The Dirac operator for the Ruelle-Koopman pair on L^p-spaces: an interplay between Connes distance and symbolic dynamics

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Abstract

Denote by $\boldsymbol{\mu}$ the maximal entropy measure for the shift σ acting on $\Omega = \{0,1\}^{\mathbb{N}}$, by $\mathcal{L}$ the associated Ruelle operator and by $\mathcal{K} = \mathcal{L}^{\dagger}$ the Koopman operator, both acting on $L^2(\boldsymbol{\mu})$. Using a diagonal representation π, the Ruelle-Koopman pair can be used for defining a dynamical Dirac operator $\mathcal{D}$, as in [6]. $\mathcal{D}$ plays the role of a derivative. In [10], the notion of a spectral triple was generalized to L^p-operator algebras; in consonance, here, we generalize results for $\mathcal{D}$ to results for a Dirac operator $\mathcal{D}_p$, and the associated Connes distance d_p, to this new L^p context, $p \geq 1$. Given the states η, ξ:

$$d_p(\eta,\xi) := \sup\{\, |\eta(a) - \xi(a)|, \text{ where } a \in \mathcal{A} \text{ and } \|[\mathcal{D}_p, \pi(a)]\| \leq 1\}.$$

In the setting of operator algebras a function $f \in \mathcal{A} = \mathcal{C}(\Omega)$ is represented by the operator M_f, where $M_f(g) = f\,g$. The operator M_f acts on $L^p(\mu)$. We explore the relationship of $\mathcal{D}_p$ with dynamics, in particular with $f \circ \sigma - f$, the discrete-time derivative of a continuous $f : \Omega \to \mathbb{R}$. Take $p, p' > 0$ satisfying $\frac{1}{p} + \frac{1}{p'} = 1$. We show for any continuous function f:

$$\|\,[\mathcal{D}_p, \pi(M_f)]\,\| = |\,\sqrt[\lambda]{\mathcal{L}\,|f\circ\sigma - f|^{\lambda}}\,|_{\infty}, \text{ where } \lambda = \max\{p, p'\}.$$

Denote by $\mathcal{P}(\sigma)$ the set of σ-invariant probabilities; then we get:

$$\|[\mathcal{D}_p, \pi(M(f))]\| \geq \sqrt[\lambda]{2}\,\sup_{\mu\in\mathcal{P}(\sigma)} \exp\left(\int \log|f\circ\sigma - f|\, \mathrm{d}\mu + \frac{h_{\mu}(\sigma)}{\lambda}\right).$$

When $p = 2$, the equality holds. We analyze the connection of d_p with transport theory. Let μ, ν be probabilities on Ω, d^{∞} a certain metric on Ω and $W_{d^{\infty}}$ its Wasserstein distance:

$$W_{d^{\infty}}(\mu,\nu) \leq d_p(\mu,\nu) \leq \sqrt[\lambda]{2}\,W_{d^{\infty}}(\mu,\nu).$$

Moreover, $d_1(\mu,\nu) = d_{\infty}(\mu,\nu) = W_{d^{\infty}}(\mu,\nu)$. d^{∞} is not compatible with the usual metric. Furthermore, we show $\|[\mathcal{D}_p, \pi(\mathcal{K}^n\mathcal{L}^n)]\| = 1$ for all $n \geq 1$. We also prove a formula analogous to the Kantorovich duality formula for minimizing the cost of tensor products.

1 Introduction

Our main goal here is to introduce a dynamical Dirac operator $\mathcal{D}_p$, $p \geq 1$ (associated to the Ruelle-Koopman pair) and to study its action on L^p-spaces of functions f defined on the symbolic space $\Omega = \{0,1\}^{\mathbb{N}}$. The $L^p(\boldsymbol{\mu})$ space concerns the maximal entropy measure $\boldsymbol{\mu}$. In the case of the space $L^2(\boldsymbol{\mu})$, the Dirac operator $\mathcal{D}_2$ plays the role of a derivative acting on self-adjoint operators.

We will present results for the associated Connes distance and the corresponding spectral triple. We will also present some explicit estimates describing the interplay of all these concepts with the dynamics of the symbolic space. Given a continuous function $f : \Omega \to \mathbb{R}$, the function $f \circ \sigma - f$ denotes the discrete-time derivative of f, where σ is the shift acting on Ω. The discrete-time derivative plays an important role in our reasoning.

The Koopman operator $\mathcal{K}$ and the Ruelle operator $\mathcal{L}$ are defined in (1). Also, d_p will denote the Connes distance for the dynamical Dirac operator $\mathcal{D}_p$, $p \geq 1$ (see expressions (6) and (7)). $\mathcal{P}(\Omega)$ denotes the set of probabilities on Ω, and $\mathcal{P}(\sigma)$ the set of σ-invariant probabilities.

Some of the results we will present here generalize our previous work [6] (see also [5]), which considered the case $p = 2$. Furthermore, we will estimate the Connes distance from $\mathcal{D}_p$ in terms of the Wasserstein distance from a cost function described in Section 6 .

In another previous work [4] (a Master dissertation), a "noncommutative generalization" of the optimal transport problem was considered and it was shown to satisfy a duality formula analogous to the Kantorovich duality formula (see the Appendix). The dual form of this noncommutative optimal transport problem is of interest to us in the present context because it can be related to the Connes distance. More precisely, the Connes distance between two states is bounded above by the noncommutative optimal transport cost between the same states (for a given "noncommutative cost function"). This is a corollary of the fact that the Connes distance is bounded above by the optimal transport cost (for a given cost function) and that the noncommutative optimal transport problem generalizes the optimal transport problem (see [16]).

There are circumstances under which the Connes distance and the optimal transport problem coincide. For instance, in this prototypical example of a spectral triple, where the commutative C*-algebra of continuous complex-valued functions of a compact manifold acts via multiplication operators on the Hilbert space of square-integrable differential forms; and the Dirac operator is the Hodge (or signature) operator [16]. In this example, the states are the Borel probability measures, and the transport cost function under consideration is the manifold's metric distance.

In general, it is known that even for commutative and finite-dimensional C*-algebras $\mathcal{A}$, the Connes distance between two probability vectors may not coincide with any[1] (as in, for any cost function) optimal transport problem between the same probability vectors (see [16]).

In a more general setting, recall that $\mathcal{A}$ is (an unital, separable, and) commutative (C*-algebra) precisely when it is (isometrically isomorphic to) the C*-algebra of continuous complex-valued functions of a given compact metric space[2]. The states of $\mathcal{A}$ can then be regarded as (Borel) probability measures on such space. One important issue is: given $\mu, \nu \in \mathcal{P}(\Omega)$, to characterize when $d_p(\mu, \nu)$ is finite (see Corollary 33).

As a means of probing the question of how the Connes distance relates to the optimal transport problem and our version of noncommutative optimal transport, we are going to carry out explicit computations in a specifically chosen example. This example concerns a form of special variation of the classical spectral triple definition because of two exceptions; the first one being our Dirac operator $\mathcal{D}_p$ is bounded, and therefore does not have compact resolvent. However, this will not be an issue, since we are interested mainly in a metric question, for which such a hypothesis is of no pertinence. Furthermore, the example will consist of a dynamically defined distance between probability measures, which is of interest *per se*.

Another interesting question regarding the Connes distance is how does it change with respect to the parameter p in the context of L^p-operator algebras. Hence, the other exception is we will also consider our algebra as an L^p-operator algebra. In doing this, we hope that our explicit computations come to offer a little bit of insight into such a question.

Results relating Spectral Triples and Ergodic Theory can be found in [14], [19], [20], [6] and [7].

2 Notation

Let $\Omega = \{0,1\}^{\mathbb{N}}$ be equipped with the product topology and the corresponding Borel σ-algebra. Typical sequences are written $x, y \in \Omega$. Consider the

[1]This follows from the observation that in [16], the set of admissible for the Connes distance is an ellipsoid, while the set of admissibles for the optimal transport problem is a rectangular prism (whichever the cost).

[2]Such a metric space can then be taken to be the so-called spectrum of $\mathcal{A}$. A more detailed description of the spectrum can be found in [11].

action of the shift map $\sigma : \Omega \longrightarrow \Omega$ defined by $x = (x_n)_{n\in\mathbb{N}} \longmapsto \sigma(x) = (x_{n+1})_{n\in\mathbb{N}}$, and denote by $\boldsymbol{\mu}$ the maximal entropy measure. We write $\boldsymbol{\mu} \sqcup \boldsymbol{\mu}$ for the measure over the disjoint space $\Omega \sqcup \Omega$ which restricts to either component as $\boldsymbol{\mu}$.

Let $L^p = L^p(\mu)$ be the Banach space of p-integrable complex-valued functions $g : \Omega \to \mathbb{C}$, with respect to the maximal entropy measure μ.

In our setting a continuous function $f : \Omega \to \mathbb{C}$ is represented by the operator M_f, given by f $g \mapsto M_f(g) := fg$. We will introduce a Dirac operator $\mathcal{D}_p$, $p \geq 1$, acting on pairs of p-integrable functions (see Section 4). Our initial focus will be on the commutator of $\mathcal{D}_p$ with operators of the form $\pi(M_f)$. That is, on:

$$[\mathcal{D}_p, \pi(M_f)],$$

where $[\, ,\,]$ means the commutator of operators, and π is a representation to be described in Section 4.

Denote by $\mathcal{C}(\Omega)$ the algebra of continuous complex-valued functions f of Ω. Let $p' \geq 1$ be the number implicitly defined given $p \geq 1$ by $\frac{1}{p} + \frac{1}{p'} = 1$, so that $L^{p'}$ is the dual space of L^p. Typical continuous functions are denoted by $f \in \mathcal{C}(\Omega)$ and p-integrable functions by $g \in L^p(\boldsymbol{\mu})$. $\mathcal{K}$ denotes the Koopman operator and $\mathcal{L}$ denotes the Ruelle operator.

The Koopman and Ruelle operators are characterized by:

$$\mathcal{K}f := f \circ \sigma, \text{ and: } \mathcal{L}[f](x) := \frac{1}{2}\left(f(0x) + f(1x)\right), \tag{1}$$

for all continuous functions $f \in \mathcal{C}(\Omega)$; they may be closed with respect to any p-norm, and we will use the same notation, $\mathcal{K}$ and $\mathcal{L}$ still.

General results on the Ruelle and Koopman operators can be found in [17]. They are dual of each other in the case of $L^2(\mu)$ (see [6])

One of our goals (see end of Section 4) is to show that

$$\|[\mathcal{D}_p, \pi(\mathcal{K}\,\mathcal{L})]\| = 1, \tag{2}$$

which is a particular case of

$$\|[\mathcal{D}, \pi(\mathcal{K}^n\mathcal{L}^n)]\| = 1. \tag{3}$$

3 Connes Distance

A Dirac operator D is necessary to define a spectral triple and a Connes distance. In [10] the authors pose the following generalization for the definition of a spectral triple:

Definition 1. *An L^p-spectral triple is an ordered triple $(A, L^p(\mu), D)$, where:*

1. *$L^p(\mu)$ is an arbitrary L^p-space;*
2. *A is an L^p-operator algebra; π is a representation of A on $L^p(\mu)$.*
3. *D is an unbounded linear operator on $L^p(\mu)$, such that:*
   (a) *$\{a \in A \mid \|[D, \pi(a)]\| < +\infty\}$ is a norm dense subalgebra of A.*
   (b) *$(\mathrm{Id} + D^2)^{-1}$ is a compact operator.*
   (c) *For any complex λ not in the spectrum of D, $(D - \lambda\,\mathrm{Id})^{-1}$ is a compact operator.*

The operator D is called the Dirac operator.

We are also going to follow [10, Definition 3.3] and define the space of states of a given L^p-operator algebra A as:

$$\mathcal{S}(A) := \{\eta \in A' \mid \|\eta\| = \eta(1) = 1\}. \tag{4}$$

The Connes distance between a pair of states $\eta, \xi \in \mathcal{S}(A)$ is defined as:

$$d_D(\eta, \xi) := \sup_{\substack{a \in A \\ \|[D,\pi(a)]\| \leq 1}} |\eta(a) - \xi(a)|. \tag{5}$$

It is an operator algebra version of the Wasserstein distance (see (15)).

In our setting, a continuous function $f \in \mathcal{C}(\Omega)$ is represented by the bounded linear operator $M_f \in \mathcal{B}(L^p(\boldsymbol{\mu}))$ which acts on $L^p(\mu)$ by $M_f(g) = fg$. We will exploit this choice to introduce a form of Dirac operator D is defined in terms of the shift dynamics over Ω (see Section 4).

Notice the parameter p governs on which space $a \in A = \mathcal{C}(\Omega)$ is being represented by π and D is acting on.

For the more precise estimates of d_D in Section 5 and 6 note that the states in (5) are Borel probabilities on Ω. In this case, when computing (5), given η, we get that $\eta(a) = \eta(f) = \int f\, d\eta$, when $a = f \in A = \mathcal{C}(\Omega)$. That is, $|\eta(a) - \xi(a)| = |\eta(f) - \xi(f)| = |\int f d\eta - \int f d\xi|$, and d_D in (5) is defined accordingly. Among other results we will estimate $d_p(\delta_x, \delta_{\sigma(x)})$ and $d_p(\delta_x, \delta_y)$, when $x, y \in \Omega$; and also $d_p(\eta, \xi)$.

Remark 2. *Note also the importance in each case to estimate whether or not* $\|[D, \pi(a)\| \leq 1$ *for a certain given* a*. This helps to find lower bounds for* $d_D(\eta, \xi)$*.*

Remark 3. *In what follows we are mostly interested in explicit computations and bounds for Connes pseudometric distance in the space of states of the* L^p*-operator algebra* $\mathcal{C}(\Omega)$*. In [8, Proposition 3, 4] A. Connes notes that for the purposes of defining a pseudometric,* D *is not required to have compact resolvent. In fact, in [18] M. Rieffel describes a considerably more general setting in which an analog of the Connes distance may be defined, and we are going to show that his setting includes ours in Section 5. In particular [18, Proposition 1.4] can be used to show that our pseudometric induces a strictly finer topology than the weak-*$*$ *topology on* $\mathcal{S}(\mathcal{C}(\Omega))$*. Section 5 will also provide insight into this matter as it describes the connected components of this topology.*

4 The Dirac Operator

Let $A := \mathcal{C}(\Omega)$. In this section we frequently identify a continuous function $f \in \mathcal{C}(\Omega)$ with the bounded linear operator $M_f \in \mathcal{B}(L^p(\boldsymbol{\mu}))$. In this way, we often think of π as a representation of $\mathcal{C}(\Omega)$, while, rigorously, it is $\pi \circ M_{()}$ that is so.

Let $\mathcal{B}(L^p(\boldsymbol{\mu}))$ act on $L^p(\boldsymbol{\mu} \sqcup \boldsymbol{\mu}) \cong L^p(\boldsymbol{\mu}) \times L^p(\boldsymbol{\mu})$ via a diagonal representation $\pi : \mathcal{B}(L^p(\boldsymbol{\mu})) \to \mathcal{B}(L^p(\boldsymbol{\mu}) \times L^p(\boldsymbol{\mu}))$, in such away that given $f \in \mathcal{C}(\Omega)$:

$$\pi(M_f) := \begin{bmatrix} M_f & 0 \\ 0 & M_f \end{bmatrix},$$

and let $D = \mathcal{D}_p$ be the linear operator acting on $L^p(\boldsymbol{\mu}) \times L^p(\boldsymbol{\mu})$ by:

$$\mathcal{D}_p := \begin{bmatrix} 0 & \mathcal{K} \\ \mathcal{L} & 0 \end{bmatrix}. \tag{6}$$

In [7] the authors considered other forms of dynamically defined Dirac operators.

Here the states are defined by

$$\mathcal{S}(A) := \{\eta \in A' \mid \|\eta\| = \eta(1) = 1\},$$

and the Connes distance for $\eta, \xi \in \mathcal{S}(A)$ by:

$$d_p(\eta, \xi) = d_{\mathcal{D}_p}(\eta, \xi) := \sup_{\substack{a \in A \\ \|[\mathcal{D}_p, \pi(a)]\| \leq 1}} |\eta(a) - \xi(a)| \,. \tag{7}$$

In order to compute expression (7) it helps to know which $f \in \mathcal{C}(\Omega)$ satisfies $\|[\mathcal{D}_p, \pi(M_f)]\| \leq 1$. We will present here some explicit estimates that will allow us to derive lower bounds for the Connes distance when $D = \mathcal{D}_p$. As a first step in that direction, notice that for any $f \in \mathcal{C}(\Omega)$:

$$\begin{aligned}
[\mathcal{D}_p, \pi(M_f)] &= \begin{bmatrix} 0 & \mathcal{K} \\ \mathcal{L} & 0 \end{bmatrix} \begin{bmatrix} M_f & 0 \\ 0 & M_f \end{bmatrix} - \begin{bmatrix} M_f & 0 \\ 0 & M_f \end{bmatrix} \begin{bmatrix} 0 & \mathcal{K} \\ \mathcal{L} & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \mathcal{K} M_f - M_f \mathcal{K} \\ \mathcal{L} M_f - M_f \mathcal{L} & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & M_{f \circ \sigma - f} \mathcal{K} \\ \mathcal{L} M_{f - f \circ \sigma} & 0 \end{bmatrix}.
\end{aligned} \tag{8}$$

Consequently:

$$\begin{aligned}
\|[\mathcal{D}_p, \pi(M_f)]\| &= \max \{ \|M_{f \circ \sigma - f} \mathcal{K}\|_p, \|\mathcal{L} M_{f - f \circ \sigma}\|_p \} \\
&= \max \{ \|M_{f \circ \sigma - f} \mathcal{K}\|_p, \|M_{f \circ \sigma - f} \mathcal{K}\|_{p'} \} \\
&= \max_{\lambda \in \{p, p'\}} \|M_{f \circ \sigma - f} \mathcal{K}\|_\lambda.
\end{aligned} \tag{9}$$

Equation (8) shows that the derivative of a given function $f \in \mathcal{C}(\Omega)$ with respect to $\mathcal{D}_p$ is completely characterized by a weighted transfer operator with weight given by a discrete-time forward dynamical derivative of f, namely $M_{f \circ \sigma - f} \mathcal{K}$. Then, the theory of weighted transfer operators applies (see [1] and [2]). In particular, there is a lower bound for the "Lipschitz Seminorm" $\|[\mathcal{D}_p, \pi(M_f)]\|$ given by the variational principle for the spectral radius:

Proposition 4. *For any continuous function $f : \Omega \to \mathbb{R}$*

$$\|[\mathcal{D}_p, \pi(M_f)]\| \geq r(M_{f \circ \sigma - f} \mathcal{K}) =$$

$$\max_{\lambda \in \{p, p'\}} \sqrt[\lambda]{2} \sup_{\mu \in \mathcal{P}(\sigma)} \exp \left(\int \log |f \circ \sigma - f| \, \mathrm{d}\mu + \frac{h_\mu(\sigma)}{\lambda} \right). \tag{10}$$

Proof. Apply [2] or [3]. □

Remark 5. *When $p = 2$, the equality holds, since the norm of an anti-selfadjoint operator is equal to its spectral radius. Note also that*

$$\frac{1}{p} \sup_{\mu \in \mathcal{P}(\sigma)} \int p \log |f \circ \sigma - f| \, d\mu + h(\mu)$$

is not exactly the classical Pressure problem (as in [17]) due to the fact that $\log |f \circ \sigma - f|$ *can take the value* $-\infty$.

Remark 6. *Combining Proposition 4 and Birkhoff's ergodic theorem, it follows that for $\boldsymbol{\mu}$-almost every $x \in \Omega$:*

$$\begin{aligned} \|[\mathcal{D}_p, \pi(M_f)]\| &\geq \exp \int \log |f \circ \sigma - f| \, \mathrm{d}\boldsymbol{\mu} \\ &= \exp \sum_{n=0}^{+\infty} \log \left| f \circ \sigma^{n+1}(x) - f \circ \sigma^n(x) \right| \\ &= \prod_{n=0}^{+\infty} \left| f \circ \sigma^{n+1}(x) - f \circ \sigma^n(x) \right|. \end{aligned}$$

Some of the present results can also be deduced from the abstract point of view of [2], [3] or [1]. For example, the reader should compare (11) and [1, Equation (98)]

Lemma 7. *For any $f \in \mathcal{C}(\Omega)$:*

$$[\mathcal{D}_p, \pi(M_f)] = 0 \iff f \circ \sigma - f = 0.$$

The latter implies that f is constant.

Proof. The proof is analogous to the one in [6]. If $f \circ \sigma - f = 0$, then:

$$\|[\mathcal{D}_p, \pi(M_f)]\| = \max_{\lambda \in \{p, p'\}} \|M_{f \circ \sigma - f} \mathcal{K}\|_\lambda = \max_{\lambda \in \{p, p'\}} \|M_0 \mathcal{K}\|_\lambda = 0.$$

In the other direction, if $[\mathcal{D}_p, \pi(M_f)] = 0$, then:

$$\max_{\lambda \in \{p, p'\}} \|M_{f \circ \sigma - f} \mathcal{K}\|_\lambda = 0,$$

and in particular:

$$\begin{aligned} \max_{\lambda \in \{p, p'\}} |M_{f \circ \sigma - f} \mathcal{K}(1)|_\lambda &= \max_{\lambda \in \{p, p'\}} |f \circ \sigma - f|_\lambda \\ &= 0, \end{aligned}$$

which means $f \circ \sigma - f = 0$. □

Lemma 8. *For any $f \in \mathcal{C}(\Omega)$:*

$$|f|_\infty \geq \sup_{|g|_p=1} |f\mathcal{K}g|_p \geq |\mathcal{L}f|_\infty .$$

Furthermore, if $f \in \mathcal{C}(\Omega)$ does not depend on the first coordinate (that is, if f is $\sigma^{-1}(\Sigma)$-measurable), then all above inequalities are equalities.

Proof. The proof is similar to the one in [6], except for the convex function to which we apply Jensen's inequality for conditional expectations is now $|\cdot|^p$ instead of $|\cdot|^2$. □

Remark 9. *Lemma 8 holds for p and p' with the same bounds.*

Theorem 10. *Replacing f by $f \circ \sigma - f$ in Lemma 8, in view of (9) and Remark 9, we get for any $f \in \mathcal{C}(\Omega)$:*

$$|\mathcal{K}f - f|_\infty \geq \|[\mathcal{D}_p, \pi(M_f)]\| \geq |f - \mathcal{L}f|_\infty .$$

Moreover, if $f \circ \sigma - f$ does not depend on the first coordinate we get the equalities:

$$|\mathcal{K}f - f|_\infty = \|[\mathcal{D}_p, \pi(M_f)]\| = |f - \mathcal{L}f|_\infty .$$

Proposition 11. *For any $f \in \mathcal{C}(\Omega)$:*

$$\|[\mathcal{D}_p, \pi(M_f)]\| = \max_{\lambda \in \{p,p'\}} \left| \sqrt[\lambda]{\mathcal{L}\,|f \circ \sigma - f|^\lambda} \right|_\infty . \tag{11}$$

Expression (11) can be written as:

$$\max_{\lambda \in \{p,p'\}} \left| \sqrt[\lambda]{\mathcal{L}\,|f \circ \sigma - f|^\lambda} \right|_\infty = \max_{\lambda \in \{p,p'\}} \sup_{x \in \Omega} \sqrt[\lambda]{\begin{array}{c}\frac{|f(x)-f(0x)|^\lambda}{2} + \\ + \frac{|f(x)-f(1x)|^\lambda}{2}\end{array}} . \tag{12}$$

The right-hand side of (12) is a form of the supremum of mean backward derivative.

Proof. Analogous to [5]. We have:

$$\sup_{|g|_\lambda=1} |f\mathcal{K}g|_\lambda = \sup_{|g|_\lambda=1} \left(\int |f\mathcal{K}g|^\lambda \, \mathrm{d}\boldsymbol{\mu} \right)^{\frac{1}{\lambda}}$$

$$
\begin{aligned}
&= \sup_{|g|_\lambda = 1} \left(\int |f|^\lambda \, |\mathcal{K} g|^\lambda \, \mathrm{d}\boldsymbol{\mu} \right)^{\frac{1}{\lambda}} \\
&= \sup_{|g|_\lambda = 1} \left(\int |f|^\lambda \left(\mathcal{K} \, |g|^\lambda \right) \mathrm{d}\boldsymbol{\mu} \right)^{\frac{1}{\lambda}} \\
&= \sup_{|g|_\lambda = 1} \left(\int \left(\mathcal{L} \, |f|^\lambda \right) |g|^\lambda \, \mathrm{d}\boldsymbol{\mu} \right)^{\frac{1}{\lambda}} \\
&= \sup_{|g|_\lambda = 1} \left| \left(\sqrt[\lambda]{\mathcal{L} \, |f|^\lambda} \right) g \right|_\lambda \\
&= \left| \sqrt[\lambda]{\mathcal{L} \, |f|^\lambda} \right|_\infty ,
\end{aligned}
$$

then we substitute f for $f \circ \sigma - f$. □

Corollary 12. *Notice that:*

$$
\max_{\lambda \in \{p, p'\}} \sqrt[\lambda]{\begin{array}{r} \frac{|f(x) - f(0x)|^\lambda}{2} + \\ + \frac{|f(x) - f(1x)|^\lambda}{2} \end{array}} = \sqrt[\max\{p,p'\}]{\begin{array}{r} \frac{|f(x) - f(0x)|^{\max\{p,p'\}}}{2} + \\ + \frac{|f(x) - f(1x)|^{\max\{p,p'\}}}{2} \end{array}} ,
$$

and that $\max\{p, p'\} \geq 2$*, so:*

$$
\begin{aligned}
\min \left\{ \begin{array}{r} |f(x) - f(0x)| , \\ |f(x) - f(1x)| \end{array} \right\} &\leq \frac{2}{\frac{1}{|f(x) - f(0x)|} + \frac{1}{|f(x) - f(1x)|}} \\
&\leq \sqrt{\begin{array}{r} |f(x) - f(0x)| \times \\ \times |f(x) - f(1x)| \end{array}} \\
&\leq \frac{1}{2} \left(\begin{array}{r} |f(x) - f(0x)| + \\ + |f(x) - f(1x)| \end{array} \right) \\
&\leq \sqrt{\begin{array}{r} \frac{|f(x) - f(0x)|^2}{2} + \\ + \frac{|f(x) - f(1x)|^2}{2} \end{array}} \\
&\leq \sup_{x \in \Omega} \sqrt{\begin{array}{r} \frac{|f(x) - f(0x)|^2}{2} + \\ + \frac{|f(x) - f(1x)|^2}{2} \end{array}} \\
&\leq \| [\mathcal{D}, \pi(M_f)] \|
\end{aligned}
$$

$$\leq \sup_{x\in\Omega} \max \left\{ \begin{array}{c} |f(x) - f(0x)|\,, \\ \quad |f(x) - f(1x)| \end{array} \right\}$$
$$= |f \circ \sigma - f|_\infty .$$

This reasoning also provides an alternative proof for the first inequality in Theorem 10.

Remark 13. *Corollary 12 shows that:*

$$\begin{aligned} \|[\mathcal{D}_p, \pi(M_f)]\| &= \max_{\lambda\in\{p,p'\}} \|M_{f\circ\sigma - f}\mathcal{K}\|_\lambda \\ &= \|M_{f\circ\sigma - f}\mathcal{K}\|_{\max\{p,p'\}}. \end{aligned}$$

Henceforth, we set $\lambda := \max\{p, p'\}$, as this will cause no confusion.

To conclude the characterization of the functions $f \in \mathcal{C}(\Omega)$ that have $\|[\mathcal{D}_p, \pi(M_f)]\| \leq 1$, first we will exhibit a sufficient condition:

Proposition 14. *For any function $f \in \mathcal{C}(\Omega)$:*

$$|f \circ \sigma - f|_\infty \leq 1 \implies \|[\mathcal{D}_p, \pi(M_f)]\| \leq 1$$

Proof. Apply Theorem 10. □

And, lastly, we present a necessary condition:

Proposition 15. *For any function $f \in \mathcal{C}(\Omega)$:*

$$\|[\mathcal{D}_p, \pi(M_f)]\| \leq 1 \implies |f \circ \sigma - f|_\infty \leq \sqrt[\lambda]{2}.$$

Proof. Notice that for any $x \in \Omega$:

$$\begin{aligned} & \|[\mathcal{D}_p, \pi(M_f)]\| \leq 1 \\ \iff \quad & \sqrt[\lambda]{\begin{array}{c} \frac{|f(x)-f(0x)|^\lambda}{2} \\ +\frac{|f(x)-f(1x)|^\lambda}{2} \end{array}} \leq 1 \\ \iff \quad & \frac{|f(x) - f(0x)|^\lambda}{2} \leq 1 - \frac{|f(x) - f(1x)|^\lambda}{2} \\ \iff \quad & |f(x) - f(0x)|^\lambda \leq 2\left(1 - \frac{|f(x) - f(1x)|^\lambda}{2}\right) \end{aligned}$$

$$
\begin{aligned}
\iff \qquad & |f(x)-f(0x)|^\lambda \le 2-|f(x)-f(1x)|^\lambda \\
\implies \qquad & |f(x)-f(0x)| \le \sqrt[\lambda]{2}.
\end{aligned}
$$

Then, exchanging 0 and 1 in the previous argument we prove our main claim. □

The specific form of $\mathcal{D}_p$ we are considering also makes it convenient to compute the Lipschitz seminorm for operators of the form $\mathcal{K}^n\mathcal{L}^n$.

We will show that

$$\|[\mathcal{D}_p, \pi(\mathcal{K}\,\mathcal{L})]\| = 1, \tag{13}$$

which is a particular case of

$$\|[\mathcal{D}, \pi(\mathcal{K}^n\mathcal{L}^n)]\| = 1. \tag{14}$$

In order to get that all the elements in the above expressions are well defined, we consider the identification of f with M_f.

In order to show (14), first notice the same computations at the end of [5, Section 2] hold for $\mathcal{D}_p$. To recall:

$$
\begin{aligned}
[\mathcal{D}_p, \pi(\mathcal{K}^n\mathcal{L}^n)] &= \begin{pmatrix} 0 & \mathcal{K}\mathcal{K}^n\mathcal{L}^n - \mathcal{K}^n\mathcal{L}^n\mathcal{K} \\ \mathcal{L}\mathcal{K}^n\mathcal{L}^n - \mathcal{K}^n\mathcal{L}^n\mathcal{L} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \mathcal{K}\mathcal{K}^n\mathcal{L}^n - \mathcal{K}^n\mathcal{L}^{n-1} \\ \mathcal{K}^{n-1}\mathcal{L}^n - \mathcal{K}^n\mathcal{L}^n\mathcal{L} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \mathcal{K}\left(\mathcal{K}^n\mathcal{L}^n - \mathcal{K}^{n-1}\mathcal{L}^{n-1}\right) \\ \left(\mathcal{K}^{n-1}\mathcal{L}^{n-1} - \mathcal{K}^n\mathcal{L}^n\right)\mathcal{L} & 0 \end{pmatrix}.
\end{aligned}
$$

Furthermore:

$$
\begin{aligned}
\left(\mathcal{K}^{n-1}\mathcal{L}^{n-1} - \mathcal{K}^n\mathcal{L}^n\right)^2 &= \begin{array}{c} \mathcal{K}^{n-1}\mathcal{L}^{n-1}\mathcal{K}^{n-1}\mathcal{L}^{n-1} - \mathcal{K}^{n-1}\mathcal{L}^{n-1}\mathcal{K}^n\mathcal{L}^n \\ -\mathcal{K}^n\mathcal{L}^n\mathcal{K}^{n-1}\mathcal{L}^{n-1} + \mathcal{K}^n\mathcal{L}^n\mathcal{K}^n\mathcal{L}^n \end{array} \\
&= \begin{array}{c} \mathcal{K}^{n-1}\mathcal{L}^{n-1} - \mathcal{K}^n\mathcal{L}^n \\ -\mathcal{K}^n\mathcal{L}^n + \mathcal{K}^n\mathcal{L}^n \end{array} \\
&= \mathcal{K}^{n-1}\mathcal{L}^{n-1} - \mathcal{K}^n\mathcal{L}^n.
\end{aligned}
$$

Also, notice that, because $\boldsymbol{\mu}$ is invariant, $\mathcal{K} : L^p \to L^p$ is an isometry for any $1 \le p \le +\infty$. Therefore, $\|\mathcal{K}T\| = \|T\|$ for any bounded linear transformation $T : L^p \to L^p$. In particular, when T is a projection, such as $\mathcal{K}^{n-1}\mathcal{L}^{n-1} - \mathcal{K}^n\mathcal{L}^n$, $\|\mathcal{K}T\| = 1$. This shows $\|[\mathcal{D}, \pi(\mathcal{K}^n\mathcal{L}^n)]\| = 1$.

We consider this fact important because it could be a starting point to eventually computing the Connes distance induced by the $\mathcal{D}_p$ on the space of states of a C*-algebra such as the Exel-Lopes C*-algebra (see [12]), since it is generated by elements of the form:

$$\sum_{i=1}^{N} M_{f_{n_i}} \mathcal{K}^{n_i} \mathcal{L}^{n_i} M_{g_{n_i}},$$

where $N, n_i \in \mathbb{N}$, and $f_{n_i}, g_{n_i} \in \mathcal{C}(\Omega)$, for every $1 \leq i \leq N$.

5 Pure States - $d_p(\delta_x, \delta_y)$

In the particular case of the L^p-operator algebra $\mathcal{C}(\Omega)$ the set of states (defined in (4)) is exactly the same as for the C*-algebra $\mathcal{C}(\Omega)$. That is because the L^p-operator norm of a multiplication operator M_f is $|f|_\infty$ regardless of which p one chooses. In this case $A = \mathcal{C}(\Omega)$; and the states are the Borel probability measures defined on Ω. In this section, we consider the particular case of pure states: the Dirac deltas δ_x on points $x \in \Omega$. In the next section, we will consider a more general case.

A natural question is to know when $d_p(\delta_x, \delta_y) < \infty$, for $x, y \in \Omega$ (see Theorem 29); which is somehow related to homoclinic equivalence relations.

There is a way to fit our results into the setting of [18]: in their notation, our normed space is $A = \mathcal{C}(\Omega)$ equipped with the supremum norm $|\cdot|_\infty$. All of our elements are Lipschitz, so $\mathcal{L} = A = \mathcal{C}(\Omega)$. Our Lipschitz seminorm is given by $L(a) = \|[\mathcal{D}_p, \pi(a)]\|$. Its zero locus is the space of constant functions $\mathcal{K} = \mathbb{C}1 \subseteq \mathcal{C}(\Omega)$, which determines η up to sign. Take it so $\eta(1) = 1$.

This means that we could apply [18, Proposition 1.4], which says our distance induces a topology finer than weak-$*$. Additionally, we will see in Example 22 that we do have states for which the distance is $+\infty$, which means it induces a *strictly* finer topology than the weak-$*$ topology in $\mathcal{S}(\mathcal{C}(\Omega))$. Notice this topology has many nontrivial connected components.

We now pass to the study of the connected components of the Connes distance. This means we want to discriminate between the pairs of states for which it is finite and the pairs of states for which it is not. First, we will restrict our attention to pure states. We begin with a simple example:

Example 16. *Let us estimate the Connes distance d_p for a pair of Dirac deltas $\delta_x, \delta_{\sigma(x)} \in \mathcal{S}(\mathcal{C}(\Omega))$. Proposition 15 implies: when $\lambda = \max\{p, p'\}$*

$$\begin{aligned} d_p(\delta_x, \delta_{\sigma(x)}) &= \sup_{\substack{f \in \mathcal{C}(\Omega) \\ \|[\mathcal{D}_p, \pi(M_f)]\| \leq 1}} |f(x) - f \circ \sigma(x)| \\ &\leq \sqrt[\lambda]{2} \\ &< +\infty. \end{aligned}$$

Example 17. *Consequently, if $x \in \Omega$ and $y \in \Omega$ are two points such that there exists two numbers $m, n \in \mathbb{N}$ for which the respective orbits meet at $\sigma^m(x) = \sigma^n(y)$, then:*

$$\begin{aligned} d_p(\delta_x, \delta_y) &\leq d_p(\delta_x, \delta_{\sigma(x)}) + d_p(\delta_{\sigma(x)}, \delta_y) \\ &\leq d_p(\delta_x, \delta_{\sigma(x)}) + d_p(\delta_{\sigma(x)}, \delta_{\sigma^2(x)}) + d_p(\delta_{\sigma^2(x)}, \delta_y) \\ &\leq d_p(\delta_x, \delta_{\sigma(x)}) + \cdots + d_p(\delta_{\sigma^{m-1}(x)}, \delta_{\sigma^m(x)}) + d_p(\delta_{\sigma^m(x)}, \delta_y) \\ &= d_p(\delta_x, \delta_{\sigma(x)}) + \cdots + d_p(\delta_{\sigma^{m-1}(x)}, \delta_{\sigma^m(x)}) + d_p(\delta_{\sigma^n(y)}, \delta_y) \\ &\leq \begin{array}{l} p(\delta_x, \delta_{\sigma(x)}) + \cdots + d_p(\delta_{\sigma^{m-1}(x)}, \delta_{\sigma^m(x)}) + \\ uad + d_p(\delta_{\sigma^n(y)}, \delta_{\sigma^{n-1}(y)}) + \cdots + d_p(\delta_{\sigma^1(y)}, \delta_y) \end{array} \\ &\leq \sqrt[\lambda]{2}\,(m+n) \\ &< +\infty. \end{aligned}$$

Now let us calculate more examples. We are looking for a function of arbitrary variation and "Lipschitz constant = 1".

Example 18. *If $f := 2\chi_{01} + 4\chi_{11} + 2\chi_{10}$, then:*

$$\begin{aligned} f \circ \sigma - f &= \begin{array}{l} (\chi_{001} + \chi_{101} + \chi_{010} + \chi_{110}) + 4\,(\chi_{011} + \chi_{111}) + \\ uad - 2\,(\chi_{010} + \chi_{011} + \chi_{100} + \chi_{101}) - 4\,(\chi_{110} + \chi_{111}) \end{array} \\ &= 2\,(\chi_{001} - \chi_{100} - \chi_{011} + \chi_{110}) + 4\,(\chi_{011} - \chi_{110}) \\ &= 2\,(\chi_{001} - \chi_{100} + \chi_{011} - \chi_{110})\,. \end{aligned}$$

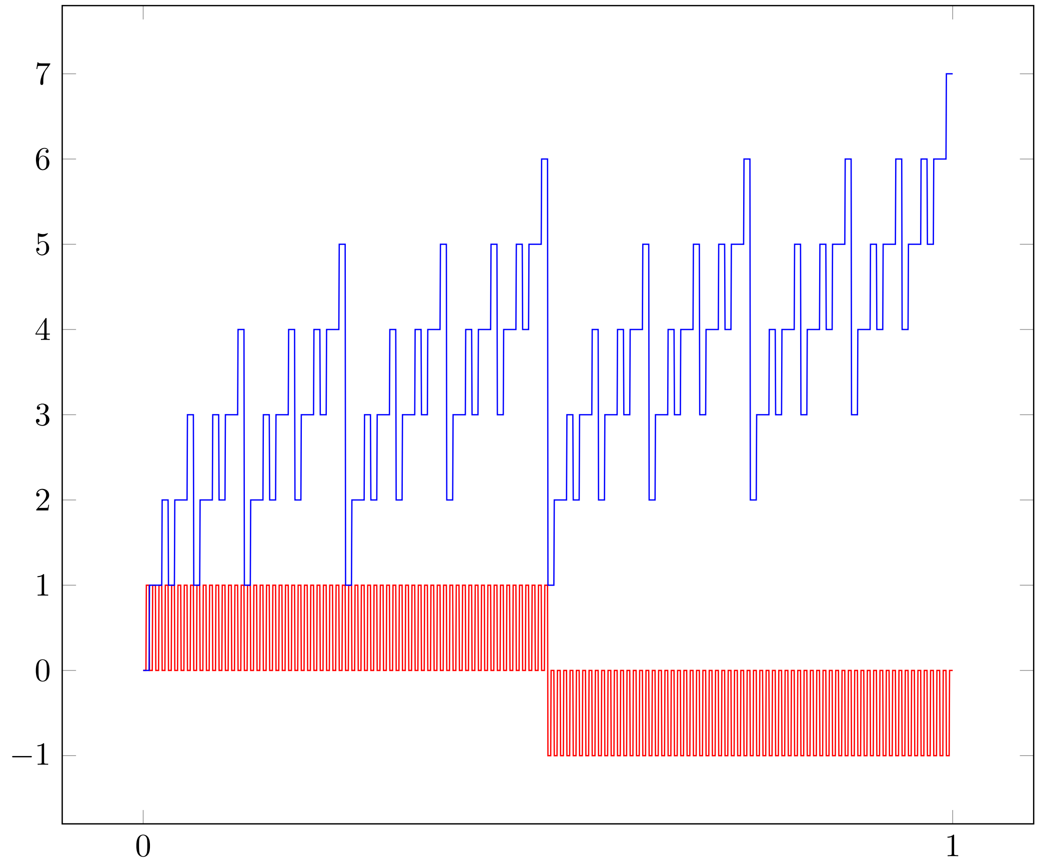
Example 19. *If $f := 2\chi_{001} + 4\chi_{011} + 6\chi_{111} + 4\chi_{110} + 2\chi_{100}$, then:*

$$f \circ \sigma - f = \begin{array}{l} 2\,(\chi_{0001} + \chi_{1001} + \chi_{0100} + \chi_{1100}) \\ \quad +4\,(\chi_{0011} + \chi_{1011} + \chi_{0110} + \chi_{1110}) \\ \qquad +6\,(\chi_{0111} + \chi_{1111}) \\ \quad -2\,(\chi_{0010} + \chi_{0011} + \chi_{1000} + \chi_{1001}) \\ \qquad -4\,(\chi_{0110} + \chi_{0111} + \chi_{1100} + \chi_{1101}) \\ \qquad\quad -6\,(\chi_{1110} + \chi_{1111}) \end{array}$$

$$
\begin{aligned}
&= \begin{array}{l} 2\left(\chi_{0001}+\chi_{0100}+\chi_{1100}-\chi_{0010}-\chi_{0011}-\chi_{1000}\right) \\ \quad +4\left(\chi_{0011}+\chi_{1011}+\chi_{1110}-\chi_{0111}-\chi_{1100}-\chi_{1101}\right) \\ \qquad +6\left(\chi_{0111}-\chi_{1110}\right) \end{array} \\
&= \begin{array}{l} 2\left(\chi_{0001}+\chi_{0100}-\chi_{1100}+\chi_{0111}-\chi_{0010}+\chi_{0011}-\chi_{1000}-\chi_{1110}\right) \\ \quad +4\left(\chi_{1011}-\chi_{1101}\right) \end{array} .
\end{aligned}
$$

Example 20. *If* $f := 2\left(\chi_{001}+\chi_{010}+\chi_{100}\right)+4\left(\chi_{011}+\chi_{101}+\chi_{110}\right)+6\chi_{111}$, *then:*

$$
\begin{aligned}
f\circ\sigma - f &= \begin{array}{l} 2\left(\chi_{0001}+\chi_{1001}+\chi_{0010}+\chi_{1010}+\chi_{0100}+\chi_{1100}\right) \\ \quad +4\left(\chi_{0011}+\chi_{1011}+\chi_{0101}+\chi_{1101}+\chi_{0110}+\chi_{1110}\right) \\ \qquad +6\left(\chi_{0111}+\chi_{1111}\right) \\ \quad -2\left(\chi_{0010}+\chi_{0011}+\chi_{0100}+\chi_{0101}+\chi_{1000}+\chi_{1001}\right) \\ \qquad -4\left(\chi_{0110}+\chi_{0111}+\chi_{1010}+\chi_{1011}+\chi_{1100}+\chi_{1101}\right) \\ \qquad\quad -6\left(\chi_{1110}+\chi_{1111}\right) \end{array} \\
&= \begin{array}{l} 2\left(\chi_{0001}+\chi_{1010}+\chi_{1100}-\chi_{0101}-\chi_{0011}-\chi_{1000}\right) \\ \quad +4\left(\chi_{0011}+\chi_{0101}+\chi_{0110}+\chi_{1110}-\chi_{0110}-\chi_{0111}-\chi_{1010}-\chi_{1100}\right) \\ \qquad +6\left(\chi_{0111}-\chi_{1110}\right) \end{array} \\
&= \begin{array}{l} 2\left(\chi_{0001}+\chi_{1010}+\chi_{1100}-\chi_{0101}-\chi_{0011}-\chi_{1000}+\chi_{0111}-\chi_{1110}\right) \\ \quad +4\left(\chi_{0011}+\chi_{0101}+\chi_{0110}-\chi_{0110}-\chi_{1010}-\chi_{1100}\right) \end{array} \\
&= 2\left(\chi_{0001}-\chi_{1010}+\chi_{1100}+\chi_{0101}-\chi_{0011}-\chi_{1000}+\chi_{0111}-\chi_{1110}\right) .
\end{aligned}
$$



f_7^γ (blue) and $f_7^\gamma \circ \sigma - f_7^\gamma$ (red). In this picture, the sequence $x \in \Omega$ corresponds to the real number $\sum x_i 2^{-i} \in [0,1]$.

Proposition 21. *If $x, y \in \Omega$ are two sequences such that:*

$$\sup_{n \in \mathbb{N}} |\# \{i \leq n \mid x_i = 1\} - \# \{i \leq n \mid y_i = 1\}| \geq N,$$

then $d_p(\delta_x, \delta_y) \geq N$. In particular, $d_p(\delta_{0^\infty}, \delta_{1^\infty}) = +\infty$.

Proof. Consider the following family of continuous functions of Ω:

$$f_k^\gamma := \sum_{w \in \hat{W}_k} \# \{i \mid w_i = 1\} \chi_w.$$

Notice that:

$$\begin{aligned}\|[\mathcal{D}_p, \pi(f_k^\gamma)]\| &\leq |a\,(f_k^\gamma \circ \sigma - f_k^\gamma)|_\infty \\ &\leq |f_k^\gamma \circ \sigma - f_k^\gamma|_\infty \\ &\leq 1,\end{aligned}$$

so that this family gives us a lower bound for the Connes distance. That is:

$$\begin{aligned}d_p(\delta_x, \delta_y) &= \sup_{\substack{f\in\mathcal{C}(\Omega)\\ \|\left[\mathcal{D}_p, \pi(M_f)\right]\|\leq 1}} |f(x) - f(y)| \\ &\geq \sup_k |f_k^\gamma(x) - f_k^\gamma(y)| \\ &\geq N.\end{aligned}$$

□

Example 22. *In particular,* $d_p(\delta_{0^\infty}, \delta_{1^\infty}) = +\infty$.

Proposition 23. *If* f_k *is a function of the form* $\sum_{w\in\hat{W}_k} \theta_w \chi_w$ *such that* $\|[\mathcal{D}_p, \pi(M_f)]\| \leq 1$*, then* $|f_k(x) - f_k(y)| \leq \sqrt[\lambda]{2}k$.

Proof. Consider the point $z = (x_1, x_2, \cdots, x_k, y_1, y_2, \cdots, y_k) \in \Omega$. It is clear that $f_k(z) = f_k(x)$ and $f_k \circ \sigma^k(z) = f_k(y)$. Now the values $f_k(x)$ and $f_k(y)$ are telescopically related as:

$$\begin{aligned}|f_k(x) - f_k(y)| &= \left|f_k(z) - f_k \circ \sigma^k(z)\right| \\ &= \left|f_k(z) - f_k \circ \sigma(z) + f_k \circ \sigma(z) - f_k \circ \sigma^k(z)\right| \\ &= \left|\begin{array}{l} f_k(z) - f_k \circ \sigma(z) + \\ \quad + f_k \circ \sigma(z) - f_k \circ \sigma^2(z) + \\ \qquad + \cdots + \\ \qquad\quad + f_k \circ \sigma^{k-1}(z) - f_k \circ \sigma^n(z) \end{array}\right| \\ &\leq \begin{array}{l} |f_k(z) - f_k \circ \sigma(z)| + \\ \quad + |f_k(\sigma(z)) - f_k \circ \sigma(\sigma(z))| + \\ \qquad + \cdots + \\ \qquad\quad + \left|f_k(\sigma^{k-1}(z)) - f_k \circ \sigma(\sigma^{k-1}(z))\right| \end{array} \\ &\leq \sqrt[\lambda]{2}k,\end{aligned}$$

where the last inequality follows from Proposition 15. □

We will now consider words of finite length on the alphabet $\{0,1\}$. For a given $k \in \mathbb{N}$, W_k is the set of words $w = [w_1, w_2, ..., w_s]$, $s \le k$, of length at most k and $\hat{W}_k \cong \{0,1\}^k$ is the set of words $w = [w_1, w_2, ..., w_s]$ of length exactly k. By abuse of language we say that $\sigma([x_1, x_2, .., x_k]) = [x_2, .., x_k]$, and words $[w_1, w_2, ..., w_s]$ can also represent cylinder sets $[w_1, w_2, ..., w_s] \subset \{0,1\}^{\mathbb{N}}$.

Given $x = (x_1, x_2, \cdots, x_n, \cdots)$, we denote by $x|_k$ the word $[x_1, x_2, \cdots, x_k]$ of length k.

It will be appropriate to define a metric d_k which is a graph distance in $\hat{W}_k$.

Proposition 24. *Consider the graph (V_k, E_k) given by:*

$$V_k = \hat{W}_k \text{ and } E_k = \left\{ (u,v) \in \hat{W}_k \times \hat{W}_k \mid (\sigma([u]) \cap [v]) \cup ([u] \cap \sigma([v])) \neq \emptyset \right\}.$$

Let d_k denote the graph distance in $V_k = \hat{W}_k$. Then,

$$\sup_k d_k(x|_k, y|_k) \le d_p(\delta_x, \delta_y) \le \sqrt[p]{2} \sup_k d_k(x|_k, y|_k).$$

Proof. Let $P = ((w_1 = x|_k, w_2), (w_2, w_3), \cdots, (w_{n-1}, w_n = y|_k))$ be a path joining $x|_k$ and $y|_k$ along E_k. Also, let f_k^θ be a function that only depends on the first k coordinates. If f_k^θ satisfies $\|\left[\mathcal{D}_p, \pi(f_k^\theta)\right]\| \le 1$, then by definition of E_k, we have that $\left|\theta_{w_i} - \theta_{w_{i+1}}\right| \le \sqrt[p]{2}$. Also notice the family of all f_k^θ is dense in $\mathcal{C}(\Omega)$. From the above we get:

$$\begin{aligned} d_p(\delta_x, \delta_y) &= \sup_{\substack{f \in \mathcal{C}(\Omega) \\ \|\left[\mathcal{D}_p, \pi(M_f)\right]\| \le 1}} |f(x) - f(y)| \\ &= \sup_k \sup_{\substack{\theta \in \mathbb{R}^{\hat{W}_k} \\ \|\left[\mathcal{D}_p, \pi(f_k^\theta)\right]\| \le 1}} \left|f_k^\theta(x) - f_k^\theta(y)\right| \\ &= \sup_{k,\theta} \left|f_k^\theta(x) - f_k^\theta(y)\right| \\ &= \sup_{k,\theta} \left|\theta_{x|_k} - \theta_{y|_k}\right| \\ &\le \sqrt[p]{2} \sup_k d_k(x|_k, y|_k). \end{aligned}$$

Now define f_k^γ by:

$$\gamma_w = d_k(w, y|_k).$$

It is clear from the definition that:

$$\| \left[\mathcal{D}_p, \pi(f_k^\theta) \right] \| \leq 1 \text{ and } |f_k^\gamma(x) - f_k^\gamma(y)| = d_k(x|_k, y|_k),$$

which gives the other inequality. ∎

Remark 25. *It can happen that:*

$$\sup_{n \in \mathbb{N}} |\# \{i \leq n \mid x_i = 1\} - \# \{i \leq n \mid y_i = 1\}| < +\infty,$$

and yet $d_p(\delta x, \delta y) = +\infty$. For instance, consider the sequences:

$$x = (0, 1, 0, 1, 0, 1, 0, 1, \cdots) \text{ and } y = (0, 0, 1, 1, 0, 0, 1, 1, \cdots).$$

Their incidences of 1's up to length n differ by at most $1 \ll +\infty$, and yet the distance between the truncations of this two points on the graph (V_k, E_k) is of the same order of magnitude as k. Then Proposition 24 gives $d(\delta_x, \delta_y) = +\infty$.

Proposition 26. *Consider the length ℓ of the longest common subword c of u and v, which are words of length k; that is, there exists $m, n \in \mathbb{N}$ such that:*

$$\begin{array}{ccccc} c_1 & = & u_m & = & v_n, \\ c_2 & = & u_{m+1} & = & v_{n+1}, \\ c_3 & = & u_{m+2} & = & v_{n+2}, \\ & & \vdots & & \\ c_\ell & = & u_{m+\ell} & = & v_{n+\ell}. \end{array}$$

Then the k^{th} graph distance from $u \in \hat{W}_k$ to $v \in \hat{W}_k$ satisfies

$$d_k(u, v) = \min \{k, k - \ell + m + 2n, k - \ell + n + 2m\}.$$

Proof. From the definition of E_k it follows that two vertices $w, w' \in V_k = \hat{W} = \{0, 1\}^k$ are "connected" if and only if they are of the form:

$$\begin{array}{ccc} w_1 & = & w'_2 \\ w_2 & = & w'_3 \\ w_3 & = & w'_4 \\ & \vdots & \\ w_{k-1} & = & w'_k \end{array} \quad \text{, or:} \quad \begin{array}{ccc} w_2 & = & w'_1 \\ w_3 & = & w'_2 \\ w_4 & = & w'_3 \\ & \vdots & \\ w_k & = & w'_{k-1} \end{array}.$$

In particular, each vertex of w has four neighbours w', uniquely characterized by one of the alternatives: $w'_1 = 0$, $w'_1 = 1$, $w'_k = 0$, or $w'_k = 1$.

We say that c is *on the same side* in u and v if $m \vee n \leq \frac{(k-\ell)}{2}$ or if $m \wedge n \geq \frac{(k-\ell)}{2}$. Otherwise, we say that c is *on opposite sides* in u and v.

Of course, the diameter of the graph is k. Now let us exhibit a shorter path from u to v when their maximal common word c has length $\ell \geq \frac{k}{2}$ and is on the same side in u and v. We are going to do the case $m < n \leq \frac{(k-\ell)}{2}$; the other possibilities are analogous.

1. Starting at $w^1 = u$, take $m-1$ steps to the neighbour w^{i+1} such that $w^{i+1}_k = 0$.
2. This will get us to $w^m = [w, u|^{m+\ell+1}, 0^{m-1}]$.
3. Take $k - \ell - 1$ steps to the neighbour w^{i+1} such that $w^{i+1}_1 = v_{n+m-i}$.
4. This will get us to $w^{m+k-\ell} = [v|_{n-1}, w, u|^{k-\ell-n+1}_{k-n+m}]$.
5. Take $n-1$ steps to the neighbour w^{i+1} such that $w^{i+1}_1 = 0$.
6. This will get us to $w^{m+k-\ell+n} = [0^{k-\ell-n+1}, v|_{n-1}, w]$.
7. Take $n-1$ steps to the neighbour w^{i+1} such that $w^{i+1}_1 = v_{k-\ell+2n+m-i}$.
8. This will get us to $w^{k-\ell+m+2n} = v$.

□

Remark 27. *In particular,* $d_k(u, v) \geq k - \ell$*,* $u, v \in \hat{W}_k$*.*

Proposition 28. *If x and y are two points such that $d_p(\delta_x, \delta_y) < +\infty$, then there exist $m, n \in \mathbb{N}$ such that $\sigma^m(x) = \sigma^n(y)$.*

Proof. Let $\ell(k)$ denote the size of the largest common word between $x|_k$ and $y|_k$. By Proposition 24 we have that $d(\delta_x, \delta_y) < +\infty \implies \sup_k k - \ell(k) < +\infty$. But this implies that there exists a k_0 such that $k - \ell(k) \equiv r = r(x, y)$ for every $k \geq k_0$. This means that x and y only differ for finitely many terms, so there exist $m, n \in \mathbb{N}$ such that $\sigma^m(x) = \sigma^n(y)$. □

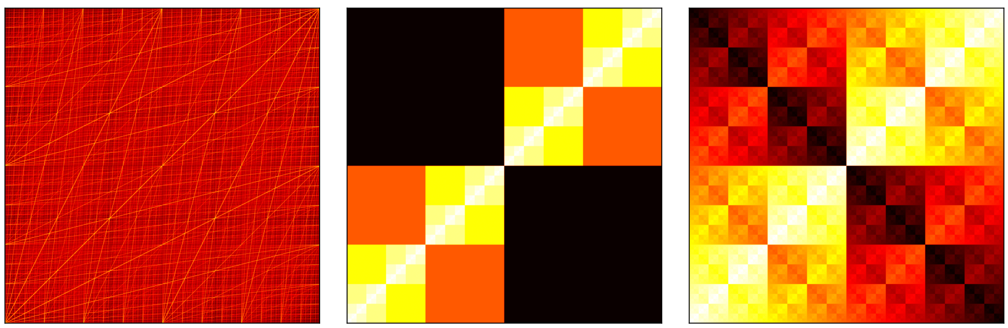
By combining Proposition 28 with Example 17 we have just proved:

Theorem 29. *For any pair of points $x, y \in \Omega$, $p \geq 1$,*

$$d_p(\delta_x, \delta_y) < +\infty \iff \text{there exist } m, n \in \mathbb{N} \text{ such that } \sigma^m(x) = \sigma^n(y).$$

□

The properties mentioned above in Theorem 29 are somehow related to the so-called *homoclinic equivalence relation* as defined in Section 6 in [9]; the particular case where $\sigma^m(x) = \sigma^m(y)$, $m \in \mathbb{N}$ (see also [12]). In this case we get $d_p(\delta_x, \delta_y) < +\infty$. For the case $m \neq n$ see Section 9 in [13] (and also [15]).



(V_k, E_k)-graph, truncated and cumulative distances on $\{0,1\}^{12}$. Or $d_k(u,v)$, $2^{-N(u,v)}$ and $\sum_{u_i=v_i} 2^{-i}$ respectively, $N(u,v)$ being the smallest index for which u differs from v. Here, we identified the word $u \in \hat{W}_k$ with the real number $\sum u_i 2^{-i} \in [0,1]$. The distance from u to v is plotted lighter if it is close to zero and darker if it is close to the diameter of Ω (1 for truncated and cumulative distances and k for d_k). This figure shows $k = 12$.

6 General States - $d_p(\mu, \nu)$

Now we pass to the question of computing and estimating the Connes distance $d_p(\mu, \nu)$, for two general states $\mu, \nu \in \mathcal{S}(\mathcal{C}(\Omega)) = \mathcal{P}(\Omega)$. First, we will exhibit an analog of Example 16:

Proposition 30. *If $\sigma_\sharp$ denotes the push-forward through σ, then for any given state $\mu \in \mathcal{P}(\Omega)$, Proposition 15 implies: when $\lambda = \max\{p, p'\}$*

$$d_p(\mu, \sigma_\sharp(\mu)) = \sup_{\substack{f \in \mathcal{C}(\Omega) \\ \|[\mathcal{D}_p, \pi(M_f)]\| \leq 1}} \left| \int f \, \mathrm{d}\mu - \int f \, \mathrm{d}\sigma_\sharp(\mu) \right|$$

$$= \sup_{\substack{f\in\mathcal{C}(\Omega)\\ \|[\mathcal{D}_p,\pi(M_f)]\|\le 1}} \left|\int f\circ\sigma - f\,\mathrm{d}\mu\right|$$
$$\le \sqrt[\lambda]{2}.$$

□

For the next definitions the compact metric space $(X,\tilde{d})$ will represent either the set $\{0,1\}^{\mathbb{N}}$, or the set $V_k = \{0,1\}^k$, $k \ge 1$.

Given a metric $\tilde{d}$ the Wasserstein distance between the probabilities μ and ν on X is (see [21])

$$W_{\tilde{d}}(\mu,\nu) = \sup_{\substack{f\in\mathcal{C}(X)\\ |f(x)-f(y)|\le\tilde{d}(x,y)}} \left|\int f\,\mathrm{d}\mu - \int f\,\mathrm{d}\nu\right|. \qquad (15)$$

Next, an analog of Proposition 24 will consider the case where $\tilde{d} = d_k$ and $X = V_k$. Later we will introduce a metric $\tilde{d} = d^\infty$ on $X = \Omega$.

Proposition 31. *Let $\mu,\nu \in \mathcal{P}(\Omega)$ be any two states. Then, given $x,y\in\Omega = \{0,1\}^{\mathbb{N}}$, when $\lambda = \max\{p,p'\}$*

$$\sup_{\substack{k\in\mathbb{N}\\ y\in\Omega}} \left|\int d_k(x|_k,y|_k)\,\mathrm{d}(\mu-\nu)\right| \le d_p(\mu,\nu) \le \sqrt[\lambda]{2}\sup_{\substack{k\in\mathbb{N}\\ y\in\Omega}} \left|\int d_k(x|_k,y|_k)\,\mathrm{d}(\mu-\nu)\right|.$$

Or:

$$\sup_{k\in\mathbb{N}} W_{d_k}(\mu,\nu) \le d_p(\mu,\nu) \le \sqrt[\lambda]{2}\sup_{k\in\mathbb{N}} W_{d_k}(\mu,\nu),$$

Proof. Analogous to the proof of Theorem 32. Observe the Wasserstein distance is equal to the supremum in y for each respective k. It is also increasing in k so that the suprema are actually limits. □

Finally, we have that:

Theorem 32. *Let $\mu,\nu \in \mathcal{P}(\Omega)$ be any two states. Then, for $p \ge 1$, when $\lambda = \max\{p,p'\}$*

$$W_{d^\infty}(\mu,\nu) \le d_p(\mu,\nu) \le \sqrt[\lambda]{2}W_{d^\infty}(\mu,\nu),$$

where d^∞ is given by:

$$d^\infty(x,y) := \min_{\substack{m,n\in\mathbb{N}\\ \sigma^m(x)=\sigma^n(y)}} m+n.$$

Proof. Consider a function $f \in \mathcal{C}(\Omega)$ such that $\|[\mathcal{D}_p, \pi(M_f)]\| \leq 1$. Proposition 15 shows that $|f \circ \sigma(x) - f(x)| \leq \sqrt[\lambda]{2}$. Now, if $m, n \in \mathbb{N}$ are such that $\sigma^m(x) = \sigma^n(y)$, then:

$$\begin{aligned}
|f(x) - f(y)| &= \left| \begin{array}{l} f(x) - f \circ \sigma(x) + \\ \quad + f \circ \sigma(x) - f(y) \end{array} \right| \\
&= \left| \begin{array}{l} f(x) - f \circ \sigma(x) + \\ \quad + f \circ \sigma(x) - f \circ \sigma^2(x) + \\ \qquad + f \circ \sigma^2(x) - f(y) \end{array} \right| \\
&= \left| \begin{array}{l} f(x) - f \circ \sigma(x) + \\ \quad + \cdots + \\ \qquad + f \circ \sigma^{m-1}(x) - f \circ \sigma^m(x) + \\ \qquad\quad + f \circ \sigma^n(y) - f \circ \sigma^{n-1}(y) + \\ \qquad\qquad + \cdots + \\ \qquad\qquad\quad + f \circ \sigma(y) - f(y) \end{array} \right| \\
&\leq \sqrt[\lambda]{2}\,(m+n),
\end{aligned}$$

which shows that $|f(x) - f(y)| \leq \sqrt[\lambda]{2} d_\infty(x, y)$. On the other hand, if $f \in \mathcal{C}(\Omega)$ is such that $|f(x) - f(y)| \leq d_\infty(x, y)$, then $|f \circ \sigma(x) - f(x)| \leq 1$. Therefore:

$$\begin{aligned}
\|[\mathcal{D}_p, \pi(M_f)]\| &= \sqrt[\lambda]{\begin{array}{l} \frac{|f(x) - f(0x)|^\lambda}{2} + \\ \quad + \frac{|f(x) - f(1x)|^\lambda}{2} \end{array}} \\
&\leq \sqrt[\lambda]{\frac{1}{2} + \frac{1}{2}} \\
&= 1.
\end{aligned}$$

□

Note that the distance d^∞ does not produce the same topology as the one obtained from the usual metric on Ω.

Corollary 33. *The Connes distance between two states $\mu, \nu \in \mathcal{P}(\Omega)$ is finite if and only if they give the same weight to each equivalence class of the relation given by $x \mathcal{R} y \iff \exists m, n \in \mathbb{N} : \sigma^m(x) = \sigma^n(y)$, that is: if $\mu(\bar{x}) = \nu(\bar{x})$ for any $x \in \Omega$, $\bar{x}$ the equivalence class of x. Each of these equivalence classes is the connected component of each of its elements with respect to distance d^∞.*

Taking the limits p in Theorem 32 gives $d_{\mathcal{D}_1} = d_{\mathcal{D}_\infty} = W_{d^\infty}$. Notice, $d_{\mathcal{D}_p} = \sqrt[p]{2} d_\infty$. This shows the family $d_{\mathcal{D}_p}$ interpolates between the Connes and Wasserstein distances; the Wasserstein distance corresponds to the cases $p = 1, +\infty$. We have numerical evidence for the inequalities in Theorem 32 being strict.

7 Appendix

The material of this section was taken from [4]. The noncommutative generalization of the optimal transport problem in [4] is a version of the Monge-Kantorovich optimal transport problem (on compact spaces with positive continuous cost) according to the following dictionary:

Real functions $f \in \mathcal{C}(X)$, $g \in \mathcal{C}(Y)$	Self-adjoint elements $a \in \mathcal{A}$, $b \in \mathcal{B}$
Probability measures $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$	C*-algebra states $\eta \in \mathcal{S}(\mathcal{A})$, $\xi \in \mathcal{S}(\mathcal{B})$
Cost function $c \in \mathcal{C}(X \times Y)$, $c \geq 0$	Cost element $c \in \mathcal{A} \otimes \mathcal{B}$, $c \geq 0$
Coupling probabilities $\rho \in \mathcal{P}(X \times Y)$ $\int f(x) + g(y)\,\mathrm{d}\rho = \int f(x)\,\mathrm{d}\mu + \int g(y)\,\mathrm{d}\nu$	Coupling states $\omega \in \mathcal{S}(\mathcal{A} \otimes \mathcal{B})$ $\omega(a \otimes 1 + 1 \otimes b) = \eta(a) + \xi(b)$
$W_c(\mu, \nu) := \inf_\rho \int c\,\mathrm{d}\rho$	$\mathcal{W}_c(\eta, \xi) := \inf_\omega \omega(c)$

Therefore, it is natural to pursue the following reasoning:
Let $\mathcal{A}$ and $\mathcal{B}$ be two unital C*-algebras.
Denote by $\mathcal{A} \otimes \mathcal{B}$ the maximal tensor product between $\mathcal{A}$ and $\mathcal{B}$.
Let $c \in (\mathcal{A} \otimes \mathcal{B})^+$ be a positive element (henceforth called *cost element*).
Let $\eta \in \mathcal{S}(\mathcal{A})$ and $\xi \in \mathcal{S}(\mathcal{B})$ be two given C*-algebra states.
Denote by $\Gamma(\eta, \xi)$ the set of all states $\omega \in \mathcal{S}(\mathcal{A} \otimes \mathcal{B})$ such that:

$$\omega(a \otimes 1_{\mathcal{B}} + 1_{\mathcal{A}} \otimes b) = \eta(a) + \xi(b) \quad \forall\, a \in \mathcal{A}, b \in \mathcal{B}.$$

These states are called the *admissible couplings of* η *and* ξ.

Denote by $\tilde{\Gamma}(c)$ the set of all pairs of self-adjoint elements $(a, b) \in \mathcal{A} \times \mathcal{B}$ such that:

$$a \otimes 1_{\mathcal{B}} + 1_{\mathcal{A}} \otimes b \leq c$$

These pairs are called the *admissible pairings for the cost c*.

The noncommutative optimal transport problem from [4] consists of minimizing the evaluation of the cost element among all admissible couplings of η and ξ. We call this value the minimum optimal transport cost from η to ξ. A value c is fixed according to convenience in each problem. Thus, we write:

$$\mathcal{W}_c(\eta, \xi) := \inf_{\omega \in \Gamma(\eta, \xi)} \omega(c) \tag{16}$$

It is possible to prove that minimizers for (16) exist by employing the direct method of the calculus of variations (see [4]). Furthermore, and also done in [4], it is possible to prove a formula analogous to the Kantorovich duality formula for (16), as we will see. Notice this recovers the existence of minimizers as a corollary.

Theorem 34. *Let $\mathcal{A}$, $\mathcal{B}$, $c \in (\mathcal{A} \otimes \mathcal{B})^+$, $\eta \in \mathcal{S}(\mathcal{A})$, and $\xi \in \mathcal{S}(\mathcal{B})$ be as above, and consider the aforementioned definitions of $\Gamma(\eta, \xi)$ and $\tilde{\Gamma}(c)$. Then:*

$$\mathcal{W}_c(\eta, \xi) := \inf_{\omega \in \Gamma(\eta, \xi)} \omega(c) = \sup_{\substack{a \in A \\ b \in \mathcal{B} \\ a \otimes 1 + 1 \otimes b \leq c}} \eta(a) + \xi(b).$$

Proof. We very closely follow [21], which amounts to employing the Fenchel-Rockafellar duality theorem. In the notation therein, our normed vector space E is the real vector space of self-adjoint elements of $\mathcal{A} \otimes \mathcal{B}$, and our convex functions $\Theta : E \to \mathbb{R}$ and $\Xi : E \to \mathbb{R}$ are given by:

$$\Theta(x) := \begin{cases} 0 & \text{if } x \geq -c, \\ +\infty & \text{otherwise.} \end{cases}$$

And:

$$\Xi(x) := \begin{cases} \eta(a) + \xi(b) & \text{if } x = a \otimes 1_{\mathcal{B}} + 1_{\mathcal{A}} \otimes b, \\ +\infty & \text{otherwise.} \end{cases}$$

The point $x_0 = 1_{\mathcal{A} \otimes \mathcal{B}}$ lies in the intersection of the effective domains of both functions (that is, $\Theta(1_{\mathcal{A} \otimes \mathcal{B}}) < +\infty$ and $\Xi(1_{\mathcal{A} \otimes \mathcal{B}}) < +\infty$), because $1_{\mathcal{A} \otimes \mathcal{B}} \geq 0 \geq -c$ and:

$$\Xi(1_{\mathcal{A} \otimes \mathcal{B}}) = \Xi(1_{\mathcal{A}} \otimes 1_{\mathcal{B}})$$

$$
\begin{aligned}
&= \Xi\left(\frac{1}{2}1_{\mathcal{A}}\otimes 1_{\mathcal{B}} + 1_{\mathcal{A}}\otimes\frac{1}{2}1_{\mathcal{B}}\right)\\
&= \eta\left(\frac{1}{2}1_{\mathcal{A}}\right) + \xi\left(\frac{1}{2}1_{\mathcal{B}}\right)\\
&= \frac{1}{2}+\frac{1}{2}\\
&= 1.
\end{aligned}
$$

The function Θ is continuous at the point $x_0 = 1_{\mathcal{A}\otimes\mathcal{B}}$ because this point lies in the interior of its effective domain. For example, the set $\mathcal{A}\otimes\mathcal{B}^{+^\circ}$ of all strictly positive elements of $\mathcal{A}\otimes\mathcal{B}$ is an open set entirely contained in the effective domain; and $1_{\mathcal{A}\otimes\mathcal{B}}$ pertains to such set.

Applying the Fenchel-Rockafellar duality, we conclude:

$$
\inf_{x\in E}\Theta(x)+\Xi(x) = \max_{\chi\in E^\star} -\Theta^\star(-\chi) - \Xi^\star(\chi). \tag{17}
$$

Now we pass to the issue of computing both sides of (17). On the left side, we have:

$$
\begin{aligned}
\inf_{\substack{x=a\otimes 1_{\mathcal{B}}+1_{\mathcal{A}}\otimes b\\ x\geq -c}} \eta(a)+\xi(b) &= -\sup_{\substack{x=a\otimes 1_{\mathcal{B}}+1_{\mathcal{A}}\otimes b\\ x\leq c}} \eta(a)+\xi(b)\\
&= -\sup_{(a,b)\in\tilde{\Gamma}(c)} \eta(a)+\xi(b).
\end{aligned}
$$

On the right side, the Legendre transform of Θ is:

$$
\Theta^\star(-\chi) = \sup_{x\in E} -\chi x - \Theta x = \sup_{x\geq -c} -\chi x.
$$

If $\chi \ngeq 0$, there must be some $x \geq 0 \geq -c$ for which $\chi(x) < 0$. Given such x, the family of positive elements nx ensures that the supremum be $+\infty$. If otherwise $\chi \geq 0$, then we can compare the evaluation of χ at $-c$ with the evaluation of χ at any other x:

$$
-\chi(-c) - [-\chi\,(x)] = -\chi(-c-x) = \chi(c+x) \geq 0,
$$

and see that the supremum must be $\chi(c)$.

Still on the right side of (17), the Legendre transform of Ξ is:

$$
\Xi^\star(\chi) = \sup_{x\in E} \chi x - \Xi x
$$

$$= \sup_{x=a\otimes 1_{\mathcal{B}}+1_{\mathcal{A}}\otimes b} \chi(a\otimes 1_{\mathcal{B}}+1_{\mathcal{A}}\otimes b) - (\eta(a)+\xi(b)).$$

If, for any of these x, the quantity $\chi(a\otimes 1_{\mathcal{B}}+1_{\mathcal{A}}\otimes b) - (\eta(a)+\xi(b))$ is not zero, then either the family nx or $-nx$ ensures the supremum be $+\infty$. In the absence of such x, $\Xi^\star(\chi) = 0$. Synthetically:

$$\Theta^\star(-\chi) = \begin{cases} \chi(c) \text{ if } \chi \geq 0, \\ +\infty \text{ otherwise.} \end{cases}$$

And:

$$\Xi^\star(\chi) = \begin{cases} \quad 0 \text{ if } \chi(a\otimes 1_{\mathcal{B}}+1_{\mathcal{A}}\otimes b) = \eta(a)+\xi(b), \\ +\infty \text{ otherwise.} \end{cases}$$

Conveniently, the intersection of the effective domains of such functions is precisely $\Gamma(\eta, \xi)$.

Finally, we rewrite Fenchel-Rockafellar duality in terms of the previous observations:

$$\inf_{a\otimes 1_{\mathcal{B}}+1_{\mathcal{A}}\otimes b \geq -c} \eta(a)+\xi(b) = \max_{\substack{\chi\geq 0 \\ \chi(a\otimes 1_{\mathcal{B}}+1_{\mathcal{A}}\otimes b)=\eta(a)+\xi(b)}} -\chi(c),$$

and exchange signs, to obtain:

$$\sup_{(a,b)\in\tilde{\Gamma}(c)} \eta(a)+\xi(b) = \min_{\omega\in\Gamma(\eta,\xi)} \omega(c).$$

□

When $\mathcal{A} = \mathcal{B} = \mathcal{C}(X)$ and the cost $c = d$ is a metric, we recover the following form of the Kantorovich duality formula:

$$\mathcal{W}_d(\mu,\nu) = \sup_{\substack{f\in\mathcal{C}(X) \\ |f(x)-f(y)|\leq d(x,y)}} \left| \int f \,\mathrm{d}\mu - \int f \,\mathrm{d}\nu \right|.$$

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