

LIE DERIVATIVES OF SECTIONS OF NATURAL VECTOR BUNDLES

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Dedicated to the memory of Joseph A. Wolf

ABSTRACT. Time derivatives of pullbacks and push forwards along smooth curves of diffeomorphism of sections of natural vector bundles are computed in terms of Lie derivatives along adapted non-autonomous vector fields by extending a key lemma in [4]. There is also the analogous result about the first non-vanishing derivative of higher order.

1. INTRODUCTION

The following is an adaptation of the rather well known method of Lie derivation along mapping $N \rightarrow M$ as explained for differential forms in [6, 31.11] and more generally for purely covariant tensor fields in [5, 12.2–12.5]. It is used in by now classical proofs of the Poincaré lemma and the theorem of Darboux, see e.g. [7] and [9]. Namely, we prove the the following corollary 3.1 of [4, Lemma 6]; the need for this result arose during the preparation of [1].

Corollary. *Let φ_t be a smooth curve of local diffeomorphisms. Then we get two time dependent vector fields*

$$X_{t_0} = T\varphi_{t_0}^{-1} \circ \partial_t|_{t_0} \varphi_t \text{ and } Y_{t_0} = \partial_t|_{t_0} \varphi_t \circ \varphi_{t_0}^{-1}$$

Then for any natural vector bundle functor F and for any section $s \in \Gamma(F(M))$ we have the first non-vanishing derivative

$$(1) \quad \partial_t \varphi_t^* s = \varphi_t^* \mathcal{L}_{Y_t} s = \mathcal{L}_{X_t} \varphi_t^* s.$$

$$(2) \quad \partial_t (\varphi_t)_* s = \partial_t (\varphi_t^{-1})^* s = -(\varphi_t)_* \mathcal{L}_{X_t} s = -\mathcal{L}_{Y_t} (\varphi_t)_* s.$$

We also include the analogous result for the first non-vanishing derivative of higher order at fixed time points in Corollary 3.2.

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2. BACKGROUND FROM [4]

2.1. Curves of local diffeomorphisms. Let $\varphi : \mathbb{R} \times M \supset U_\varphi \rightarrow M$ be a smooth mapping where U_φ is an open neighborhood of $\{0\} \times M$ in $\mathbb{R} \times M$, such that

- $U_\varphi \cap (\mathbb{R} \times \{x\})$ is connected for each $x \in M$,
- φ_t is a diffeomorphism on its domain $U_{\varphi_t} := U_\varphi \cap (\{t\} \times M)$,
- and $\varphi_0 = Id_M$.

We say that φ_t is a *curve of local diffeomorphisms* through Id_M . From Lemma 2.2 below we see that if $\frac{\partial^j}{\partial t^j}|_0 \varphi_t = 0$ for all $1 \leq j < k$, then $X := \frac{1}{k!} \frac{\partial^k}{\partial t^k}|_0 \varphi_t$ is a well defined vector field on M . We say that X is the *first non-vanishing derivative* at 0 of the curve φ_t of local diffeomorphisms. We may paraphrase this as $(\partial_t^k|_0 \varphi_t^*)f = k! \mathcal{L}_X f$.

2.2. Lemma. *Let $c : \mathbb{R} \rightarrow M$ be a smooth curve. If $c(0) = x \in M$, $c'(0) = 0, \dots, c^{(k-1)}(0) = 0$, then $c^{(k)}(0)$ is a well defined tangent vector in $T_x M$ which is given by the derivation $f \mapsto (f \circ c)^{(k)}(0)$ at x .*

Moreover, if ψ is a local diffeomorphism defined near $x \in M$, then $\psi \circ c$ has again first non-vanishing derivative at 0 which is given by $(\psi \circ c)^{(k)} = T_x \psi \cdot c^{(k)}(0)$.

Proof. The first claim is [4, Lemma 2]. The second claim follows since tangent vectors at x are mapped by $T_x \psi$. \square

2.3. Natural vector bundles. See [2, 6.14]. Let $\mathcal{M}f_m$ denote the category of all smooth m -dimensional manifolds and local diffeomorphisms between them. A *vector bundle functor* or *natural vector bundle* is a functor F which associates a vector bundle $(F(M), p_M, M)$ to each manifold M and a vector bundle homomorphism

$$\begin{array}{ccc} F(M) & \xrightarrow{F(f)} & F(N) \\ \downarrow p_M & & \downarrow p_N \\ M & \xrightarrow{f} & N \end{array}$$

to each $f : M \rightarrow N$ in $\mathcal{M}f_m$, which covers f and is fiber wise a linear isomorphism. If f is the embedding of an open subset of N then this diagram turns out to be a pullback diagram. We also point out that $f \mapsto F(f)$ maps smoothly parameterized families to smoothly parameterized families, see [2, 14.8]. Assuming this property all vector bundle functors were classified by [8]: They correspond to linear representations of higher jet groups, they are associated vector bundles to higher order frame bundles, see also [2, 14.8].

Examples of vector bundle functors are tangent and cotangent bundles, tensor bundles, densities, $M \mapsto L(TM, TM)$, and also the trivial bundle $M \times \mathbb{R}$.

2.4. Pullback of sections. Let F be a vector bundle functor on $\mathcal{M}f_m$ as described in 2.3. Let M be an m -manifold and let φ_t be a curve of local diffeomorphisms through Id_M on M . Then φ_t , for fixed t , is a diffeomorphism defined on an open subset U_{φ_t} of M . The mapping

$$\begin{array}{ccccccc} F(M) & \xleftarrow{\supseteq} & F(U_{\varphi_t}) & \xrightarrow[\cong]{F(\varphi_t)} & F(\varphi_t(U_{\varphi_t})) & \xrightarrow{\subseteq} & F(M) \\ \downarrow p_M & & \downarrow & & \downarrow & & \downarrow p_M \\ M & \xleftarrow{\supseteq} & U_{\varphi_t} & \xrightarrow[\cong]{\varphi_t} & \varphi_t(U_{\varphi_t}) & \xrightarrow{\subseteq} & M \end{array}$$

is then a local vector bundle isomorphism.

We consider a section $s \in \Gamma(F(M))$ of the vector bundle $(F(M), p_M, M)$ and we define for $t \in \mathbb{R}$ *pullback* and *push forward* as

$$\varphi_t^* s := F(\varphi_t^{-1}) \circ s \circ \varphi_t, \quad (\varphi_t)_* s = (\varphi_t^{-1})^* s = F(\varphi_t) \circ s \circ \varphi_t^{-1}.$$

These are local sections of the bundle $F(M)$. If φ_t is smooth curve of diffeomorphisms these are global sections. For each $x \in M$ the value $(\varphi_t^* s)(x) \in F(M)_x := p_M^{-1}(x)$ is defined, if t is small enough. So in the vector space $F(M)_x$ the expression $\frac{d}{dt}|_0(\varphi_t^* s)(x)$ makes sense. These fit together to a smooth section $\frac{d}{dt}|_0(\varphi_t)_* s$ which is globally defined and is an element of $\Gamma(F(M))$, by the following argument:

For $x \in M$ there exists $\varepsilon > 0$ and an open neighborhoods $U \subset V$ of x with the closure \bar{U} compact in V such that $\varphi_t^* s$ is a smooth section in $\Gamma(F(V))$ for each $t \in (-\varepsilon, \varepsilon)$. Since $(-\varepsilon, \varepsilon) \times V \ni (t, y) \mapsto \varphi_t^* s(y) \in F(V)$ is smooth, the curve $t \mapsto \varphi_t^* s \in \Gamma(F(V))$ is smooth into the Fréchet space $\Gamma(F(V))$ by [3, Lemma 30.8.1] with derivative $\partial_t \varphi_t^* s \in \Gamma(F(V))$.

If $\varphi_t = \text{Fl}_t^X$ is the flow of a vector field X on M the section

$$\mathcal{L}_X s := \frac{d}{dt}|_0(\text{Fl}_t^X)^* s$$

is called the *Lie derivative* of s along X . The Lie derivative satisfies $\mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X = \mathcal{L}_{[X, Y]}$; see [2, 6.20].

2.5. Lemma. [4, Lemma 6] *Let φ_t be a smooth curve of local diffeomorphisms through Id_M with first non-vanishing derivative $k!X = \partial_t^k|_0 \varphi_t$. Then for any vector bundle functor F and for any section $s \in \Gamma(F(M))$ we have the first non-vanishing derivative*

$$k! \mathcal{L}_X s = \partial_t^k|_0 \varphi_t^* s.$$

3. THE RESULTS

3.1. Corollary. *Let φ_t be a smooth curve of (local) diffeomorphisms. Consider the two time dependent (locally defined) vector fields*

$$X_{t_0} = T\varphi_{t_0}^{-1} \circ \partial_t|_{t_0} \varphi_t \text{ and } Y_{t_0} = \partial_t|_{t_0} \varphi_t \circ \varphi_{t_0}^{-1}$$

Then for any vector bundle functor F and for any section $s \in \Gamma(F(M))$ we have

$$(1) \quad \partial_t \varphi_t^* s = \varphi_t^* \mathcal{L}_{Y_t} s = \mathcal{L}_{X_t} \varphi_t^* s.$$

$$(2) \quad \partial_t(\varphi_t)_* s = \partial_t(\varphi_t^{-1})^* s = -(\varphi_t)_* \mathcal{L}_{X_t} s = -\mathcal{L}_{Y_t}(\varphi_t)_* s.$$

Proof. Let $\tilde{\varphi}_t = \varphi_{t_0}^{-1} \circ \varphi_{t+t_0}$, a smooth curve of (local) diffeomorphisms through Id_M . We have

$$\partial_t|_0 \tilde{\varphi}_t = \partial_t|_0 \varphi_{t_0}^{-1} \circ \varphi_{t+t_0} = T\varphi_{t_0}^{-1} \circ \partial_t|_0 \varphi_{t+t_0} = T\varphi_{t_0}^{-1} \circ \partial_t|_{t_0} \varphi_t = X_{t_0}.$$

By Lemma 2.5 we we get that

$$\begin{aligned} \mathcal{L}_{X_{t_0}} s &= \partial_t|_{t_0} \tilde{\varphi}_t^* s = \partial_t|_{t_0} (\varphi_{t_0}^{-1} \circ \varphi_t)^* s = \partial_t|_{t_0} \varphi_t^* (\varphi_{t_0}^{-1})^* s \\ \implies \mathcal{L}_{X_{t_0}} \varphi_{t_0}^* s &= \partial_t|_{t_0} \varphi_t^* (\varphi_{t_0}^{-1})^* s \quad \text{which is part of (1).} \end{aligned}$$

For the second part of (1) we consider $\bar{\varphi}_t = \varphi_{t+t_0} \circ \varphi_{t_0}^{-1}$, another smooth curve of local diffeomorphisms through Id_M . Here we have, again by Lemma 2.5,

$$\begin{aligned} \partial_t|_0 \bar{\varphi}_t &= \partial_t|_0 \varphi_{t+t_0} \circ \varphi_{t_0}^{-1} = \partial_t|_{t_0} \varphi_t \circ \varphi_{t_0}^{-1} = Y_{t_0}. \\ \mathcal{L}_{Y_{t_0}} s &= \partial_t|_0 \bar{\varphi}_t^* s = \partial_t|_0 (\varphi_{t+t_0} \circ \varphi_{t_0}^{-1})^* s = \partial_t|_0 (\varphi_{t_0}^{-1})^* \varphi_{t+t_0}^* s \\ &= (\varphi_{t_0}^{-1})^* \partial_t|_0 \varphi_{t+t_0}^* s, \\ &\quad \text{since } (\varphi_{t_0}^{-1})^* : \Gamma(F(M)) \rightarrow \Gamma(F(M)) \text{ is bounded linear,} \\ &= (\varphi_{t_0}^{-1})^* \partial_t|_{t_0} \varphi_t^* s \text{ which implies the second part of (1).} \end{aligned}$$

To show (2) note first that

$$\begin{aligned} 0 &= \partial_t(\text{Id}) = \partial_t(\varphi_t^{-1} \circ \varphi_t) = (\partial_t \varphi_t^{-1}) \circ \varphi_t + T\varphi_t^{-1} \circ \partial_t \varphi_t \\ \partial_t(\varphi_t^{-1}) &= -T\varphi_t^{-1} \circ (\partial_t \varphi_t) \circ \varphi_t^{-1} \\ T\varphi_t \circ \partial_t(\varphi_t^{-1}) &= -(\partial_t \varphi_t) \circ \varphi_t^{-1} = -Y_t \\ (\partial_t \varphi_t^{-1}) \circ \varphi_t &= -T\varphi_t^{-1} \circ (\partial_t \varphi_t) = -X_t \end{aligned}$$

Hence, replacing φ_t by φ_t^{-1} in (1) replaces X_t by $-Y_t$ and Y_t by $-X_t$ and noting that $(\varphi_t)_* s = (\varphi_t^{-1})^* s$ transforms (1) into (2). \square

First non-vanishing derivatives of order higher than one makes sense only at discrete time points and not along whole curves of diffeomorphism which would turn out to be constant. The result is as follows.

3.2. Corollary. *Let φ_t be a smooth curve of diffeomorphisms which for a fixed time t_0 has a first non-vanishing derivative $k!\Xi = \partial_t^k|_{t_0}\varphi_t$; it is a vector field along the diffeomorphism φ_{t_0}*

$$\begin{array}{ccc} & & TM \\ & \nearrow \Xi & \downarrow \pi_M \\ M & \xrightarrow{\varphi_{t_0}} & M \end{array}$$

Consider the two (now autonomous) vector fields

$$X = T\varphi_{t_0}^{-1} \circ \Xi \text{ and } Y = \Xi \circ \varphi_{t_0}^{-1}$$

Then for any vector bundle functor F and for any section $s \in \Gamma(F(M))$ we have the first non-vanishing derivatives at time t_0

$$(3) \quad \partial_t^k|_{t_0}\varphi_t^*s = k!\varphi_{t_0}^*\mathcal{L}_Ys = k!\mathcal{L}_X\varphi_{t_0}^*s.$$

$$(4) \quad \partial_t^k|_{t_0}(\varphi_t)_*s = \partial_t^k|_{t_0}(\varphi_t^{-1})^*s = -k!(\varphi_{t_0})_*\mathcal{L}_Xs = -k!\mathcal{L}_Y(\varphi_{t_0})_*s.$$

Proof. We consider $\tilde{\varphi}_t = \varphi_{t_0}^{-1} \circ \varphi_{t+t_0}$, a smooth curve of local diffeomorphisms through Id_M . By the second part of Lemma 2.2 it has the following first non-vanishing derivatives at t_0

$$\partial_t^k|_0\tilde{\varphi}_t = \partial_t^k|_0(\varphi_{t_0}^{-1} \circ \varphi_{t+t_0}) = T\varphi_{t_0}^{-1} \circ \partial_t^k|_{t_0}\varphi_t = k!T\varphi_{t_0}^{-1} \circ \Xi = k!X.$$

Then Lemma 2.5 may be applied as follows:

$$\begin{aligned} k!\mathcal{L}_Xs &= \partial_t^k|_{t_0}\tilde{\varphi}_t^*s = \partial_t^k|_{t_0}(\varphi_{t_0}^{-1} \circ \varphi_t)^*s = \partial_t^k|_{t_0}\varphi_t^*(\varphi_{t_0}^{-1})^*s \\ \implies k!\mathcal{L}_X\varphi_{t_0}^*s &= \partial_t^k|_{t_0}\varphi_t^*(\varphi_{t_0}^{-1})^*s \quad \text{which is part of (3).} \end{aligned}$$

For the second part of (3) we consider $\bar{\varphi}_t = \varphi_{t+t_0} \circ \varphi_{t_0}^{-1}$, another smooth curve through Id_M . As above we have:

$$\begin{aligned} \partial_t^k|_0\bar{\varphi}_t &= \partial_t^k|_0\varphi_{t+t_0} \circ \varphi_{t_0}^{-1} = \partial_t^k|_{t_0}\varphi_t \circ \varphi_{t_0}^{-1} = k!\Xi \circ \varphi_{t_0} = k!Y. \\ k!\mathcal{L}_Ys &= \partial_t^k|_0\bar{\varphi}_t^*s = \partial_t^k|_0(\varphi_{t+t_0} \circ \varphi_{t_0}^{-1})^*s = \partial_t^k|_0(\varphi_{t_0}^{-1})^*\varphi_{t+t_0}^*s \\ &= (\varphi_{t_0}^{-1})^*\partial_t^k|_0\varphi_{t+t_0}^*s, \\ &\quad \text{since } (\varphi_{t_0}^{-1})^* : \Gamma(F(M)) \rightarrow \Gamma(F(M)) \text{ is bounded linear,} \\ &= (\varphi_{t_0}^{-1})^*\partial_t^k|_{t_0}\varphi_t^*s \text{ which implies the second part of (3).} \end{aligned}$$

To show (4) note first that for the first non-vanishing derivatives at t_0 we have

$$\begin{aligned} 0 &= \partial_t^k|_{t_0}(\text{Id}) = \partial_t^k|_{t_0}(\varphi_t^{-1} \circ \varphi_t) = (\partial_t^k|_{t_0}\varphi_t^{-1}) \circ \varphi_{t_0} + T\varphi_{t_0}^{-1} \circ \partial_t^k|_{t_0}\varphi_t \\ \partial_t^k|_{t_0}(\varphi_t^{-1}) &= -T\varphi_{t_0}^{-1} \circ (\partial_t^k|_{t_0}\varphi_t) \circ \varphi_{t_0}^{-1} \\ T\varphi_{t_0} \circ \partial_t^k|_{t_0}(\varphi_t^{-1}) &= -(\partial_t^k|_{t_0}\varphi_t) \circ \varphi_{t_0}^{-1} = -Y \\ (\partial_t^k|_{t_0}\varphi_t^{-1}) \circ \varphi_{t_0} &= -T\varphi_{t_0}^{-1} \circ (\partial_t^k|_{t_0}\varphi_t) = -X \end{aligned}$$

Hence, replacing φ_{t_0} by $\varphi_{t_0}^{-1}$ in (3) replaces X by $-Y$ and Y by $-X$ and noting that $(\varphi_{t_0})_*s = (\varphi_{t_0}^{-1})^*s$ transforms (3) into (4). \square

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