Predictive Synthesis of Control Barrier Functions and its Application to Time-Varying Constraints

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Abstract—This paper presents a systematic method for synthesizing a Control Barrier Function (CBF) that encodes predictive information into a CBF. Unlike other methods, the synthesized CBF can account for changes and time-variations in the constraints even when constructed for time-invariant constraints. This avoids recomputing the CBF when the constraint specifications change. The method provides an explicit characterization of the extended class $\mathcal{K}_e$ function α that determines the dynamic properties of the CBF, and α can even be explicitly chosen as a design parameter in the controller synthesis. The resulting CBF further accounts for input constraints, and its values can be determined at any point without having to compute the CBF over the entire domain. The synthesis method is based on a finite horizon optimal control problem inspired by Hamilton-Jacobi reachability analysis and does not rely on a nominal control law. The synthesized CBF is time-invariant if the constraints are. The method poses mild assumptions on the controllability of the dynamic system and assumes the knowledge of at least a subset of some control invariant set. The paper provides a detailed analysis of the properties of the synthesized CBF, including its application to time-varying constraints. A simulation study applies the proposed approach to various dynamic systems in the presence of time-varying constraints. The paper is accompanied by an online available parallelized implementation of the proposed synthesis method.

Index Terms—Control Barrier Functions, Constrained Control, Time-Varying Systems, Safety.

I. INTRODUCTION

The dynamic capabilities of a system with respect to a given state constraint can be effectively characterized by a Control Barrier Function (CBF). Thereby, CBFs constitute an important control theoretic tool for ensuring constraint satisfaction and the systematic construction of safety filters [1], [2]. Having their origin in the optimization literature as barrier functions [3], [4], CBFs became a by now well-established and wide-spread tool within the domain of control [5], [6]. The application of CBFs has been explored in a wide range of areas such as vehicle control [7], [8], vehicle coordination [9]–[11], the control of vessels [12]–[14] and underwater vehicles [15], for air- and spacecrafts [16]–[18], to handle sensor limitations [19], as well as in robotics [20]–[24]. Moreover, CBFs have been suggested for handling various classes of spatiotemporal constraints [25]–[28]. Once a CBF has been found, the design of a controller ensuring constraint satisfaction is rather straightforward. A challenge, however, remains the

systematic synthesis of CBFs — in particular for systems with input constraints or subject to time-varying state constraints.

The problem of deriving a CBF is stated as follows: Let a dynamic system $\dot{x} = f(x, u)$ and a state constraint $x \in \mathcal{H} := \{x \,|\, h(x) \geqslant 0\}$ be given. Then, a Lipschitz-continuous function b shall be derived such that $\mathcal{C} := \{x \,|\, b(x) \geqslant 0\} \subseteq \mathcal{H}$ is a control-invariant subset of $\mathcal{H}$ with respect to the dynamic system. In addition, a certain ascend condition on b yet to be specified must be satisfied. Such a function b is called a CBF.

If the dynamics are locally controllable on the boundary of $\mathcal{H}$ and sufficiently large control inputs u are admitted, then the design of a CBF is straightforward. In particular, a CBF is then directly given by h. Beyond such favorable cases, CBFs are, like all other value functions as well, notoriously hard to compute. This is especially true for systems with weak controllability properties, such as systems that are *not* locally controllable, or that are subject to input constraints. Then more sophisticated synthesis methods are needed. We start by providing a survey on available methods.

A. Related Work

The synthesis of CBFs has developed into a thriving research branch in its own right and a wide variety of construction methods have been proposed, each of them with its own advantages. The proposed methods can be roughly subdivided into analytical and numerical approaches, even though some of them combine both analytical and numerical elements.

At first, we review analytical construction approaches. As previously discussed, the constraint function h readily constitutes a CBF in some favorable cases. In less favorable cases, backstepping as known from nonlinear control design may still lead to a CBF [29]–[31]. Starting with h, which is then called a high-order CBF, an actual CBF can be derived. Yet it is important to note that with the order of the CBF also its sensitivity with respect to model uncertainties increases. A related, though model-free approach is prescribed performance control (PPC) [32], [33]. It is applicable to systems that possess relatively strong controllability properties. Other works develop a systematic analytical CBF construction for particular classes of dynamic systems [34], [35], or augment a function that almost everywhere (a.e.) satisfies the CBF-properties with a logic to handle the remaining critical states [36]. In order to account for input constraints, analytical approaches commonly require, if at all possible, a meticulous construction of the CBF.

Often, the explicit consideration of input constraints and more generic classes of dynamic systems are possible with numerical approaches. As such, a sum-of-squares (SOS) approach is taken in [37]–[39]. Here, a CBF is determined by

solving an optimization problem over a polynomial basis. Approaches based on sum-of-squares are limited to polynomial dynamics. Due to the complexity of the optimization problem, there is no guarantee that a CBF is found even if it exists.

Since most control laws based on CBFs are gradient-based, they result in a reactive behavior. Therefore, if a more anticipatory control performance is required from a CBF-based feedback control law, predictive information needs to be encoded into the CBF upon synthesis. Some control approaches circumvent this problem by combining CBFs with model predictive control (MPC) [40]–[42], yet, these assume that a CBF is readily provided. In [43], an a-priori known control-invariant set is extended with finite horizon predictions, however without constructing a CBF. The first works to construct a CBF via predictions were [44], [45]. These approaches simulate the system dynamics controlled by some given nominal control law over an infinite time horizon. By additionally employing an a-priori known CBF, [46] reduces the prediction horizon to a finite one. The hereby resulting CBF is called a backup CBF in the literature. A predictive CBF is proposed in [47] using a finite, though not further specified, prediction horizon. The method is based on a sensitivity analysis by varying a nominal control law.

The need for nominal control laws can be circumvented by choosing an approach via Hamilton-Jacobi reachability analysis as in [48]. Despite the time-invariance of the state constraint, a time-dependent barrier function is obtained here. Only with an infinite number of iterations, a time-invariant barrier function can be asymptotically obtained. Its zero super-level set, however, is the largest control-invariant set that ensures the satisfaction of the original state constraint.

The last category to point out among the numerical CBF synthesis approaches are the learning based ones, see [49]–[51] and references therein. Based on the observed trajectories of a dynamical system, a neural network can be trained to approximate a CBF. Learning-based approaches inherently take input constraints into account and the class $\mathcal{K}$ in the ascend condition of the CBF is often obtained along with the learnt CBF. A challenge of such approaches is the generation of the large amount of trajectories needed for the training as well as the verification of the learnt function as a CBF.

A major problem in the CBF synthesis, also within the above mentioned literature, is that CBFs are derived for a particular (static) constraint. Any change in the constraint requires a recomputation of the CBF, which is usually computationally expensive. At the same time, it is important to note that a CBF constructed for a given dynamic system and a given state constraint is far from being unique.

In this paper, we deliberately take advantage of this freedom in order to construct a CBF with various favorable properties. The here proposed construction method yields a CBF that encodes predictive information and accounts for potential time-variations in the state constraint. Thereby, the resulting CBF does not require an expensive recomputation if the state constraint varies over time but can be adapted. In our follow-up paper [52], we furthermore leverage the here proposed approach to the synthesis of CBFs for equivariant systems [53], [54]. Our particular contributions are as follows.

B. Contributions

Given a dynamic system, a state constraint, and a control-invariant set (or one of its subsets) satisfying the state constraint, our method synthesizes a CBF that encodes predictive information and has the following favorable properties:

- The CBF accounts for time-varying constraints. While the synthesized CBF, in the sequel denoted by b, is time-invariant, the function

$$b(x) + \lambda(t)$$

  is guaranteed to be still a CBF for any function λ that varies within certain bounds. These bounds are a design parameter to the proposed synthesis method. Following the terminology in [55], the constructed CBF is *shiftable*. To the best of our knowledge, no other available CBF synthesis method provides this property.
- The CBF can be determined on any domain containing its zero super-level set. Most synthesis methods in the literature yield only CBFs on a domain equal to their zero super-level set.
- Our synthesis method yields an explicit characterization of the extended class $\mathcal{K}_e$ function in the ascend condition along with the corresponding CBF. It is furthermore indifferent to the relative degree of the constraint.
- Our method allows to compute the numeric values of the CBF at some given state without the need to compute the entire CBF. This is particularly advantageous for exploiting the equivariances of some dynamics in the CBF construction as detailed in our follow-up work [52].

Beyond these, our synthesized CBF explicitly accounts for input constraints and can handle a general class of nonlinear dynamics which it has in common with other predictive CBF synthesis methods. However, our synthesis method does not rely on an auxiliary nominal controller in contrast to other predictive methods. In contrast, our method is rather inspired by Hamilton-Jacobi reachability as in [48].

A preliminary version of our synthesis method has been presented in [56]. However, the domain of the CBF constructed there was confined to the zero super-level set of the CBF. Moreover, our earlier paper did not account for time-varying constraints. Additionally, we accompany the theoretic results in this paper with detailed implementation remarks, and an elaborate simulation study for various dynamic systems subject to static and time-varying constraints. Furthermore, this paper comes along with a Python toolbox implementing the proposed synthesis method. It supports the parallelized CBF computation on multiple cores.

C. Outline

The remainder is structured as follows. Section II introduces some preliminaries including the definition of a CBF in the Dini sense. Section III states fundamental assumptions on a control invariant set and on the required controllability properties before stating the objective of the paper formally. Section IV presents our CBF synthesis method, and Section V provides a detailed analysis and discussion of its properties. Section VI remarks on the implementation of our method.

Finally, in Section VII, we apply our method to various dynamic systems and time-varying constraints, and present numerical results. Section VIII draws a conclusion.

D. Notation

Let $\mathcal{X} \subseteq \mathbb{R}^n$, $x \in \mathbb{R}^n$. We denote sets by calligraphic upper case letters, while trajectories $\boldsymbol{x} : \mathbb{R} \to \mathcal{X}$ are denoted by boldface lower case letters. The set of all trajectories $\boldsymbol{x}$ defined on $[t_1, t_2]$ is denoted by $\boldsymbol{\mathcal{X}}_{[t_1,t_2]}$, and $\boldsymbol{\mathcal{X}}$ is used for brevity whenever the interval of definition is clear from the context. The complement and boundary of $\mathcal{X}$ are denoted by $\mathcal{X}^c$ and $\partial \mathcal{X}$, respectively, and the Euclidean norm and Hausdorff distance by $||\cdot||$ and $d_H(x, \mathcal{X}) := \inf_{x' \in \mathcal{X}} ||x - x'||$. A ball around x_0 with radius r is defined as $\mathcal{B}_r(x_0) := \{x \,|\, ||x_0 - x|| < r\}$. For two sets $\mathcal{X}_1, \mathcal{X}_2 \subseteq \mathbb{R}^n$, the Minkowski sum is defined as $\mathcal{X}_1 \oplus \mathcal{X}_2 = \{x_1 + x_2 \,|\, x_1 \in \mathcal{X}_1, \, x_2 \in \mathcal{X}_2\}$, and the Pontryagin difference as $\mathcal{X}_1 \ominus \mathcal{X}_2 := \{x_1 \in \mathbb{R}^n \,|\, x_1 + x_2 \in \mathcal{X}_1, \, \forall x_2 \in \mathcal{X}_1\}$. A class $\mathcal{K}$ function is defined as a continuous, strictly increasing function $\alpha : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ with $\alpha(0) = 0$; if the function is defined on $\mathbb{R}$ as $\alpha : \mathbb{R} \to \mathbb{R}$, then it is called an extended class $\mathcal{K}_e$ function. At last, if a property holds everywhere except on a set of measure zero, we say that it holds almost everywhere (a.e.).

II. PRELIMINARIES

We consider the dynamic system

$$\dot{x} = f(x, u), \qquad x(0) = x_0 \tag{1}$$

where $x, x_0 \in \mathbb{R}^n$, $u \in \mathcal{U} \subseteq \mathbb{R}^m$, and $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n$ is Lipschitz continuous in both of its arguments. The system is subject to the state constraint

$$x \in \mathcal{H} := \{x \,|\, h(x) \geqslant 0\} \tag{2}$$

where $h : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz continuous function. For a given input trajectory $\boldsymbol{u} : \mathbb{R}_{\geqslant 0} \to \boldsymbol{\mathcal{U}}$, continuous a.e., the solution to (1) up to some time T is given by $\boldsymbol{\varphi} : [0, T] \to \mathbb{R}^n$ where $\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}) := x_0 + \int_0^T f(x(\tau), \boldsymbol{u}(\tau))d\tau$. The forward completeness of $\boldsymbol{\varphi}$ is assumed for the considered input trajectory $\boldsymbol{u}$. We call a set $\mathcal{S}$ *forward control invariant* with respect to system (1) if there exist $\boldsymbol{u} \in \boldsymbol{\mathcal{U}}_{[0,\infty)}$ such that $\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}) \in \mathcal{S}$ for all $t \geqslant 0$. Furthermore, we call $\mathcal{S}$ *forward invariant* under input $\boldsymbol{u} \in \boldsymbol{\mathcal{U}}_{[0,\infty)}$ with respect to (1) if $\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}) \in \mathcal{S}$ for all $t \geqslant 0$.

A. Control Barrier Functions in the Dini Sense

Control Barrier Functions (CBF) have been introduced as the system theoretic analogue to Contol Lyapunov Functions (CLF) for forward set invariance [1], [2]. While these works introduce them as differentiable functions, requiring differentiability can be limiting. The need for non-differentiable CBFs arises in the context of constraints with non-differentiable outer bounds (e.g. box constraints), or due to certain dynamics as it is the case for CLFs [57], [58]. For this reason, we state CBFs more generally in terms of the Dini derivative analogously to CLFs in the Dini sense [58], [59].

Definition 1 (CBF in the Dini Sense). Consider $\mathcal{D} \subseteq \mathbb{R}^n$ and a locally Lipschitz continuous function $b : \mathbb{R}^n \to \mathbb{R}$ such that $\mathcal{C}$ defined as

$$\mathcal{C} := \{x \,|\, b(x) \geqslant 0\} \tag{3}$$

is compact and it holds $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathbb{R}^n$. We call such b a *CBF in the Dini sense* on $\mathcal{D}$ with respect to (1) if there exists an extended class $\mathcal{K}_e$ function α such that for all $x \in \mathcal{D}$

$$\sup_{u \in \mathcal{U}} \{db(x; f(x, u))\} \geqslant -\alpha(b(x)) \tag{4}$$

where $d\phi(x; v)$ with $\phi : \mathbb{R}^n \to \mathbb{R}$ locally Lipschitz continuous denotes the *Dini derivative* at x in the direction of v as

$$d\phi(x; v) := \liminf_{\sigma \downarrow 0} \frac{\phi(x + \sigma v) - \phi(x)}{\sigma}.$$

Remark 1. In this paper, we explicitly allow for a domain $\mathcal{D}$ that is larger than $\mathcal{C}$. This is in contrast to our earlier work [56].

For convenience, we define the other superlevel sets of b as

$$\mathcal{C}_\lambda := \{x \,|\, b(x) \geqslant -\lambda\}$$

where $\lambda \in \mathbb{R}$. Next, we briefly show that CBFs in the Dini sense characterize control inputs that render $\mathcal{C}$ forward invariant; this is analogous to the differentiable case. The subsequent result is a consequence of the Comparison Lemma.

Theorem 1. Let $\boldsymbol{u} \in \boldsymbol{\mathcal{U}}_{[0,T]}$ be continuous a.e. with the corresponding state trajectory $\boldsymbol{x}(t) := \boldsymbol{\varphi}(t; x_0, \boldsymbol{u})$ starting in some initial state $x_0 \in \mathcal{C}$. If

$$db(\boldsymbol{x}(t); f(\boldsymbol{x}(t), \boldsymbol{u}(t))) \geqslant -\alpha(b(\boldsymbol{x}(t))) \qquad \forall t \in [0, T],$$

then $\mathcal{C}$ is forward invariant such that $\boldsymbol{x}(t) \in \mathcal{C}$ for all $t \in [0, T]$.

Proof. Let the intervals, where $\boldsymbol{u}$ is continuous, be w.l.o.g. given as $[\tau_i, \tau_{i+1})$ where $\tau_i \in \{\tau_i\}_{i=1,\dots,N-1}$ with

$$\tau_0 = 0 < \cdots < \tau_i < \tau_{i+1} < \cdots < \tau_N = T.$$

Next, assuming that $b(\boldsymbol{x}(\tau_i)) \geqslant 0$, it follows together with $db(\boldsymbol{x}(t); f(\boldsymbol{x}(t), \boldsymbol{u}(t))) \geqslant -\alpha(b(\boldsymbol{x}(t)))$ from the Comparison Lemma [60, Lemma 3.4] that $b(\boldsymbol{x}(t)) \geqslant 0$ for all $t \in [\tau_i, \tau_{i+1})$. Due to the continuity of b and $\boldsymbol{x}$, we further obtain

$$b(\boldsymbol{x}(\tau_{i+1})) = \liminf_{\tau \uparrow \tau_{i+1}} b(\boldsymbol{\varphi}(\tau; \boldsymbol{x}(\tau_i), \boldsymbol{u})) \geqslant 0.$$

Since $x_0 \in \mathcal{C}$ and thus $b(x_0) = b(\boldsymbol{x}(\tau_0)) \geqslant 0$, it follows inductively that $b(\boldsymbol{x}(t)) \geqslant 0$ and therefore $\boldsymbol{x}(t) \in \mathcal{C}$ for all $t \in [0, T]$. $\qquad \square$

B. Reachability and Controllability

A state x_1 is called *T-reachable* from x_0 under dynamics (1) if $\boldsymbol{\varphi}(T; x_0, \boldsymbol{u}) = x_1$ for some bounded measurable input trajectory $\boldsymbol{u} \in \boldsymbol{\mathcal{U}}_{[0,T]}$. We define the set of all such points as $\mathcal{R}_T(x_0) := \{x_1 \,|\, \exists \boldsymbol{u} \in \boldsymbol{\mathcal{U}}_{[0,T]} : \boldsymbol{\varphi}(T; x_0, \boldsymbol{u}) = x_1\}$. System (1) is *controllable* on $\mathcal{M} \subseteq \mathbb{R}^n$ if $\mathcal{M} \subseteq \bigcup_{t \in [0,\infty)} \mathcal{R}_t(x_0)$ for all $x_0 \in \mathcal{M}$ [61]. Moreover, we call system (1) *locally-locally controllable* on $\mathcal{M} \subseteq \mathbb{R}^n$ [62] if there exist $\varepsilon \geqslant \delta_\varepsilon > 0$ such that for any state $x_f \in \mathcal{B}_{\delta_\varepsilon}(x_0)$ there exists a $\boldsymbol{u} \in \boldsymbol{\mathcal{U}}$ and $t \geqslant 0$ such that $\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}) = x_f$ and $\boldsymbol{\varphi}(\tau; x_0, \boldsymbol{u}) \in \mathcal{B}_\varepsilon(x_0)$ for all $\tau \in [0, t]$ ("any state in a δ_ε-neighborhood can be reached without leaving a certain ε-neighborhood").

Fig. 1: Illustration of Assumption 1.

(a)   (b)

Fig. 2: Kinematic bicycle model: (a) schematic sketch; (b) construction of a set with its viable points via turning radius r.

III. PROBLEM SETTING

Let us consider the constraint set

$$\mathcal{H} := \{x \mid h(x) \geqslant 0\}$$

where h is a Lipschitz continuous function. The CBF synthesis method presented in this paper is based on the idea that if a system is initialized with x_0 sufficiently far from the boundary of constraint set $\mathcal{H}$, then it can be expected that there exists an input trajectory which ensures that the system state stays within $\mathcal{H}$ for all times. This property holds, for example, for all locally-locally controllable systems, but also for any more general system that possesses a forward control invariant set contained in $\mathcal{H}$. The particular premises, under which a CBF is constructed in the sequel, are formalized by the following assumptions. These parallel the setting in our previous work [56].

A. Assumption on Forward Control Invariant Sets

First, we assume the existence of a forward control invariant set $\mathcal{V}$ which does not need to be explicitly known. Instead, we assume that only one of its subsets, denoted by $\mathcal{F}$, is known.

Assumption 1. There exists a forward control invariant subset $\mathcal{V} \subset \mathcal{H}$ where $\mathcal{H} := \{x \in \mathbb{R}^n \mid h(x) \geqslant 0\}$ such that $h(x) \geqslant \delta$ for all $x \in \mathcal{V}$ and some $\delta > 0$. While $\mathcal{V}$ is not required to be explicitly known, we assume that a subset $\mathcal{F} \subseteq \mathcal{V}$ is known.

The assumption is illustrated in Figure 1. Often, the construction of $\mathcal{F}$ is more straightforward than that of $\mathcal{V}$, and an intuitive understanding of the system dynamics can be taken as a starting point.

Example 1. The kinematic model of a vehicle modeled as a bicycle [63] (see Fig. 2a) is given as

$$\dot{x} = v \cos(\psi + \beta(\zeta)) \tag{5a}$$

$$\dot{y} = v \sin(\psi + \beta(\zeta)) \tag{5b}$$

$$\dot{\psi} = \frac{v \cos(\beta(\zeta)) \tan(\zeta)}{L} \tag{5c}$$

where $\beta(\zeta) = \arctan(\frac{1}{2}\tan(\zeta))$. The position of the center of mass C is denoted by the states $\mathbf{x}_{\text{pos}} = [x, y]^T$, and the vehicle's orientation by ψ; inputs are velocity v and steering angle ζ. The vector of the system states is denoted by $\mathbf{x} = [x, y, \psi]^T$. The vehicle is also subject to input constraints $0 < v_{\text{min}} \leqslant v \leqslant v_{\text{max}}$ and $|\zeta| \leqslant \zeta_{\text{max}}$. The minimum turning radius can be directly obtained from the dynamics as $R = \frac{L}{\cos(\beta(\zeta_{\text{max}})) \tan(\zeta_{\text{max}})}$. We let the vehicle move in a plane with an obstacle as shown in Figure 2b. The obstacle is specified as a set $\mathcal{H}^c \subset \mathbb{R}^n$. Correspondingly, we define $h(\mathbf{x}) = d_H(\mathbf{x}, \mathcal{H}^c)$. By geometrical considerations, as shown in Fig. 2b, the set $\mathcal{F} = \{\mathbf{x} \mid h(\mathbf{x}) \geqslant \delta + 2R\}$ is determined as a subset of the control-invariant set $\mathcal{V}$. It is however important to note that $\mathcal{F}$ is not forward control invariant. While for the construction of $\mathcal{F}$ it is sufficient to consider the vehicle's position, the construction of $\mathcal{V}$ also requires orientation ψ.

In the example, we exploit the fact that a bicycle can always return to its initial state by moving on a circle. More generally, this relates to the following observation.

Proposition 2. A set $\mathcal{F}$ is a subset of a forward control invariant set if for any state $x_0 \in \partial\mathcal{F}$ there exists a control input $\mathbf{u} \in \mathcal{U}$ such that $\boldsymbol{\varphi}(t_f; x_0, \mathbf{u}) \in \mathcal{F}$ for some $t_f \geqslant 0$.

Proof. For a given $x_0 \in \partial\mathcal{F}$, let there be a control input $\mathbf{u}_{x_0} \in \mathcal{U}$ and a final time $t_{f,x_0} \geqslant 0$ such that $\boldsymbol{\varphi}(t_{f,x_0}; x_0, \mathbf{u}_{x_0}) \in \mathcal{F}$; their existence is a consequence of the premises of the proposition. Then, note that

$$\mathcal{V} := \bigcup_{x_0 \in \partial\mathcal{F}} \{x \mid x = \boldsymbol{\varphi}(t; x_0, \mathbf{u}_{x_0}), \ t \in [0, t_{f,x_0}]\} \cup \mathcal{F}$$

is forward control invariant and $\mathcal{F} \subseteq \mathcal{V}$. □

This however is not a necessary condition, as any trajectory through x_0 might converge to some point or limit cycle outside of $\mathcal{F}$. For the construction of a set $\mathcal{F}$ satisfying Assumption 1, it is of particular interest if any such trajectory starting at an arbitrary $x_0 \in \partial\mathcal{F}$ can be bounded into an ε-neighborhood of x_0. This gives rise to the following sufficient conditions.

Proposition 3. The set $\mathcal{F} := \{x \mid h(x) \geqslant \delta\} \ominus B_\varepsilon(0)$ for some $\varepsilon > 0$ is a subset of a forward control invariant set $\mathcal{V}$ that satisfies Assumption 1 if at least one of the subsequent conditions hold:

(1) System (1) is locally-locally controllable on $\partial\mathcal{F}$ with constants $\varepsilon \geqslant \delta_\varepsilon > 0$.
(2) For any state $x_0 \in \partial\mathcal{F}$, there exists a control input $\mathbf{u} \in \mathcal{U}$ such that $\boldsymbol{\varphi}(t; x_0, \mathbf{u}) \in B_\varepsilon(x_0)$ for all $t \geqslant 0$ (e.g. $B_\varepsilon(x_0)$ contains an attractor, or $\boldsymbol{\varphi}$ converges to a periodic solution in $B_\varepsilon(x_0)$).
(3) For any state $x_0 \in \partial\mathcal{F}$, there exists a control input $\mathbf{u} \in \mathcal{U}$ such that $\boldsymbol{\varphi}(t_f; x_0, \mathbf{u}) \in \mathcal{F}$ for some $t_f \geqslant 0$ and $\boldsymbol{\varphi}(\tau; x_0, \mathbf{u}) \in \mathcal{F} \oplus B_\varepsilon(0)$ for all $\tau \in [0, t_f]$.

Also, any subset of $\mathcal{F}$ satisfies Assumption 1.

Proof. The proof can be found in the appendix. □

This proposition is of practical relevance. For instance, the construction of $\mathcal{F}$ in Example 1 is based on condition (2) of the proposition. More generally, the proposition points out that the

knowledge on a system's locally-locally controllability or on the existence of attractors and periodic solutions are valuable properties that can be exploited for the construction of $\mathcal{F}$.

B. Assumption on Controllability and Reachability

We call a trajectory starting at some point x_0 *viable* if there exists an input trajectory $\boldsymbol{u} : \mathbb{R}_{\geqslant 0} \to \mathcal{U}$ such that $\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}) \in \mathcal{H}$ for all $t \geqslant 0$. By the next assumption, we ensure that the viability of a trajectory can be determined by predicting the trajectories of a system over a finite time horizon. To this end, we note that a trajectory that ends in $x_1 \in \mathcal{F}$ can be feasibly continued for all times as $\mathcal{F}$ is a subset of a forward control invariant set $\mathcal{V} \subset \mathcal{H}$. Thus, the time horizon required in order to decide if a trajectory starting at x_0 is viable, is the minimal time $\tau(x_0)$ that it takes to reach $\mathcal{F}$. It is formally defined as

$$\tau(x_0) := \min_{\tau \geqslant 0} \tau \qquad (6a)$$

$$\text{s.t.} \quad \dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), \boldsymbol{u}(t)) \quad (a.e.), \qquad (6b)$$

$$\boldsymbol{x}(0) = x_0, \quad \boldsymbol{u}(t) \in \mathcal{U}, \quad \boldsymbol{x}(\tau) \in \mathcal{F}. \qquad (6c)$$

Note that trajectory $\boldsymbol{x}$ is not required to stay in $\mathcal{H}$ for all times. Here, the states $x_0 \in \mathcal{D}$ are of particular interest as a CBF on the domain $\mathcal{D}$ shall be constructed. While for $x_0 \in \mathcal{F}$ it clearly holds $\tau(x_0) = 0$, further assumptions are required to ensure that $\tau(x_0)$ is finite for any other $x_0 \in \mathcal{D} \backslash \mathcal{F}$. To this end, we impose the following controllability assumption.

Assumption 2. Let either of the following statements hold:

A2.1 System (1) is controllable on the closure of $\mathcal{F}^{\mathrm{c}}$; or
A2.2 For all $x_0 \in \mathcal{D} \backslash \mathcal{F}$, there exist $t \geqslant 0$ such that $\mathcal{R}_t(x_0) \cap \mathcal{F} \neq \varnothing$.

Proposition 4 ([56], Proposition 2)**.** Let Assumption 2 hold. Then there exists for all $x_0 \in \mathcal{D} \backslash \mathcal{F}$ a finite time $\tau(x_0) \in \mathbb{R}_{>0}$ that minimizes (6).

Let us denote the upper bound of $\tau(\cdot)$ on $\mathcal{D} \backslash \mathcal{F}$ by

$$\tau := \sup_{x \in \mathcal{D} \backslash \mathcal{F}} \tau(x) \qquad (7)$$

Upper bounds to $\tau(\cdot)$ can often be analytically found. An illustrative example on the bicycle model can be found in [56].

C. Objective

Let Assumptions 1 and 2 hold, and let a constraint set $\mathcal{H}$, a domain $\mathcal{D} \supseteq \mathcal{H}$, and a set $\mathcal{F}$ as defined in Assumption 1 be given. Then, construct a CBF with respect to (1) on domain $\mathcal{D}$ such that its zero-superlevel set $\mathcal{C}$ is a subset of constraint set $\mathcal{H}$, i.e., $\mathcal{C} \subseteq \mathcal{H}$.

IV. PREDICTIVE CBF SYNTHESIS

We now present our synthesis approach and prove that the synthesized function constitutes a CBF in the Dini sense. Thereafter, we refine the established result in order to relax imposed conditions and to explicitly incorporate the extended class $\mathcal{K}_e$ function α as a design parameter into the synthesis.

A. Synthesis Approach

The CBF is determined pointwise by solving an optimal control problem over a finite prediction horizon T. Thus, the CBF can be computed on parts of its domain without having to compute it on the entire domain. This is, for instance, advantageous when the overall CBF can be induced from its values computed on a subset of its domain as it is the case for equivariant systems [52].

In order to become more specific, let us choose the prediction horizon T as $T \geqslant \tau$, where τ is defined in (7). Moreover, let us define a function $H_T : \mathcal{D} \to \mathbb{R}$ as

$$H_T(x_0) := \max_{\boldsymbol{u}(\cdot) \in \boldsymbol{\mathcal{U}}_{[0,T]}} \min_{t \in [0,T]} h(\boldsymbol{x}(t)) - \gamma t \qquad (8a)$$

$$\text{s.t.} \quad \boldsymbol{x}(0) = x_0, \qquad (8b)$$

$$\dot{\boldsymbol{x}}(s) = f(\boldsymbol{x}(s), \boldsymbol{u}(s)) \quad (a.e.), \qquad (8c)$$

$$\boldsymbol{u}(s) \in \mathcal{U}, \qquad \forall s \in [0, T] \qquad (8d)$$

$$\boldsymbol{x}(\vartheta) \in \mathcal{F}, \qquad \text{for some } \vartheta \in [0, T], \qquad (8e)$$

where $\gamma > 0$ is some positive constant. In the sequel, H_T turns out to be a CBF in the Dini sense. For later reference, we denote the input trajectory $\boldsymbol{u}$ and the times t and ϑ that solve optimization problem (8) for initial value x_0 by $\boldsymbol{u}_{x_0}^*$, $t_{x_0}^*$ and $\vartheta_{x_0}^*$.

The intuition behind optimal control problem (8) is as follows. Let us assume for a moment that $\gamma = 0$. In the optimal control problem, we consider a state trajectory $\boldsymbol{x}(\cdot) = \boldsymbol{\varphi}(\cdot; x_0, \boldsymbol{u})$ that starts in x_0 and evolves according to some input trajectory $\boldsymbol{u}$ over a time horizon T such that (8e) is satisfied at some time $\vartheta \in [0, T]$. The minimization determines that point of time $t_{x_0}^*$ when the trajectory $\boldsymbol{x}$ takes the smallest value on h, whereas the maximization aims at increasing this value as much as possible with a suitable input trajectory. Thereby for $\gamma = 0$, the optimal control problem (8) defines the function H_T such that it can be interpreted as a measure for how close the system state gets to the boundary of set $\mathcal{H}$, or to which extent the state trajectory leaves set $\mathcal{H}$ when it evolves over time. Constraint (8e) ensures that $\boldsymbol{\varphi}(\cdot; x_0, \boldsymbol{u}_{x_0}^*)$ can be always feasibly continued within $\mathcal{H}$ even beyond time horizon T. For $\gamma = 0$, (8) is identical to the optimal control problem proposed in [56]. Next, let us consider γ to be strictly positive, which renders $-\gamma t$ non-positive on the time-interval $[0, T]$ and penalizes the magnitude of $t_{x_0}^*$. Thereby, it is ensured that the value of H_T increases along trajectory $\boldsymbol{\varphi}(\cdot; x_0, \boldsymbol{u}_{x_0}^*)$. In particular, we formally derive later on in the proof of Theorem 6 that for any $x_0 \in \mathcal{D} \backslash \mathcal{F}$ it holds

$$dH_T(x; f(x, \boldsymbol{u}_{x_0}^*)) \Big|_{x = \boldsymbol{\varphi}(t; x_0, \boldsymbol{u}_{x_0}^*)} > 0.$$

The latter inequality eventually allows us to establish that H_T constitutes a CBF even on domains $\mathcal{D} \supset \mathcal{C}$. This is in contrast to most existing works on CBF synthesis, which only allow for $\mathcal{D} = \mathcal{C}$.

Before formally establishing this result, we show that H_T is well-defined and the zero super-level set of H_T, in the remainder of the paper denoted by $\mathcal{C} := \{x \mid H_T(x) \geqslant 0\}$, is a subset of the state constraint set $\mathcal{H}$, i.e., $\mathcal{C} \subseteq \mathcal{H}$. Thereby, any point in $\mathcal{C}$ satisfies state constraint (2).

Fig. 3: Relation of the defined sets. The particular shapes of the sets are only schematic and have no further meaning.

Proposition 5. Let Assumptions 1 and 2 hold. Then for all $x_0 \in \mathcal{D}$, H_T defined in (8) is well-defined, i.e., there exists a solution to (8). Moreover, $\mathcal{C} \subseteq \mathcal{H}$.

Proof. In order to show the first part, consider a state $x_0 \in \mathcal{D}$. As Assumptions 1–2 hold, it follows from Proposition 4 that there exists a finite time horizon T with $T \geqslant \tau$. Thus, by the definition of τ, there also exists an input trajectory $\boldsymbol{u}_{x_0}^* \in \mathcal{U}_{[0,T]}$ and times $t_{x_0}^*$ and $\vartheta_{x_0}^*$ which solve (8). Thus, H_T is well-defined.

For the second part, we note that

$$
\begin{aligned}
H_T(x_0) &\overset{(8a)}{=} \max_{\boldsymbol{u}(\cdot)} \min_{t \in [0,T]} h(\boldsymbol{\varphi}(t; x_0, \boldsymbol{u})) - \gamma t \\
&\leqslant \max_{\boldsymbol{u}(\cdot)} \min_{t \in [0,T]} h(\boldsymbol{\varphi}(t; x_0, \boldsymbol{u})) \\
&\leqslant \max_{\boldsymbol{u}(\cdot)} h(\boldsymbol{\varphi}(0; x_0, \boldsymbol{u})) = h(x_0).
\end{aligned}
$$

Thus, if $H_T(x_0) \geqslant 0$, then $h(x_0) \geqslant 0$ and $\mathcal{C} \subseteq \mathcal{H}$. $\square$

Thereby, the relation of the so far defined sets can be summarized as

$$
\mathcal{F} \subseteq \mathcal{V} \subseteq \mathcal{C} \subseteq \mathcal{H} \subseteq \mathcal{D} \subseteq \mathbb{R}^n
$$

which is illustrated in Figure 3. Now we are ready to prove that H_T constitutes a CBF in the Dini sense.

Theorem 6. Let Assumptions 1 and 2 hold, let h be Lipschitz-continuous and let $T \geqslant \tau$. Moreover, let $H_T : \mathcal{D} \to \mathbb{R}$ be defined by (8) on some domain $\mathcal{D} \subseteq \mathbb{R}^n$ with $\mathcal{H} \subset \mathcal{D}$; the parameter γ is chosen such that

$$
\gamma < \frac{\delta}{T} \tag{9}
$$

where δ is determined as part of Assumption 1. Furthermore, let f be bounded on $\mathcal{C}$ in the sense that for all $x \in \mathcal{C}$ there exists a $u \in \mathcal{U}$ such that $\|f(x, u)\| \leqslant M$ for some constant $M > 0$. If H_T is locally Lipschitz-continuous, then H_T constitutes a CBF in the Dini sense on the domain $\mathcal{D}$ with respect to the dynamics (1).

Proof. To start with, note that at any $x_0 \in \mathcal{D}$, $H_T(x_0)$ is well-defined as by Proposition 5. The value of H_T at x_0 is then given by $H_T(x_0) = h(\boldsymbol{\varphi}(t_{x_0}^*; x_0, \boldsymbol{u}_{x_0}^*))$ where $\boldsymbol{\varphi}(\cdot; x_0, \boldsymbol{u}_{x_0}^*) : [0, T] \to \mathbb{R}^n$ is the state trajectory that starts in x_0 and is induced by $\boldsymbol{u}_{x_0}^*$.

In order to prove that H_T is a CBF in the Dini sense, it needs to be shown that for some extended class $\mathcal{K}_e$ function α it holds

$$
\sup_{u \in \mathcal{U}} \{dH_T(x_0; f(x_0, u))\} \geqslant -\alpha(H_T(x_0)) \tag{10}
$$

for all $x_0 \in \mathcal{D}$. To this end, we introduce $\varepsilon > 0$ such that

$$
\gamma < \frac{\delta}{(\varepsilon + 1)T} < \frac{\delta}{T} \tag{11}
$$

which exists due to (9). As such, we can choose any $\varepsilon \in (0, \frac{\delta}{\gamma T} - 1)$. From here on, we proceed in two steps: at first, we consider those x_0 with $H_T(x_0) \leqslant \delta - \gamma(\varepsilon + 1)T$ (marked blue in Figure 3); thereafter, we consider the remaining x_0, namely those with $H_T(x_0) > \delta - \gamma(\varepsilon + 1)T$ (marked in green).

Step 1: Let x_0 be such that $H_T(x_0) \leqslant \delta - \gamma(\varepsilon + 1)T$. Note that the right-hand side $\delta - \gamma(\varepsilon + 1)T$ is strictly positive due to (11). In this way, Step 1 considers all states x_0 outside of $\mathcal{C}$ ($x_0 \notin \mathcal{C}$), together with those contained in $\mathcal{C}$ within in some neighborhood of its boundary (see Figure 3). Now let us extend input trajectory $\boldsymbol{u}_{x_0}^*$ by an input trajectory $\boldsymbol{u}_{e,x_0} \in \mathcal{U}_{[T,\infty)}$ that renders $\mathcal{V}$ forward invariant for all times $t \geqslant T$. In particular,

$$
\boldsymbol{u}_{e,x_0}^*(t) := \begin{cases} \boldsymbol{u}_{x_0}^*(t) & \text{if } t \in [0, \vartheta_{x_0}^*] \\ \boldsymbol{u}_{e,x_0}(t) & \text{if } t \in (\vartheta_{x_0}^*, T(\varepsilon + 1)] \end{cases} \tag{12}
$$

where $\boldsymbol{u}_{e,x_0} \in \mathcal{U}_{(\vartheta_{x_0}^*, T(\varepsilon+1)]}$ such that $\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}_{e,x_0}^*) \in \mathcal{V}$ for all $t > \vartheta_{x_0}^*$. Since $\mathcal{V}$ is forward control invariant, such $\boldsymbol{u}_{e,x_0}$ exists. Moreover, we note that

$$
\begin{aligned}
H_T(x_0) &= h(\boldsymbol{\varphi}(t_{x_0}^*; x_0, \boldsymbol{u}_{x_0}^*)) - \gamma t_{x_0}^* \tag{13a} \\
&\leqslant \delta - \gamma(\varepsilon + 1)T \tag{13b} \\
&\leqslant h(\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}_{e,x_0}^*)) - \gamma t \quad \forall t \in [\vartheta_{x_0}^*, T(\varepsilon + 1)] \tag{13c}
\end{aligned}
$$

where the first inequality holds by assumption for those x_0 considered in *Step 1*; for the second inequality, we note that $\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}_{e,x_0}^*) \in \mathcal{V}$ for all $t \geqslant \vartheta_{x_0}^*$ and thus $\delta \leqslant h(\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}_{e,x_0}^*))$, and $-\gamma(\varepsilon + 1)T \leqslant -\gamma t$ for $t \leqslant T(\varepsilon + 1)$. In particular, (13) implies that

$$
\min_{t \in [0, \vartheta_{x_0}^*]} h(\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}_{x_0}^*)) - \gamma t \leqslant \min_{t \in [\vartheta_{x_0}^*, (\varepsilon+1)T]} h(\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}_{x_0}^*)) - \gamma t \tag{14}
$$

where the left-hand side equals (13a), and the right-hand side is the minimum of (13c). By using the latter result, we derive

$$
\begin{aligned}
H_T(x_0) &= \min_{t \in [0, T]} h(\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}_{x_0}^*)) - \gamma t \\
&\overset{(14)}{=} \min_{t \in [0, (\varepsilon+1)T]} h(\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}_{e,x_0}^*)) - \gamma t, \tag{15}
\end{aligned}
$$

where the last equality follows as the extension of the time interval by εT does not impact the value of H_T due to (14). Based on this, and by introducing the auxiliary time variable $t' \in (0, \varepsilon T)$ (it may take arbitrary values on its interval of definition), we derive that

$$
\begin{aligned}
H_T(x_0) &\overset{(15)}{=} \min_{t \in [0, (\varepsilon+1)T]} h(\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}_{e,x_0}^*)) - \gamma t \tag{16a} \\
&\leqslant \min_{t \in [\varepsilon T, (\varepsilon+1)T]} h(\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}_{e,x_0}^*)) - \gamma t \tag{16b}
\end{aligned}
$$

$$= \min_{t\in[0,T]} h(\boldsymbol{\varphi}(t+\varepsilon T; x_0, \boldsymbol{u}^*_{e,x_0})) - \gamma t - \gamma\varepsilon T \tag{16c}$$

$$= \min_{t\in[0,T]} h(\boldsymbol{\varphi}(t; \boldsymbol{\varphi}(\varepsilon T; x_0, \boldsymbol{u}^*_{x_0}), \boldsymbol{u}^*_{e,x_0}(*-\varepsilon T))) - \gamma t - \gamma\varepsilon T \tag{16d}$$

$$\leqslant \max_{\substack{\boldsymbol{u}(\cdot)\in\boldsymbol{\mathcal{U}}_{[0,T]} \\ \text{s.t. } \boldsymbol{x}(0)=\boldsymbol{\varphi}(\varepsilon T; x_0, \boldsymbol{u}^*_{x_0}), \\ (8c)-(8e)}} \min_{t\in[0,T]} h(\boldsymbol{x}(t)) - \gamma t - \gamma\varepsilon T \tag{16e}$$

$$\overset{(8)}{=} H_T(\boldsymbol{\varphi}(\varepsilon T; x_0, \boldsymbol{u}^*_{x_0})) - \gamma\varepsilon T, \tag{16f}$$

where we obtain (16c) by employing a time-shift in the argument of the min-operator, and in (16d) we indicate that the argument of the input trajectory is shifted by a time Δt by writing $\boldsymbol{u}^*_{e,x_0}(*-\Delta t)$. We point out that the inequality in (16e) follows from the suboptimality of $\boldsymbol{u}^*_{e,x_0}$ with respect to the shifted initial state, which is $\boldsymbol{\varphi}(\varepsilon T; x_0, \boldsymbol{u}^*_{x_0})$. To summarize the reasoning in *Step 1* so far, we have shown that

$$\frac{H_T(\boldsymbol{\varphi}(\varepsilon T; x_0, \boldsymbol{u}^*_{x_0})) - H_T(x_0)}{\varepsilon T} \geqslant \gamma \qquad \forall t' \in [0,T]. \tag{17}$$

Furthermore, we observe that

$$\liminf_{\sigma\downarrow 0} \frac{H_T(\boldsymbol{\varphi}(\sigma; x_0, \boldsymbol{u})) - H_T(x_0)}{\sigma} \tag{18a}$$

$$= \liminf_{\sigma\downarrow 0} \frac{H_T(x_0 + \sigma f(x_0, \boldsymbol{u}(0))) - H_T(x_0)}{\sigma}$$
$$+ \liminf_{\sigma\downarrow 0} \frac{H_T(\boldsymbol{\varphi}(\sigma; x_0, \boldsymbol{u})) - H_T(x_0 + \sigma f(x_0, \boldsymbol{u}(0)))}{\sigma} \tag{18b}$$

$$= \liminf_{\sigma\downarrow 0} \frac{H_T(x_0 + \sigma f(x_0, \boldsymbol{u}(0))) - H_T(x_0)}{\sigma} \tag{18c}$$

where the second limit in (18b) equals zero as

$$\boldsymbol{\varphi}(\sigma; x_0, \boldsymbol{u}) = \boldsymbol{\varphi}(0; x_0, \boldsymbol{u}) + \sigma f(x_0, \boldsymbol{u}(0)) + \mathcal{O}(\sigma^2)$$
$$= x_0 + \sigma f(x_0, \boldsymbol{u}(0)) + \mathcal{O}(\sigma^2)$$

and thus, by employing the local Lipschitz continuity of H_T in terms of the local Lipschitz constant L,

$$\liminf_{\sigma\downarrow 0} \frac{|H_T(\boldsymbol{\varphi}(\sigma; x_0, \boldsymbol{u})) - H_T(x_0 + \sigma f(x_0, \boldsymbol{u}(0)))|}{\sigma}$$

$$= \liminf_{\sigma\downarrow 0} \frac{|H_T(x_0+\sigma f(x_0,\boldsymbol{u}(0))+\mathcal{O}(\sigma^2)) - H_T(x_0+\sigma f(x_0,\boldsymbol{u}(0)))|}{\sigma}$$

$$\leqslant \liminf_{\sigma\downarrow 0} \frac{L\mathcal{O}(\sigma^2)}{\sigma} = 0.$$

Equipped with these results, we can finally show that for any state x_0 with $H_T(x_0) \leqslant \delta - \gamma(\varepsilon+1)T$ there exists an input trajectory, namely $\boldsymbol{u}^*_{x_0}$, that results in an ascend on H_T in the direction of the state trajectory $\boldsymbol{\varphi}(\cdot; x_0, \boldsymbol{u}^*_{x_0})$ at point x_0. More precisely, it holds

$$\sup_{u\in\mathcal{U}} \{dH_T(x_0; f(x_0, u))\} \geqslant dH_T(x_0; f(x_0, \boldsymbol{u}^*_{x_0}(0))) \tag{19a}$$

$$= \liminf_{\sigma\downarrow 0} \frac{H_T(x_0 + \sigma f(x, \boldsymbol{u}^*_{x_0}(0))) - H_T(x_0)}{\sigma} \tag{19b}$$

$$\overset{(18)}{=} \liminf_{\sigma\downarrow 0} \frac{H_T(\boldsymbol{\varphi}(\sigma; x_0, \boldsymbol{u}^*_{x_0})) - H_T(x_0))}{\sigma} \tag{19c}$$

$$\overset{(17)}{\geqslant} \lim_{\sigma\downarrow 0} \frac{\gamma\sigma}{\sigma} = \gamma \tag{19d}$$

$$\geqslant -\alpha(H_T(x_0)) \tag{19e}$$

Fig. 4: Schematic sketch of the extended class $\mathcal{K}_e$ function α.

for all x_0 with $H_T(x_0) \leqslant \delta - \gamma(\varepsilon+1)T$ where α is some suitable extended class $\mathcal{K}_e$ function that we choose as

$$\alpha(\zeta) = \begin{cases} \alpha_1(\zeta) & \zeta \geqslant 0 \\ \alpha_2(\zeta) & \zeta < 0 \end{cases} \tag{20}$$

such that

$$-\gamma \leqslant \alpha(\zeta) \tag{21}$$

holds. To satisfy condition (21), we choose $\alpha_1 : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ as an arbitrary class $\mathcal{K}$ function which trivially satisfies (21) as α_1 is by definition non-negative and $\gamma > 0$. In order to complement α in (20) as an extended class $\mathcal{K}_e$ function, $\alpha_2 : \mathbb{R}_{<0} \to \mathbb{R}_{<0}$ needs to be chosen as some strictly increasing function with $\alpha_2(0) = 0$. Because also (21) needs to be satisfied, we choose α_2 as a sigmoid function but also other choices are feasible. For example, we choose $\alpha_2(\zeta) = 2\gamma(\text{sig}(\zeta)-0.5)$ where $\text{sig}(\zeta) = \frac{1}{1+e^{-\zeta}}$. A schematic sketch of α can be found in Figure 4.

Now, we have shown that (10) holds for all $x_0 \in \mathbb{R}^n$ with $H_T(x_0) \leqslant \delta - \gamma(\varepsilon+1)T$. It remains to show that (10) also holds for those $x_0 \in \mathcal{D}$ where $H_T(x_0) \geqslant \delta-\gamma(\varepsilon+1)T$, which is done in the next step.

Step 2: Let us now consider states x_0 with $H_T(x_0) > \delta - \gamma(\varepsilon+1)T$. By (9), we then have $H_T(x_0) > 0$ and $x_0 \in \mathcal{C}$. Therefore, for any such x_0, it holds by assumption that there exists some $u \in \mathcal{U}$ such that $||f(x_0, u)|| \leqslant M$. Because $x(t) = x_0 + \int_0^t f(x(s), u(s))ds$, it follows for sufficiently small t' that

$$||x_0 - \boldsymbol{\varphi}(t'; x_0, \boldsymbol{u})|| = \left\|\int_0^{t'} f(\boldsymbol{\varphi}(s; x_0, \boldsymbol{u}), \boldsymbol{u}(s)) \, ds\right\|$$

$$\leqslant \int_0^{t'} ||f(\boldsymbol{\varphi}(s; x_0, \boldsymbol{u}), \boldsymbol{u}(s))|| \, ds \leqslant \int_0^{t'} M \, ds = M \, t'. \tag{22}$$

By invoking the local Lipschitz continuity of H_T with local Lipschitz constant L, we lower-bound $H_T(\boldsymbol{\varphi}(t'; x_0, \boldsymbol{u}'))$ for some input trajectory $\boldsymbol{u}' \in \boldsymbol{\mathcal{U}}_{[0,t']}$ as

$$H_T(\boldsymbol{\varphi}(t'; x_0, \boldsymbol{u})) \geqslant H_T(x_0) - L \, ||x_0 - \boldsymbol{\varphi}(t'; x_0, \boldsymbol{u}')||$$
$$\overset{(22)}{\geqslant} H_T(x_0) - L \, M \, t'. \tag{23}$$

Now it follows analogously to (19) that

$$\sup_{u\in\mathcal{U}} \{dH_T(x_0; f(x_0, u))\} \geqslant dH_T(x_0; f(x_0, \boldsymbol{u}^*_{x_0}(0))) \tag{24a}$$

$$= \liminf_{\sigma\downarrow 0} \frac{H_T(x_0 + \sigma f(x, \boldsymbol{u}^*_{x_0}(0))) - H_T(x_0)}{\sigma} \tag{24b}$$

$$\overset{(18)}{=} \liminf_{\sigma\downarrow 0} \frac{H_T(\boldsymbol{\varphi}(\sigma; x_0, \boldsymbol{u}^*_{x_0})) - H_T(x_0))}{\sigma} \tag{24c}$$

$$\overset{(23)}{\geqslant} \lim_{\sigma\downarrow 0} \frac{-L \, M \, \sigma}{\sigma} = -L \, M. \tag{24d}$$

Thus, by continuously extending the class $\mathcal{K}$ function α_1 in (20) such that $\alpha_1(\zeta) \geqslant L\,M$ for all $\zeta > \delta - \gamma(\varepsilon + 1)T$, (10) also holds for all x_0 with $H_T(x_0) > \delta - \gamma(\varepsilon + 1)T$. The choice of α_1 in this step does not conflict with the choice of α_1 in Step 1 as α_1 is only required to be an arbitrary class $\mathcal{K}$ function.

Altogether, we have now shown that (10) holds for all $x_0 \in \mathcal{D}$. Together with the assumption that H_T is locally Lipschitz continuous, it follows that H_T is a CBF in the Dini sense according to Definition 1, which concludes the proof. $\square$

Conceptually, we showed in the first step of the proof that for all states marked blue in Figure 3 at least an ascend of γ can be achieved on H_T. In the second step, we lower-bounded the minimal possible descend on H_T for all states marked green in Figure 3. Based on these bounds, we established that H_T is a CBF in the Dini sense by constructing an extended class $\mathcal{K}_e$ function satisfying (4). The transition between *no descend* (for $H_T(x_0) = 0$) and *limited descend* (for $H_T(x_0) = \delta - \gamma(\varepsilon + 1)T$) on H_T takes place on the interval $0 \leqslant \zeta \leqslant \delta - \gamma(\varepsilon + 1)T$ and is characterized by the choice of $\alpha_1(\zeta)$ as a continuous function.

Remark 2. The assumption of Theorem 6 that f is bounded on $\mathcal{C}$ is reasonable as the time-variation of practically relevant systems is bounded within their domain of operation.

Remark 3. It is known for certain types of nonlinear systems that finite horizon optimal control problems such as (8) may be sensitive to variations in the initial condition. While this seems to be a minor practical problem, it can be still observed in some examples [64]. This can be mitigated by varying the design parameters T, γ, δ or choosing $\mathcal{F}$ closer to $\mathcal{V}$. In Theorem 6, we account for this by including local Lipschitz continuity into the premises. This can be observed after the computation of H_T.

B. Relaxed Upper-Bound on γ

The proof of the previous theorem unraveled the role of parameter γ in the CBF synthesis. As it becomes evident from (19d), γ constitutes a lower bound on the maximum possible ascend on H_T at any x_0 contained in the set marked blue in Figure 3. In Theorem 6, we formulated a condition on the choice of γ in (9). It can be relaxed as follows:

- If

$$t_{x_0}^* := \arg\min_{t \in [0, \overline{T}]} h(\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}_{x_0}^*)) - \gamma t \in [0, \overline{T}] \quad (25)$$

  for some time $\overline{T} < T$ and any x_0, where $t_{x_0}^*$ is defined as the time that solves the inner minimization in (8a), then condition (9) relaxes to $\gamma < \frac{\delta}{\overline{T}}$.
- Or, introducing the extended class $\mathcal{K}_e$ function α into (8) instead of γ allows us to drop condition (9).

While the first approach requires additional knowledge, the second adds slightly to the complexity of the computation of H_T. We investigate both approaches in the following.

Theorem 7. Let (25) hold for some time $\overline{T} < T$. Then Theorem 6 holds with

$$\gamma < \frac{\delta}{\overline{T}} \quad (26)$$

instead of condition (9).

Proof. Let us consider $\varepsilon \in (0, \min\{\frac{\delta}{\gamma\overline{T}}, \frac{T}{\overline{T}}\} - 1)$. Then, $\gamma < \frac{\delta}{(\varepsilon+1)\overline{T}} < \frac{\delta}{\overline{T}}$ and $(\varepsilon + 1)\overline{T} < T$ hold. Analogously to before, we divide the proof into two steps.

Step 1: Let x_0 be such that $H_T(x_0) \leqslant \delta - \gamma(\varepsilon + 1)\overline{T}$. We note that the right-hand side is strictly positive due to the above choice of ε. Furthermore, let us consider $\boldsymbol{u}_{e,x_0}^*$ as defined in (12). Based on (25), we derive

$$H_T(x_0) = \min_{t \in [0,T]} h(\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}_{e,x_0}^*)) - \gamma t \quad (27a)$$

$$\overset{(25)}{=} \min_{t \in [0,\overline{T}]} h(\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}_{e,x_0}^*)) - \gamma t \quad (27b)$$

$$\overset{(25)}{=} \min_{t \in [0,(\varepsilon+1)\overline{T}]} h(\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}_{e,x_0}^*)) - \gamma t \quad (27c)$$

$$\leqslant \min_{t \in [\varepsilon T,(\varepsilon+1)\overline{T}]} h(\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}_{e,x_0}^*)) - \gamma t \quad (27d)$$

$$\overset{(25)}{=} \min_{t \in [\varepsilon T,(\varepsilon+1)T]} h(\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}_{e,x_0}^*)) - \gamma t \quad (27e)$$

where (27b) and (27e) are immediate consequences of (25); in (27c), we additionally employed that for the chosen ε it holds $(\varepsilon + 1)\overline{T} < T$. We note that (27e) coincides with (16b) in the proof of Theorem 6, and we follow the same steps from here onward. Thereby, (10) follows for all x_0 considered in Step 1.

Step 2: We follow the proof of Theorem 6, Step 2, for all x_0 with $H_T(x_0) > \delta - \gamma(\varepsilon + 1)\overline{T}$ and finally choose α_1 such that $\alpha_1(\zeta) \geqslant L\,M$ for all $\zeta > \delta - \gamma(\varepsilon + 1)\overline{T}$. $\square$

Remark 4. The construction of α in the proof of Theorem 7 stays the same as in Figure 4, however, with T replaced by $\overline{T}$.

Let us assume a differentiable constraint function h. Then according to the KKT-conditions, it holds at time $t_{x_0}^*$ in (25) that $\nabla h(\boldsymbol{x}(t_{x_0}^*))\,f(\boldsymbol{x}(t_{x_0}^*), u) \geqslant \gamma$ and $\frac{d}{dt}(\nabla h(\boldsymbol{x}(t_{x_0}^*))\,f(\boldsymbol{x}(t_{x_0}^*), u)) > 0$ for some $u \in \mathcal{U}$. We apply this insight in the next example.

Fig. 5

Example 2. Let us reconsider the bicycle dynamics in Example 1. Furthermore, let h be a differentiable function with $||\nabla h(x)|| = 1$ specifying some convex constraint. Now we note that the bicycle can be steered from any initial point x_0 to a point x where $\nabla h(x)$ and $f(x, u)$ are aligned as depicted in Figure 5. This can be done by steering the bicycle on a circular trajectory with radius R for at most a half round; R denotes the turning radius as previously specified. Thus if $v_{\max} \geqslant \gamma$, we obtain that $\nabla h(x)\,f(x, u) \geqslant ||\nabla h(x)||\,||f(x, u)|| \geqslant v_{\max} \geqslant \gamma$ and hence that the KKT conditions are satisfied for some $t_{x_0}^* \leqslant \overline{T} = \frac{\pi R}{v_{\max}}$.

C. Incorporating α as Explicit Design Parameter

Alternatively, the extended class $\mathcal{K}_e$ function α can be used as a design parameter in the synthesis of H_T instead of γ. To this end, we modify (8) and replace γ by

$$\bar{\alpha}(\zeta) \coloneqq \begin{cases} \alpha(\zeta) & \text{if } \zeta \leqslant 0 \\ 0 & \text{if } \zeta > 0 \end{cases} \tag{28}$$

where α is an extended class $\mathcal{K}_e$ function. The modified optimization problem is then

$$H_T(x_0) \coloneqq \max_{\boldsymbol{u}(\cdot) \in \boldsymbol{\mathcal{U}}_{[0,T]}} \min_{t \in [0,T]} h(\boldsymbol{x}(t)) + \bar{\alpha}(h(\boldsymbol{x}(t))) \, t \tag{29a}$$

$$\text{s.t. } (8\text{b})\text{-}(8\text{e}) \text{ hold.} \tag{29b}$$

By following similar arguments as before, we can show that also H_T as defined in (29) constitutes a CBF in the Dini sense.

Theorem 8. Let Assumptions 1 and 2 hold, and let h and T be as in Theorem 6. Moreover, let $H_T : \mathcal{D} \to \mathbb{R}$ be defined on some domain $\mathcal{D} \subseteq \mathbb{R}^n$ with $\mathcal{H} \subset \mathcal{D}$ in (29) where $\bar{\alpha}$ is defined as in (28) via some extended class $\mathcal{K}_e$ function α. Furthermore, let f be bounded in the sense that for all x with $H_T(x) \geqslant \delta$ there exists a $u \in \mathcal{U}$ such that $||f(x, u)|| \leqslant M$ for some constant $M > 0$. If H_T is locally Lipschitz continuous with Lipschitz constant L, and α satisfies $\alpha(\zeta) \geqslant L \, M$ for all $\zeta \geqslant \delta$, then

$$\sup_{u \in \mathcal{U}} \{dH_T(x_0; f(x_0, u))\} \geqslant -\alpha(H_T(x_0)) \tag{30}$$

and H_T constitutes a CBF in the Dini sense on the domain $\mathcal{D}$ with respect to the dynamics (1).

Proof. Using the same arguments as in Proposition 5, we conclude that also H_T defined in (29) is well-defined for all $x_0 \in \mathcal{D}$. We conduct this proof, as the previous ones, in two steps: at first, we consider those x_0 with $H_T(x_0) \leqslant \delta$; in a second step, x_0 with $H_T(x_0) > \delta$ are considered. In each of the steps, we mostly follow the line of arguments in the proof of Theorem 6 and therefore only point out important intermediate results. In this proof, we denote the input trajectory $\boldsymbol{u}$ and the times t and ϑ that solve optimization problem (29) by $\boldsymbol{u}_{x_0}^*$, $t_{x_0}^*$ and $\vartheta_{x_0}^*$ analogously to before.

Step 1: Let x_0 be such that $H_T(x_0) \leqslant \delta$. As in the proof of Theorem 6, we obtain analogously to (13)

$$H_T(x_0) = h(\boldsymbol{\varphi}(t_{x_0}^*; x_0, \boldsymbol{u}_{x_0}^*)) + \bar{\alpha}(h(\boldsymbol{\varphi}(t_{x_0}^*; x_0, \boldsymbol{u}_{x_0}^*))) \, t_{x_0}^*$$
$$\leqslant \delta \leqslant h(\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}_{e,x_0}^*)) \qquad \forall t \in [\vartheta_{x_0}^*, T(\varepsilon+1)]$$

where $\boldsymbol{u}_{e,x_0}^* \coloneqq \begin{cases} \boldsymbol{u}_{x_0}^*(t) & \text{if } t \in [0, \vartheta_{x_0}^*] \\ \boldsymbol{u}_{e,x_0}(t) & \text{if } t \in (\vartheta_{x_0}^*, T(\varepsilon+1)] \end{cases}$ with $\boldsymbol{u}_{e,x_0} \in \mathcal{U}_{(\vartheta_{x_0}^*, T(\varepsilon+1)]}$ such that $\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}_{e,x_0}^*) \in \mathcal{V}$ for all $t > \vartheta_{x_0}^*$. Thus the analogue to (15) becomes

$$H_T(x_0) = \min_{t \in [0,T]} h(\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}_{x_0}^*)) + \bar{\alpha}(h(\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}_{x_0}^*))) \, t$$
$$\leqslant \min_{t \in [0, T(\varepsilon+1)]} h(\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}_{e,x_0}^*)) + \bar{\alpha}(h(\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}_{e,x_0}^*))) \, t$$

which is the same as (15) when replacing γ with $-\bar{\alpha}$; note that $-\bar{\alpha}$ is by definition non-negative. This ultimately leads to

$$\frac{H_T(\boldsymbol{\varphi}(\varepsilon; x_0, \boldsymbol{u}_{x_0}^*)) - H_T(x_0)}{\varepsilon} \geqslant -\bar{\alpha}(H_T(\boldsymbol{\varphi}(\varepsilon; x_0, \boldsymbol{u}_{x_0}^*))) \tag{31}$$

for all $\varepsilon \in [0, T]$, and further to

$$\sup_{u \in \mathcal{U}} \{dH_T(x_0; f(x_0, u))\} \geqslant dH_T(x_0; f(x_0, \boldsymbol{u}_{x_0}^*(0))) \tag{32a}$$

$$= \liminf_{\varepsilon \downarrow 0} \frac{H_T(x_0 + \sigma f(x, \boldsymbol{u}_{x_0}^*(0))) - H_T(x_0)}{\sigma} \tag{32b}$$

$$\overset{(18)}{=} \liminf_{\sigma \downarrow 0} \frac{H_T(\boldsymbol{\varphi}(\sigma; x_0, \boldsymbol{u}_{x_0}^*)) - H_T(x_0))}{\sigma} \tag{32c}$$

$$\overset{(31)}{\geqslant} \lim_{\sigma \downarrow 0} -\frac{\bar{\alpha}(H_T(\boldsymbol{\varphi}(\sigma; x_0, \boldsymbol{u}_{x_0}^*))) \, \sigma}{\sigma} = -\bar{\alpha}(H_T(x_0)) \tag{32d}$$

$$\geqslant -\alpha(H_T(x_0)). \tag{32e}$$

Step 2: Replacing γ with $-\bar{\alpha}$ does not change Step 2 in the proof of Theorem 6. Thus, we analogously conclude

$$\sup_{u \in \mathcal{U}} \{dH_T(x_0; f(x_0, u))\} \geqslant -L \, M \geqslant -\alpha(H_T(x_0)). \tag{33}$$

Combining Step 1 and 2, it follows that (30) holds and H_T defined by (29) is a CBF in the Dini sense. $\square$

In the remainder of the paper, we refer to (8) whenever we write H_T unless otherwise specified. Nevertheless, results analogous to the following ones also apply to H_T in (29).

V. Properties of H_T

Next, we characterize some of the properties of the determined CBF H_T.

A. Impact of the Prediction Horizon

At first, we investigate the impact of the prediction horizon T on the size of the zero-superlevel set $\mathcal{C}$ as well as on the size of the other superlevel sets $\mathcal{C}_\lambda$ of H_T with $\lambda \geqslant 0$. To avoid ambiguities in the notation, we include the time horizon as an argument to these sets. In particular, we define $\mathcal{C}_{0,T} \coloneqq \{x \mid H_T(x) \geqslant 0\}$ and $\mathcal{C}_{\lambda,T} \coloneqq \{x \mid H_T(x) \geqslant -\lambda\}$. We show next that the sets $\mathcal{C}_{0,T}$ and $\mathcal{C}_{\lambda,T}$, $\lambda \geqslant 0$, can be enlarged by increasing time horizon T.

Proposition 9. Let Assumptions 1 and 2 hold, let h be Lipschitz continuous and let $T_2 \geqslant T_1 \geqslant \tau$. Moreover, let $H_{T_1} : \mathcal{D} \to \mathbb{R}$ and $H_{T_2} : \mathcal{D} \to \mathbb{R}$ be defined as in (8) for some $\gamma > 0$ where

$$\gamma < \frac{\delta}{T_2}. \tag{34}$$

Then $\mathcal{C}_{0,T_1} \subseteq \mathcal{C}_{0,T_2}$ and, more generally, $\mathcal{C}_{\lambda,T_1} \subseteq \mathcal{C}_{\lambda,T_2}$ for any $\lambda \geqslant 0$ with $\mathcal{C}_{\lambda,T_1}, \mathcal{C}_{\lambda,T_2} \subseteq \mathcal{D}$.

Proof. The proof can be found in the appendix. $\square$

Analogous results hold for $\gamma < \frac{\delta}{\max\{\overline{T}_1, \overline{T}_2\}}$, or if H_T is synthesized based on (29).

B. Changing and Time-Varying Constraint Specifications

Independent of whether H_T was synthesized based on Theorem 6, 7 or 8, further CBFs can be derived based on it. In particular, the CBF property of H_T is preserved when adding – within bounds still to be further specified – a constant or a time-varying trajectory. In this way, the

synthesized CBF H_T can account for corresponding changes in the constraint specifications and time-variations.

To become more specific, let us define the largest superlevel set of H_T contained in $\mathcal{D}$ as $\mathcal{C}_{\Lambda,T} := \{x \mid H_T(x) \geqslant -\Lambda\}$ where $\Lambda := \max\{\lambda \mid \mathcal{C}_{\lambda,T} \subseteq \mathcal{D}\}$. Based on this,

$$H_{\lambda,T}(x) := H_T(x) + \lambda$$

is also a CBF for any $\lambda \in [0, \Lambda]$. This is a direct implication of the fact that[1] $\sup_{u \in \mathcal{U}} \{dH_T(x_0; f(x_0, u))\} \geqslant \gamma \geqslant 0$ in any x_0 where $H_T(x_0) \in [-\Lambda, 0]$ as shown in (19) as part of the proof of Theorem 6. In the light of [55], an even stronger property holds. Assuming that H_T is not only locally Lipschitz continuous but differentiable, then H_T constitutes a so-called Λ-shiftable CBF on the domain $\mathcal{D}$ with respect to (1); for details refer to [55]. This characteristic leads to the following property: for a time-varying trajectory $\boldsymbol{\lambda} : \mathbb{R}_{\geqslant 0} \rightarrow [0, \Lambda]$ satisfying some additional condition, the function

$$H_{\boldsymbol{\lambda}(\cdot),T}(t,x) := H_T(x) + \boldsymbol{\lambda}(t) \tag{35}$$

constitutes a differentiable CBF [2] with respect to the dynamics augmented in time. This is formally stated as follows.

Proposition 10. Let $H_{\boldsymbol{\lambda}(\cdot),T}$ be defined as in (35), and let the same premises hold as in Theorem 6. Moreover, let H_T as defined in (8) be differentiable and let $\boldsymbol{\lambda} : \mathbb{R}_{\geqslant 0} \rightarrow [0, \Lambda]$ be a differentiable function that satisfies

$$\frac{\partial \boldsymbol{\lambda}}{\partial t}(t) \geqslant -\widetilde{\alpha}(\boldsymbol{\lambda}(t)) \tag{36}$$

where $\widetilde{\alpha}$ is an either convex or concave class $\mathcal{K}$ function and it holds $\alpha(-\zeta) \leqslant -\widetilde{\alpha}(\zeta)$ for all $\zeta \in [0, \Lambda]$. Then, $H_{\boldsymbol{\lambda}(\cdot),T}$ is a CBF on the domain $\mathbb{R}_{\geqslant 0} \times \mathcal{D}$ with respect to dynamics (1) augmented by time $\begin{bmatrix} \dot{t} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 1 \\ f(x,u) \end{bmatrix}$.

Proof. By Theorem 6, H_T is a CBF on the domain $\mathcal{C}_\Lambda \subseteq \mathcal{D}$. Hence, as H_T is assumed to be differentiable, it follows by [55, Definition 2] that H_T is a Λ-shiftable CBF. Based on this, the proposition is a direct implication of [55, Theorem 4]. $\square$

Remark 5. The same result follows if the premises of Theorem 7 hold instead of those of Theorem 6, or alternatively, if H_T is defined in (29) with the premises of Theorem 8 satisfied.

The construction of H_T in the previous section has the following favorable properties with respect to the application of Proposition 10:

1.) While most CBF construction approaches in the literature only apply to the determination of CBFs on a domain $\mathcal{C}$, our proposed method allows for the construction on larger domains $\mathcal{D} \supseteq \mathcal{C}$, which gives rise to the "shiftability"-property.

2.) Our method comes along with a concrete construction of the extended class $\mathcal{K}_e$ function α invoking the design parameter γ as outlined in the proofs of Theorems 6 and 7, see Figure 4. Alternatively, in the synthesis of H_T according to (29), α can be even chosen as a design parameter.

3.) Regardless of the particular approach, function α can here always be chosen to be convex (see also Figure 4), thus

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[1]In the case of H_T defined by (29), it holds analogously $\sup_{u \in \mathcal{U}} \{dH_T(x_0; f(x_0, u))\} \geqslant -\bar{\alpha}(H_T(x_0)) \geqslant 0$ according to (32) for any x_0 with $H_T(x_0) \in [-\Lambda, 0]$.

$\alpha(-\zeta) = -\widetilde{\alpha}(\zeta)$ for all $\zeta \in [0, \Lambda]$ in Proposition 10 is a feasible choice. Then, (36) becomes

$$\frac{\partial \boldsymbol{\lambda}}{\partial t}(t) \geqslant \alpha(-\boldsymbol{\lambda}(t)). \tag{37}$$

4.) Our CBF construction approach accounts for input constraints.

C. Saturated H_T

By introducing a saturated version of H_T, the computation of a CBF in the Dini sense can be simplified. To this end, we make the following observation.

Proposition 11. Let the saturated value function $\bar{H}_T : \mathcal{D} \rightarrow \mathbb{R}$ be defined as

$$\bar{H}_T(x_0) := \min\{H_T(x_0), \delta - \gamma T\}$$

where H_T is defined by (8), and let the same premises hold as in Theorem 6. Then, $\bar{H}_T$ is a CBF in the Dini sense.

Proof. For x_0 with $\bar{H}_T(x_0) \leqslant \delta - \gamma T$, we trivially have $\bar{H}_T(x_0) = H_T(x_0)$ and the proof of Theorem 6 still applies. For x_0 with $\bar{H}_T(x_0) > \delta - \gamma T$, it holds $d\bar{H}_T(x_0; f(x_0, u)) = 0$ for any $u \in \mathcal{U}$ as $\bar{H}_T$ is constant in the ϵ-neighborhood of x. Thus, (24) still holds when substituting H_T by $\bar{H}_T$ because $\sup_{u \in \mathcal{U}} \{d\bar{H}_T(x_0; f(x_0, u))\} = 0 \geqslant -LM$. From this, we conclude that the proof of Theorem 6 also applies to $\bar{H}_T$ and we conclude that $\bar{H}_T$ is a CBF in the Dini sense. $\square$

Remark 6. If the premises of Theorem 7 are considered instead of those in Theorem 6, then the above results holds for $\bar{H}_T(x_0) := \min\{H_T(x_0), \delta - \gamma \overline{T}\}$. If H_T as defined by (29) is considered and the premises of Theorem 8 hold, then the above result holds for $\bar{H}_T(x_0) := \min\{H_T(x_0), \delta\}$. The proofs in each of the cases are analogous to the one of Proposition 11.

VI. IMPLEMENTATION REMARKS

The implementation of the optimization problems in (8) and (29) as a max-min-problem is not entirely straightforward and deserves a discussion on its own. While their formulation is well-suited for analysis, we propose in the sequel some simplifications with regard to their practical implementation. These yield an arbitrarily close approximation of $H_T(x_0)$ while giving rise to a computationally efficient implementation. We focus on H_T defined in (8), and note that analogous remarks apply to H_T defined in (29).

A. Discrete-Time Implementation

For the practical implementation of the optimization problems in (8) and (29), dynamics and trajectories need to be discretized. To this end, we discretize time-horizon T into $N + 1$ time-steps using discretization time $\Delta t = T/N$. Correspondingly, state and input trajectories in (8) become

$$\mathbf{x}_N := \begin{bmatrix} \boldsymbol{x}(0\Delta T) & \boldsymbol{x}(1\Delta T) & \boldsymbol{x}(2\Delta T) & \ldots & \boldsymbol{x}(N\Delta T) \end{bmatrix} \in \mathbb{R}^{n,N+1},$$

$$\mathbf{u}_{N-1} := \begin{bmatrix} \boldsymbol{u}(0\Delta T) & \boldsymbol{u}(1\Delta T) & \boldsymbol{u}(2\Delta T) & \ldots & \boldsymbol{u}((N-1)\Delta T) \end{bmatrix} \in \mathbb{R}^{m,N}.$$

The k-th column of $\mathbf{x}_N$ is referred to as $\mathbf{x}_N[k]$. The values of h corresponding to $\mathbf{x}_N[\cdot]$ are

$$\mathbf{h}_N(\mathbf{x}_N) \coloneqq \begin{bmatrix} h(\mathbf{x}_N[0]) \ h(\mathbf{x}_N[1]) \ h(\mathbf{x}_N[2]) \ \ldots \ h(\mathbf{x}_N[N]) \end{bmatrix}^T \in \mathbb{R}^{N+1}.$$

The max-min-problem (8) can be then rewritten as

$$H_T(x_0) := \max_{\mathbf{u}_{N-1}} \min_{k=0,1,\ldots,N} h(\mathbf{x}_N[k]) - \gamma k \Delta t \tag{38a}$$

$$\text{s.t. } \mathbf{x}_N[0] = x_0, \tag{38b}$$

$$\mathbf{x}_N[k+1] = f_d(\mathbf{x}_N[k], \mathbf{u}_{N-1}[k]), \ \forall k=0,1,\ldots,N-1, \tag{38c}$$

$$\mathbf{u}_{N-1}[k] \in \mathcal{U}, \quad \forall k = 0,1,\ldots,N-1, \tag{38d}$$

$$\mathbf{x}_N[\kappa] \in \mathcal{F}, \quad \text{for some } \kappa \in \{0,1,\ldots,N\}, \tag{38e}$$

where $f_d : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ denotes the discretized dynamics of system (1). A good approximation of the discretized dynamics can be obtained via numerical integration algorithms, e.g., a Runge-Kutta method. The input trajectory $\mathbf{u}_{N-1}$ and the times k and κ that solve (38) for initial value x_0 are denoted by $\mathbf{u}_{N-1,x_0}^*$, $k_{x_0}^*$ and $\kappa_{x_0}^*$. Note that due to the incremental definition of the discrete-time dynamics in (38c), $\mathbf{u}_{N-1}$ has one entry less than $\mathbf{x}_N$. By increasing N, (38) becomes an arbitrarily close approximation of (8).

B. Bypassing the Nested Optimization Problem

Max-min-problem (8) allows for an efficient implementation bypassing the nested optimization problem. The essence of this approach is to approximate the inner minimization using a p-norm.

Let us first consider the discrete-time approximation (38). By drawing a factor (-1) out from the nested optimization in (38a), the max- and min-operators are reversed and we obtain

$$H_T(x_0) := - \min_{\substack{\mathbf{u}_{N-1} \\ \text{s.t. (38b)-(38e)}}} \max_{k=0,1,\ldots,N} -(h(\mathbf{x}_N[k]) - \gamma k \Delta t). \tag{39}$$

In the following, we first determine $\mathbf{u}_{N-1,x_0}^*$ and $k_{x_0}^*$ before actually computing $H_T(x_0)$. To this end, we observe that for some strictly positive function f it holds

$$\arg\max_x -f(x) = \arg\max_x \frac{1}{f(x)}$$

since $-f(x_1) > -f(x_2) \Leftrightarrow f(x_1) < f(x_2) \Leftrightarrow \frac{1}{f(x_2)} < \frac{1}{f(x_1)}$ (replacing the negation with the multiplicative inverse is order-preserving). Furthermore, it is $\arg\max \frac{1}{f(x)} = \arg\max \frac{1}{f(x)+c}$ for any positive constant $c \in \mathbb{R}_{>0}$. With this, we obtain $\mathbf{u}_{N-1,x_0}^*$ and $k_{x_0}^*$ from (39) as

$$\mathbf{u}_{N-1,x_0}^*, k_{x_0}^* \leftarrow \min_{\substack{\mathbf{u}_{N-1} \\ \text{s.t. (38b)-(38e)}}} \max_{k=0,1,\ldots,N} \frac{1}{h(\mathbf{x}_N[k]) - \gamma k \Delta t + \tilde{h}} \tag{40}$$

where $\tilde{h} > \max\{0, \min_{x \in \mathcal{D}} h(x) + \gamma T\}$ is some constant that ensures the strict positivity of the denominator. At last for bypassing the nested optimization, we approximate the maximization by a p-norm with a sufficiently high p. Therefore, we note that for any (finite-dimensional) vector $x = [x_1, \ldots, x_n] \in \mathbb{R}_{>0}^n$ it holds that $||x||_p \approx \max_k x_k$ with

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Algorithm 1: Computation of $H_T(x_0)$

Parameters: h, f_d, T, N, $\mathcal{F}$, γ
Input : x_0
Output : $H_T(x_0)$

1 $\Delta t \leftarrow T/N$;
2 $\mathbf{u}_{N-1,x_0}^* \leftarrow$ solve (41);
3 $\mathbf{x}_{N,x_0}^*[0] \leftarrow x_0$;
4 $\mathbf{x}_{N,x_0}^*[k+1] \leftarrow f_d(\mathbf{x}_{N,x_0}^*[k], \mathbf{u}_{N-1,x_0}^*[k])$ for $k = 0,1,\ldots,N-1$;
5 $H_T(x_0) \leftarrow \min_{k=0,1,\ldots,N} h(\mathbf{x}_{N,x_0}^*[k]) - \gamma k \Delta t$;

$p \gg 0$ where $||x||_p := \sqrt[p]{\sum_{k=1}^n x_k^p}$. Thus, for a sufficiently large p, we obtain

$$\mathbf{u}_{N-1,x_0}^* \approx \underset{\substack{\mathbf{u}_{N-1} \\ \text{s.t. (38b)-(38e)}}}{\arg\min} \left\| \begin{bmatrix} \frac{1}{h(\mathbf{x}_N[0]) - 0\,\gamma\Delta t + \tilde{h}} \\ \vdots \\ \frac{1}{h(\mathbf{x}_N[k]) - k\,\gamma\Delta t + \tilde{h}} \\ \vdots \\ \frac{1}{h(\mathbf{x}_N[N]) - N\,\gamma\Delta t + \tilde{h}} \end{bmatrix} \right\|_p. \tag{41}$$

From this, we can approximate $\mathbf{x}_{N,x_0}^*[k+1] = f_d(\mathbf{x}_{N,x_0}^*[k], \mathbf{u}_{N-1,x_0}^*[k])$, $k = 0,1,\ldots,N-1$, with initial condition $\mathbf{x}_{N,x_0}^*[0] := x_0$, and

$$k_{x_0}^* = \underset{k=0,1,\ldots,N}{\arg\max} \frac{1}{h(\mathbf{x}_{N,x_0}^*[k]) - \gamma k \Delta t + \tilde{h}},$$

or equivalently

$$k_{x_0}^* = \underset{k=0,1,\ldots,N}{\arg\min} h(\mathbf{x}_{N,x_0}^*[k]) - \gamma k \Delta t. \tag{42}$$

From this, we finally obtain

$$H_T(x_0) = h(\mathbf{x}_{N,x_0}^*[k_{x_0}^*]) - \gamma k_{x_0}^* \Delta t. \tag{43}$$

The algorithm is summarized in Algorithm 5.

C. Terminal Constraint

Optimization problem (38) is a mixed integer problem due to (38e). When $\mathcal{F}$ however is readily given as a forward control invariant set, a mixed integer problem can be avoided and (38) becomes

$$H_T(x_0) := \max_{\mathbf{u}_{N-1}} \min_{k=0,1,\ldots,N} h(\mathbf{x}_N[k]) - \gamma k \Delta t$$

$$\text{s.t. (38b)-(38d) hold,} \tag{44}$$

$$\mathbf{x}_N[N] \in \mathcal{F}.$$

If $\mathcal{F}$ is forward control invariant, Assumption 1 is still satisfied and Theorem 6 applies.

D. Computation of the Saturated CBF $\bar{H}_T$

The computation of $\bar{H}_T$ reduces the effort for computing a CBF in the Dini sense, as $\bar{H}_T(x_0) = \delta - \gamma T$ for $x_0 \in \mathcal{F}$. Thus, $\bar{H}_T$ is only required to be explicitly computed on $\mathcal{D} \backslash \mathcal{F}$.

VII. NUMERICAL EXAMPLES

In the following, we apply our CBF synthesis method to the design of safety filters for various dynamic systems in the presence of both static and time-varying constraints. In particular, we consider

A. single and double integrators as examples for first and second order systems;

B. the bicycle model with minimum velocity > 0 as an example for a system that is not locally controllable;

C. the unicycle as an example for a system with a non-holonomic constraint and state-dependent relative degree.

For each of the systems, we compute the function H_T based on (8) for the static constraint over a grid and approximate all further values by linear interpolation. The approximated function coincides in each of the grid points with H_T, while all further points are close approximations. Also other methods for fitting a function into the set of computed values can be applied including the training of a neural network. The subsequent simulation examples however indicate that also elementary approximation methods yield promising results. An implementation of our CBF synthesis method in Python using Casadi [65] is provided on Github[2]. It allows for the parallelized computation of H_T. Videos to the examples can be found on Youtube[3].

The constraint function under consideration is

$$h(x) = \sqrt{(x - x_c)^2 + (y - y_c)^2} - r$$

describing a circular obstacle with center (x_c, y_c) and radius r; the square root ensures that h scales linearly with the distance. The extended class $\mathcal{K}_e$ functions are chosen as convex functions of the form $\alpha(\zeta) := \begin{cases} c\zeta & \text{if } \zeta \geqslant 0 \\ 2\gamma \left(\text{sig}\left(\frac{c\zeta}{4} \right) - 0.5 \right) & \text{if } \zeta < 0 \end{cases}$ according to Theorem 6 with $c > 0$. To vary the constraint function h and the synthesized CBF H_T over time, we add the periodic time-varying function

$$\boldsymbol{\lambda}(t) = -r_{\max} \left| \sin\left(\frac{\pi t}{\tau_p} - \sigma \right) \right| + r$$

where $r_{\max} \leqslant r$ denotes the maximum radius, τ_p is the period and σ some shift. Its parameters are chosen such that (37) holds and thereby Proposition 10 applies. Thus, $H_{\boldsymbol{\lambda}(\cdot),T}(t, x) := H_T(x) + \boldsymbol{\lambda}(t)$ is a CBF. The control task can be now stated as follows: track a straight line while avoiding all possibly time-varying obstacles. For tracking, we employ a feedback controller that generates a baseline input u_{baseline}. For obstacle avoidance, we use a standard safety filter based on the time-varying CBF $H_{\boldsymbol{\lambda}(\cdot),T}$ defined as

$$u_{\text{safe}}(t) = \arg \min_{u \in \mathcal{U}} (u - u_{\text{baseline}}(t))^T P (u - u_{\text{baseline}}(t))$$

$$\text{s.t. } dH_T(\boldsymbol{x}(t); f(\boldsymbol{x}(t), \boldsymbol{u}(t))) + \dot{\boldsymbol{\lambda}}(t)$$
$$\geqslant -\alpha(H_T(\boldsymbol{x}(t)) + \boldsymbol{\lambda}(t)) + c_\alpha$$

where P is some positive definite matrix, and $c_\alpha \geqslant 0$ some constant that can be set positive to add robustness with view

(a)

(b)

(c)

(d)

(e) CBF for the single integrator with $\dot{x} \in [1, 2]$. Points mark explicitly computed values.

(f) Difference between constraint function h and the CBF depicted in Fig. 6e.

Fig. 6: Simulation results for single and double integrators: trajectories and the corresponding CBF values are depicted in (a), (b) for static and in (c), (d) for time-varying constraints (maximum expansion is indicated by the dotted circle). The single integrator with $\mathcal{U} = [-2, 2]^2$ is marked blue, the one with $\mathcal{U} = [1, 2] \times [-2, 2]$ in green, and the double integrator in yellow. (e), (f) depict a CBF and compare it to h.

to the discretization. An overview over all parameters and computation times of the various CBFs can be found in Table I. The CBFs have been computed on a 12th Gen Intel Core i9-12900K with 64GB RAM.

A. Single and Double Integrator

First, we consider single and double integrators in a plane with dynamics $\dot{\mathbf{x}} = u$ and $\ddot{\mathbf{x}} = u$ where $\mathbf{x} = [x, y]^T$. We consider two single integrators where the first (marked blue in Figure 6) has input constraints $\mathcal{U} = [-2, 2]^2$ and the second (green) $\mathcal{U} = [1, 2] \times [-2, 2]$. While the first single integrator can stop or move backwards, the second always moves into the positive x-direction. For the double integrator (yellow), the input constraint is $\mathcal{U} = [-1, 1]^2$, and we limit its velocity to $\dot{\mathbf{x}} \in [-2, 2]^2$ using the modified constraint function $\tilde{h}(\mathbf{x}, \dot{\mathbf{x}}) := \eta \min\{\frac{1}{\eta} h(\mathbf{x}), 2 + \dot{x}, 2 + \dot{y}, 2 - \dot{x}, 2 - \dot{y}\}$ with some positive constant $\eta \in \mathbb{R}_{>0}$. The simulation results are depicted in Figure 6 for static (a,b) and time-varying constraints (c,d). While in the static case, both single integrators have an almost identical trajectory, in the time-varying case they become

	constr.	CBF params.				class $\mathcal{K}_e$ fcn.		N	numerical computation			
	r	γ	δ	$\overline{T}$	T	c	c_α		domain	discretization	#points	comp. time
single integrator	9	2	1	0	10	2	0.2	25	$[-10, 10]^2$	41, 41	1681	0:01:24
single i. ($v_{x,\min} > 0$)	9	2	1	0	10	2	0.2	25	$[-10, 10]^2$	41, 41	1681	0:01:11
double integrator	9	1	5	3	12	0.5	0.2	30	$[-14, 14]^2 \times [-2.5, 2.5]^2$	29, 29, 15, 15	189 225	2:49:46
bicycle (less agile)	6	2	9	4.4	12	1	0.1	30	$[-15, 15]^2 \times [-\pi, \pi]$	61, 61, 41	152 561	5:01:42
bicycle (more agile)	6	2	4	1.8	10	1	0.1	30	$[-10, 10]^2 \times [-\pi, \pi]$	41, 41, 41	68 921	2:33:33
unicycle	6	2	4	1.8	10	0.5	0.1	30	$[-10, 10]^2 \times [-\pi, \pi]$	41, 41, 41	68 921	1:38:12

TABLE I: Constraint specs, parameters and computation times [h:mm:ss] for the CBFs of each considered dynamic system.

distinct as the first single integrator uses its capability to move backwards. The positive CBF values in Figures 6b and 6d indicate constraint satisfaction.

Even in the case of single integrators, the CBF synthesis can be a non-trivial task. While the constraint function h is directly a CBF for the unconstrained single integrator, the same is not necessarily true in the presence of input constraints. This can be clearly seen in the case of the second single integrator for which $\dot{x} \in [1, 2]$; its numerically computed CBF is depicted in Figure 6e. It can be seen that, in contrast to h, the CBF is nonsmooth along the negative x-axis. The difference between h and the CBF is depicted in Figure 6f.

B. Kinematic Bicycle Model

Let us reconsider the kinematic bicycle model from Example 1 with inputs $u = [v, \zeta]^T$ and input constraints $\mathcal{U} = [1, 2] \times [-\zeta_{\max}, \zeta_{\max}]$ where $\zeta_{\max}$ specifies the limitations of the steering angle. In Figure 7, we consider bicycles with two different input constraints: the first is less agile with $\zeta_{\max} = \frac{20}{180}\pi$ (blue), while the second is more agile with $\zeta_{\max} = \frac{40}{180}\pi$ (green). As it can be seen from Figures 7a and 7b, the more agile bicycle stays closer to the obstacles than the less agile one. Each of the obstacles is encoded via a separate CBF that can be varied in time independently of the others. The sinusoidal signal varying the size of each of the obstacles is shifted by a third period; initial and maximum expansions are indicated by green and light-blue dashed circles in Figure 7b. For the case with time-varying obstacles, the difference between the safe and the baseline input, the CBF values and the vehicles' distances to the obstacles are depicted in Figure 7e. Figure 7f shows the corresponding control inputs. It indicates that the time-varying constraints could be satisfied with finite inputs that stay within the input constraints. Because $v > 0$, the CBF of both bicycles is nonsmooth as well as in the previous example, which is shown in Figure 7c for $\zeta_{\max} = \frac{20}{180}\pi$ and a fixed orientation. As it can be seen from Figure 7d, the zero super-level sets are highly dependent on the vehicle's orientation.

C. Kinematic Unicycle Model

As last example, we consider the unicycle dynamics

$$\dot{x} = v\cos(\psi), \quad \dot{y} = v\sin(\psi), \quad \dot{\psi} = \omega$$

with input vector $u = [v, \omega]^T$ and input constraint $\mathcal{U} = [-0.9, 0.9] \times [1, 2]$. The dynamic properties of the kinematic unicycle are of particular interest, as it firstly involves a nonholonomic constraint, and secondly its relative degree is state-dependent. As such, the relative degree with respect to state x is one for all $\psi \neq \pi/2$. Yet, given an orientation of $\psi = \pi/2$, one obtains $\dot{x} = 0$ and the system becomes a second-order system with respect to x. An analogous observation holds for state y and orientation $\psi = 0$. As our approach, however, is indifferent to the order of the system, the CBF synthesis method stays the same as for the bicycle dynamics in the previous example. Hence, no high-order CBF concepts need to be employed. Simulations in scenarios analogous to those for the kinematic bicycle model have been conducted; the results are shown in Figure 8. As indicated by Figure 8c, constraint satisfaction is also ensured for the unicycle despite the more challenging dynamic properties. We point out that the values of the CBF and the distance to the obstacle are not trivially correlated due to the dynamic properties of the system.

VIII. Conclusion

In this work, we presented a systematic method for synthesizing CBFs that encode predictive information. We showed how this information can be advantageously used to account for changes in the constraint specifications and to derive time-varying CBFs. In particular, we presented three synthesis methods that allow to specify the time-varying capabilities of the CBF in terms of a design parameter. The theoretical analysis of the synthesized CBFs was complemented by a detailed discussion of its properties and practical implementation remarks. The proposed method was applied to multiple dynamic systems to demonstrate its applicability. This work is accompanied by a python implementation of the synthesis method that allows for parallelization. The challenge that CBF synthesis methods generally do not scale well with dimensionality remains and has not been addressed in this work. In our follow-up work, we show how the synthesis method presented here can be advantageously applied to equivariant systems in order to reduce the time complexity of the CBF synthesis. Furthermore, compositional controller design approaches may be of interest in this direction.

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(a) Trajectories of the kinematic bicycle model for static constraints. A marker is depicted every 2s.

(b) Trajectories of the kinematic bicycle model for time-varying constraints. Green circles mark the maximum and light-blue ones the initial expansion of the obstacles. A marker is depicted every 2s.

(c) CBF at $\psi = 0$ for $\zeta_{max} = 20\pi/180$ and one circular obstacle. Points mark the explicitly computed values.

(d) Zero super-level sets in the x-y-plane in dependence of orientation ψ for $\zeta_{max} = 20\pi/180$.

(e) Correction of steering angle, CBF values and distance to obstacle over time (time-varying constraints).

(f) Input trajectories (time-varying constraints).

Fig. 7: Simulation results for the kinematic bicycle model: the bicycle with $\zeta_{max} = 20\pi/180$ is indicated in blue, the one with $\zeta_{max} = 45\pi/180$ in green.

(a) Trajectories of the kinematic unicycle model for static constraints. A marker is depicted every 2s.

(b) Trajectories of the kinematic unicycle model for time-varying constraints. Green circles mark the maximum and light-blue ones the initial expansion of the obstacles. A marker is depicted every 2s.

(c) CBF values and distance to obstacle over time (time-varying constraints).

Fig. 8: Simulation results for the kinematic unicycle model.

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APPENDIX

Proof of Proposition 3. Let us construct in the following for each $x_0 \in \partial\mathcal{F}$ an input trajectory $\boldsymbol{u}_{x_0} \in \mathcal{U}$ and its corresponding state trajectory $\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}_{x_0})$, which we consider over a time horizon $t \in [0, t_{f,x_0}]$ where t_{f,x_0} is to be specified later. As before, we consider $\mathcal{V} := \bigcup_{x_0 \in \partial\mathcal{F}} \{x \mid x = \boldsymbol{\varphi}(\tau; x_0, \boldsymbol{u}_{x_0}), \ \tau \in [0, t_{f,x_0}]\} \cup \mathcal{F}$ for which clearly $\mathcal{F} \subseteq \mathcal{V}$. It remains to show that for each condition, $\boldsymbol{u}_{x_0}$ and t_{f,x_0} can be chosen such that $\mathcal{V}$ is forward control invariant and $h(x) \geqslant \delta$ for all $x \in \mathcal{V}$.

Let condition (1) hold. Consider a state $x_{f,x_0} \in \mathcal{F} \cap B_{\delta_\varepsilon}(x_0)$ in the neighborhood of an arbitrary $x_0 \in \partial\mathcal{F}$. As system (1) is locally-locally controllable in any $x_0 \in \partial\mathcal{F}$, there exists by definition a $\boldsymbol{u}_{x_0}$ and t_{f,x_0} such that $\boldsymbol{\varphi}(t_{f,x_0}; x_0, \boldsymbol{u}_{x_0}) = x_{f,x_0}$ and $\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}_{x_0}) \in B_\varepsilon(x_0)$ for all $t \in [0, t_{f,x_0}]$. Then by Proposition 2, the above defined $\mathcal{V}$ is forward control invariant. Moreover, as for any $x_0 \in \mathcal{F}$ it holds $\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}_{x_0}) \in \mathcal{F} \oplus B_\varepsilon(x_0)$ for all $t \in [0, t_{f,x_0}]$, we have $\mathcal{V} \subseteq \mathcal{F} \oplus B_\varepsilon(0)$. From this, we conclude that $\mathcal{V} \subseteq \mathcal{F} \oplus B_\varepsilon(0) \subseteq \{x \mid h(x) \geqslant \delta\}$ where the last subset relation holds by [66, (3.1.12)]. Hence, Assumption 1 holds.

Let now condition (2) hold. For each $x_0 \in \partial\mathcal{F}$, choose $\boldsymbol{u}_{x_0}$ as suggested by the condition and let $t_{f,x_0} \to \infty$. Then, $\mathcal{V}$ is clearly forward control invariant and $\mathcal{V} \subseteq \mathcal{F} \oplus B_\varepsilon(0) \subseteq \{x \mid h(x) \geqslant \delta\}$ as previously. Thus, Assumption 1 holds.

Let at last condition (3) hold. Choose u_{x_0} and t_{f,x_0} as suggested by the condition, and it follows from analogous arguments as in the previous case that Assumption 1 holds. □

Proof of Proposition 9. Let us consider a trajectory $\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}^*_{x_0,T_1})$ defined for $t \in [0, T_1]$ starting in an arbitrary point $x_0 \in \mathcal{D}$. The input trajectory and those times that solve (8) for time horizon T_1 are denoted by $\boldsymbol{u}^*_{x_0,T_1}$, $t^*_{x_0,T_1}$ and $\vartheta^*_{x_0,T_1}$, respectively. Similarly to before, we define an extended input trajectory as $\boldsymbol{u}^*_{e,x_0,T_1}(t) := \begin{cases} \boldsymbol{u}^*_{x_0,T_1}(t) & \text{if } t \in [0, \vartheta^*_{x_0,T_1}] \\ \boldsymbol{u}_{e,x_0,T_1}(t) & \text{if } t > \vartheta^*_{x_0,T_1} \end{cases}$ where $\boldsymbol{u}_{e,x_0,T_1} \in \mathcal{U}_{(\vartheta^*_{x_0},\infty)}$ such that $\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}^*_{e,x_0,T_1}) \in \mathcal{V}$ for all $t > \vartheta^*_{x_0,T_1}$. Since $\mathcal{V}$ is forward control invariant, such a trajectory exists.

Let us first consider those states x_0 with $H_{T_1}(x_0) < \delta - \gamma T_2$, where the right-hand side intentionally employs T_2 instead of T_1. Furthermore, note that the right-hand side is strictly positive due to (34). We now derive analogously to (13) that

$$\begin{aligned} H_{T_1}(x_0) &= h(\boldsymbol{\varphi}(t^*_{x_0,T_1}; x_0, \boldsymbol{u}^*_{x_0,T_1})) - \gamma t^*_{x_0,T_1} \\ &\leqslant \delta - \gamma T_2 \\ &\leqslant h(\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}^*_{e,x_0,T_1})) - \gamma t \quad \forall t \in [\vartheta^*_{x_0,T_1}, T_2]. \end{aligned} \quad (45)$$

Based on this, we further obtain

$$H_{T_1}(x_0) = h(\boldsymbol{\varphi}(t^*_{x_0,T_1}; x_0, \boldsymbol{u}^*_{x_0,T_1})) - \gamma t^*_{x_0,T_1} \quad (46a)$$

$$= \min_{t \in [0,T_1]} h(\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}^*_{x_0,T_1})) - \gamma t \quad (46b)$$

$$\overset{(45)}{=} \min_{t \in [0,T_2]} h(\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}^*_{e,x_0,T_1})) - \gamma t \quad (46c)$$

$$\leqslant \min_{t \in [0,T_2]} h(\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}^*_{x_0,T_2})) - \gamma t = H_{T_2}(x_0) \quad (46d)$$

where (46c) follows analogously to (15) as it holds for T_2 that $\vartheta^*_{x_0,T_1} \leqslant T_1 \leqslant T_2$, (46d) follows due to the suboptimality of $\boldsymbol{u}^*_{e,x_0,T_1}$, and $\boldsymbol{u}^*_{x_0,T_2}$ denotes the input trajectory that solves (8) for time horizon T_2 at state x_0.

At last, we note that for the remaining states x_0 with $H_{T_1}(x_0) > \delta - \gamma T_2$ we have that $H_{T_2}(x_0) > 0$ as

$$H_{T_2}(x_0) = \min_{t \in [0,T_2]} h(\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}^*_{x_0,T_2})) - \gamma t \quad (47a)$$

$$\geqslant \min_{t \in [0,T_2]} h(\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}^*_{e,x_0,T_1})) - \gamma t \quad (47b)$$

$$= \min\left\{ H_{T_1}(x_0), \min_{t \in [T_1,T_2]} h(\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}^*_{e,x_0,T_1})) - \gamma t \right\} \quad (47c)$$

$$\geqslant \delta - \gamma T_2 \overset{(34)}{\geqslant} 0 \quad (47d)$$

where (47b) follows from the suboptimality of $\boldsymbol{u}^*_{e,x_0,T_1}$, and (47d) holds as $\boldsymbol{\varphi}(t; x_0, \boldsymbol{u}^*_{e,x_0,T_1}) \in \mathcal{V}$ for all $t \in [T_1, T_2]$, $h(x) > \delta$ for all $x \in \mathcal{V}$ and $\gamma t < \gamma T_2$ for all $t \in [T_1, T_2]$.

We have now shown that the following holds: $H_{T_1}(x_0) \geqslant 0$ implies $H_{T_2}(x_0) \geqslant 0$, and thus $\mathcal{C}_{0,T_1} \subseteq \mathcal{C}_{0,T_2}$; moreover, as for all $x_0 \in \mathcal{D}$ with $H_{T_1}(x_0) < \delta - \gamma T_2$, and in particular for all $x_0 \in \mathcal{D}$ with $H_{T_1} \leqslant 0$, it holds $H_{T_1}(x_0) \leqslant H_{T_2}(x_0)$ as by (46), we conclude that also $\mathcal{C}_{\lambda,T_1} \subseteq \mathcal{C}_{\lambda,T_2}$ for all $\lambda \geqslant 0$ with $\mathcal{C}_{\lambda,T_1}, \mathcal{C}_{\lambda,T_2} \subseteq \mathcal{D}$. □