# No-Regret Model Predictive Control with Online Learning of Koopman Operators

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Abstract—We study a problem of simultaneous system identification and model predictive control of nonlinear systems. Particularly, we provide an algorithm for systems with unknown residual dynamics that can be expressed by Koopman operators. Such residual dynamics can model external disturbances and modeling errors, such as wind and wave disturbances to aerial and marine vehicles, or inaccurate model parameters. The algorithm has finite-time near-optimality guarantees and asymptotically converges to the optimal non-causal controller. Specifically, the algorithm enjoys sublinear dynamic regret, defined herein as the suboptimality against an optimal clairvoyant controller that knows how the unknown dynamics will adapt to its states and actions. To this end, we assume the algorithm is given Koopman observable functions such that the unknown dynamics can be approximated by a linear dynamical system. Then, it employs model predictive control based on the current learned model of the unknown residual dynamics. This model is updated online using least squares in a self-supervised manner based on the data collected while controlling the system. We validate our algorithm in physics-based simulations of a cart-pole system aiming to maintain the pole upright despite inaccurate model parameters.

#### I. INTRODUCTION

In the future, mobile robots will leverage their on-board control capabilities to complete tasks such as package delivery [1], target tracking [2], and inspection and maintenance [3]. Such tasks require accurate and efficient tracking control under uncertainty, particularly, under unknown dynamics and external disturbances. This is challenging since the uncertainty is versatile across different environments and is possibly adaptive to robots' actions and states. Examples of such tasks are: quadrotors to (i) pick up and carry packages of unknown weight, (ii) chase a moving target at high speeds where the induced aerodynamic drag is hard to model, and (iii) inspect and maintain outdoor infrastructure exposed to turbulence and wind gusts.

State-of-the-art methods for control under unknown dynamics and disturbances include: robust control [4]–[8]; adaptive control and disturbance compensation [9]–[13]; and online learning [14]–[21]. The robust control methods, given a known upper bound on the magnitude of the noise, can be conservative due to assuming worse-case disturbance realization [22], instead of planning based on an accurate predictive model of the disturbance. Similarly, the adaptive control and the online learning control methods may exhibit sub-optimal performance due to only reacting to the history



Fig. 1: Overview of Pipeline for Model Predictive Control with Online Learning of Koopman Operator. The pipeline is composed of two interacting modules: (i) a model predictive control (MPC) module, and (ii) an online Koopman learning module with predefined Koopman observable functions. The MPC module uses the estimated unknown dynamics from the Koopman learning module to calculate the next control input. Given the control input and the observed new state, the online Koopman learning module then updates the estimate of the unknown dynamics.

of observed disturbances, instead of planning based on an accurate predictive model of the disturbance [9], [10], [14]. A relevant work on simultaneous system identification and model predictive control is [21], where the model of unknown dynamics and disturbances are learned by random Fourier features [23]. The method is suitable when we have no prior knowledge about the unknown dynamics, *i.e.*, no knowledge about what features are representative to approximating the unknown dynamics, since random Fourier features can be sampled from predefined distributions. In the case where we can identify suitable features, control performance can be benefited by using such features to learn unknown dynamics.

In this paper, we leverage the success of Koopman operator in modeling and learning of nonlinear dynamics [24]. Koopman operator represents nonlinear dynamics as a (potentially infinite-dimensional) linear system by evolving functions of the state of interest, i.e., Koopman observables, in time. Such representations can be incrementally updated computationally efficiently, enabling online learning of Koopman operator. Moreover, due to the linear representation, the learned model can be incorporated into model predictive control for real-time applications. Therefore, we propose a self-supervised method to learn online a predictive model of the unknown uncertainties approximated by Koopman operator (Fig. 1). The proposed method promises to enable: one-shot online learning (as opposed to offline or episodic learning); online adaptation to the actual disturbance realization (as opposed to the worst-case);

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and control planned over a look-ahead horizon of predicted system dynamics and disturbances (as opposed to the their past). We elaborate on our contributions next.

**Contributions.** We provide a real-time and asymptotically optimal algorithm for the simultaneous Koopman learning and control of nonlinear systems. The algorithm is self-supervised and learns unknown residual dynamics that can be expressed by Koopman operator. Specifically, the algorithm is composed of two interacting modules (Fig. 1): (i) a Model Predictive Control (MPC) module, and (ii) an online Koopman learning module with pre-specified Koopman observable functions. At each time step, the MPC module uses the estimated unknown residual dynamics from the Koopman learning module to calculate the next control input. Given the control input and the observed new state, the online Koopman learning module then updates the estimate of the unknown dynamics by least-squares estimation via online gradient descent (OGD) [25].

The algorithm has *no dynamic regret*, that is, it asymptotically matches the performance of the optimal controller that knows a priori the unknown dynamics.<sup>1</sup> Particularly, we provide the following finite-time performance guarantee for Algorithm 1 (Theorem 1):

Regret of MPC 
$$\leq \mathcal{O}\left(T^{\frac{3}{4}}\right)$$
.

**Numerical Evaluations.** We validate the algorithm in physics-based simulations in simulated scenarios of a cartpole system that aims to stabilize at a setpoint despite inaccurate model parameters (Section VI). We compare our algorithm with a nominal MPC (Nominal MPC) that ignores the unknown dynamics or disturbances, the Gaussian process MPC (GP-MPC) [26], and the MPC with random Fourier features (RFF-MPC) [21]. Our method achieves better tracking performance than GP-MPC and RFF-MPC.

#### II. RELATED WORK

We discuss work on adaptive control; robust control; nonstochastic control; and Koopman operator for control.

Adaptive control: Adaptive control methods often assume parametric uncertainty additive to the known system dynamics, *e.g.*, parametric uncertainty in the form of unknown coefficients multiplying known basis functions [9], [10], or directly estimate the value of unknown disturbances [11]–[13]. They update online the coefficients or the value of unknown disturbances, and generate an adaptive control input to compensate for the estimated disturbances. In contrast, we leverage the Koopman operator to learn a model of the residual dynamics online to enable adaptive model predictive control. *Robust control:* Robust control algorithms select control inputs based on the assumption of worst-case realization of disturbances with an upper bound of its magnitude [4]–[8]. However, assuming the worst-case disturbances can be conservative. In this paper, instead, we incorporate a learned model of disturbances to reduce conservativeness caused by the worst-case disturbances assumption.

Non-stochastic control: Online learning algorithms, also known as non-stochastic control algorithms, provide controllers with bounded regret guarantees, upon employing the OCO framework to capture the control problem as a sequential game between a controller and an adversary [14]– [20]. They typically select control inputs based on past information [14]-[19], instead of using MPC as in this paper. [20] provides a MPC method for target tracking, focusing on learning unknown trajectories of the target, as opposed to learning unknown dynamics/disturbances in our paper. Additionally, it focuses on linear systems and linear MPC. Instead, we utilize nonlinear MPC for controlaffine systems. [21] learns the residual dynamics with random Fourier features [23] for MPC, where those features are sampled randomly from prespecified distributions. The method is suitable when we have no prior knowledge about the unknown dynamics and what features are representative to approximating the unknown dynamics. By contrast, we leverage Koopman operator theory to learn residual dynamics with properly selected Koopman observable functions, and demonstrate better performance than [21].

*Koopman Operator for control:* Koopman operator [24] represents a nonlinear dynamical system as a (potentially infinite-dimensional) linear system by evolving functions of the state, namely, observables, in time. Such Koopman dynamics can be found through data-driven methods [27]-[29] that generate a finite-dimensional approximation to the theoretical infinite-dimensional Koopman operator. Their methods typically aim to represent the whole system dynamics by Koopman operator learned offline [30]-[36], while we focus only on the online learning of residual dynamics. We can enable online adaptation in the case of distribution shift between the data collected offline and encountered during employment, upon combining with those methods of offline Koopman operator learning (Remark 3). We also provide a no-dynamic-regret guarantee for the provided simultaneous Koopman learning and MPC algorithm.

# III. MODEL PREDICTIVE CONTROL WITH ONLINE LEARNING OF KOOPMAN OPERATOR

We formulate the problem of *Model Predictive Control* with Online Learning of Koopman Operator (Problem 1). To this end, we use the following framework and assumptions.

**Control-Affine Dynamics.** We consider control-affine system dynamics of the form

$$x_{t+1} = f(x_t) + g(x_t)u_t + w_t, \quad t \ge 1,$$
(1)

where  $x_t \in \mathbb{R}^{d_x}$  is the state,  $u_t \in \mathbb{R}^{d_u}$  is the control input,  $f : \mathbb{R}^{d_x} \to \mathbb{R}^{d_x}$ ,  $g : \mathbb{R}^{d_x} \to \mathbb{R}^{d_x} \times \mathbb{R}^{d_u}$  are known locally Lipschitz functions,  $w_t \triangleq h(w_{t-1}, z_t) : \mathbb{R}^{d_x} \times \mathbb{R}^{d_z} \to \mathbb{R}^{d_x}$  is an

<sup>&</sup>lt;sup>1</sup>This definition of dynamic regret differs from the standard one in the non-stochastic control literature, *e.g.*, [14]–[17], where the optimal controller and the Algorithm experienced the same realization of disturbances, instead of different realizations given that the unknown dynamics can be adaptive to states and actions (Remark 2).

unknown locally Lipschitz function with bounded magnitude, and  $z_t \in \mathbb{R}^{d_z}$  is a vector of features chosen as a subset of  $[x_t^\top \ u_t^\top]^\top$ .

We refer to the undisturbed  $x_{t+1} = f(x_t) + g(x_t) u_t$  as the *nominal dynamics*, and  $w_t$  or h as the *residual dynamics*.  $w_t$  represents unknown disturbances or system dynamics. Examples of such unknown disturbances or system dynamics are given in [37].

**Model Predictive Control** (MPC). MPC selects a control input  $u_t$  by simulating the nominal system dynamics over a look-ahead horizon N, *i.e.*, MPC selects  $u_t$  by solving the optimization problem [38]:

$$\min_{u_t, \dots, u_{t+N-1}} \sum_{k=t}^{t+N-1} c_k(x_k, u_k)$$
(2a)

subject to 
$$x_{k+1} = f(x_k) + g(x_k) u_k,$$

$$u_k \in \mathcal{U}, \quad k \in \{t, \dots, t+N-1\}, \quad (2c)$$

(2b)

where  $c_t(\cdot, \cdot) : \mathbb{R}^{d_x} \times \mathbb{R}^{d_u} \to \mathbb{R}$  is the cost function, and  $\mathcal{U}$  is a compact set that represents constraints on the control input due to, *e.g.*, controller saturation.

The optimization problem in eq. (2) ignores the residual dynamics h. To improve performance in the presence of h, in this paper we propose a method to estimate h online so eq. (2) can be adapted to the optimization problem:

$$\min_{u_t, \dots, u_{t+N-1}} \sum_{k=t}^{t+N-1} c_k(x_k, u_k)$$
(3a)

subject to 
$$x_{k+1} = f(x_k) + g(x_k) u_k + \hat{w}_t$$
, (3b)

$$u_k \in \mathcal{U}, \quad k \in \{t, \dots, t+N-1\}, \quad (3c)$$

where  $\hat{w}_t$  is the estimate of  $w_t$ . Specifically,  $\hat{w}_t \triangleq h(\cdot, \cdot; \hat{\alpha})$  where  $\hat{\alpha}$  is the parameter that is updated online by our proposed method to improve the control performance.

We define the notion of MPC's *value function* and state the assumption on the cost function and value function.

**Definition 1** (Value Function [39]). Given a state x and parameter  $\hat{\alpha}$ , the value function  $V_t(x; \hat{\alpha})$  is defined as the optimal value of eq. (4):

$$\min_{u_{t}, \dots, u_{t+N-1}} \sum_{k=t}^{t+N-1} c_{k}(x_{k}, u_{k})$$
(4a)

subject to 
$$x_{k+1} = f(x_k) + g(x_k)u_k + \hat{w}_t$$
, (4b)

$$x_t = x, \ u_k \in \mathcal{U}, \ k \in \{t, \dots, t+N-1\}.$$
(4c)

Assumption 1 (Bounds on Cost Function and Value Function [39]). There exist positive scalars  $\underline{\lambda}$ ,  $\overline{\lambda}$ , and a continuous function  $\sigma : \mathbb{R}^{d_x} \to \mathbb{R}_+$ , such that (i)  $c_t(x, u) \geq \underline{\lambda}\sigma(x)$ ,  $\forall x, u, t$ ; (ii)  $V_t(x; \hat{\alpha}) \leq \overline{\lambda}\sigma(x)$ ,  $\forall x, t$ , and (iii)  $\lim_{\|x\|\to\infty} \sigma(x) \to \infty$ .

Under Assumption 1, the MPC policy in eq. (3) can be proved to ensure that the system in eq. (3b) is globally asymptotic stable [39].

A cost function that satisfies Assumption 1 is the quadratic cost  $c_t(x_t, u_t) = x_t Q x_t^\top + u_t R u_t^\top$  when, for example, the system dynamics is linear [39, Lemma 1], or when the quadratic cost is (exponentially/asymptotically) controllable to zero with respect to  $\sigma : \mathbb{R}^{d_x} \to \mathbb{R}_+$  [39, Section. III].

**Assumption 2** (Lipschitzness). We Assume that  $c_t(x, u)$  is locally Lipschitz in x and u,  $\hat{h}(\cdot)$  is locally Lipschitz in  $\hat{\alpha}$ .

Assumption 2 will be used to establish the Lipschitzness of the value function  $V_t(x; \hat{\alpha})$  with respect to the initial state x and parameter  $\hat{\alpha}$ .

**Koopman operator.** A Koopman operator [24] represents a nonlinear dynamical system h by a (potentially infinitedimensional) linear system. Consider first the residual dynamics without dependence on  $z_t: w_t \triangleq h(w_{t-1})$ ; and define the observable function  $\Phi \in \mathcal{O}$ , where  $\mathcal{O}$  is the infinitedimensional function space of all observation functions. Then, the Koopman operator  $\kappa : \mathcal{O} \to \mathcal{O}$  is an operator acting on  $\Phi$  such that

$$\Phi\left(w_{t+1}\right) = \kappa \Phi\left(w_t\right). \tag{5}$$

Typically,  $\kappa$  cannot be implemented due to infinite dimensionality. Instead, a finite subspace approximation  $A_k \in \mathbb{R}^{d_{\Phi}} \times \mathbb{R}^{d_{\Phi}}$  acting on a subspace  $\mathcal{F} \subset \mathcal{O}$  is used such that

$$\Phi\left(w_{t+1}\right) = A_k \Phi\left(w_t\right) + r,\tag{6}$$

where  $r \in \mathcal{O}$  is the error due to a finite dimensional approximation of  $\kappa$ . In principle, the error  $r \to 0$  as  $\mathcal{F} \to \mathcal{O}$  [40], [41]. If  $\mathcal{F}$  is an invariant subspace, then r can be zero [42].

To recover  $w_t$  from  $\Phi(w_t)$ , we often use  $w_t$  as part of observables such that  $w_t = C_k \Phi(w_t)$ , where  $C_k = [\mathbf{I}_{n_x}, \mathbf{0}]$ .

In the case where h also depends on  $z_t$ , eq. (6) can be extended as [35], [37]

$$\Phi(w_{t+1}) = A_k \Phi(w_t) + B_k \Psi(w_t, z_{t+1}) + r.$$
(7)

We assume the following for the residual dynamics h.

Assumption 3 (Linear Representation of h by Koopman Operator). There exist nonlinear functions  $\Phi : \mathbb{R}^{d_x} \to \mathbb{R}^{d_{\Phi}}$ and  $\Psi : \mathbb{R}^{d_x} \times \mathbb{R}^{d_z} \to \mathbb{R}^{d_{\Psi}}$  such that

$$\Phi(w_t) = A_k \Phi(w_{t-1}) + B_k \Psi(w_{t-1}, z_t), \qquad (8)$$

where  $A_k : \mathbb{R}^{d_{\Phi}} \times \mathbb{R}^{d_{\Phi}}$ ,  $B_k : \mathbb{R}^{d_{\Phi}} \times \mathbb{R}^{d_{\Psi}}$ , and  $\Phi(\cdot)$  are locally Lipschitz, and  $\Psi(\cdot, \cdot)$  is uniformly bounded.

We require  $\Psi(\cdot, \cdot)$  to be uniformly bounded as we assume that  $w_t$  is bounded, and therefore  $\Phi(w_t)$  is bounded, which means  $\Psi(\cdot, \cdot)$  is also bounded. Examples of such function include, *e.g.*,  $\sin(\cdot)$ ,  $\cos(\cdot)$ , and  $\tanh(\cdot)$ .

Assumption 3 assumes that r = 0 and that we can represent h by a linear model given in eq. (7).

**Remark 1** (Approximation Error). When the approximation error  $r \neq 0$ , we can generalize the regret guarantee in Theorem 1 using [21, Corollary 1] such that it depends on the approximation error.

**Control Performance Metric.** We design  $u_t$  to ensure a control performance that is comparable to an optimal clair-voyant (non-causal) policy that knows the residual dynamics a priori. Particularly, we consider the metric below.

**Definition 2** (Dynamic Regret). Assume a total time horizon of operation T, and loss functions  $c_t$ , t = 1, ..., T. Then, dynamic regret is defined as

$$\operatorname{Regret}_{T}^{D} = \sum_{t=1}^{T} c_t \left( x_t, u_t, w_t \right) - \sum_{t=1}^{T} c_t \left( x_t^{\star}, u_t^{\star}, w_t^{\star} \right), \quad (9)$$

where the cost  $c_t$  explicitly dependent on the residual dynamics  $w_t$ ,  $u_t^*$  is the optimal control input in hindsight, i.e., the optimal (non-causal) input given a priori knowledge of the unknown h, and  $x_{t+1}^*$  is the state reached by applying the optimal control inputs  $(u_1^*, \ldots, u_t^*)$ , and  $w_t^*$  is the residual dynamics experienced by the optimal controller.

**Remark 2** (Adaptivity of h). In the definition of regret in eq. (9), h adapts (possibly differently) to the state and control sequences  $(x_1, u_1), \ldots, (x_T, u_T)$  and  $(x_1^*, u_1^*), \ldots, (x_T^*, u_T^*)$  since h is a function of the state and the control input. This is in contrast to previous definitions of dynamic regret, e.g., [14]–[17] and references therein, where the optimal state  $x_{t+1}^*$  is reached given the same realization of w as of  $x_{t+1}$ , i.e.,  $x_{t+1}^* = f(x_t^*) + g(x_t^*) u_t^* + w_t$ .

**Problem 1** (Model Predictive Control with Online Learning of Koopman Operator). At each t = 1, ..., T, estimate the residual dynamics function h, and identify a control input  $u_t$  by solving eq. (3), such that  $\operatorname{Regret}_T^D$  is sublinear.

A sublinear regret means  $\lim_{T\to\infty} \operatorname{Regret}_T^D/T \to 0$ , which implies the algorithm asymptotically converges to the optimal (non-causal) controller.

# IV. Algorithm

We present the algorithm for Problem 1 (Algorithm 1). The algorithm is sketched in Figure 1. The algorithm is composed of two interacting modules: (i) an MPC module, and (ii) an online Koopman operator learning module. At each  $t = 1, 2, \ldots$ , the MPC module uses the estimated h from the Koopman operator learning module to calculate the control input  $u_t$ . Given the current control input  $u_t$  and the observed new state  $x_{t+1}$ , the Koopman operator learning module updates the estimate h. To this end, it employs online least-squares estimation via online gradient descent, where h is parameterized as a linear system by Koopman operator.

To present the algorithm, we first next provide background information on online gradient descent for estimation.

#### A. Online Least-Squares Estimation

Given a data point  $(w_{t-1}, z_t, w_t)$  observed at time t, we employ an online least-squares algorithm that updates the parameters  $\hat{\alpha}_t \triangleq \begin{bmatrix} \hat{A}_{k,t}, \hat{B}_{k,t} \end{bmatrix}$  to minimize the approximation error  $l_t = \|\Phi(w_t) - \Phi(\hat{w}_t)\|_2^2$ , where  $\Phi(\hat{w}_t) = \hat{A}_{k,t}\Phi(w_{t-1}) + \hat{B}_{k,t}\Psi(w_{t-1}, z_t)$ . Specifically, the algorithm uses the online gradient descent algorithm (OGD) [25]. At each  $t = 1, \ldots, T$ , it makes the steps: **Algorithm 1:** Simultaneous Koopman Learning and Model Predictive Control (Koopman-MPC).

**Input:** Koopman observable functions  $\Phi$  and  $\Psi$ ; domain set  $\mathcal{D}$ ; gradient descent learning rate  $\eta$ .

**Output:** At each time step t = 1, ..., T, control input  $u_t$ .

- 1: Initialize  $x_1, \hat{\alpha}_1 \in \mathcal{D};$
- 2: for each time step  $t = 1, \ldots, T$  do
- 3: Apply control input  $u_t$  by solving eq. (3) with  $h(\cdot, \cdot; \hat{\alpha}_t) \triangleq C_k \left( \hat{A}_{k,t} \Phi(\cdot) + \hat{B}_{k,t} \Psi(\cdot, \cdot) \right);$
- 4: Observe state  $x_{t+1}$ , and calculate residuals via  $w_t = x_{t+1} f(x_t) g(x_t)u_t$ ;
- 5: Formulate estimation loss  $l_{t}(\hat{\alpha}_{t}) \triangleq \left\| \Phi(w_{t}) - \hat{A}_{k,t} \Phi(w_{t-1}) - \hat{B}_{k,t} \Psi(w_{t-1}, z_{t}) \right\|^{2};$ 6: Calculate gradient  $\nabla_{t} \triangleq \nabla_{\hat{\alpha}_{t}} l_{t}(\hat{\alpha}_{t});$
- 7: Update  $\hat{\alpha}'_{t+1} = \hat{\alpha}_t \eta \nabla_t$ ;
- 8: Project  $\hat{\alpha}'_{t+1}$  onto  $\mathcal{D}$ , *i.e.*,  $\hat{\alpha}_{t+1} = \Pi_{\mathcal{D}}(\hat{\alpha}'_{t+1});$

9: end for

• Given  $(w_{t-1}, z_t, w_t)$ , formulate the estimation loss function (approximation error):

$$l_{t}\left(\hat{\alpha}_{t}\right) \triangleq \left\|\Phi\left(w_{t}\right) - \hat{A}_{k,t}\Phi\left(w_{t-1}\right) - \hat{B}_{k,t}\Psi\left(w_{t-1}, z_{t}\right)\right\|^{2}$$

• Calculate the gradient of  $l_t(\hat{\alpha}_t)$  with respect to  $\hat{\alpha}_t$ :

$$\nabla_t \triangleq \nabla_{\hat{\alpha}_t} l_t \left( \hat{\alpha}_t \right).$$

• Update using gradient descent with learning rate  $\eta$ :

$$\hat{\alpha}_{t+1}' = \hat{\alpha}_t - \eta \nabla_t$$

• Project  $\hat{\alpha}_{t+1}'$  onto a convex and compact domain set  $\mathcal{D}$ :

$$\hat{\alpha}_{t+1} = \prod_{\mathcal{D}} (\hat{\alpha}'_{t+1}) \triangleq \underset{\alpha \in \mathcal{D}}{\operatorname{argmin}} \|\alpha - \hat{\alpha}'_{t+1}\|_{2}^{2}.$$

The above online least-squares estimation enjoys an  $\mathcal{O}\left(\sqrt{T}\right)$  regret bound, per the regret bound of OGD [25].

**Proposition 1** (Regret Bound of Online Least-Squares Estimation [25]). Assume  $\eta = O\left(1/\sqrt{T}\right)$ . Then,

$$\operatorname{Regret}_{T}^{S} \triangleq \sum_{t=1}^{T} l_{t}\left(\alpha_{t}\right) - \sum_{t=1}^{T} l_{t}\left(\alpha^{\star}\right) \leq \mathcal{O}\left(\sqrt{T}\right), \quad (10)$$

where  $\alpha^* \triangleq \underset{\alpha \in \mathcal{D}}{\operatorname{argmin}} \sum_{t=1}^{T} l_t(\alpha)$  is the optimal parameter that achieves lowest cumulative loss in hindsight.

The online least-squares estimation algorithm thus asymptotically achieves the same estimation error as the optimal parameter  $\alpha^*$  since  $\lim_{T\to\infty} \text{Regret}_T^S/T = 0$ .

#### B. Algorithm for Problem 1

We describe the algorithm for Problem 1. The pseudocode is in Algorithm 1. The algorithm is composed of three steps: initialization, control, and estimation. The control and estimation steps are interacting and influence each other at each time step (Fig. 1):

Initialization step: Algorithm 1 first initializes the system state x₁ and parameter â₁ ∈ D (line 1).

- Control steps: Then, at each t, given the current estimate  $h(\cdot, \cdot; \hat{\alpha}_t) \triangleq C_k \left( \hat{A}_{k,t} \Phi(\cdot) + \hat{B}_{k,t} \Psi(\cdot, \cdot) \right)$ , Algorithm 1 applies the control inputs  $u_t$  by solving eq. (3) (line 3).
- Estimation steps: The system then evolves to state  $x_{t+1}$ , and,  $w_t$  is calculated upon observing  $x_{t+1}$  (line 4). Afterwards, the algorithm formulates the loss  $l_t(\hat{\alpha}_t) \triangleq \left\| \Phi(w_t) - \hat{A}_{k,t} \Phi(w_{t-1}) - \hat{B}_{k,t} \Psi(w_{t-1}, z_t) \right\|^2$  (line 5), and calculates the gradient  $\nabla_t \triangleq \nabla_{\hat{\alpha}_t} l_t(\hat{\alpha}_t)$  (line 6). Algorithm 1 then updates the parameter  $\hat{\alpha}_t$  to  $\hat{\alpha}'_{t+1}$  (line 8) and projects  $\hat{\alpha}'_{t+1}$  back to the domain  $\mathcal{D}$  (line 9).

**Remark 3** (Combination with Offline Learned Koopman Operator). The proposed online learning method can be combined with offline learning of Koopman operator. For example, we can learn a linear representation of the nominal dynamics as  $\bar{\Phi}(x_{t+1}) = \bar{A}_k \bar{\Phi}(x_t) + \bar{B}_k u_k$  [33] and use OGD to learn online the residual dynamics in the lift space of the form  $\Delta \bar{A}_k \bar{\Phi}(x_t) + \Delta \bar{B}_k u_k$ . To obtain Theorem 1, we require Assumption 1, Assumption 2, and uniform boundedness of  $x_t$ . We replace Assumption 3 by the assumption of uniform boundedness of  $x_t$  since  $\bar{\Phi}(\cdot)$  typically is not an uniformly bounded function and can violate the assumption of bounded  $w_t$ , which may cause the loss and gradient (lines 5-6 in Algorithm 1) to be unbounded.

## V. NO-REGRET GUARANTEE

We present the sublinear regret bound of Algorithm 1.

**Theorem 1** (No-Regret). Assume Algorithm 1's learning rate is  $\eta = O\left(1/\sqrt{T}\right)$ . Then, Algorithm 1 achieves

$$\operatorname{Regret}_{T}^{D} \leq \mathcal{O}\left(T^{\frac{3}{4}}\right). \tag{11}$$

The proof follows the steps in the proof of [21, Theorem 1]

Theorem 1 serves as a finite-time performance guarantee as well as implies that Algorithm 1 converges to the optimal (non-causal) control policy since  $\lim_{T\to\infty} \text{Regret}_T^D/T \to 0$ .

## VI. NUMERICAL EVALUATIONS

We evaluate Algorithm 1 in simulated scenarios of control under uncertainty, where the controller aims to track a reference setpoint despite unknown residual dynamics. Specifically, we consider a cart-pole aiming to stabilize around a setpoint despite inaccurate model parameters, *i.e.*, inaccurate cart mass, pole mass, and pole length.

**Simulation Setup.** We consider a cart-pole system, where a cart of mass  $m_c$  connects via a prismatic joint to a 1Dtrack, while a pole of mass  $m_p$  and length 2l is hinged to the cart. The state vector **x** includes the horizontal position of the cart x, the velocity of the cart  $\dot{x}$ , the angle of the pole with respect to vertical  $\theta$ , and the angular velocity of the pole  $\dot{\theta}$ . The control input is the force F applied to the center of mass of the cart. The goal of the cart-pole is to stabilize







(b) Sample trajectory.



Fig. 2: Simulation Results of the Cart-Pole Stabilization Experiment under 25% Inaccurate Model Parameters. (a) Average stabilization error over 20 runs with random initialization. (b) Sample trajectory. The results demonstrate that Algorithm 1 (Koopman-MPC) achieves the fastest stabilization of the system among all tested algorithms. (c) Estimation error of the residual dynamics. The results demonstrate that the quick convergence of online learning of the Koopman operator with appropriately chosen observables.



Fig. 3: Sample Trajectory of the Cart-Pole Stabilization Experiment under 45% Inaccurate Model Parameters. Algorithm 1 (Koopman-MPC) successfully achieves stabilization of the system despite 45% inaccuracy of the nominal model, while GP-MPC and RFF-MPC fail.

at 
$$(x, \dot{x}, \theta, \theta) = (0, 0, 0, 0)$$
. The cart-pole dynamics are:

$$\ddot{x} = \frac{m_p l \left(\dot{\theta}^2 \sin \theta - \ddot{\theta} \cos \theta\right) + F}{m_c + m_p},$$

$$\ddot{\theta} = \frac{g \sin \theta + \cos \theta \left(\frac{-m_p l \dot{\theta}^2 \sin \theta - F}{m_c + m_p}\right)}{l \left(\frac{4}{3} - \frac{m_p \cos^2 \theta}{m_c + m_p}\right)},$$
(12)

where g is the acceleration of gravity.

To control the system, we will employ MPC at 15Hzwith a look-ahead horizon N = 20. We use quadratic cost functions with Q = diag([5.0, 0.1, 5.0, 0.1]) and R = 0.1. We use the fourth-order Runge-Kutta method for discretizing the above dynamics. The true system parameters are  $m_c =$  $1.0, m_p = 0.1$ , and l = 0.5, but the parameters for the nominal dynamics are scaled to 75% and 55% of the said true values, which corresponding to 25% and 45% inaccuracy.

We use  $\Phi(w_t) = w_t$ ,  $\Psi(w_{t-1}, z_t) = [\tanh x_t, \tanh \dot{x}_t, \tanh \theta_t, \tanh \dot{\theta}_t, \tanh \dot{\theta}_t, u_t, \tanh \theta_t \tanh \dot{\theta}_t, (\tanh \dot{\theta}_t)^2, u_t \tanh \theta_t, u_t \tanh \dot{\theta}_t]^\top$ ,  $\eta = 0.01$ , and initialize  $\hat{\alpha}$  as zero. We simulate the setting for 6 seconds, performing the simulation 20 times with random initialization of the state sampled uniformly from  $x \in [-1, 1]$ ,  $\dot{x} \in [-0.1, 0.1]$ ,  $\theta \in [-0.2, 0.2]$ ,  $\dot{\theta} \in [-0.1, 0.1]$ .

We use the physics-based simulation environment from [43] in PyBullet [44].

**Compared Algorithms.** We compare Algorithm 1 (Koopman-MPC) with an MPC that uses the nominal system parameters (Nominal MPC), the Gaussian process MPC (GP-MPC) [26], and MPC with random Fourier Features (RFF-MPC) [21]. The Nominal MPC uses the nominal dynamics to select control input by

solving eq. (2). The GP-MPC learns  $\hat{h}(\cdot)$  with a sparse Gaussian process (GP) [45] whose data points are collected online, *i.e.*, GP fixes its hyperparameters and collects data points ( $w_{t-1}$ ,  $z_t$ ,  $w_t$ ) online. The RFF-MPC learns  $\hat{h}(\cdot)$  with random Fourier features [23] also with data points ( $w_{t-1}$ ,  $z_t$ ,  $w_t$ ) collected online.

**Performance Metric.** We evaluate the performance of Nominal MPC, GP-MPC, RFF-MPC, and Algorithm 1 in terms of their stabilization error  $\|\mathbf{x}_t\|^2$ .

**Results.** The results are given in Figure 2 and Figure 3.

In the case of 25% inaccurate model parameters (Figure 2), we observe that Algorithm 1 achieves stabilization the fastest. RFF-MPC comes second, possibly because it estimates residual dynamics with randomly sampled features while our method uses suitable observables which enables better learning of residual dynamics (Figure 2(c)). GP-MPC is able to stabilize the system at the end but it incurs a larger deviation during the transition period from the stabilization goal (0, 0, 0, 0) than Algorithm 1.

In the case of 45% inaccurate model parameters (Figure 3), Algorithm 1 successfully stabilizes the system in all 20 runs, while GP-MPC and RFF-MPC fail. The result demonstrates robustness of Algorithm 1 under such extreme disturbances.

# VII. CONCLUSION

Summary. We provided Algorithm 1 for the problem of Model Predictive Control with Online Learning of Koopman Operator (Problem 1). Algorithm 1 guarantees no-regret against an optimal clairvoyant policy that knows the residual dynamics h a priori. (Theorem 1). The algorithm uses Koopman operator to approximate the residual dynamics. Then, it employs model predictive control based on the current learned model of h. The model of the unknown dynamics is updated online in a self-supervised manner using least squares based on the data collected while controlling the system. We validate Algorithm 1 in physics-based PyBullet simulations of a cart-pole aiming to maintain the pole upright despite inaccurate model parameters (Section VI). We demonstrate that our method achieves better tracking performance than the state-of-the-art methods GP-MPC [26] and RFF-MPC [21].

**Future Work.** The Koopman observable functions in the paper are manually selected, and may not generalize to settings with different unknown dynamics or disturbances. We expect this can be resolved by using meta-learning (with neural networks) to automate the discovery of Koopman observable functions that can generalize to different  $h(\cdot)$ .

#### REFERENCES

- E. Ackerman, "Amazon promises package delivery by drone: Is it for real?" *IEEE Spectrum, Web*, 2013.
- [2] J. Chen, T. Liu, and S. Shen, "Tracking a moving target in cluttered environments using a quadrotor," in 2016 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS). IEEE, 2016, pp. 446–453.
- [3] D. Seneviratne, L. Ciani, M. Catelani, D. Galar *et al.*, "Smart maintenance and inspection of linear assets: An industry 4.0 approach," *Acta Imeko*, vol. 7, pp. 50–56, 2018.

- [4] D. Q. Mayne, M. M. Seron, and S. Raković, "Robust model predictive control of constrained linear systems with bounded disturbances," *Automatica*, vol. 41, no. 2, pp. 219–224, 2005.
- [5] G. Goel and B. Hassibi, "Regret-optimal control in dynamic environments," *arXiv preprint:2010.10473*, 2020.
- [6] A. Martin, L. Furieri, F. Dörfler, J. Lygeros, and G. Ferrari-Trecate, "On the guarantees of minimizing regret in receding horizon," *IEEE Transactions on Automatic Control*, 2024.
- [7] A. Didier, J. Sieber, and M. N. Zeilinger, "A system level approach to regret optimal control," *IEEE Control Systems Letters (L-CSS)*, 2022.
- [8] H. Zhou and V. Tzoumas, "Safe control of partially-observed linear time-varying systems with minimal worst-case dynamic regret," in 2023 62nd IEEE Conference on Decision and Control (CDC). IEEE, 2023, pp. 8781–8787.
- [9] J.-J. E. Slotine, "Applied nonlinear control," *PRENTICE-HALL google schola*, vol. 2, pp. 1123–1131, 1991.
- [10] M. Krstic, P. V. Kokotovic, and I. Kanellakopoulos, Nonlinear and adaptive control design. John Wiley & Sons, Inc., 1995.
- [11] E. Tal and S. Karaman, "Accurate tracking of aggressive quadrotor trajectories using incremental nonlinear dynamic inversion and differential flatness," *IEEE Transactions on Control Systems Technology*, vol. 29, no. 3, pp. 1203–1218, 2020.
- [12] Z. Wu, S. Cheng, P. Zhao, A. Gahlawat, K. A. Ackerman, A. Lakshmanan, C. Yang, J. Yu, and N. Hovakimyan, "L1 quad: L1 adaptive augmentation of geometric control for agile quadrotors with performance guarantees," arXiv preprint arXiv:2302.07208, 2023.
- [13] E. Das and J. W. Burdick, "Robust control barrier functions using uncertainty estimation with application to mobile robots," *arXiv preprint arXiv:2401.01881*, 2024.
- [14] E. Hazan and K. Singh, "Introduction to online nonstochastic control," arXiv preprint arXiv:2211.09619, 2022.
- [15] H. Zhou and V. Tzoumas, "Safe non-stochastic control of linear dynamical systems," in 2023 62nd IEEE Conference on Decision and Control (CDC). IEEE, 2023, pp. 5033–5038.
- [16] H. Zhou, Y. Song, and V. Tzoumas, "Safe non-stochastic control of control-affine systems: An online convex optimization approach," *IEEE Robotics and Automation Letters*, 2023.
- [17] H. Zhou, Z. Xu, and V. Tzoumas, "Efficient online learning with memory via frank-wolfe optimization: Algorithms with bounded dynamic regret and applications to control," in 2023 62nd IEEE Conference on Decision and Control (CDC). IEEE, 2023, pp. 8266–8273.
- [18] N. M. Boffi, S. Tu, and J.-J. E. Slotine, "Regret bounds for adaptive nonlinear control," in *Learning for Dynamics and Control*. PMLR, 2021, pp. 471–483.
- [19] M. Nonhoff, J. Köhler, and M. A. Müller, "Online convex optimization for constrained control of nonlinear systems," *arXiv preprint* arXiv:2412.00922, 2024.
- [20] A. Tsiamis, A. Karapetyan, Y. Li, E. C. Balta, and J. Lygeros, "Predictive linear online tracking for unknown targets," *arXiv preprint* arXiv:2402.10036, 2024.
- [21] H. Zhou and V. Tzoumas, "Simultaneous system identification and model predictive control with no dynamic regret," *arXiv preprint* arXiv:2407.04143, 2024.
- [22] K. Zhou and J. C. Doyle, *Essentials of robust control*. Prentice hall Upper Saddle River, NJ, 1998, vol. 104.
- [23] A. Rahimi and B. Recht, "Random features for large-scale kernel machines," Advances in neural information processing systems, vol. 20, 2007.
- [24] B. O. Koopman, "Hamiltonian systems and transformation in hilbert space," *Proceedings of the National Academy of Sciences*, vol. 17, no. 5, pp. 315–318, 1931.
- [25] E. Hazan et al., "Introduction to online convex optimization," Foundations and Trends in Optimization, vol. 2, no. 3-4, pp. 157–325, 2016.
- [26] L. Hewing, J. Kabzan, and M. N. Zeilinger, "Cautious model predictive control using gaussian process regression," *IEEE Transactions on Control Systems Technology*, vol. 28, no. 6, pp. 2736–2743, 2019.
- [27] S. L. Brunton, M. Budišić, E. Kaiser, and J. N. Kutz, "Modern koopman theory for dynamical systems," arXiv preprint arXiv:2102.12086, 2021.
- [28] L. Shi, M. Haseli, G. Mamakoukas, D. Bruder, I. Abraham, T. Murphey, J. Cortes, and K. Karydis, "Koopman operators in robot learning," arXiv preprint arXiv:2408.04200, 2024.
- [29] E. Kaiser, J. N. Kutz, and S. L. Brunton, "Data-driven approximations of dynamical systems operators for control," *The Koopman operator*

in systems and control: concepts, methodologies, and applications, pp. 197–234, 2020.

- [30] S. L. Brunton, J. L. Proctor, and J. N. Kutz, "Sparse identification of nonlinear dynamics with control (sindyc)," *IFAC-PapersOnLine*, vol. 49, no. 18, pp. 710–715, 2016.
- [31] J. L. Proctor, S. L. Brunton, and J. N. Kutz, "Dynamic mode decomposition with control," *SIAM Journal on Applied Dynamical Systems*, vol. 15, no. 1, pp. 142–161, 2016.
- [32] —, "Generalizing koopman theory to allow for inputs and control," SIAM Journal on Applied Dynamical Systems, vol. 17, no. 1, pp. 909– 930, 2018.
- [33] M. Korda and I. Mezić, "Linear predictors for nonlinear dynamical systems: Koopman operator meets model predictive control," *Automatica*, vol. 93, pp. 149–160, 2018.
- [34] D. Bruder, X. Fu, R. B. Gillespie, C. D. Remy, and R. Vasudevan, "Data-driven control of soft robots using koopman operator theory," *IEEE Transactions on Robotics*, vol. 37, no. 3, pp. 948–961, 2020.
- [35] I. Abraham and T. D. Murphey, "Active learning of dynamics for data-driven control using koopman operators," *IEEE Transactions on Robotics*, vol. 35, no. 5, pp. 1071–1083, 2019.
- [36] C. Folkestad and J. W. Burdick, "Koopman nmpc: Koopman-based learning and nonlinear model predictive control of control-affine systems," in 2021 IEEE International Conference on Robotics and Automation (ICRA). IEEE, 2021, pp. 7350–7356.
- [37] J. Jia, W. Zhang, K. Guo, J. Wang, X. Yu, Y. Shi, and L. Guo, "Evolver: Online learning and prediction of disturbances for robot control," *IEEE Transactions on Robotics*, 2023.
- [38] J. B. Rawlings, D. Q. Mayne, and M. Diehl, *Model predictive control: Theory, computation, and design.* Nob Hill Publishing, 2017, vol. 2.
- [39] G. Grimm, M. J. Messina, S. E. Tuna, and A. R. Teel, "Model predictive control: for want of a local control lyapunov function, all is not lost," *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 546–558, 2005.
- [40] M. Budišić, R. Mohr, and I. Mezić, "Applied koopmanism," Chaos: An Interdisciplinary Journal of Nonlinear Science, vol. 22, no. 4, 2012.
- [41] I. Mezić, "On applications of the spectral theory of the koopman operator in dynamical systems and control theory," in 2015 54th IEEE Conference on Decision and Control (CDC). IEEE, 2015, pp. 7034– 7041.
- [42] S. L. Brunton, B. W. Brunton, J. L. Proctor, and J. N. Kutz, "Koopman invariant subspaces and finite linear representations of nonlinear dynamical systems for control," *PloS one*, vol. 11, no. 2, p. e0150171, 2016.
- [43] Z. Yuan, A. W. Hall, S. Zhou, L. Brunke, M. Greeff, J. Panerati, and A. P. Schoellig, "Safe-control-gym: A unified benchmark suite for safe learning-based control and reinforcement learning in robotics," *IEEE Robotics and Automation Letters*, vol. 7, no. 4, pp. 11142–11149, 2022.
- [44] E. Coumans and Y. Bai, "Pybullet, a python module for physics simulation for games, robotics and machine learning," 2016.
- [45] J. Quinonero-Candela and C. E. Rasmussen, "A unifying view of sparse approximate gaussian process regression," *The Journal of Machine Learning Research*, vol. 6, pp. 1939–1959, 2005.