

k -INDUCTIVE AND INTERPOLATION-INSPIRED BARRIER CERTIFICATES FOR STOCHASTIC DYNAMICAL SYSTEMS

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ABSTRACT. In this paper we introduce two notions of barrier certificates that use multiple functions to provide a lower bound on the probabilistic satisfaction of safety for stochastic dynamical systems. A barrier certificate for a stochastic dynamical system is defined as a nonnegative real-valued function over states of the system whose expectation is nonincreasing as the system evolves. This ensures that a barrier certificate acts as a nonnegative supermartingale, and thus provides a lower bound on the probability that the system is safe. The promise of such certificates is that their search can be effectively automated. Typically, one may use optimization or satisfiability modulo theory (SMT) solvers to find such barrier certificates of a given fixed template. When such approaches fail, a typical approach is to instead change the template. We propose an alternative approach inspired by the notion of interpolation. We dub these certificates as interpolation-inspired barrier certificates. An interpolation-inspired barrier certificate consists of a set of functions that jointly provide a lower bound on the probability of satisfying safety. We show how one may find such certificates of a fixed template, even when we fail to find standard barrier certificates of the same template. However, we note that such certificates still need to ensure a supermartingale guarantee for one function in the set. To address this challenge, we consider the use of k -induction in conjunction with these interpolation-inspired certificates. The recent use of k -induction in barrier certificates allows one to relax the supermartingale requirement at every time step to a combination of a supermartingale requirement every k steps and a c -martingale requirement for the intermediate steps. We provide a generic, more permissive formulation of a barrier certificate that we refer to as k -inductive interpolation-inspired barrier certificate. The formulation allows for several combinations of the interpolation-inspired barrier certificate and k -induction. We present two examples among the possible combinations. We finally present sum-of-squares (SOS) programming to synthesize this set of functions and demonstrate the effectiveness of our proposed methods in case studies.

1. INTRODUCTION

Barrier certificates are a prominent method to verify the safety of dynamical systems. The authors of [PJ04], propose the notion of a barrier certificate for continuous-time systems as a real-valued function whose zero level set separates the unsafe region from all the reachable region. The value of the barrier certificate is positive over a given set of unsafe states, nonpositive over a given set of initial states, and nonincreasing as a system evolves. Thus, it acts as an inductive proof of safety. This certificate is also used for stochastic systems [PJP04, PJP07] where it provides probabilistic lower bounds on safety guarantees. Here, a barrier certificate is a nonnegative real-valued function, whose value is greater than or equal to 1 for the unsafe states, less than 1 for the initial states and it remains nonincreasing in *expectation* along the trajectory of the system. Thus, the certificate acts as a supermartingale as a system evolves and provides lower bounds on the probability of the system being safe. Although the search for such a certificate is effectively automatable, it relies on users fixing a template. The search for barrier certificates is carried out effectively using SMT-based approaches [DMB11] or optimization [Par03, PJ04]. In both of these approaches, we search for a function in a fixed template satisfying the conditions characterizing barrier certificates. Typically, when such a function is not successfully found for a given template, a different template is considered. We instead address this challenge by proposing generalized notions of barrier certificates inspired by interpolation [McM03] and k -induction [SSS00, DHKR11], which are typically used in establishing inductive invariants for hardware and software systems.

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A property over the states of a system is said to be an inductive invariant if *i*) it holds true for the set of initial states, and *ii*) it holds true for the next state (following the transition function) if it is true for the current state. Thus, via an inductive argument, an inductive invariant is true for all the reachable states of a system. Ensuring that the negation of this property is true for the set of unsafe states, thus acts as a proof of safety. Unfortunately, while a property may be true for all the reachable states, it often is not inductive. In such cases, a common approach is to *strengthen* it (using methods such as interpolation or considering alternative inductive formulations such as k -induction) as a conjunction of multiple properties to establish an inductive invariant.

A barrier certificate is a discretization-free functional inductive invariant and, as such, the search for such functions suffers from similar issues. Typically, the conditions of barrier certificates are imposed over a single function. However, these conditions can be restrictive. One option to change the notion of induction, as shown in [AMTZ22], is through the use of k -induction. Here, the authors still rely on a single function to act as a k -inductive barrier certificate, where they relax the supermartingale condition at each time step to a c -martingale condition for less than k steps and a supermartingale condition for every k steps.

Another option to relax the standard conditions is by allowing multiple functions to act as barrier certificates via interpolation [McM03], which we employ in this work. We consider a notion of interpolation-inspired barrier certificates that allows us to use a broader range of function templates as proofs of safety, and can be searched for similar to standard barrier certificates. Next, we generalize the notion of k -inductive barrier certificates proposed in [AMTZ22] by considering a k -inductive barrier certificate as a set of functions. Finally, as both k -induction and interpolation provide valid but incomparable advantages when trying to find barrier certificates, we show that they can be combined to form a more general formulation of barrier certificates, which we call k -inductive interpolation-inspired barrier certificates.

Contributions. The contributions of the paper are listed below.

- (1) This work proposes a novel notion of interpolation-inspired barrier certificates that have more permissive conditions compared to the standard formulation of barrier certificates.
- (2) We relax the conditions given in [AMTZ22] to allow a set of functions to be considered a k -inductive barrier certificate.
- (3) We show that the conditions in interpolation-inspired barrier certificate and k -inductive barrier certificate can be combined in several ways to form a more relaxed formulation of barrier certificates.
- (4) We present SOS programming to search for the proposed barrier certificates using a fixed template.
- (5) We show that even when a standard barrier certificate of a given template fails to ensure safety, one can find functions of the same template using the proposed notions of barrier certificates.
- (6) We demonstrate that by selecting specific values for certain hyperparameters, our proposed relaxed formulations retrieve the conventional conditions of barrier certificates.

Related works. Inductive invariants and incremental inductive proofs are important tools in verifying the safety of finite state-transition systems as seen in [ZPH04, CNQ08, Bra11]. Typically, such systems are described as a set of logical variables, where the initial and unsafe states, as well as the transition relation, are described as logical formulae. The safety verification goal is to ensure that the negation of the formula representing the unsafe states is an inductive invariant (we refer the interested reader to Appendix A for more details). Unfortunately, this is often not inductive. Thus, a prominent approach to make it inductive is to incrementally strengthen this formula via interpolation-based approaches [McM03, Bra12]. Given a property of interest, such proofs check whether it is an inductive invariant. If not, they try to incrementally constrain it till an inductive proof is obtained. In the context of bounded model checking, interpolation unrolls the transition function a fixed number of times, for example $\ell \in \mathbb{N}$, and finds intermediate formulae called interpolants until an inductive invariant formula is found. IC3 [Bra11, Bra12] uses overapproximating frames and counterexamples to build incremental formulae one step at a time until an inductive invariant is found. Both of these approaches use multiple formulae to form an inductive invariant.

Since barrier certificates are functional inductive invariants, we now compare our work with works that used multiple functions as barrier certificates. The design and utilization of standard barrier certificates for stochastic systems has been carried out in various works such as [PJP04, JSZ20, HCL⁺17, Cla21, Xue24a, Xue24b]. The authors of [MS19, BMT12, SGTP18, OMTZ24] consider notions of multiple barrier certificates for safety verification of nonstochastic systems, while the authors of [AJZ19, FCX⁺20] consider multiple functions for stochastic systems. Our work differs from the latter as follows. The authors of [AJZ19] consider multiple barrier certificates for the case of switched stochastic systems. They design different standard barrier certificates for each mode of a switched system. In contrast, we consider multiple functions that replace the purpose of a standard barrier certificate. For instance, an interpolation-inspired barrier certificate can be designed for each mode of a switched system. The authors of [FCX⁺20] propose a similar idea as [SGTP18] for continuous time stochastic dynamical systems. The similarity with our work is in the use of multiple functions and exploiting supermartingale properties for bounding the probabilities. The first difference is that the conditions for the initial and unsafe states are imposed over all the functions in their work while our proposed method imposes the conditions over all the functions only for the unsafe states. Furthermore, in their study, they assign the relationship between the functions to a Metzler matrix, whereas our approach does not have to be represented in this manner. Instead of all the functions, we only need one function to act as a supermartingale as the system evolves. The authors of [LSZ24] considered a notion of time-varying k -inductive barrier certificates. We use a similar argument to propose a k -inductive barrier certificate that uses multiple functions. These functions can be considered time-varying such that we pick the i^{th} function for every $t = rk + i$ time-step for some nonnegative integer r . We refer the interested readers to Section 3.2 for more details.

Organization. In Section 2, we outline notations, define stochastic dynamical systems, and review standard barrier certificates. In Section 3, we introduce the first key theoretical result of our paper by proposing interpolation-inspired barrier certificates for safety verification. We then generalize k -inductive barrier certificates and present the second key theoretical result: combining them with interpolation-inspired barrier certificates. An implementation of the proposed techniques is discussed in Section 4, followed by case studies in Section 5. Note that inductive invariants and interpolation are formally covered in the Appendix.

2. PRELIMINARIES

We consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is the sample space, \mathcal{F} is the σ -algebra on Ω that contains subsets of Ω as events in the probability space, and \mathbb{P} is the probability measure that assigns to each event in the event space a probability, which is a number between 0 and 1. We consider random variables to be measurable functions of the form $X : (\Omega, \mathcal{F}) \rightarrow (S_X, \mathcal{F}_X)$ from the sample space Ω to another measurable space S called the state space. Each random variable X is associated with a probability measure on (S_X, \mathcal{F}_X) as $\Pr\{Z\} = \mathbb{P}\{X^{-1}(Z)\}$ for any $Z \in \mathcal{F}_X$.

Let X be topological space. The collection of all Borel sets on X forms the Borel σ -algebra $B(X)$. The map $f : X \rightarrow Y$ is said to be measurable when it is Borel-measurable.

2.1. Notations. We use \mathbb{N} and \mathbb{R} to denote the set of natural numbers and reals, respectively. For $k \in \mathbb{R}$, we use $\mathbb{R}_{\geq k}$ and $\mathbb{R}_{> k}$ to denote the intervals $[k, \infty)$ and (k, ∞) , respectively. Similarly, for any natural number $n \in \mathbb{N}$, we use $\mathbb{N}_{\geq n}$ to denote the set of natural numbers greater than or equal to n . The n -dimensional Euclidean space is denoted by \mathbb{R}^n .

We use \exists and \forall to denote the existential and universal quantifiers, respectively. We use logical operators \vee , \wedge , \neg and \implies for disjunction (logical OR), conjunction (logical AND), negation (logical NOT) and implication, respectively.

For a function $f : \mathcal{X} \times \mathcal{A} \rightarrow \mathcal{X}$ and $k \in \mathbb{N}_{\geq 1}$, we use $f^k : \mathcal{X} \times \mathcal{A}^k \rightarrow \mathcal{X}$ to denote the composition of the function f by itself k -times (i.e. given a set of k values $(a_0, a_1, \dots, a_{k-1})$, we define $f^k(x, a_0) = f(x, a_0)$ for $k = 1$ and $f^k(x, (a_0, a_1, \dots, a_{k-1})) = f(f^{k-1}(x, (a_0, \dots, a_{k-2})), a_{k-1})$ for $k > 1$). Given a collection of

sets \mathcal{X}_i , $i = \{0, 1, \dots, N\}$, we use $\bigcup_{i=0}^N \mathcal{X}_i$ to denote the union of the sets \mathcal{X}_i . Given two sets \mathcal{X} and \mathcal{Y} , $\mathcal{X} \setminus \mathcal{Y} := \{x : x \in \mathcal{X} \text{ and } x \notin \mathcal{Y}\}$. We use $\mathcal{N}(\mu, \sigma^2)$ to denote a normal distribution with mean μ and variance σ^2 .

2.2. Stochastic Dynamical System.

Definition 2.1. A discrete-time stochastic dynamical system (dt-SS) \mathcal{S} is given by the tuple:

$$(1) \quad \mathcal{S} = (\mathcal{X}, \mathcal{X}_0, w, f),$$

where

- $\mathcal{X} \subseteq \mathbb{R}^n$ is a Borel space that represents the state set of \mathcal{S} ;
- $\mathcal{X}_0 \subseteq \mathcal{X}$ denotes a set of initial states;
- $w := \{w(t) : \Omega \rightarrow \mathcal{W}, t \in \mathbb{N}\}$ is a sequence of i.i.d random variables from a sample space Ω to the measurable space $(\mathcal{W}, \mathcal{F}_w)$, commonly interpreted as system noise; and
- $f : \mathcal{X} \times \mathcal{W} \rightarrow \mathcal{X}$ is a measurable function that describes the state evolution of \mathcal{S} .

For $x(t)$, the state of the system at time step $t \in \mathbb{N}$, the state of the system in the next time step is given by the following stochastic difference equation:

$$(2) \quad x(t+1) = f(x(t), w(t)), \quad \forall x(t) \in \mathcal{X}.$$

We use $\mathbf{x}_{x_0} = (x(0), x(1), x(2), \dots)$ to denote the solution process generated by \mathcal{S} starting from the initial state $x(0) = x_0 \in \mathcal{X}_0$. We denote the state at time step $t \in \mathbb{N}$ for the solution process \mathbf{x}_{x_0} as $\mathbf{x}_{x_0}(t)$. Now we define reachable states of a dt-SS.

Definition 2.2 (Reachability). We say a state $x(t_1)$ of a dt-SS \mathcal{S} given in Definition 2.1 is reachable from the state $x(t_0)$ if there exists a solution process $\mathbf{x}_{x(t_0)}$ which contains $x(t_1)$. That is, $x(t_1) = f^i(x(t_0), w_i(t_0))$, for some $i \in \mathbb{N}$ and some $w_i(t_0) = [w(t_0); \dots; w(t_0 + i - 1) = w(t_1 - 1)]$, which is a vector of noise terms from t_0 to $t_1 - 1$.

Since the codomain of the map f is \mathcal{X} , this implicitly implies that the state set \mathcal{X} is forward invariant, which might seem conservative when dealing with unbounded noise, especially when \mathcal{X} is bounded. Following the convention introduced in [Kus67, Xue24b, AMTZ22], to ensure the forward invariance of \mathcal{X} , we adopt the standard assumption of stopping the stochastic process if it reaches the boundary of \mathcal{X} :

Assumption 2.3. For any solution process \mathbf{x}_{x_0} of dt-SS \mathcal{S} starting from some initial state $x_0 \in \mathcal{X}_0$, we have $\mathbf{x}_{x_0}(t) \in \mathcal{X}$ for all $t \in \mathbb{N}$. This is ensured by considering a “stopped process” $\bar{\mathbf{x}}_{x_0}(t)$ given as:

$$(3) \quad \bar{\mathbf{x}}_{x_0}(t) = \begin{cases} \mathbf{x}_{x_0}(t) & \forall t < \tau, \\ \mathbf{x}_{x_0}(\tau - 1) & \forall t \geq \tau. \end{cases}$$

where $\tau \in \mathbb{N}$ is the first exit time of \mathbf{x}_{x_0} from \mathcal{X} .

From this point onward, we assume that the state set \mathcal{X} is forward invariant, without explicitly referring to the stopped process, for the sake of simplicity in presentation.

As we deal with stochastic systems with noise which consist of unbounded support, we are interested in obtaining probabilistic guarantees over the satisfaction of safety for a given dt-SS \mathcal{S} . Particularly, we would like to compute a tight lower bound on the probability of satisfying safety. We present the definition of probabilistic satisfaction of safety below.

Definition 2.4 (Safety Probability). Let $\mathcal{X}_0 \subseteq \mathcal{X}$ and $\mathcal{X}_u \subseteq \mathcal{X}$ represent the set of initial states and unsafe states, respectively for a dt-SS \mathcal{S} given in Definition 2.1. We say \mathcal{S} satisfies safety with a probability bound of λ if the solution processes of \mathcal{S} starting from any $x_0 \in \mathcal{X}_0$ never reach \mathcal{X}_u with a probability of at least λ , i.e.

$$\mathbb{P}\{\mathbf{x}_{x_0}(t) \notin \mathcal{X}_u, \forall t \in \mathbb{N}\} \geq \lambda, \quad \forall x_0 \in \mathcal{X}_0.$$

The goal of safety verification for a dt-SS \mathcal{S} is to compute the probability bound constant $0 \leq \lambda \leq 1$.

2.3. Barrier Certificates. For safety verification of a dt-SS \mathcal{S} as in Definition 2.1, we now discuss the notion of barrier certificates [PJ04] that provide sufficient conditions for safety.

Definition 2.5 (Barrier Certificate). *A function $\mathcal{B} : \mathcal{X} \rightarrow \mathbb{R}$ is a barrier certificate for a dt-SS \mathcal{S} with respect to a set of initial states \mathcal{X}_0 and a set of unsafe states \mathcal{X}_u if there exists a constant $0 \leq \gamma \leq 1$ such that:*

$$\begin{aligned}
 (4) \quad & \mathcal{B}(x) \geq 0 && \forall x \in \mathcal{X}, \\
 (5) \quad & \mathcal{B}(x) \leq \gamma && \forall x \in \mathcal{X}_0, \\
 (6) \quad & \mathcal{B}(x) \geq 1 && \forall x \in \mathcal{X}_u, \\
 (7) \quad & \mathbb{E}[\mathcal{B}(f(x, w))|x] \leq \mathcal{B}(x) && \forall x \in \mathcal{X} \setminus \mathcal{X}_u.
 \end{aligned}$$

Observe that condition (7) ensures that \mathcal{B} acts as a supermartingale, *i.e.*, the expected value of the function is nonincreasing at every time step. Definition 2.5 can be used to obtain the lower bound on the probability that the dt-SS \mathcal{S} satisfies safety.

Theorem 2.6 (Barrier certificates imply safety [PJP07]). *Consider a dt-SS \mathcal{S} as in Definition 2.1. Let there exist a function $\mathcal{B} : \mathcal{X} \rightarrow \mathbb{R}$ for \mathcal{S} such that it is a barrier certificate as in Definition 2.5 for some $0 \leq \gamma \leq 1$. The probability that the solution process \mathbf{x}_{x_0} starting from an initial state $x_0 \in \mathcal{X}_0$ does not reach unsafe states \mathcal{X}_u is bounded by*

$$(8) \quad \mathbb{P}\{\mathbf{x}_{x_0}(t) \notin \mathcal{X}_u, \forall t \in \mathbb{N}\} \geq 1 - \gamma.$$

We remark that Definition 2.5 is only useful if the initial set \mathcal{X}_0 and the unsafe set \mathcal{X}_u are disjoint.

The common approach to search for such barrier certificates relies on first fixing their template, where one considers the certificate to be a linear combination of some fixed basis functions. For example, if the template is a polynomial of a given degree, then we consider the basis functions to be monomials. If the template is a neural network [AAE⁺21], one may fix the number of nodes and layers. Then one employs search techniques, such as Satisfiability Modulo Theory (SMT) solvers [DMB11] or Sum-of-Squares (SOS) programming [Par03], to search for the coefficients satisfying conditions (4)-(7). Unfortunately, if no barrier certificate is found, the common approach adopted is to change the template. For example, one may choose a higher degree polynomial or change the network architecture to reflect a change in the template. Such changes typically make searching for barrier certificates computationally demanding and may lead to inconclusive results. For example, the authors of [BGS24] compute certificates in 2 minutes but have a time-out of 6 hours to verify these certificates through the SMT solver z3 [DMB08]. We now illustrate with the help of the following example, an alternative approach to remedy this issue.

Example 1. *Consider a one-dimensional dt-SS*

$$(9) \quad \mathcal{S} : x(t+1) = 0.5x(t) + 0.05w(t).$$

The state set, initial, and unsafe sets are given by $\mathcal{X} = [0, 3]$, $\mathcal{X}_0 = [2, 2.3]$, and $\mathcal{X}_u = [1.6, 1.9]$, respectively. We take $\gamma = 0.5$ and $w(t) \sim \mathcal{N}(0, 1)$. We consider a cubic barrier certificate template. Based on the following set of inequalities that are infeasible, one can verify that there exists no standard cubic barrier certificate.

$$\begin{aligned}
 & 0 \leq B(x = 0), \\
 & 0 \leq B(x = 3), \\
 & 0 \leq B(x = 2) \leq \gamma, \\
 & 1 \leq B(x = 1.9), \text{ and} \\
 & 0 \leq B(x = 2) - \mathbb{E}[B(f(x = 2, w))].
 \end{aligned}$$

In the next section, we define a notion of multiple barrier certificates that we refer to as interpolation-inspired barrier certificates. We show how the conditions for these are less restrictive and can be used to give a nontrivial probabilistic bound on safety for Example 1.

3. INTERPOLATION AND k -INDUCTIVE BARRIER CERTIFICATES

Here, we introduce a notion of interpolation-inspired barrier certificates and discuss their relation to k -inductive barrier certificates. First, we discuss how ideas from interpolation may be used to consider different conditions for barrier certificates. For a comprehensive explanation of interpolation related to inductive invariants in the context of hardware verification as discussed in [McM03, Bra11], the interested reader is referred to Appendix A.

3.1. Interpolation-Inspired Barrier Certificate (IBC). Here, we introduce a notion of interpolation-inspired barrier certificates (IBC) and demonstrate their efficacy.

Definition 3.1 (IBC). Consider a dt-SS \mathcal{S} as in Definition 2.1. A set of functions $\mathcal{B}_i : \mathcal{X} \rightarrow \mathbb{R}$, for all $0 \leq i \leq \ell$, is an IBC for \mathcal{S} if there exist constants $0 \leq \gamma \leq 1$, $\ell \in \mathbb{N}$, $\alpha_i \in \mathbb{R}_{>0}$, $0 \leq i < \ell$ such that:

$$\begin{aligned}
 (10) \quad & \mathcal{B}_i(x) \geq 0 && \forall x \in \mathcal{X}, 0 \leq i \leq \ell, \\
 (11) \quad & \mathcal{B}_0(x) \leq \gamma && \forall x \in \mathcal{X}_0, \\
 (12) \quad & \mathcal{B}_i(x) \geq 1 && \forall x \in \mathcal{X}_u, 0 \leq i \leq \ell, \\
 (13) \quad & \mathbb{E}[\mathcal{B}_{i+1}(f(x, w)) | x] \leq \alpha_i \mathcal{B}_i(x) && \forall x \in \mathcal{X} \setminus \mathcal{X}_u, 0 \leq i < \ell, \\
 (14) \quad & \mathbb{E}[\mathcal{B}_\ell(f(x, w)) | x] \leq \mathcal{B}_\ell(x) && \forall x \in \mathcal{X} \setminus \mathcal{X}_u.
 \end{aligned}$$

Definition 3.1 can be used to obtain the lower bound on the probability that a dt-SS \mathcal{S} is safe.

Theorem 3.2 (IBCs imply safety). Consider a dt-SS \mathcal{S} as in Definition 2.1. Let there exist a set of functions $\mathcal{B}_i : \mathcal{X} \rightarrow \mathbb{R}$, $0 \leq i \leq \ell$, for \mathcal{S} such that it is an IBC as in Definition 3.1 for some $0 \leq \gamma \leq 1$, $\alpha_i \in \mathbb{R}_{>0}$, $0 \leq i < \ell$. The probability that the solution process \mathbf{x}_{x_0} starting from an initial state $x_0 \in \mathcal{X}_0$ does not reach unsafe set \mathcal{X}_u is lower bounded by

$$(15) \quad \mathbb{P}\{\mathbf{x}_{x_0}(t) \notin \mathcal{X}_u, \forall t \in \mathbb{N}\} \geq 1 - \gamma \left(1 + \prod_{i=0}^{\ell-1} \alpha_i + \sum_{t=0}^{\ell-2} \prod_{i=0}^t \alpha_i \right).$$

Observe that smaller values of γ and α_i provide better bounds. In particular, if we have $\alpha_i = 1$ for all $0 \leq i < \ell$, then the probability of safety is upper bounded by $1 - \gamma(1 + \ell)$.

Proof. Following condition (12), we have $\mathcal{X}_u \subseteq \{x \in \mathcal{X} : \mathcal{B}_i(x) \geq 1, 0 \leq i \leq \ell\}$. Now, we can separate the probability of visiting an unsafe state into visiting the unsafe state in less than and after ℓ time steps as follows:

$$\begin{aligned}
 (16) \quad & \mathbb{P}\{\exists t \in \mathbb{N} : \mathbf{x}_{x_0}(t) \in \mathcal{X}_u\} \leq \mathbb{P}\{\exists 0 \leq t < \ell : \mathbf{x}_{x_0}(t) \in \mathcal{X}_u\} + \mathbb{P}\{\exists t \geq \ell : \mathbf{x}_{x_0}(t) \in \mathcal{X}_u\} \\
 (17) \quad & \leq \mathbb{P}\{\exists 0 \leq t < \ell : \mathcal{B}_t(x(t)) \geq 1\} + \mathbb{P}\{\exists t \geq \ell : \mathcal{B}_\ell(x(t)) \geq 1\}.
 \end{aligned}$$

Each of these terms can be upper bounded with the use of Boole's inequality and Markov's inequality as follows:

$$\begin{aligned}
 (18) \quad & \mathbb{P}\{\exists 0 \leq t < \ell : \mathcal{B}_t(x(t)) \geq 1\} \\
 (19) \quad & \leq \mathbb{P}\left\{ \bigcup_{t=0}^{\ell-1} (\mathcal{B}_t(x(t)) \geq 1) \right\} \leq \sum_{t=0}^{\ell-1} \mathbb{P}\{\mathcal{B}_t(x(t)) \geq 1\} \\
 (20) \quad & \leq \mathbb{E}[\mathcal{B}_0(x(0))] + \sum_{t=1}^{\ell-1} \mathbb{E}[\mathcal{B}_t(x(t))].
 \end{aligned}$$

Using the law of total expectation and condition (13) inductively, the expectations can be upper bounded as follows for $1 \leq j \leq \ell$:

$$\begin{aligned}
 (21) \quad & \mathbb{E}[\mathcal{B}_j(x(j))] = \mathbb{E}(\mathbb{E}[\mathcal{B}_j(x(j))|x(j-1)]) \\
 (22) \quad & \leq \alpha_{j-1} \mathbb{E}[\mathcal{B}_{j-1}(x(j-1))] \leq \dots \\
 (23) \quad & \leq \mathbb{E}[\mathcal{B}_0(x(0))] \prod_{i=0}^{j-1} \alpha_i \leq \gamma \prod_{i=0}^{j-1} \alpha_i.
 \end{aligned}$$

Then, it follows that:

$$(24) \quad \mathbb{P}\{\exists 0 \leq t < \ell : \mathcal{B}_t(x(t)) \geq 1\} \leq \gamma + \gamma \sum_{t=0}^{\ell-2} \prod_{i=0}^t \alpha_i.$$

Conditions (10) and (14) show that \mathcal{B}_ℓ is a nonnegative supermartingale. By use of Ville's inequality and (21)-(23),

$$(25) \quad \mathbb{P}\{\exists t \geq \ell : \mathcal{B}_\ell(x(t)) \geq 1\} \leq \mathbb{E}[\mathcal{B}_\ell(x(\ell))] \leq \gamma \prod_{i=0}^{\ell-1} \alpha_i.$$

By complementation of the sum of (24) and (25) in (17), we get the lower bound in (15). □

Note that by setting $\ell = 0$ in Definition 3.1, conditions (10), (11), (12) and (14) reduce to the standard barrier certificate conditions as in Definition 2.5 while condition (13) is no longer applicable. This is relevant for the implementation as we first start with $\ell = 0$ to find a standard barrier certificate. We then increment ℓ by one only if we fail, and check for satisfaction of conditions (10)-(14). We repeat the above until we find an IBC or we reach a maximum number ℓ_{max} . Any IBC found for $\ell > 0$ indicates that a standard barrier certificate with the given template could not be found. Also observe that once an IBC is found for a given $\ell \in \mathbb{N}$, we guarantee that an IBC can be found for all $j > \ell$. Thus, ℓ is the minimum integer that forms an IBC for a given fixed template.

We now show that even if a standard barrier certificate cannot be found, one might be able to find an IBC based on Example 1.

Example 1 (Continued). *We consider a set of cubic functions as the template for $\mathcal{B}_i(x)$. We attempt to compute the coefficients of the functions, with an upper bound $i \leq \ell_{max} = 3$ such that the collection of $\mathcal{B}_i(x)$ is an IBC as given in Definition 3.1. We take $\gamma = 0.5$ and $\alpha_i = 0.75$ for all $0 \leq i < \ell$. We utilize TSSOS [WML21] in Julia to solve conditions (10)-(14) and find the coefficients. See Section 4.1 for the SOS solver formulation.*

An IBC was found for $\ell = 1$ with $\mathcal{B}_0(x) = -10.475x^3 + 77.522x^2 - 187.854x + 149.919$ and $\mathcal{B}_1(x) = 0.639x^3 - 0.864x^2 + 0.375x + 0.00133$. Figure 1 shows the IBC computed along with the relevant initial and unsafe sets. Function $\mathcal{B}_0(x)$ is always between 0 and γ for the initial states and larger than 1 for the unsafe states as expected. Observe that the set of states that are reachable in one step or more from the initial state are within the set $\{x : 0 \leq \mathcal{B}_1(x) < 1\}$ but not in the set $\{x : 0 \leq \mathcal{B}_0(x) \leq \gamma\}$. Based on Theorem 3.2, the lower bound on the probability of safety is 0.125.

While our probabilistic guarantee for Example 1 is low, we illustrate this on more detailed examples in Section 5. The low guarantee can be explained due to the proximity of reachable regions to the unsafe set as shown in Figure 1.

Our view of interpolation-inspired barrier certificates tackles a similar problem to that of k -inductive barrier certificates introduced in [AMTZ22]. We discuss these similarities and differences in the following subsection.

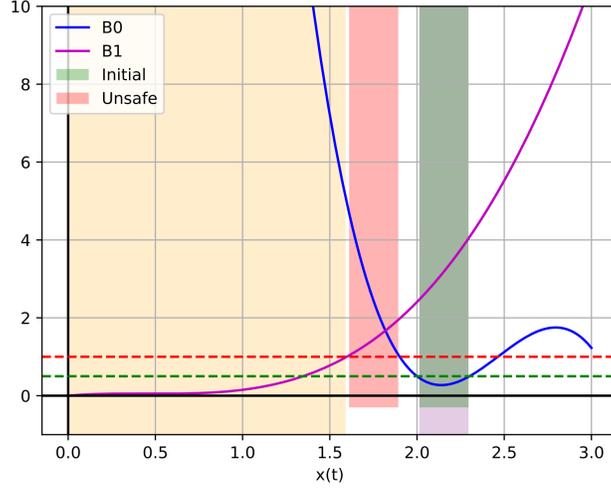


FIGURE 1. IBC with $\ell = 1$. The purple (overlapping green) and orange shaded regions represent the sets $\{x : 0 \leq \mathcal{B}_0(x) \leq \gamma\}$ and $\{x : 0 \leq \mathcal{B}_1(x) < 1\}$, respectively. They together overapproximate all the reachable states with probability of at least 0.125. The green and red dashed horizontal lines indicate γ and 1, respectively.

3.2. Relaxing k -Inductive Barrier Certificates. To discuss k -inductive barrier certificates, we first discuss some details on notation. For a dt-SS \mathcal{S} as in Definition 2.1, the value of the solution process after the i^{th} time step is given by $x(t+i) = f^i(x(t), w_i(t))$, where $w_i(t) = [w(t); \dots; w(t+i-1)]$ is the vector containing all the noise terms from time t to $t+i-1$.

As stated in [AMTZ22], a function $\mathcal{B} : \mathcal{X} \rightarrow \mathbb{R}$ is a k -inductive barrier certificate for dt-SS \mathcal{S} for some constants $k \in \mathbb{N}_{\geq 1}$, $0 \leq \lambda_0 \leq 1$, and $c \geq 0$ if:

$$(26) \quad \mathcal{B}(x) \geq 0 \quad \forall x \in \mathcal{X},$$

$$(27) \quad \mathcal{B}(x) \leq \lambda_0 \quad \forall x \in \mathcal{X}_0,$$

$$(28) \quad \mathcal{B}(x) \geq 1 \quad \forall x \in \mathcal{X}_u,$$

$$(29) \quad \mathbb{E}[\mathcal{B}(f(x, w)) | x] \leq \mathcal{B}(x) + c \quad \forall x \in \mathcal{X} \setminus \mathcal{X}_u,$$

$$(30) \quad \mathbb{E}[\mathcal{B}(f^k(x, w_k)) | x] \leq \mathcal{B}(x) \quad \forall x \in \mathcal{X} \setminus \mathcal{X}_u.$$

We propose the following definition of k -inductive barrier certificates that has a relaxed set of conditions compared to the above. The following formulation is similar to the formulation of IBCs, in that they both use multiple functions. However, the two formulations are in general incomparable.

Definition 3.3. Consider a dt-SS \mathcal{S} as in Definition 2.1. A set of functions $\mathcal{B}_i : \mathcal{X} \rightarrow \mathbb{R}$, $0 \leq i < k$, is a k -inductive barrier certificate for \mathcal{S} if there exist constants $k \in \mathbb{N}_{\geq 1}$ and $\lambda_i \in \mathbb{R}_{>0}$, $0 \leq i < k$, such that:

$$(31) \quad \mathcal{B}_i(x) \geq 0 \quad \forall x \in \mathcal{X}, 0 \leq i < k,$$

$$(32) \quad \mathcal{B}_0(x) \leq \lambda_0 \quad \forall x \in \mathcal{X}_0,$$

$$(33) \quad \mathcal{B}_i(x) \geq 1 \quad \forall x \in \mathcal{X}_u, 0 \leq i < k,$$

$$(34) \quad \mathbb{E}[\mathcal{B}_i(f^i(x_0, w_i)) | x_0] \leq \lambda_i \quad \forall x_0 \in \mathcal{X}_0, 1 \leq i < k,$$

$$(35) \quad \mathbb{E}[\mathcal{B}_i(f^k(x, w_k)) | x] \leq \mathcal{B}_i(x) \quad \forall x \in \mathcal{X} \setminus \mathcal{X}_u, 0 \leq i < k.$$

The lower bound on the probability that the dt-SS \mathcal{S} is safe can be derived from Definition 3.3 using the following result.

Theorem 3.4. Consider a dt-SS \mathcal{S} as in Definition 2.1. Let there exist a set of functions $\mathcal{B}_i : \mathcal{X} \rightarrow \mathbb{R}$, $0 \leq i < k$, for \mathcal{S} such that it is a k -inductive barrier certificate as in Definition 3.3 for some $k \in \mathbb{N}_{\geq 1}$ and $\lambda_i \geq 0, 0 \leq i < k$. The probability that the solution process \mathbf{x}_{x_0} starting from an initial state $x_0 \in \mathcal{X}_0$ does not reach unsafe set \mathcal{X}_u is bounded by

$$(36) \quad \mathbb{P}\{\mathbf{x}_{x_0}(t) \notin \mathcal{X}_u, \forall t \in \mathbb{N}\} \geq 1 - \sum_{i=0}^{k-1} \lambda_i.$$

To get a meaningful probability, one requires the sum of the values of λ_i to be below 1. One can select λ_i to be decision variables and try to minimize them as much as possible.

Proof. Consider k systems sampled after every k steps, each starting from initial conditions $x(0), \dots, x(k-1)$, respectively. The dynamic's equations are given as follows:

$$\begin{aligned} x(t+k) &= f^k(x(t), w_k(t)), \\ x(t+k+1) &= f^k(x(t+1), w_k(t+1)), \\ &\vdots \\ x(t+2k-1) &= f^k(x(t+k-1), w_k(t+k-1)). \end{aligned}$$

Condition (35) implies that the \mathcal{B}_i satisfy the supermartingale condition for each of these systems. Via Boole's inequality and Ville's inequality, we get

$$(37) \quad \begin{aligned} &\mathbb{P}\{\exists t = i + jk, j \in \mathbb{N}, 0 \leq i < k : \mathcal{B}_i(x(t)) \geq 1\} \\ &\leq \sum_{i=0}^{k-1} \mathbb{P}\{\exists t = jk, j \in \mathbb{N} : \mathcal{B}_i(x(t+i)) \geq 1\} \end{aligned}$$

$$(38) \quad \leq \mathbb{E}[B_0(x(0))] + \sum_{i=1}^{k-1} \mathbb{E}[B_i(x(i))].$$

Following conditions (32) and (34), and using law of total expectation for each term on the right hand side of the inequality above, we have

$$\begin{aligned} \mathbb{P}\{\exists t \in \mathbb{N} : \mathbf{x}_{x_0}(t) \in \mathcal{X}_u\} &\leq \lambda_0 + \sum_{i=1}^{k-1} \mathbb{E}(\mathbb{E}[B_i(f^i(x_0, w_i)) | x_0]) \\ &\leq \lambda_0 + \sum_{i=1}^{k-1} \mathbb{E}(\lambda_i) = \sum_{i=0}^{k-1} \lambda_i. \end{aligned}$$

By complementation, we get the lower bound in (36). □

Note that a k -inductive barrier certificate satisfying conditions (26)-(30) also satisfies conditions (31)-(35) by choosing $\mathcal{B}_i = \mathcal{B}$ and $\lambda_i = \lambda_0 + ic$. The lower bound on probability of safety via this choice is the same as the one given in [AMTZ22] based on the following simplification:

$$\sum_{i=0}^{k-1} \lambda_i = \sum_{i=0}^{k-1} (\lambda_0 + ic) = k\lambda_0 + \frac{k(k-1)c}{2}.$$

The relaxation of Definition 3.3 follows from condition (34), where the conditional is only over an initial state $x_0 \in \mathcal{X}_0$ while condition (29) requires the inequality to hold for any given state $x \in \mathcal{X} \setminus \mathcal{X}_u$.

3.3. Combining Interpolation and k -Induction. We note that while both k -inductive barrier certificates and IBCs provide similar benefits, they are generally incomparable. IBCs still require a supermartingale condition at every step, but rely on different functions which are interrelated but may themselves not be supermartingales. On the other hand, k -inductive barrier certificates rely on providing a supermartingale at every k^{th} step but bound the expected value by constants λ_i for all $0 \leq i < k$. Thus, it is possible that k -inductive barrier certificates are not found while IBCs are found for the same template. In fact, we did not find cubic k -inductive barrier certificates for the system in Example 1 via SOS with $\gamma = 0.5, c \leq 0.5$, and $k \leq 4$. This demonstrates that IBCs can be a better option for certain systems. Still, since both of these notions of barrier certificates are incomparable, we show that there is a method of combining them methodically.

As mentioned earlier, the function \mathcal{B}_ℓ from Definition 3.1 is a nonnegative supermartingale for every time step starting at ℓ . This condition could be restrictive and make finding suitable barrier certificates challenging. As discussed in [AMTZ22], the supermartingale requirement at each time step for probabilistic safety guarantees could be relaxed for bounded-time horizon using c -martingale and combined with k -induction for unbounded time guarantees. Motivated by this relaxation, we combine the IBC formulation from Definition 3.1 with the principle of k -induction to formulate a notion of what we call k -inductive interpolation-inspired barrier certificate (k -IBC).

We should note that there are many possible ways of formulating k -IBCs. Let ℓ denote the number of functions considered for interpolation and k be the bound on k -induction. Then, the number of ways of combining them reduces to the number of ways of uniquely finding a supermartingale argument. For ℓ number of functions in an IBC, we can apply k -induction to m of these functions in $\binom{\ell}{m}$ ways, where $1 \leq m \leq \ell$. Then for each of these m options, we can select the last function to use for interpolation. This gives us a total of $\mathcal{O}\left(\sum_{m=1}^{\ell} \binom{\ell}{m} m\right)$ ways of combining interpolation and k -induction without considering potentially redundant formulations. However, not all of them may provide the same benefit, and further analysis is required to determine if a certain combination will lead to better probabilistic guarantees or simpler templates than others. We now propose two notions of k -IBC.

The first notion of k -IBC derived from Definition 3.1 and conditions (29) and (30) is defined as follows.

Definition 3.5 (k -IBC v1). *Consider a dt-SS \mathcal{S} as in Definition 2.1. A set of functions $\mathcal{B}_i : \mathcal{X} \rightarrow \mathbb{R}, 0 \leq i \leq \ell$, is a k -IBC for \mathcal{S} if there exist constants $0 \leq \gamma \leq 1, \ell \in \mathbb{N}, k \in \mathbb{N}_{\geq 1}, c \in \mathbb{R}_{\geq 0}, \alpha_i \in \mathbb{R}_{>0}, 0 \leq i < \ell$, such that:*

$$\begin{aligned}
(39) \quad & \mathcal{B}_i(x) \geq 0 && \forall x \in \mathcal{X}, 0 \leq i \leq \ell, \\
(40) \quad & \mathcal{B}_0(x) \leq \gamma && \forall x \in \mathcal{X}_0, \\
(41) \quad & \mathcal{B}_i(x) \geq 1 && \forall x \in \mathcal{X}_u, 0 \leq i \leq \ell, \\
(42) \quad & \mathbb{E}[\mathcal{B}_{i+1}(f(x, w))|x] \leq \alpha_i \mathcal{B}_i(x) && \forall x \in \mathcal{X} \setminus \mathcal{X}_u, 0 \leq i < \ell, \\
(43) \quad & \mathbb{E}[\mathcal{B}_\ell(f(x, w))|x] \leq \mathcal{B}_\ell(x) + c && \forall x \in \mathcal{X} \setminus \mathcal{X}_u, \\
(44) \quad & \mathbb{E}[\mathcal{B}_\ell(f^k(x, w_k))|x] \leq \mathcal{B}_\ell(x) && \forall x \in \mathcal{X} \setminus \mathcal{X}_u.
\end{aligned}$$

Note that condition (43) requires \mathcal{B}_ℓ to be a c -martingale at each time step and condition (44) requires \mathcal{B}_ℓ sampled after every k^{th} step to be a supermartingale. Note that $c = 0$ gives us back IBC. We now present the probabilistic bound of safety for k -IBC v1.

Theorem 3.6 (k -IBC v1 implies safety). *Consider a dt-SS \mathcal{S} as in Definition 2.1. Let there exist a set of functions $\mathcal{B}_i : \mathcal{X} \rightarrow \mathbb{R}, 0 \leq i \leq \ell$, for \mathcal{S} such that it is a k -IBC as in Definition 3.5 for some $0 \leq \gamma \leq 1, \ell \in \mathbb{N}, k \in \mathbb{N}_{\geq 1}, c \in \mathbb{R}_{\geq 0}, \alpha_i \in \mathbb{R}_{>0}, 0 \leq i < \ell$. The probability that the solution process \mathbf{x}_{x_0} starting from an initial state $x_0 \in \mathcal{X}_0$ does not reach unsafe set \mathcal{X}_u is lower bounded by*

$$(45) \quad \mathbb{P}\{\mathbf{x}_{x_0}(t) \notin \mathcal{X}_u, \forall t \in \mathbb{N}\} \geq 1 - \gamma \left(1 + k \prod_{i=0}^{\ell-1} \alpha_i + \sum_{t=0}^{\ell-2} \prod_{i=0}^t \alpha_i \right) - \frac{k(k-1)c}{2}.$$

Proof. Expressions (16) and (17) still hold and the proof for less than ℓ time steps holds per (24). For more than or equal to ℓ time steps, consider k systems sampled after every k steps, each starting from initial conditions $x(\ell), \dots, x(\ell + k - 1)$, respectively. The dynamic's equations are given as follows:

$$(46) \quad x(t + \ell + k) = f^k(x(t + \ell), w_k(t + \ell)),$$

$$(47) \quad x(t + \ell + k + 1) = f^k(x(t + \ell + 1), w_k(t + \ell + 1)),$$

\vdots

$$(48) \quad x(t + \ell + 2k - 1) = f^k(x(t + \ell + k - 1), w_k(t + \ell + k - 1)).$$

Condition (44) implies that \mathcal{B}_ℓ satisfies the supermartingale condition for each of these systems. Via Boole's inequality and Ville's inequality, we get

$$(49) \quad \begin{aligned} & \mathbb{P}\{\exists t \geq \ell : \mathcal{B}_\ell(x(t)) \geq 1\} \\ & \leq \sum_{i=0}^{k-1} \mathbb{P}\{\exists t = j\ell, j \in \mathbb{N} : \mathcal{B}_\ell(x(t + \ell + i)) \geq 1\} \end{aligned}$$

$$(50) \quad \leq \sum_{i=0}^{k-1} \mathbb{E}[B_\ell(x(\ell + i))].$$

Using the law of total expectation, condition (43) and expressions (21)-(23), the expectations can be bounded as follows:

$$(51) \quad \mathbb{E}[B_\ell(x(\ell + i))] = \mathbb{E}(\mathbb{E}[\mathcal{B}_\ell(x(\ell + i)) | x(\ell + i - 1)])$$

$$(52) \quad \leq \mathbb{E}[\mathcal{B}_\ell(x(\ell + i - 1))] + c \leq \dots$$

$$(53) \quad \leq \mathbb{E}[\mathcal{B}_\ell(x(\ell))] + ic \leq \gamma \prod_{j=0}^{\ell-1} \alpha_j + ic.$$

Then, it follows that:

$$(54) \quad \mathbb{P}\{\exists t \geq \ell : \mathcal{B}_\ell(x(t)) \geq 1\} \leq \sum_{i=0}^{k-1} \left(\gamma \prod_{j=0}^{\ell-1} \alpha_j + ic \right)$$

$$(55) \quad \leq k\gamma \prod_{i=0}^{\ell-1} \alpha_i + \frac{k(k-1)c}{2}.$$

By complementation of the sum of (24) and (55) in (17), we get the lower bound in (45). □

We now show that k -IBC's could help improve the probabilistic lower bound of safety using Example 1.

Example 1 (Continued). *We consider a set of cubic functions as the template for $\mathcal{B}_i(x)$. We attempt to compute the coefficients of the functions, with an upper bound $i \leq \ell_{max} = 3$ and $k_{max} = 3$ such that the collection of $\mathcal{B}_i(x)$ is a k -IBC v1 as given in Definition 3.5. We take $\gamma = 0.35, c = 0.00026$ and $\alpha_i = 0.3$ for all $0 \leq i < \ell$. We utilize TSSOS [WML21] in Julia to solve conditions (39)-(44) and find the coefficients. See Section 4.2 for the SOS solver formulation.*

A k -IBC v1 was found for $\ell = 3, k = 3$ with $\mathcal{B}_0(x) = -12.457x^3 + 93.486x^2 - 228.960x + 183.983$, $\mathcal{B}_1(x) = 2.123x^3 - 4.498x^2 + 2.377x + 0.0155$, $\mathcal{B}_2(x) = 0.515x^3 - 0.514x^2 + 0.131x + 0.00043$ and $\mathcal{B}_3(x) = 0.321x^3 - 0.131x^2 + 0.0135x$. Figure 2 shows the IBC computed along with the relevant initial and unsafe sets. The function $\mathcal{B}_0(x)$ is always between 0 and γ for the initial states and larger than 1 for the unsafe states. Observe that the set of states that are reachable in one or more steps from the initial state are within the set $\bigcup_{i=1}^3 \{x : 0 \leq \mathcal{B}_i(x) < 1\}$ but not in the set $\{x : 0 \leq \mathcal{B}_0(x) \leq \gamma\}$. Based on Theorem 3.6, the improved lower bound on the probability of safety is 0.484.

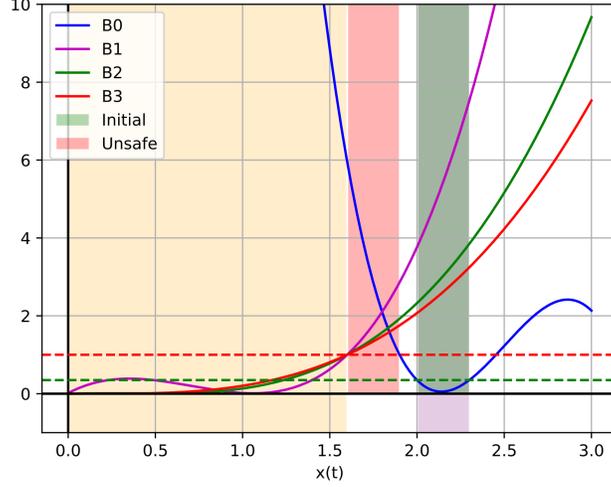


FIGURE 2. k -IBC v1 with $\ell = 3$. The purple (overlapping green) and orange shaded regions represent the sets $\{x : 0 \leq \mathcal{B}_0(x) \leq \gamma\}$ and $\bigcup_{i=1}^3 \{x : 0 \leq \mathcal{B}_i(x) < 1\}$, respectively. They together overapproximate all the reachable states with probability of at least 0.484. The green and red dashed horizontal lines indicate γ and 1, respectively.

While our probabilistic guarantee for Example 1 is by itself low, it is improved compared to the probabilistic bound obtained from only an IBC due to the addition of k -induction.

The second instance of k -IBC derived from Definitions 3.1 and 3.3 is defined as follows.

Definition 3.7 (k -IBC v2). *Consider a dt-SS \mathcal{S} as in Definition 2.1. A set of functions $\mathcal{B}_i : \mathcal{X} \rightarrow \mathbb{R}$, for all $0 \leq i < \ell + k$, is a k -IBC for \mathcal{S} if there exist constants $0 \leq \gamma \leq 1$, $\ell \in \mathbb{N}$, $k \in \mathbb{N}_{\geq 1}$, $\alpha_i \in \mathbb{R}_{>0}$, $0 \leq i < \ell$ and $\beta_j \in \mathbb{R}_{>0}$, $1 \leq j < k$, such that:*

$$(56) \quad \mathcal{B}_i(x) \geq 0 \quad \forall x \in \mathcal{X}, \quad 0 \leq i < \ell + k,$$

$$(57) \quad \mathcal{B}_0(x) \leq \gamma \quad \forall x \in \mathcal{X}_0,$$

$$(58) \quad \mathcal{B}_i(x) \geq 1 \quad \forall x \in \mathcal{X}_u, \quad 0 \leq i < \ell + k,$$

$$(59) \quad \mathbb{E}[\mathcal{B}_{i+1}(f(x, w)) | x] \leq \alpha_i \mathcal{B}_i(x) \quad \forall x \in \mathcal{X} \setminus \mathcal{X}_u, \quad 0 \leq i < \ell,$$

$$(60) \quad \mathbb{E}[\mathcal{B}_{\ell+j}(f^{2j+1}(x, w_{2j+1})) | x] \leq \beta_j \mathcal{B}_{\ell-j-1}(x) \quad \forall x \in \mathcal{X} \setminus \mathcal{X}_u, \quad 1 \leq j < \ell,$$

$$(61) \quad \mathbb{E}[\mathcal{B}_{\ell+j}(f^{\ell+j}(x, w_{\ell+j})) | x] \leq \beta_j \mathcal{B}_0(x) \quad \forall x \in \mathcal{X} \setminus \mathcal{X}_u, \quad \ell \leq j < k,$$

$$(62) \quad \mathbb{E}[\mathcal{B}_{\ell+j}(f^k(x, w_k)) | x] \leq \mathcal{B}_{\ell+j}(x) \quad \forall x \in \mathcal{X} \setminus \mathcal{X}_u, \quad 0 \leq j < k.$$

Observe that in the above formulation, from Definition 3.3, $\lambda_0 = \alpha_{\ell-1} \mathcal{B}_{\ell-1}(x)$, $\lambda_j = \beta_j \mathcal{B}_{\ell-j-1}(x)$ for $1 \leq j < \ell$ or $\lambda_j = \beta_j \mathcal{B}_0(x)$ for $\ell \leq j < k$. Additionally, condition (61) is not applicable if $k < \ell$ and condition (60) will only apply for $1 \leq j < k$. When ℓ steps are reached through interpolation, k -induction is utilized. The functions derived from interpolation are then employed to bound the functions involved in the k -induction process. Specifically, for this scenario, the expected value of $\mathcal{B}_{\ell+j}$ at the $(\ell + j)^{th}$ step is bounded by the value of $\mathcal{B}_{\ell-j-1}$ as stated in condition (60). This results in a step difference of $\ell + j - (\ell - j - 1) = 2j + 1$ to transition from the state overestimated by $\mathcal{B}_{\ell-j-1}$ to that of $\mathcal{B}_{\ell+j}$, justifying the use of $2j + 1$ in condition (60). An analogous rationale explains the adoption of $\ell + j$ in condition (61).

We now present the usefulness of k -IBC v2.

Theorem 3.8 (k -IBC v2 implies safety). *Consider a dt-SS \mathcal{S} as in Definition 2.1. Let there exist a set of functions $\mathcal{B}_i : \mathcal{X} \rightarrow \mathbb{R}$, $0 \leq i < \ell + k$, for \mathcal{S} such that it is a k -IBC as in Definition 3.7 for some $0 \leq \gamma \leq 1$,*

$\ell \in \mathbb{N}$, $k \in \mathbb{N}_{\geq 1}$, $\alpha_i \in \mathbb{R}_{>0}$, $0 \leq i < \ell$, and $\beta_j \in \mathbb{R}_{>0}$, $1 \leq j < k$. The probability that the solution process \mathbf{x}_{x_0} starting from an initial state $x_0 \in \mathcal{X}_0$ does not reach unsafe set \mathcal{X}_u is bounded by

$$(63) \quad \mathbb{P}\{\mathbf{x}_{x_0}(t) \notin \mathcal{X}_u, \forall t \in \mathbb{N}\} \geq 1 - \gamma \left(1 + \prod_{i=0}^{\ell-1} \alpha_i + \sum_{t=0}^{\ell-2} \prod_{i=0}^t \alpha_i + \sum_{j=1}^{\ell-1} \left(\beta_j \prod_{i=0}^{\ell-j-1} \alpha_i \right) + \sum_{j=\ell}^{k-1} \beta_j \right).$$

Proof. Once again, equation (16) still holds and the proof for less than ℓ time steps holds per (24). For more than or equal to ℓ time steps, consider k systems given in equations (46)-(48). Condition (62) implies that each $\mathcal{B}_{\ell+j}$ satisfies the supermartingale condition for each of these systems. Via Boole's inequality and Ville's inequality, we get

$$(64) \quad \begin{aligned} & \mathbb{P}\{\exists t = i\ell + j, i \in \mathbb{N}_{\geq 1}, 0 \leq j < k : \mathcal{B}_{\ell+j}(x(t)) \geq 1\} \\ & \leq \sum_{j=0}^{k-1} \mathbb{P}\{\exists t = i\ell, i \in \mathbb{N} : \mathcal{B}_{\ell+j}(x(t + \ell + j)) \geq 1\} \end{aligned}$$

$$(65) \quad \leq \mathbb{E}[B_\ell(x(\ell))] + \sum_{j=1}^{k-1} \mathbb{E}[B_{\ell+j}(x(\ell + j))].$$

$\mathbb{P}\{\exists t = i + jk, j \in \mathbb{N}, 0 \leq i < k : \mathcal{B}_i(x(t)) \geq 1\}$ The first term from the right hand side of the above inequality can be upper bounded using inequality (23). Each term of the summation can be upper bounded using conditions (60), (61), inequality (23) and law of total expectation as follows. For $1 \leq j < \ell$,

$$(66) \quad \mathbb{E}[B_{\ell+j}(x(\ell + j))] = \mathbb{E}(\mathbb{E}[\mathcal{B}_{\ell+j}(f^{2j+1}(x(\ell - j - 1), w_{2j+1}) | x(\ell - j - 1)])$$

$$(67) \quad \leq \beta_j \mathbb{E}[B_{\ell-j-1}(x(\ell - j - 1))] \leq \gamma \beta_j \prod_{i=0}^{\ell-j-1} \alpha_i.$$

For $\ell \leq j < k$,

$$(68) \quad \mathbb{E}[B_{\ell+j}(x(\ell + j))] = \mathbb{E}(\mathbb{E}[\mathcal{B}_{\ell+j}(f^{\ell+j}(x(0), w_{\ell+j}) | x(0)])$$

$$(69) \quad \leq \beta_j \mathbb{E}[B_0(x(0))] \leq \gamma \beta_j.$$

Then,

$$(70) \quad \sum_{j=0}^{k-1} \mathbb{P}\{\exists t \geq \ell : \mathcal{B}_{\ell+j}(x(t)) \geq 1\} \leq \gamma \prod_{i=0}^{\ell-1} \alpha_i + \gamma \sum_{j=1}^{\ell-1} \left(\beta_j \prod_{i=0}^{\ell-j-1} \alpha_i \right) + \gamma \sum_{j=\ell}^{k-1} \beta_j.$$

By complementation of the sum of (24) and (70) in (16), we get the lower bound in (63). \square

Remark 3.9. Note that for a k -IBC, a choice of $\ell = 0, k = 1$ boils down to a standard barrier certificate, a choice of $k = 1$ boils down to an IBC and a choice of $\ell = 0$ boils down to a k -inductive barrier certificate.

4. SYNTHESIZING IBC AND k -IBC

Here, we provide computational methods for synthesizing IBCs and k -IBC based on Definitions 3.1 and 3.5, respectively. To do so, we first note that a set $V \subseteq \mathbb{R}^n$ is semi-algebraic if it can be defined with a vector of polynomial inequalities of $h(x)$ as $V = \{x \in \mathbb{R}^n : h(x) \geq 0\}$, where the inequalities are element-wise.

The technique of using semidefinite programming [Par03] and framing the search for standard barrier certificates as SOS polynomials [PJP07] is usually simpler and takes less time when compared to SMT based approaches. In this section, we provide an SOS formulation as we found it to be the most effective for our case studies.

To do so, we make the following assumption over dt-SS \mathcal{S} .

Assumption 4.1. *The dt-SS \mathcal{S} has a continuous state set $\mathcal{X} \subseteq \mathbb{R}^n$, and its transition function $f : \mathcal{X} \times \mathcal{W} \rightarrow \mathcal{X}$ is a polynomial function of the state variable x and noise variable w . The sets \mathcal{X} , \mathcal{X}_0 and \mathcal{X}_u are semi-algebraic with corresponding vectors of polynomials $g(x)$, $g_0(x)$ and $g_u(x)$, respectively.*

We now show how one may utilize a SOS approach to find IBCs and k -IBC.

4.1. **IBC.** Under Assumption 4.1, the IBC conditions (10)-(14) can be formulated as a set of SOS constraints in order to compute a polynomial IBC of a predefined degree per the following lemma.

Lemma 4.2. *Consider a dt-SS \mathcal{S} . Suppose Assumption 4.1 holds for \mathcal{S} . Suppose there exist constants $0 \leq \gamma \leq 1$, $\ell \in \mathbb{N}$, $\alpha_i \in \mathbb{R}_{>0}$, $0 \leq i < \ell$, polynomials of same degree $\mathcal{B}_i(x)$ and SOS polynomials $\hat{\eta}_i(x)$, $\eta_0(x)$, $\eta_{u,i}(x)$, $\eta_i(x)$, $\hat{\eta}(x)$ of appropriate dimensions such that the following expressions are SOS polynomials:*

$$(71) \quad \mathcal{B}_i(x) - \hat{\eta}_i^T(x)g(x) \quad \forall 0 \leq i \leq \ell,$$

$$(72) \quad \gamma - \mathcal{B}_0(x) - \eta_0^T(x)g_0(x),$$

$$(73) \quad \mathcal{B}_i(x) - 1 - \eta_{u,i}^T(x)g_u(x) \quad \forall 0 \leq i \leq \ell,$$

$$(74) \quad \alpha_i \mathcal{B}_i(x) - \mathbb{E}[\mathcal{B}_{i+1}(f(x, w))|x] - \eta_i^T(x)g(x) \quad \forall 0 \leq i < \ell,$$

$$(75) \quad \mathcal{B}_\ell(x) - \mathbb{E}[\mathcal{B}_\ell(f(x, w))|x] - \hat{\eta}^T(x)g(x),$$

where x is the state variable over the set \mathcal{X} and w is the noise variable over the set \mathcal{W} . Then, the set of functions $\mathcal{B}_i(x)$, $0 \leq i \leq \ell$, is an IBC following Definition 3.1.

4.2. **k -IBC.** The SOS formulations for the two instances of k -IBCs discussed in Section 3.3 are given as follows.

4.2.1. **k -IBC v1.** Under Assumption 4.1, the k -IBC v1 conditions (39)-(44) can be formulated as a set of SOS constraints per the following lemma.

Lemma 4.3. *Consider a dt-SS \mathcal{S} . Suppose Assumption 4.1 holds for \mathcal{S} . Suppose there exist constants $0 \leq \gamma \leq 1$, $\ell \in \mathbb{N}$, $k \in \mathbb{N}_{\geq 1}$, $c \in \mathbb{R}_{\geq 0}$, $\alpha_i \in \mathbb{R}_{>0}$, $0 \leq i < \ell$, polynomials of same degree $\mathcal{B}_i(x)$ and SOS polynomials $\hat{\eta}_i(x)$, $\eta_0(x)$, $\eta_{u,i}(x)$, $\eta_i(x)$, $\hat{\eta}(x)$, $\hat{\eta}_k(x)$ of appropriate dimensions such that the following expressions are SOS polynomials:*

$$(76) \quad \mathcal{B}_i(x) - \hat{\eta}_i^T(x)g(x) \quad \forall 0 \leq i \leq \ell,$$

$$(77) \quad \gamma - \mathcal{B}_0(x) - \eta_0^T(x)g_0(x),$$

$$(78) \quad \mathcal{B}_i(x) - 1 - \eta_{u,i}^T(x)g_u(x) \quad \forall 0 \leq i \leq \ell,$$

$$(79) \quad \alpha_i \mathcal{B}_i(x) - \mathbb{E}[\mathcal{B}_{i+1}(f(x, w))|x] - \eta_i^T(x)g(x) \quad \forall 0 \leq i < \ell,$$

$$(80) \quad \mathcal{B}_\ell(x) + c - \mathbb{E}[\mathcal{B}_\ell(f(x, w))|x] - \hat{\eta}^T(x)g(x),$$

$$(81) \quad \mathcal{B}_\ell(x) - \mathbb{E}[\mathcal{B}_\ell(f^k(x, w))|x] - \hat{\eta}_k^T(x)g(x),$$

where x is the state variable over the set \mathcal{X} and w is the noise variable over the set \mathcal{W} . Then the set of functions $\mathcal{B}_i(x)$, $0 \leq i \leq \ell$, is a k -IBC v1 following Definition 3.5.

4.2.2. **k -IBC v2.** Similarly, under Assumption 4.1, the k -IBC v2 conditions (56)-(62) can be formulated as a set of SOS constraints per the following lemma.

Lemma 4.4. *Consider a dt-SS \mathcal{S} . Suppose Assumption 4.1 holds for \mathcal{S} . Suppose there exist constants $0 \leq \gamma \leq 1$, $\ell \in \mathbb{N}$, $k \in \mathbb{N}_{\geq 1}$, $\alpha_i \in \mathbb{R}_{>0}$, $0 \leq i < \ell$, and $\beta_j \in \mathbb{R}_{>0}$, $1 \leq j < k$, polynomials of same degree*

$\mathcal{B}_i(x)$ and SOS polynomials $\hat{\eta}_i(x), \eta_0(x), \eta_{u,i}(x), \eta_i(x), \eta_{\ell,j}(x), \hat{\eta}_{j,k}(x)$ of appropriate dimensions such that the following expressions are SOS polynomials:

$$(82) \quad \mathcal{B}_i(x) - \hat{\eta}_i^T(x)g(x) \quad \forall 0 \leq i < \ell + k,$$

$$(83) \quad \gamma - \mathcal{B}_0(x) - \eta_0^T(x)g_0(x),$$

$$(84) \quad \mathcal{B}_i(x) - 1 - \eta_{u,i}^T(x)g_u(x) \quad \forall 0 \leq i < \ell + k,$$

$$(85) \quad \alpha_i \mathcal{B}_i(x) - \mathbb{E}[\mathcal{B}_{i+1}(f(x, w))|x] - \eta_i^T(x)g(x) \quad \forall 0 \leq i < \ell,$$

$$(86) \quad \beta_j \mathcal{B}_{\ell-j-1}(x) - \mathbb{E}[\mathcal{B}_{\ell+j}(f^{2j+1}(x, w_{2j+1}))|x] - \eta_{\ell,j}^T(x)g(x) \quad \forall 1 \leq j < \ell,$$

$$(87) \quad \beta_j \mathcal{B}_0(x) - \mathbb{E}[\mathcal{B}_{\ell+j}(f^{\ell+j}(x, w_{\ell+j}))|x] - \eta_{\ell,j}^T(x)g(x) \quad \forall \ell \leq j < k,$$

$$(88) \quad \mathcal{B}_{\ell+j}(x) - \mathbb{E}[\mathcal{B}_{\ell+j}(f^k(x, w_k))|x] - \hat{\eta}_{j,k}^T(x)g(x) \quad \forall 0 \leq j < k,$$

where x is the state variable over the set \mathcal{X} and w is the noise variable over the set \mathcal{W} . Then the set of functions $\mathcal{B}_i(x)$, $0 \leq i < \ell + k$, is a k -IBC v2 following Definition 3.7.

5. CASE STUDIES

We experimentally demonstrate the utility of IBCs and k -IBCs on a Lotka–Volterra-type model and a logistic map model. Given a state $x = x(t)$, we use x_+ to denote $x(t + 1)$. The resulting certificates are provided in Appendix B.

5.1. Lotka–Volterra Model. For our first case study, we consider the discrete-time Lotka–Volterra type prey–predator model with state variables v, p denoting the victim/prey and the predator, respectively. The dynamics is given by the following difference equations:

$$(89) \quad \begin{cases} v_+ = v + T(\theta v(1 - v) - \phi vp) + G_0 w, \\ p_+ = p - T(\psi p - \delta vp) + G_1 w, \end{cases}$$

where $T = 0.1s$ is the sampling time, $\theta = 1.1$ is the growth rate of the prey, $\phi = 0.4$ is the death rate of the prey, $\psi = 0.4$ is the death rate of the predator, $\delta = 0.1$ is the growth rate of the predator, and $G_0 = 0.01$, $G_1 = 0.005$ are the noise coefficients. The state set, initial set, and unsafe set are given by $\mathcal{X} = [0, 10] \times [0, 5]$, $\mathcal{X}_0 = [6, 7] \times [2, 3]$, and $\mathcal{X}_u = [3, 5] \times [0, 3]$, respectively. We first consider a degree five polynomial function in two variables as our parametric template of the barrier certificate $\mathcal{B}(v, p)$ and attempt to compute suitable coefficients such that $\mathcal{B}(v, p)$ is a standard barrier certificate as in Definition 2.5 (i.e. IBC with $\ell = 0$). We used TSSOS [WML21] in Julia to reformulate conditions (4)–(7) as SOS optimization problem as described in the previous section with $\gamma = 0.1$. However, we were unable to find a standard barrier certificate.

We then reformulated conditions (10)–(14) as an SOS optimization problem via Lemma 4.2. We set $\alpha_i = 0.44$, $\ell_{max} = 3$, $\gamma = 0.1$ and considered the same parametric form as above for $\mathcal{B}_i(v, p)$. We obtained an IBC with $\ell = 1$. Figure 3 shows the resulting plot of the functions along with an overapproximation of the reachable regions of the system. Observe that the blue and purple shaded regions denote the sublevel sets of the functions $\mathcal{B}_0(v, p)$ and $\mathcal{B}_1(v, p)$, respectively. Together, they contain all reachable states of the system with a probability of at least 0.856, per Theorem 3.2.

5.2. Logistic Map Model. The logistic map is one of the most common models used in chaos theory to demonstrate how complex, chaotic behaviour can arise from very simple nonlinear dynamical equations. We considered a discrete-time logistic map model with state variable x denoting the ratio of existing population to the maximum possible population. The dynamics is given by the following difference equation:

$$(90) \quad x_+ = rx(1 - x) + Gw,$$

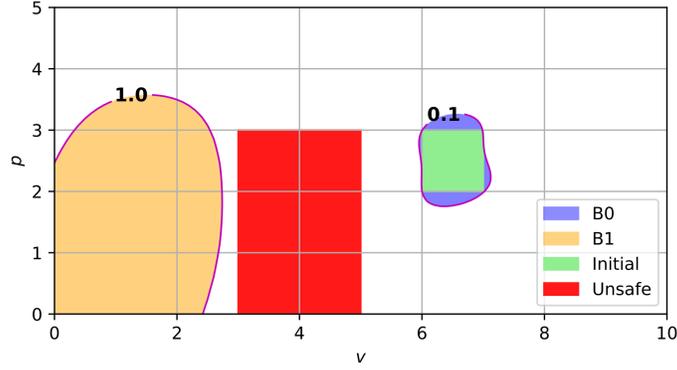


FIGURE 3. IBC with $\ell = 1$ for Lotka-Volterra type model. The axes show the state variables v and p . The blue and purple shaded regions show the sets $\{x : 0 \leq \mathcal{B}_0(v, p) \leq \gamma\}$ and $\{x : 0 \leq \mathcal{B}_1(v, p) < 1\}$, respectively.

where $r = 1.5$ is a parameter that dictates the behavior of the model and $G = 0.0005$ is the noise coefficient. The state set, initial set, and unsafe set are given by $\mathcal{X} = [0, 1]$, $\mathcal{X}_0 = [0.85, 0.95]$, and $\mathcal{X}_u = [0.6, 0.7]$, respectively. We consider a degree five polynomial function as our parametric template of the barrier certificate function(s). We attempted to compute suitable coefficients such that we get a standard barrier certificate, an IBC or a k -inductive barrier certificate following Definitions 2.5, 3.1 or conditions (26)-(30), respectively. We formulated their corresponding conditions as SOS optimization problem using TSSOS in Julia but we found that no suitable coefficients exist. Therefore, one cannot verify the safety of this system using either of these options for a barrier certificate.

We then reformulated conditions (39)-(44) as an SOS optimization problem via Lemma 4.3. By taking all $\alpha_i = 0.3$, $\ell_{max} = 2$, $k_{max} = 3$, $c = 0.005$, $\gamma = 0.01$ and the same parametric form as above for $\mathcal{B}_i(x)$, we obtained a k -IBC v1 with $\ell = 1$, $k = 2$. Figure 4 shows the resulting plot of the functions along with an overapproximation of the reachable regions of the system. Note again that the purple and orange shaded regions denote the sublevel sets of the functions $\mathcal{B}_0(x)$ and $\mathcal{B}_1(x)$, respectively. They jointly contain all

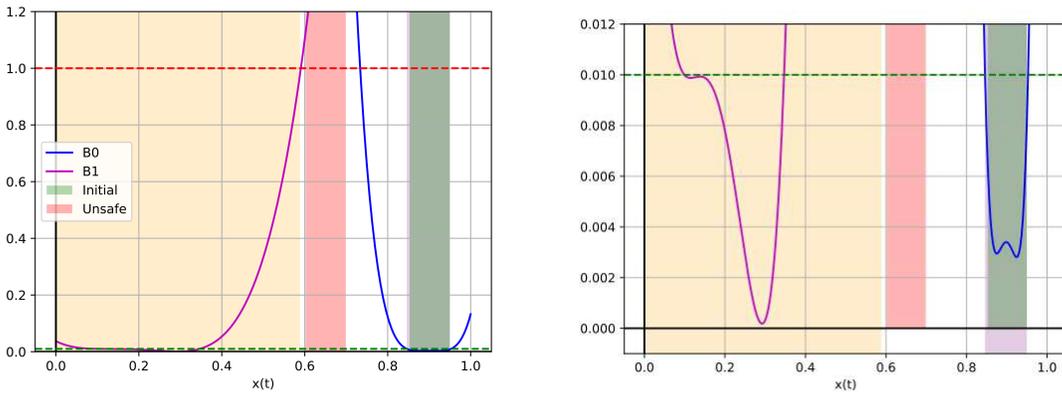


FIGURE 4. k -IBC v1 for logistic map model. The purple (mostly overlapping green) and orange shaded regions show the sets $\{x : 0 \leq \mathcal{B}_0(x) \leq \gamma\}$ and $\{x : 0 \leq \mathcal{B}_1(x) < 1\}$, respectively. The green and red dashed horizontal lines indicate γ and 1, respectively. The figure on the right shows a close-up of the functions near the zero region.

reachable states of the system with a probability of at least 0.979, per Theorem 3.6. The close-up is provided for better clarity near the zero value of the vertical axis.

6. CONCLUSION

We proposed a notion of interpolation-inspired barrier certificate (IBC) and k -inductive interpolation-inspired barrier certificate (k -IBC) for stochastic dynamical systems. These certificates relax the conditions of a standard barrier certificate by incrementally finding functions that together guarantee safety. We presented SOS optimization as a prominent technique of computing this set of functions under mild assumptions. Using an example and case studies, we demonstrated that given a barrier certificate template, one may find IBC and k -IBC even when standard barrier certificates do not exist for a system. Given that SOS-based approaches do not computationally perform well for systems with high dimensions, we hope that the potential to find multiple low degree polynomials via IBC and k -IBC will alleviate these concerns. As future work, we plan to investigate data-driven and neural network based approaches to find IBCs and k -IBC as well as explore their use in controller synthesis. We also plan to investigate how to automate the search for combinations of interpolation and k -induction with the help of counterexamples.

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APPENDIX A. INDUCTIVE INVARIANTS AND INTERPOLATION

We describe the notion of inductive invariants as discussed in [Bra11]. Consider a finite-state system (which is not stochastic), where the state set is a set of logical values while the initial set of states and transition map are described by propositional logical formulae. That is, $\mathcal{X} \subseteq \{\text{true}, \text{false}\}^n$, $\mathcal{X}_0 = \{x : I(x) = \text{true}\}$, and $x' = f(x, w = 0)$ such that $T(x, x') = \text{true}$, where the formula $I(x)$ is the initial condition over the system’s states x , and $T(x, x')$ is the transition relation from the current state x to the next state x' .

We look at a safety property expressed by a logical formula $P(x)$ described over the state variable $x \in \mathcal{X}$. We say that such a system satisfies a safety property if, for every reachable state $x \in \mathcal{X}$ from the initial set, we have $P(x) = \text{true}$. A prominent, and effective approach to prove safety is through the use of inductive invariants. A formula Q is said to be an inductive invariant, if:

- $\forall x \in \mathcal{X}$, we have $I(x) \implies Q(x)$, and
- $\forall x, x' \in \mathcal{X}$, we have $Q(x) \wedge T(x, x') \implies Q(x')$.

Observe that any reachable state x satisfies an inductive invariant formula Q . Thus, showing that a safety property P is an inductive invariant acts as a proof of safety. When we fail to prove P to be an inductive invariant (that is $I(x) \not\implies P(x)$ and/or $P(x) \wedge T(x, x') \not\implies P(x')$), we try to *strengthen* P . Property \tilde{P} is said to be an inductive strengthening of a non-inductive safety property P if there exists a formula F such that $\tilde{P} = F \wedge P$ is inductive. In [MP12], two strengthening strategies are discussed: i) using a stronger property, or ii) performing an incremental proof using previously computed formulae. Interpolation [McM03] is one of these incremental techniques used in the inductive strengthening process.

For interpolation, we unroll the transition relation for some $\ell \in \mathbb{N}$ times and construct a formula representing all possible execution paths from an initial state (with the assumption that all states before the ℓ^{th} step are safe). Let's say x_i is the state after the i^{th} transition. Then the sequence of states for this unrolling is given by:

$$(91) \quad I(x_0) \wedge T(x_0, x_1) \wedge \cdots \wedge T(x_{\ell-1}, x_\ell) \wedge \neg P(x_\ell),$$

where logical formula $P(x)$ describes a safety property. For $\ell = 0$, the formula reduces down to $I(x_0) \wedge \neg P(x_0)$.

If formula (91) is satisfied, then we conclude that the system is unsafe. Otherwise, we utilize interpolation to try to prove safety by finding an intermediate logical formula or a series of formulae called interpolants via Craig's interpolation theorem [McM03] as follows.

Theorem A.1 (Craig's interpolation theorem). *Given a pair of clauses (a disjunction of boolean variables or their negation) E and G such that $E \wedge G$ is unsatisfiable, there exists an intermediate interpolant clause F such that:*

- $E \implies F$,
- $F \wedge G$ is unsatisfiable, and
- F refers to the common variables of E and G .

The proof of Theorem A.1 can be found in [TS00].

Based on this theorem, when formula (91) is unsatisfiable, there exists intermediate formulae F_j such that formula (91) can be broken down as follows:

$$(92) \quad \underbrace{I(x_0)}_{E_0(x_0)} \wedge \underbrace{T(x_0, x_1) \wedge \cdots \wedge T(x_{\ell-1}, x_\ell) \wedge \neg P(x_\ell)}_{G_0(x_0, x_1, \dots, x_\ell)} \text{ is unsatisfiable, then}$$

$$\left\{ \begin{array}{l} I(x_0) \implies F_0(x_0), \text{ and} \\ \underbrace{F_0(x_0) \wedge T(x_0, x_1)}_{E_1(x_0, x_1)} \wedge \cdots \wedge \underbrace{T(x_{\ell-1}, x_\ell) \wedge \neg P(x_\ell)}_{G_1(x_1, \dots, x_\ell)} \text{ is unsatisfiable.} \end{array} \right.$$

G_1 can iteratively be separated as follows:

$$(93) \quad \left\{ \begin{array}{l} I(x_0) \implies F_0(x_0) \\ F_0(x_0) \wedge T(x_0, x_1) \implies F_1(x_1) \\ \vdots \\ F_{\ell-1}(x_{\ell-1}) \wedge T(x_{\ell-1}, x_\ell) \implies F_\ell(x_\ell), \text{ and} \\ F_\ell(x_\ell) \wedge \neg P(x_\ell) \text{ is unsatisfiable.} \end{array} \right.$$

Condition (93) says that the set of $x_j \in \mathcal{X}$ where the formula $F_j(x_j)$ is true is an over-approximation of states reachable in j steps and states satisfying $F_j(x_j)$ will not violate the safety property after $(\ell - j)$ transitions. The interpolants can be computed as shown in [Bra11, Bra12]. We start with $\ell = 0$ and incrementally compute a sequence of interpolants $F_0(x_0) = I(x_0), F_1(x_1), \dots, F_\ell(x_\ell)$ by setting $E(x_i, x_{i+1}) = F(x_i) \wedge T(x_i, x_{i+1})$ and $G(x_{i+1}, \dots, x_\ell) = T(x_{i+1}, x_{i+2}) \wedge \cdots \wedge T(x_{\ell-1}, x_\ell) \wedge \neg P(x_\ell)$ according to Theorem A.1. This iterative process is stopped when the union of the initial formula and all computed interpolants contains all reachable states.

APPENDIX B. CASE STUDY RESULTS

The simulations were conducted on a Windows 11 device equipped with an AMD Ryzen 9 4900HS Mobile Processor and 16 GB of RAM.

The IBC in Section 5.1 took 13.7952s to compute. The certificates we find are:

$$\begin{aligned}\mathcal{B}_0(v, p) &= 0.00241v^5 + 0.0115v^4p - 0.057v^3p^2 + 0.0989v^2p^3 - 0.0671vp^4 + 0.0353p^5 - 0.0865v^4 - 0.116v^3p \\ &\quad + 0.595v^2p^2 - 1.061vp^3 + 0.509p^4 + 1.0195v^3 + 0.0716v^2p + 0.367vp^2 - 0.285p^3 - 4.412v^2 - 3.139vp \\ &\quad + 2.242p^2 + 6.03v + 2.937p + 10.558, \\ \mathcal{B}_1(v, p) &= 0.0998v^5 - 0.0256v^4p + 0.081v^3p^2 + 0.00266v^2p^3 + 0.0217vp^4 + 0.0138p^5 - 0.437v^4 - 0.229v^3p \\ &\quad - 0.22v^2p^2 - 0.188vp^3 - 0.12p^4 + 0.714v^3 + 0.865v^2p + 0.576vp^2 + 0.46p^3 - 0.402v^2 - 1.096vp \\ &\quad - 0.789p^2 - 0.0827v + 0.783p + 0.153.\end{aligned}$$

For Section 5.2 (*k*-IBC v1), the computation time was 20.0444s, and the certificates were:

$$\begin{aligned}\mathcal{B}_0(x) &= 193.77232x^5 + 541.60798x^4 - 3515.71744x^3 + 5445.90085x^2 - 3474.43221x + 809.0011, \\ \mathcal{B}_1(x) &= 2.1214x^5 + 37.98675x^4 - 28.37654x^3 + 7.07388x^2 - 0.73407x + 0.03721.\end{aligned}$$

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