the corresponding steady calculation. An example for O_{F5} at $A_0 = 10^{-5}$ for the two optimisations are shown in figure 16(b,c). Such roll structures can improve Nu more at a larger force amplitude, similar to the observation in the optimisation of shear turbulence. Similar conclusions were drawn at $C = 32$.

(a)

(b)

Figure 15: (*a*) Time evolution of instantaneous Nusselt number optimised at different force amplitude for O_{F4} at $Re = 3000, C = 8$. A vertical dashed line indicates the optimisation target time $\mathcal{T} = 200$. (*b*) Comparison between O_{Fm} optimised in unsteady convective turbulence (solid line) and a steady two-dimensional laminar state (dash line).

(a)

(b)

(c)

Figure 16: (*a*) Comparison between O_{Fm} optimised in unsteady convective turbulence (solid line) and a steady two-dimensional laminar state (dash line) at $A_0 = 10^{-5}, C = 16, Re = 3000$. (*b*) The optimal force O_{F5} optimised for a steady laminar state.(*c*) The optimal force O_{F5} optimised for the convective turbulence state with target time $\mathcal{T} = 100$. The largest arrow has magnitude 1.4×10^{-3} in (*b*) and 1.9×10^{-3} in (*c*).

4. Conclusions

In this work we have developed a heat transfer optimisation method, based on a variational technique (Pringle & Kerswell 2010; Marensi *et al.* 2020), designed to identify the optimal body force that maximises heat transfer, in particular, in the presence of time-dependent flow states, and limited ability to influence the flow, measured by the amplitude of the force. Focussing primarily on the feasibility and practicality of the method, optimisations have been conducted only at $Re = 3000$ with the constant temperature boundary condition, but the method has been applied to flows initially in each of the typical states of heated pipe flow, i.e. laminarised flow ($C = 3$), shear turbulence ($C = 1$) and convective turbulence ($C \geqslant 4$).

Preliminary optimisations reveal that the optimal body force is predominantly governed by near-wall vortex structures that are uniform in the streamwise direction, consistent with results for optimised steady flows, e.g. Meng *et al.* (2005). The target time is an extra parameter here, and forces optimised with different target times exhibit different rotational symmetry. Specifically, the short-time optimal forces correspond to larger azimuthal wave numbers, while long-time optimal forces have smaller azimuthal wave numbers. This flow pattern aligns with linear optimal perturbations that aim to maximise flow perturbation growth in

isothermal flow (Schmid 2007). Forcing such modes efficiently modifies the flow, and may lead to either turbulence or laminarisation.

In the laminarisation regime of vertical pipe flow ($C = 3$), heat transfer increases with force amplitude A_0, as expected, then increases significantly at the point at which time-dependent flow is triggered by the force. Different initial guesses for the body force were tested to examine convergence behaviour and revealed that forces characterised by the azimuthal wave numbers of rotational symmetry are local optimals of the method. To reduce computational costs considerably, we opted to compare the Nusselt number of flows forced by optimals constrained in the Fourier space, O_m and O_{Fm} (the former keeping modes $0, m, 2m, \ldots$ and the latter only keeping modes 0 and m). This method is found to be efficient, since a comparison between the optimal forces O_m and O_{Fm} reveals that O_m only provides significantly larger Nu than O_{Fm} when that of a subspace, e.g. O_3 within O_1, has greater Nu. Results indicate that the global optimal force is not static but varies with changes in force amplitude: O_{F1} is the global optimal force at the lowest amplitude ($A_0 < 10^{-8}$), then O_{F2}, and m increases further with A_0. However, rather than the enhancement in Nu spiking for a particular m, the enhancement is similar over a broad range in m. Nu increases when turbulence is first triggered, but at larger A_0 turbulence does not necessarily lead to an increase in Nu. Visualisations of heat flux reveal that the streamwise vortices in forced turbulent states are not as efficient for heat transfer as those in forced laminar states. At larger force amplitude, unsteadiness of the rolls inhibits heat transfer. Three-dimensional travelling wave solutions were also identified for O_{F1} forcing.

Optimisation in the shear-turbulence regime ($C = 1$) is most challenging, as the flow is highly time-dependent and chaotic, preventing long target times. However, the method is found to still be effective for much shorter times, with $\mathcal{T} = 50$. Despite the short $\mathcal{T}$, the number of initial velocity conditions N was found to have little effect on the resulting Nusselt number. Comparing the O_{Fm} for different m, the heat transfer is again only relatively weakly dependent on m, at least for sufficiently large m (i.e. $m \geqslant 3$). Comparing optimisations with the short $\mathcal{T}$ and optimisations with the artificial steady flow assumption (and the same rotational symmetry O_{Fm}), it is found that including time-dependence results in a force with rolls located closer to the wall, which lead to flows with greater Nu.

For weak convective turbulence states ($C = 4 - 6$), the flow is rapidly laminarised by the force, even for small $A_0 = 10^{-7}$. Optimisations at $C = 4 - 7$ are well converged within a target time $\mathcal{T} = 600$, and show similar behaviour to optimisation for laminar state at $C = 3$. The forced flow is only weakly chaotic at $C = 8$, so that optimisations still show similar results to those for a steady laminar state. In the more chaotic convective state at $C = 16$ and C=32, optimisations with short $\mathcal{T}$ show roll structures closer to the wall than with the steady assumption, similar to optimisations in the shear-turbulent state, leading to larger Nu.

An important consideration is that the optimisation presented in this study primarily focuses on feasibility of the method and maximising heat transfer, without factoring in the associated pumping power required for the flow, nor the power expended by the force. However, the fundamental form of the force, dominated by the rolls, has been shown to extend beyond time-independent laminar flows to the turbulent convective and shear turbulent states, and is expected to extend also to higher Reynolds numbers. While it is acknowledged that accurately inducing the desired flow in practice is challenging, optimisations under the laminar steady state assumption, such as those by Meng *et al.* (2005), have inspired designs like the alternating elliptical axis tube, discrete double-inclined ribs tubes (Li *et al.* 2009) and many other applications (Liu & Sakr 2013; Sheikholeslami *et al.* 2015). Our results suggest that additional factors are worthy of consideration – the distance of the rolls from the wall affects the heat transfer, the triggering or potential laminarisation of turbulence does not necessarily have the expected result on heat transfer, and here, the vertical orientation of

the flow affects also the laminarisation. On the other hand, enhancement of *Nu* is found to be relatively robust to the selection of the azimuthal symmetry *m* of induced rolls, at a fixed value of *Re*. For a pipe of shorter length, however, supposing that this corresponds to shorter $\mathcal{T}$, our results suggest that larger *m* might be favoured. Dependence of *m* on *Re* is another parameter worthy of future consideration.

Declaration of interests. The authors report no conflict of interest.

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