Non-Hermitian wave turbulence

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Wave turbulence describes the long-time statistical behavior of out-of-equilibrium systems composed of weakly interacting waves. Non-Hermitian media ranging from open quantum systems to active materials can sustain wave propagation in so-called $\mathcal{PT}$-symmetric states where gain and loss are effectively balanced. Here, we derive the kinetic equations governing wave turbulence in a prototypical non-Hermitian medium: a three-dimensional fluid with odd viscosity. We calculate its exact anisotropic solution, the so-called Kolmogorov-Zakharov spectrum, and validate the existence of this regime using direct numerical simulations. This non-Hermitian wave turbulence generates a direct cascade that is sustained down to the smallest scales, suppressing the transition to strong turbulence typically observed in rotating fluids and electron magnetohydrodynamics. Beyond odd viscous fluids, this qualitative mechanism applies to any non-linear system of waves where non-Hermitian effects are enhanced at small scales through gradient terms in the dynamical equations, e.g. via odd elastic moduli or other non-reciprocal responses.

Wave turbulence describes the long-time statistical behavior of out-of-equilibrium systems composed of weakly interacting waves [1–6]. Non-Hermitian media ranging from open quantum systems to active materials can sustain wave propagation in so-called $\mathcal{PT}$-symmetric states where gain and loss are effectively balanced [7–35]. Here we ask: what happens to such non-Hermitian media when their non-linear dynamics is dominated by weakly interacting waves? We dub this realm *non-Hermitian wave turbulence*.

We exemplify our approach by studying this wave turbulence in a prototypical non-linear non-Hermitian system: the Navier-Stokes equations of a fluid with additional dissipationless viscosity coefficients variously known as odd, Hall or gyroviscosity [36–51]. Non-Hermitian media often feature activity (i.e. gain) at the microscopic level in addition to dissipation (i.e. loss), both of which can balance to generate waves in the system. As a case in point, experimental realizations of odd-viscous fluids, ranging from magnetized polyatomic gases [52] to magnetized graphene [53] and spinning colloids [54], are all characterized by the presence of rotational drive at the microscopic level.

If a fluid is rotated as a whole around a fixed axis, anisotropic wave turbulence is typically observed at large scales, while isotropic strong turbulence is observed at small scales [55–57], see Fig. 1a. In our non-Hermitian fluid, by contrast, wave turbulence is sustained all the way down to the smallest active scales of the flow (Fig. 1b). Intuitively, this non-Hermitian effect occurs because odd viscosity acts on velocity gradients. Unlike chiral body forces, e.g. Coriolis, the generation of waves by odd viscosity is thus enhanced as wavenumber increases, suppressing the potential transition to strong turbulence. Strong turbulence is observed in odd fluids as a distinct regime (Fig. 1c), studied in Refs. [58, 59], where the description of the odd fluid in terms of weakly interacting waves breaks down, in a similar way as the quasi-particle picture breaks down in non-Fermi liquids [60]. Beyond odd viscous fluids, the qualitative mechanism investigated here applies to any non-linear system of waves where non-Hermitian effects are enhanced at small scales through gradient terms in the dynamical equations, e.g. via odd elastic moduli or other non-reciprocal responses [16, 37, 61].

Non-Hermitian chiral fluid—Consider an incompressible chiral fluid with cylindrical symmetry in a direction $\boldsymbol{e}_\parallel$ [39], along which all the particles are assumed to be spinning. The flow $\boldsymbol{u}$ is described by the Navier-Stokes equations including odd viscosity,

$$D_t\boldsymbol{u} = -\boldsymbol{\nabla}p + \begin{pmatrix} \nu & \nu_{\mathrm{odd}} & 0 \\ -\nu_{\mathrm{odd}} & \nu & 0 \\ 0 & 0 & \nu \end{pmatrix} \Delta\boldsymbol{u} + \boldsymbol{f} \qquad (1)$$

in which $D_t\boldsymbol{u} \equiv \partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}$ and $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$. Here, p is the reduced pressure, $\boldsymbol{f}$ the driving force of the flow, ν the regular viscosity, and ν_{odd} is odd viscosity. The matrix in Eq. (1) is not symmetric when $\nu_{\mathrm{odd}} \neq 0$, manifesting the non-Hermitian character of odd viscosity. The wavevector $\boldsymbol{k}$ is decomposed as $\boldsymbol{k} \equiv (k_\perp\cos\theta, k_\perp\sin\theta, k_\parallel)$ in the basis $(\boldsymbol{e}^1_\perp, \boldsymbol{e}^2_\perp, \boldsymbol{e}_\parallel)$ in which the matrix in Eq. (1) is expressed. Odd viscosity can be seen as a wavenumber-dependent Coriolis force $-\nu_{\mathrm{odd}}k^2\boldsymbol{e}_\parallel \times \boldsymbol{u}$ that is stronger at large $k \equiv |\boldsymbol{k}|$, as manifested in the dispersion relation $\omega_{\boldsymbol{k}} = \pm\nu_{\mathrm{odd}}k_\parallel k$ of the odd waves it induces [36, 58, 59].

Conditions for wave turbulence—In order to as-

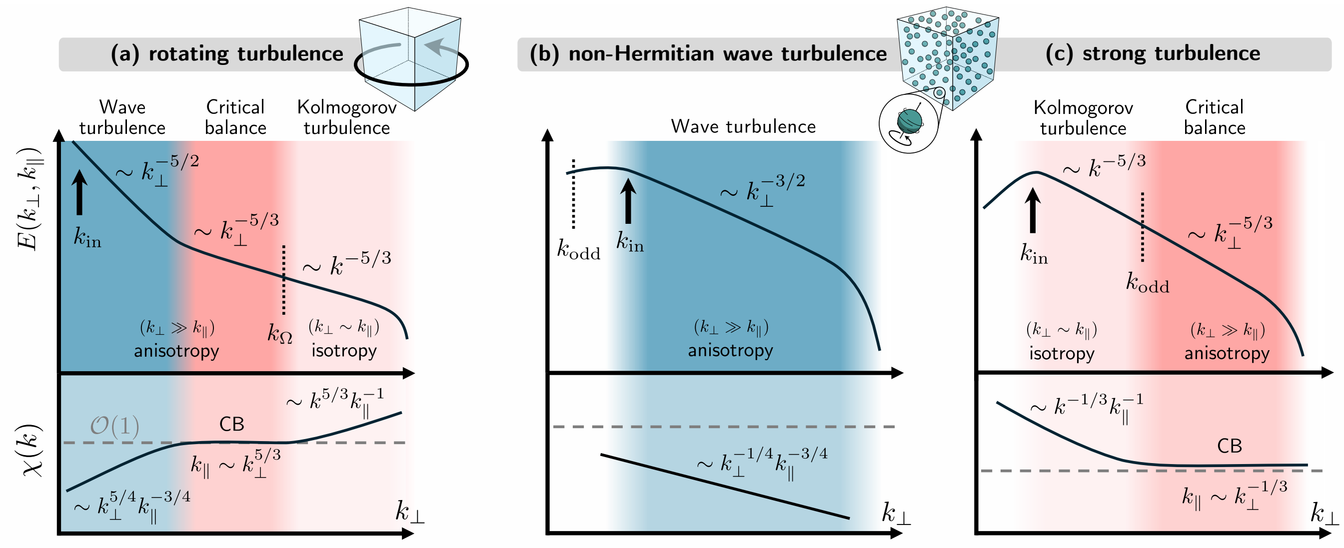

FIG. 1. Wave turbulence in rotating fluids (a) and in fluids with odd viscosity (b). The top row depicts the anisotropic energy spectra $E(k_\perp, k_\parallel)$, while the bottom row depicts the time scale ratio $\chi \equiv \tau_{\rm lin}/\tau_{\rm NL}$ for some $k_\parallel > 0$. The dashed line in the bottom panel denotes the critical value $\chi \approx \mathcal{O}(1)$ below which wave turbulence is expected. For rotating wave turbulence (a), the dynamics gradually becomes less weak and ultimately recovers strong turbulence and isotropization at small scales beyond the Zeman wavenumber k_Ω [6, 55–57]. By contrast, in odd turbulence, in the range $k > k_{\rm in} > k_{\rm odd}$ (b), weak wave turbulence is sustained all the way down to the smallest active scales, while strong turbulence is only obtained if energy is injected at large scales $k_{\rm in} < k_{\rm odd}$ (c).

sess whether wave turbulence may arise, we compare the time scale of non-linear interactions $\tau_{\rm NL}$ with the period of the linear waves $\tau_{\rm lin}$. Here, we estimate $\tau_{\rm NL}$ as the eddy turnover time, which can be expressed in terms of the kinetic energy spectrum $E(k_\perp, k_\parallel)$ as $\tau_{\rm NL} \sim 1/(k\sqrt{E(k_\perp, k_\parallel)k_\perp k_\parallel})$, while the period $\tau_{\rm lin} \sim 1/\omega_{\boldsymbol{k}}$ of the waves is given by their dispersion relation. The ratio of these timescales is

$$\chi \equiv \frac{\tau_{\rm lin}}{\tau_{\rm NL}} = \frac{\sqrt{E(k_\perp, k_\parallel)k_\perp k_\parallel}}{\nu_{\rm odd} k_\parallel}. \qquad (2)$$

While regular, eddy-dominated, *strong* turbulence is typically obtained when $\chi \geq \mathcal{O}(1)$, wave or *weak* turbulence can emerge when $\chi \ll \mathcal{O}(1)$.

The phenomenology of turbulence that is encountered then depends on the scale at which energy is injected. When energy is injected at large scales $k_{\rm in} < k_{\rm odd} \equiv \varepsilon^{1/4}\nu_{\rm odd}^{-3/4}$ where ε is the turbulent energy injection rate [58], we encounter Kolmogorov-like turbulence in the range $k < k_{\rm odd}$ where waves are slow ($\chi > \mathcal{O}(1)$). When the crossover scale $k_{\rm odd}$ is passed, by construction, the eddy timescale and wave timescale become of the same order and we enter a state which in wave turbulence is known as critical balance [6, 57, 62–65] in the range $k > k_{\rm odd}$ where $\chi \sim \mathcal{O}(1)$, see Fig. 1c. In this range, earlier work has revealed energy accumulation leading to pattern formation and spectral scaling $E(k) \sim k^{-1}$ [58, 59].

On the other hand, when injecting energy at smaller scales $k_{\rm in} > k_{\rm odd}$, an inverse cascade of kinetic energy is known to emerge in the 2D manifold (with $k_\parallel = 0$) in the range $k < k_{\rm in}$ as detailed in Ref. [58]. However, only a part of the kinetic energy is cascaded upscale through the 2D manifold. A remaining part of the energy flux is still cascaded forward to the range $k > k_{\rm in}$ (a property also observed in inertial wave turbulence [66]). This range has not been studied in detail in earlier works. As we will show, this is precisely the range where the dynamics is slow enough so that $\chi \ll \mathcal{O}(1)$ for all modes, indicating that we may encounter wave turbulence in this regime in the 3D manifold (with $k_\parallel > 0$). To enter these flow conditions, we therefore need to force the flow at scales $k_{\rm in} > k_{\rm odd}$ and then focus on the range $k > k_{\rm in}$ (blue region in Fig. 1b, see also Fig. 5 in the End Matter).

Kinetic equation—We now use the methods of wave turbulence to derive the kinetic equation for the energy spectrum, from which physical properties (e.g. stationary spectra, cascade direction) can be obtained [1–4, 6, 67–69]. The main idea is to take the wave amplitude as a small parameter in a multiple time scales method. The derivation is outlined below while key technical steps are provided in the End Matter and detailed in the SM.

We first perform a modal expansion of the Navier-Stokes equation to put it in the symbolic form

$$i\frac{\partial A_k^{s_k}}{\partial t} = s_k \omega_k A_k^{s_k} + M_{kpq}^{s_k s_p s_q} A_p^{s_p} A_q^{s_q}, \qquad (3)$$

in which summation over s_p and s_q and integration over $\boldsymbol{p}$ and $\boldsymbol{q}$ are implied (see SM for full expression). Here, $A_k^{s_k}$ represents a mode with polarity $s_k = \pm 1$ and wavenumber k, and $M_{kpq}^{s_k s_p s_q}$ are nonlinear mode coupling coefficients. A similar equation would describe any set of waves interacting with quadratic nonlinearities including mode coupling theories of spatially extended quantum systems or optical cavities. More precisely, $A_k^{s_k} = k\hat{\psi}_k - s_k k^2 \hat{\phi}_k$ is defined from the Fourier components of the decomposition $\boldsymbol{u} = \boldsymbol{\nabla} \times (\psi \boldsymbol{e}_\parallel) + \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times (\phi \boldsymbol{e}_\parallel))$ of the velocity field into toroidal (ψ) and poloidal (ϕ) scalar fields ($\hat{X}_k$ denotes the Fourier transform of a field X). We can show that $|A_k^+|^2 + |A_k^-|^2 = 2|\hat{\boldsymbol{u}}_k|^2$.

Introducing the interaction representation for weak amplitude waves ($0 < \epsilon \ll 1$) $A_k^{s_k} = \epsilon a_k^{s_k} e^{-i s_k \omega_k t}$, we find the wave amplitude equation

$$\frac{\partial a_k^{s_k}}{\partial t} = \epsilon L_{kpq}^{s_k s_p s_q} a_p^{s_p} a_q^{s_q} e^{i\Omega_{k,pq} t} \delta_{k,p+q}, \tag{4}$$

in which summation over s_p and s_q and integration over $\boldsymbol{p}$ and $\boldsymbol{q}$ are implied, where $\Omega_{k,p+q} \equiv s_k\omega_k - s_p\omega_p - s_q\omega_q$, and where $L_{kpq}^{s_k s_p s_q}$ is an interaction coefficient given in the End Matter. Therefore, the long-time statistical behavior is governed by the resonance condition for three-wave interactions, $s_k\omega_k + s_p\omega_p + s_q\omega_q = 0$ and $\boldsymbol{k} + \boldsymbol{p} + \boldsymbol{q} = 0$. These relationships can be written as follows

$$\frac{s_q q - s_p p}{k_\parallel} = \frac{s_k k - s_q q}{p_\parallel} = \frac{s_p p - s_k k}{q_\parallel}. \tag{5}$$

For local interactions, $k \simeq p \simeq q$, we obtain $(s_q - s_p)/k_\parallel \simeq (s_k - s_q)/p_\parallel \simeq (s_p - s_k)/q_\parallel$, which means that the associated cascade is necessarily anisotropic with a negligible cascade along the parallel direction. This situation is reminiscent of inertial wave turbulence [70, 71] and kinetic Alfvén wave turbulence [72] where the waves are also helical. In the following, we take advantage of this property and consider the anisotropic limit $k_\perp \gg k_\parallel$.

Assuming statistically homogeneous and anisotropic turbulence [73], one can use a multiple time scale method to derive a kinetic equation of the form

$$\partial_t E(k_\perp, k_\parallel) = -\partial_{k_\perp} \Pi_\perp - \partial_{k_\parallel} \Pi_\parallel, \tag{6}$$

describing the evolution of the energy spectrum $E(k_\perp, k_\parallel)$, which is related to the amplitudes in Eq. (4) through $E(\boldsymbol{k})\delta(\boldsymbol{k} + \boldsymbol{k}') = \langle a_k^+ a_{k'}^+ \rangle + \langle a_k^- a_{k'}^- \rangle$ in the absence of kinetic helicity, where $\langle . \rangle$ is an ensemble average. The quantities $\Pi_\perp$ and $\Pi_\parallel$ represent the energy fluxes in the perpendicular and parallel direction and an explicit version of Eq. (6) is given in the End Matter. Assuming axisymmetric turbulence, the 2D spectrum $E(k_\perp, k_\parallel) = 2\pi k_\perp E(\boldsymbol{k})$ scales as $k_\perp^{-3/2} k_\parallel^{-1/2}$, which corresponds to the Kolmogorov-Zakharov (KZ) spectrum. This is an exact solution of the kinetic equation when odd wave turbulence is stationary. The energy flux can be shown to be positive, and thus the associated cascade is direct (End Matter).

Phenomenology—The KZ spectrum derived in the previous paragraph can be recovered with simple phenomenological arguments. Wave turbulence can be expected when $\chi \ll 1$, see Eq. (2). As explained above, wave turbulence develops through resonant triadic wave interactions. This happens with a transfer time much longer than the wave period by a factor $\chi^{-2} \sim \epsilon^{-2}$ [3]. This leads to the wave interaction transfer time [74, 75]

$$\tau_{\rm tr} \sim \frac{\tau_{\rm NL}^2}{\tau_{\rm lin}} \sim \frac{\nu_{\rm odd}}{k k_\perp E(k_\perp, k_\parallel)}. \tag{7}$$

We can then use a Kolmogorov-type argument and assume that in the wave turbulent regime, the energy in each wavenumber shell $E(k_\perp, k_\parallel) k_\perp k_\parallel$ is transported on a timescale $\tau_{\rm tr}$ at a constant energy transfer rate

$$\varepsilon \sim \frac{E(k_\perp, k_\parallel) k_\perp k_\parallel}{\tau_{\rm tr}}. \tag{8}$$

Then using Eq. (7) and with $k \sim k_\perp$ (assuming $k_\parallel \ll k_\perp$), we can retrieve the energy spectrum as

$$E(k_\perp, k_\parallel) \sim \sqrt{\varepsilon \nu_{\rm odd}}\, k_\perp^{-3/2} k_\parallel^{-1/2}. \tag{9}$$

This prediction leads to the timescale ratio

$$\chi \sim \left(\frac{\varepsilon}{k_\perp k_\parallel^3 \nu_{\rm odd}^3} \right)^{1/4}. \tag{10}$$

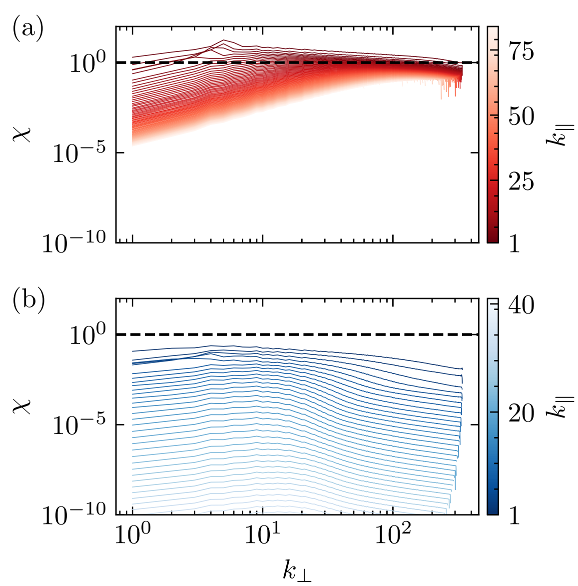

FIG. 2. The timescale ratio $\chi \equiv \tau_{\rm lin}/\tau_{\rm NL}$ that compares the timescales of odd waves and eddy turbulence for the cases of strong turbulence (a) and weak turbulence (b). Different lines represent different $k_\parallel$. The weak wave turbulence regime is found to be entered when all modes go below $\chi < \mathcal{O}(1)$ (dashed line).

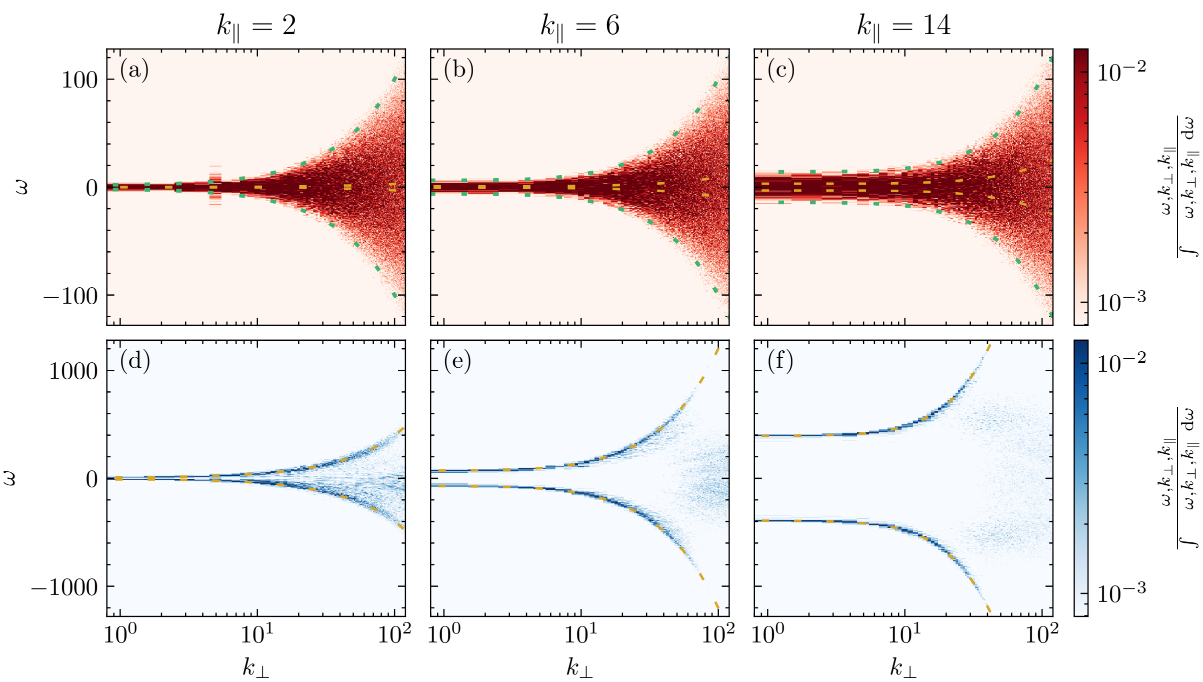

FIG. 3. Space-time energy spectra $F(\omega, k_\perp, k_\parallel)$ for the case of strong (a-c) and weak wave turbulence (d-f), and for different $k_\parallel$ of 2 (a,d), 6 (b,e) and 14 (c,f). Yellow dashed lines indicate the dispersion relation of the odd waves $\omega_k = \pm\nu_{\rm odd} k_\parallel k$. Green dotted lines in (a-c) indicate the corresponding eddy turnover frequency $\omega \sim k\sqrt{kE(k)} \sim k$ [58], which captures the envelope of the space-time spectra for strong turbulence. Spectra are normalized by their corresponding integral over the frequency space for each $k_\perp$, $k_\parallel$.

As a result, the larger $k_\perp$, the weaker the cascade. We point out that this is in contrast with rotating (inertial wave) turbulence, where strong turbulence is always recovered at the smallest scales [70, 76]. For this non-Hermitian wave turbulence, instead, the turbulence remains weak down to the smallest scales, see Fig. 1.

Numerical simulations—We numerically solve the 3D Navier-Stokes Eqs. (1) with odd viscosity through direct numerical simulation (DNS) in a periodic box using a pseudo-spectral code (see also Ref. [58]). The system is forced using a Gaussian noise $\boldsymbol{f}(\boldsymbol{k}, t)$ that is delta-correlated in time and is applied in a narrow band of wavenumbers around $k_{\rm in} = 5$, restricted to the 3D manifold. We thus force exactly those wavenumbers with simultaneously $k_{\rm in} \le |\boldsymbol{k}| < k_{\rm in} + 1$ and $k_\parallel \neq 0$. To maximize the size of the inertial range, the viscosity term is replaced by a hyperviscosity term of the form $\nu_\alpha \Delta^\alpha \boldsymbol{u}$. We also introduce a hypoviscous term of the form $\nu_h \Delta^{-\alpha_h} \boldsymbol{u}$ to dissipate the inverse flux that develops in the 2D manifold at low wavenumbers. The input parameters used in the simulations are provided in Tab. I (End Matter).

Numerical results—In order to diagnose whether we have entered the conditions that permit wave turbulence, we assess the timescale ratio χ in Eq. (2) for all modes in our DNS (Fig. 2). It can be seen that indeed, while for the strong turbulence run in Fig. 2(a), modes tend to remain around $\chi \approx 1$, the weak turbulence run in Fig. 2(b) has $\chi \ll 1$ for all modes. There, the slowest mode has $\chi \approx 0.1$, which is in the range where wave turbulence is usually observed in wave-dominated flow systems [72, 77].

To confirm that we have indeed entered the regime of non-Hermitian wave turbulence, it is crucial to check the spatio-temporal energy spectra [72, 78–84]. To that extent, we compute the temporal Fourier transform $\mathcal{F}_t\{...\}$ of time series of selected spatial Fourier modes $\hat{\boldsymbol{u}}(k_\perp, k_\parallel, t)$, yielding the spatio-temporal spectrum $F(\omega, k_\perp, k_\parallel) \equiv \frac{1}{2}|\mathcal{F}_t\{\hat{\boldsymbol{u}}(k_\perp, k_\parallel, t)\}|^2$. The results are provided in Fig. 3. For strong turbulence, this indeed shows a broad range of active modes, its envelope being well captured by the scaling of the eddy turnover frequency $\omega \sim k\sqrt{kE(k)} \sim k$ [58]. For weak turbulence, on the other hand, the kinetic energy in the inertial range is very strongly concentrated around the dispersion relation for odd waves $\omega_{\rm odd} = \nu_{\rm odd} k_\parallel k$ throughout the inertial range. This is a clear signature that the weak turbulence regime is indeed attained in this case.

The temporally averaged anisotropic kinetic energy spectra is shown in Fig. 4. In the case of strong turbulence, the assumption of critical balance discussed earlier predicts an anisotropic spectrum that scales as $E(k_\perp, k_\parallel) \sim k_\perp^{-5/3} k_\parallel^{-1}$ [6, 57, 62–65]. Indeed, we observe that $\chi \approx \mathcal{O}(1)$ in the range $k > k_{\rm odd}$ (Fig. 2a) and re-

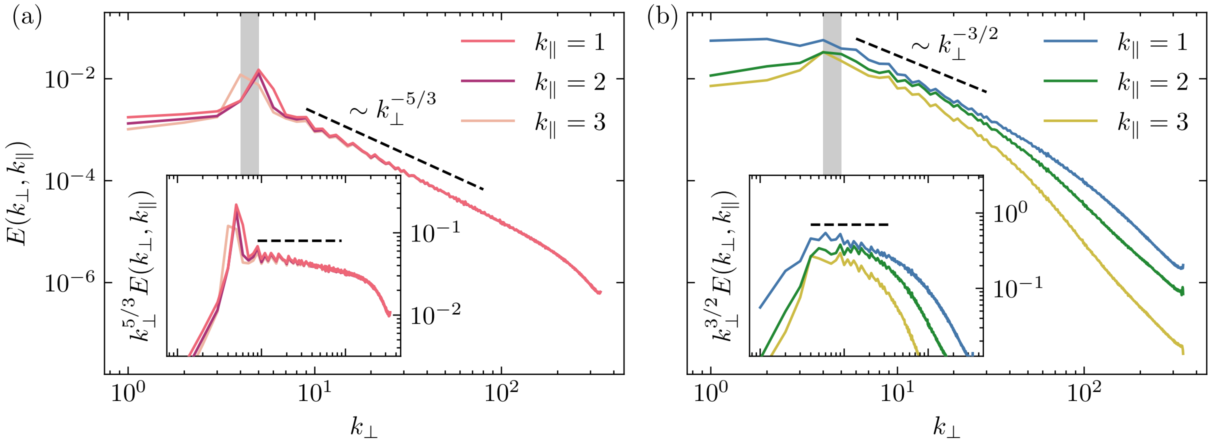

FIG. 4. The anistropic kinetic energy spectra $E(k_\perp, k_\parallel)$ for strong (a) and weak wave turbulence (b) of the first three $k_\parallel$ modes. Insets show the spectra compensated by their respective scaling predictions. The shaded area depicts the forcing range. For the corresponding isotropic energy spectra, see SM.

trieve the hypothesized critical balance spectral scaling of $E \sim k_\perp^{-5/3}$ (Fig. 4a). In the weak turbulence case, we find that we are in close agreement with the odd wave turbulence prediction in Eq. (9) that predicts $E \sim k_\perp^{-3/2}$ (Fig. 4b), although the inertial range is limited (recall that the theoretical prediction of weak turbulence is only valid for $k_\perp \gg k_\parallel$). Finally, we confirm that the spectra are compatible with a direct cascade in the perpendicular direction. We could not test numerically the scaling prediction for the parallel direction as the inertial range in the parallel direction is too narrow, owing to its very small flux.

Outlook—To sum up, we have illustrated the application of wave turbulence to incompressible odd fluids. This work paves the way to applications in other non-Hermitian systems such as open quantum systems [7, 12, 18, 24, 85–89], overdamped elasticity with odd elastic moduli [37], which could be compared with elastic wave turbulence in active plates [90, 91] and polymers solutions [92–94].

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End Matter

TABLE I. Input parameters used for the simulations in this work. Provided are the box size L of the simulation cube, the forcing wavenumber $k_{\rm in}$, the kinetic energy injection rate ϵ, the odd viscosity $\nu_{\rm odd}$, the corresponding odd wavenumber $k_{\rm odd}$, the hypoviscosity ν_h with power α_h, the hyperviscosity ν_α with power α, the integration timestep dt and spatial resolution $N_x \times N_y \times N_z$.

	L	$k_{\rm in}$	ϵ	$\nu_{\rm odd}$	$k_{\rm odd}$	ν_h	α_h	ν_α	α	dt	$N_x \times N_y \times N_z$
Strong turbulence	2π	5	0.11	0.015	13.4	0	N/A	1.5×10^{-14}	3	2×10^{-5}	$1024 \times 1024 \times 256$
Weak turbulence	2π	5	0.11	2.0	0.34	0.2	2	4.0×10^{-15}	3	5×10^{-6}	$1024 \times 1024 \times 128$

Theory of weak wave turbulence

The wave turbulence theory is mainly composed of three steps (see SM for detailed derivations). The first key step is the derivation of the wave amplitude equation (4), where the interacting coefficient reads (in the anisotropic limit, ie. $k_\perp \gg k_\parallel$)

$$L_{kpq}^{ss_ps_q} \equiv \frac{\boldsymbol{e}_\parallel \cdot (\boldsymbol{p}_\perp \times \boldsymbol{q}_\perp)}{8k_\perp p_\perp q_\perp}(s_p p_\perp - s_q q_\perp)(s k_\perp + s_p p_\perp + s_q q_\perp).$$

The wave amplitude equation describes the slow evolution of odd waves of weak amplitude. We note that the (quadratic) nonlinear coupling vanishes when the wave vectors $\boldsymbol{p}_\perp$ and $\boldsymbol{q}_\perp$ are collinear, or when the wave numbers $p_\perp$ and $q_\perp$ are equal if their associated directional polarities, s_p, s_q respectively, are also equal. Interestingly, these properties are also found for inertial wave turbulence [70, 95], and more generally for helical waves [96–100].

The second key step is the derivation of the kinetic equation. This describes the long-time statistical behavior of the dynamics, which is governed by the resonance conditions (5) for three-wave interactions. Assuming statistically homogeneous and anisotropic turbulence, and the absence of kinetic helicity, the use of the multiple time scale technique leads to

$$\begin{aligned}\frac{\partial E_k}{\partial t} = & \frac{\epsilon^2 k_\parallel}{128\nu_{\rm odd}} \sum_{ss_ps_q} \int_{\Delta_\perp} \left(\frac{\sin\theta_k}{k_\perp}\right) \frac{1}{k_\perp p_\perp q_\perp} \left(\frac{s_p p_\perp - s_q q_\perp}{k_\parallel}\right)^2 (s k_\perp + s_p p_\perp + s_q q_\perp)^2 \\ & \times \left[\omega_k E_p E_q + \omega_p E_k E_q + \omega_q E_k E_p\right] \delta(\Omega_{kpq}) \delta(k_\parallel + p_\parallel + q_\parallel) dp_\perp dq_\perp dp_\parallel dq_\parallel,\end{aligned} \tag{11}$$

with $E_k \equiv E(k_\perp, k_\parallel)$, θ_k the angle opposite to $\boldsymbol{k}_\perp$ in the triangle $\boldsymbol{k}_\perp + \boldsymbol{p}_\perp + \boldsymbol{q}_\perp = \boldsymbol{0}$, and $\Delta_\perp$ an integration domain limited to the resonance conditions. The density spectrum is defined as $e^s(\boldsymbol{k})\delta(\boldsymbol{k}+\boldsymbol{k}') \equiv \langle a_k^s a_{k'}^s \rangle$, where $\langle\rangle$ is an ensemble average. In the absence of kinetic helicity, $e^+ = e^- \equiv e$ and the energy spectrum becomes $E(\boldsymbol{k}) = 2e(\boldsymbol{k})$. Expression (11) is the kinetic equation for odd wave turbulence. It is an asymptotically exact equation that does not describe the slow mode ($k_\parallel = 0$), which implies strong turbulence.

The third key step is the derivation of physical properties, the most important of which is the KZ spectrum. This exact solution is obtained by introducing $E_k = A k_\perp^n k_\parallel^m$ into expression (11). Using the Zakharov transformation [3] and assuming stationarity, we find $n = -3/2$ and $m = -1/2$. The cascade direction can be obtained by analyzing the sign of the energy flux. Neglecting the flux in the parallel direction, we obtain $\partial E_k/\partial t = -\partial \Pi_\perp(k_\perp, k_\parallel)/\partial k_\perp$. Using the kinetic equation, we find $\Pi_\perp^{KZ} = \frac{\epsilon^2 A^2}{384\nu_{\rm odd}} \frac{1}{k_\parallel} I_\perp$, with

$$
\begin{aligned}
I_\perp \equiv \sum_{s s_p s_q} \int_{\Delta_\perp} & \sin\theta_k \left(s_q \tilde{q}_\perp - s_p \tilde{p}_\perp\right)^2 \left(s + s_p \tilde{p}_\perp + s_q \tilde{q}_\perp\right)^2 \\
& \times \tilde{p}_\perp^{-5/2} \tilde{q}_\perp^{-5/2} \tilde{p}_\parallel^{-1/2} \tilde{q}_\parallel^{-1/2} (\tilde{p}_\parallel \ln \tilde{p}_\perp + \tilde{q}_\parallel \ln \tilde{q}_\perp) \\
& \times \left(1 + \tilde{p}_\perp^{5/2} \tilde{p}_\parallel^{3/2} + \tilde{q}_\perp^{5/2} \tilde{q}_\parallel^{3/2}\right) \delta\left(s + s_p \tilde{p}_\perp \tilde{p}_\parallel + s_q \tilde{q}_\perp \tilde{q}_\parallel\right) \\
& \times \delta\left(1 + \tilde{p}_\parallel + \tilde{q}_\parallel\right) d\tilde{p}_\perp d\tilde{q}_\perp d\tilde{p}_\parallel d\tilde{q}_\parallel, \qquad (12)
\end{aligned}
$$

and with $\tilde{p}_i \equiv p_i/k_i$ and $\tilde{q}_i \equiv q_i/k_i$ ($i = \perp, \parallel$). A numerical evaluation of the sign of $I_\perp$ shows that the perpendicular energy flux is positive, so this energy cascade is direct.

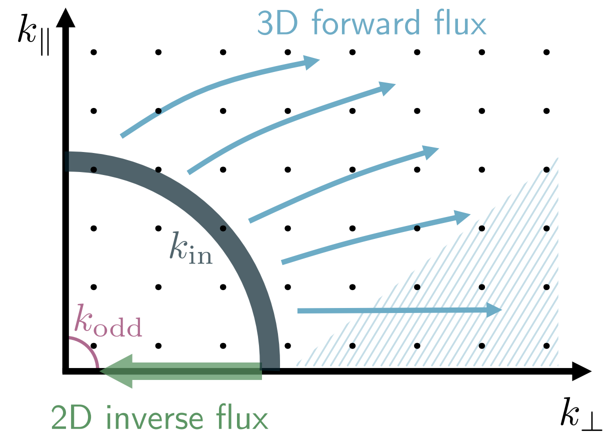

FIG. 5. Sketch of the dominant fluxes in the $(k_\perp, k_\parallel)$ space for the weak turbulence case (see Fig 1b). Wave turbulence gives rise to the 3D forward flux depicted here. The region $k_\perp \gg k_\parallel$ where the KZ-spectrum is derived is dashed in blue.

Supplemental Material: Non-Hermitian wave turbulence

THEORETICAL RESULTS

Wave amplitude equation

The Navier-Stokes equations with odd viscosity $\nu_{\rm odd}$ can be written

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla} p + \nu \Delta \boldsymbol{u} + \nu_{\rm odd} \boldsymbol{e}_\parallel \times \Delta \boldsymbol{u}, \tag{S1}$$

where $\boldsymbol{u}$ is a solenoidal velocity ($\boldsymbol{\nabla}\cdot\boldsymbol{u}=0$), p the reduced pressure and ν the classical viscosity. A parallel ($\parallel$) direction appears in the odd viscous term that will be taken along the z-direction. It is convenient to rewrite this system for the vorticity field; we obtain

$$\frac{\partial \boldsymbol{w}}{\partial t} + \nu_{\rm odd}(\boldsymbol{e}_\parallel\cdot\boldsymbol{\nabla})\Delta\boldsymbol{u} = (\boldsymbol{w}\cdot\boldsymbol{\nabla})\boldsymbol{u} - (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{w} + \nu\Delta\boldsymbol{w}. \tag{S2}$$

Hereafter, we will neglect the term proportional to the viscosity ν.

Canonical variables

We introduce the toroidal (ψ) and poloidal (ϕ) scalar fields in the following manner

$$\boldsymbol{u} = \boldsymbol{\nabla}\times(\psi\boldsymbol{e}_\parallel) + \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times(\phi\boldsymbol{e}_\parallel)), \tag{S3}$$

whose Fourier transform writes

$$\hat{\boldsymbol{u}}_k = i\hat\psi_k \boldsymbol{k}\times\boldsymbol{e}_\parallel - \hat\phi_k \boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{e}_\parallel) = i\hat\psi_k\boldsymbol{k}\times\boldsymbol{e}_\parallel + \hat\phi_k(k^2\boldsymbol{e}_\parallel - k_\parallel\boldsymbol{k}), \tag{S4}$$

from which we deduce the vorticity vector ($|\boldsymbol{k}|=k$)

$$\hat{\boldsymbol{w}}_k = \hat\psi_k(k^2\boldsymbol{e}_\parallel - k_\parallel\boldsymbol{k}) + ik^2\hat\phi_k\boldsymbol{k}\times\boldsymbol{e}_\parallel. \tag{S5}$$

It is straightforward to show in Fourier space that the linear contribution of equation (S2) leads, after projection, to

$$\frac{\partial\hat\psi_k}{\partial t} = i\nu_{\rm odd}k_\parallel k^2\hat\phi_k, \tag{S6a}$$

$$\frac{\partial\hat\phi_k}{\partial t} = i\nu_{\rm odd}k_\parallel\hat\psi_k. \tag{S6b}$$

The linear solutions are (helical) odd waves with the angular frequency ($\partial_t^2 = -\omega_k^2$ can be used)

$$\omega_k^2 = \nu_{\rm odd}^2 k_\parallel^2 k^2. \tag{S7}$$

From this property, we introduce the canonical variables

$$A_k^s \equiv A^s(\boldsymbol{k}) = k\hat\psi_k - sk^2\hat\phi_k, \tag{S8}$$

with $s=\pm$ the directional polarity. With such a choice of canonical variables, we have

$$|A_k^+|^2 + |A_k^-|^2 = 2|\hat{\boldsymbol{u}}_k|^2 \tag{S9}$$

and at the linear level

$$\frac{\partial A_k^s}{\partial t} + is\omega_k A_k^s = 0. \tag{S10}$$

Resonance condition

The resonance condition for three-wave interactions can be written [1]

$$
\begin{aligned}
s\omega_k + s_p\omega_p + s_q\omega_q &= 0, && \text{(S11a)}\\
\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q} &= 0. && \text{(S11b)}
\end{aligned}
$$

In the case of odd waves, these relations are equivalent to the conditions

$$
\frac{s_q q - s_p p}{k_\parallel} = \frac{sk - s_q q}{p_\parallel} = \frac{s_p p - sk}{q_\parallel}. \tag{S12}
$$

Assuming that the system is initially excited at large scale in a narrow isotropic domain in Fourier space, a situation often considered in DNS, the dynamics will initially be dominated by local interactions such that $k \simeq p \simeq q$. As the locality of the interactions is a property of turbulence that is generally verified, we can extend its use beyond the initial instant and we obtain

$$
\frac{s_q - s_p}{k_\parallel} \simeq \frac{s - s_q}{p_\parallel} \simeq \frac{s_p - s}{q_\parallel}. \tag{S13}
$$

From this expression, we can show that the associated cascade is necessarily anisotropic. Indeed, if $k_\parallel$ is non-zero, the left-hand term will only give a non-negligible contribution when $s_p = -s_q$. The immediate consequence is that either the middle or the right-hand term has its numerator which cancels (to leading order), which implies that the associated denominator must also cancel (to leading order) to satisfy the equality: for example, if $s = s_p$ then $q_\parallel \simeq 0$. This condition means that the transfer in the parallel direction is negligible: indeed, the integration of the wave amplitude equation in the parallel direction (see below) is then reduced to a few modes (since $p_\parallel \simeq k_\parallel$) which strongly limits the transfer between parallel modes. The cascade in the parallel direction is thus possible but relatively weak compared to that in the perpendicular direction. In the following, we will take advantage of this property and consider the anisotropic limit $k_\perp \gg k_\parallel$ to simplify the derivation. Note that once turbulence is anisotropic, we can still use the locality condition with $k \sim k_\perp$; we then obtain $k_\perp \sim p_\perp \sim q_\perp$, whereas the parallel wavenumbers are limited to a narrow domain.

Wave amplitude equation

In the derivation of the wave amplitude equation, we will consider a continuous medium which can lead to mathematical difficulties connected with infinite dimensional phase spaces. For this reason, it is preferable to assume a variable spatially periodic over a box of finite size L. However, in the derivation of the kinetic equation, the limit $L \to +\infty$ is finally taken (before the long time limit). As both approaches lead to the same kinetic equation, for simplicity, we anticipate this result and follow the original approach of Benney and Saffman [2]. Note that the anisotropic limit ($k_\perp \gg k_\parallel$) will also be taken before the (asymptotic) long time limit.

The first non-linear term of equation (S2) writes

$$
\begin{aligned}
\widehat{(\boldsymbol{w}\cdot\boldsymbol{\nabla})\boldsymbol{u}_k} &= i\int (\hat{\boldsymbol{w}}_p\cdot\boldsymbol{q})\hat{\boldsymbol{u}}_q\delta_{k,pq}d\boldsymbol{p}d\boldsymbol{q}\\
&= i\int \Big[i\hat{\phi}_p\hat{\phi}_q p^2\left(\boldsymbol{q}\cdot(\boldsymbol{p}\times\boldsymbol{e}_\parallel)\right)(q^2\boldsymbol{e}_\parallel - q_\parallel\boldsymbol{q}) - \hat{\phi}_p\hat{\psi}_q p^2\left(\boldsymbol{q}\cdot(\boldsymbol{p}\times\boldsymbol{e}_\parallel)\right)(\boldsymbol{q}\times\boldsymbol{e}_\parallel)\\
&\quad - \hat{\psi}_p\hat{\phi}_q\left(p_\parallel\boldsymbol{p}\cdot\boldsymbol{q} - p^2 q_\parallel\right)(q^2\boldsymbol{e}_\parallel - q_\parallel\boldsymbol{q}) - i\hat{\psi}_p\hat{\psi}_q\left(p_\parallel\boldsymbol{p}\cdot\boldsymbol{q} - p^2 q_\parallel\right)(\boldsymbol{q}\times\boldsymbol{e}_\parallel)\Big]\\
&\quad \times \delta_{k,pq}d\boldsymbol{p}d\boldsymbol{q},
\end{aligned} \tag{S14}
$$

with $\delta_{k,pq} \equiv \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})$. In the anisotropic limit ($k_\perp \gg k_\parallel$), a first simplification arises

$$
\begin{aligned}
\widehat{(\boldsymbol{w}\cdot\boldsymbol{\nabla})\boldsymbol{u}_k} &= i\int \Big[i\hat{\phi}_p\hat{\phi}_q p_\perp^2 q_\perp^2\left(\boldsymbol{e}_\parallel\cdot(\boldsymbol{q}_\perp\times\boldsymbol{p}_\perp)\right)\boldsymbol{e}_\parallel - \hat{\phi}_p\hat{\psi}_q p_\perp^2\left(\boldsymbol{e}_\parallel\cdot(\boldsymbol{q}_\perp\times\boldsymbol{p}_\perp)\right)(\boldsymbol{q}_\perp\times\boldsymbol{e}_\parallel)\\
&\quad - \hat{\psi}_p\hat{\phi}_q q_\perp^2\left(p_\parallel\boldsymbol{p}_\perp\cdot\boldsymbol{q}_\perp - p_\perp^2 q_\parallel\right)\boldsymbol{e}_\parallel - i\hat{\psi}_p\hat{\psi}_q\left(p_\parallel\boldsymbol{p}_\perp\cdot\boldsymbol{q}_\perp - p_\perp^2 q_\parallel\right)(\boldsymbol{q}_\perp\times\boldsymbol{e}_\parallel)\Big]\\
&\quad \times \delta_{k,pq}d\boldsymbol{p}d\boldsymbol{q}.
\end{aligned} \tag{S15}
$$

The second non-linear term of equation (S2) reads

$$\begin{aligned}
\widehat{(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{w}}_k &= i\int(\hat{\boldsymbol{u}}_p\cdot\boldsymbol{q})\hat{\boldsymbol{w}}_q\delta_{k,pq}d\boldsymbol{p}d\boldsymbol{q}\\
&= i\int\Big[i\hat{\phi}_p\hat{\phi}_q\left(p^2q_\parallel-p_\parallel\boldsymbol{p}\cdot\boldsymbol{q}\right)q^2(\boldsymbol{q}\times\boldsymbol{e}_\parallel)+\hat{\phi}_p\hat{\psi}_q\left(p^2q_\parallel-p_\parallel\boldsymbol{p}\cdot\boldsymbol{q}\right)\left(q^2\boldsymbol{e}_\parallel-q_\parallel\boldsymbol{q}\right)\\
&\quad-\hat{\psi}_p\hat{\phi}_q\left(\boldsymbol{q}\cdot(\boldsymbol{p}\times\boldsymbol{e}_\parallel)\right)q^2(\boldsymbol{q}\times\boldsymbol{e}_\parallel)+i\hat{\psi}_p\hat{\psi}_q\left(\boldsymbol{q}\cdot(\boldsymbol{p}\times\boldsymbol{e}_\parallel)\right)\left(q^2\boldsymbol{e}_\parallel-q_\parallel\boldsymbol{q}\right)\Big]\\
&\quad\times\delta_{k,pq}d\boldsymbol{p}d\boldsymbol{q},
\end{aligned}\tag{S16}$$

which simplifies in the anisotropic limit to

$$\begin{aligned}
\widehat{(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{w}}_k &= i\int q_\perp^2\Big[i\hat{\phi}_p\hat{\phi}_q\left(p_\perp^2q_\parallel-p_\parallel\boldsymbol{p}_\perp\cdot\boldsymbol{q}_\perp\right)(\boldsymbol{q}_\perp\times\boldsymbol{e}_\parallel)+\hat{\phi}_p\hat{\psi}_q\left(p_\perp^2q_\parallel-p_\parallel\boldsymbol{p}_\perp\cdot\boldsymbol{q}_\perp\right)\boldsymbol{e}_\parallel\\
&\quad-\hat{\psi}_p\hat{\phi}_q\left(\boldsymbol{e}_\parallel\cdot(\boldsymbol{q}_\perp\times\boldsymbol{p}_\perp)\right)(\boldsymbol{q}_\perp\times\boldsymbol{e}_\parallel)+i\hat{\psi}_p\hat{\psi}_q\left(\boldsymbol{e}_\parallel\cdot(\boldsymbol{q}_\perp\times\boldsymbol{p}_\perp)\right)\boldsymbol{e}_\parallel\Big]\\
&\quad\times\delta_{k,pq}d\boldsymbol{p}d\boldsymbol{q}.
\end{aligned}\tag{S17}$$

The addition of these two non-linear contributions leads to the simplified expression

$$\begin{aligned}
\widehat{NL}(\boldsymbol{k}) &= \widehat{(\boldsymbol{w}\cdot\boldsymbol{\nabla})\boldsymbol{u}}_k-\widehat{(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{w}}_k\\
&= \int\hat{\phi}_p\hat{\phi}_qp_\perp^2q_\perp^2\left(\boldsymbol{e}_\parallel\cdot(\boldsymbol{p}_\perp\times\boldsymbol{q}_\perp)\right)\boldsymbol{e}_\parallel\delta_{k,pq}d\boldsymbol{p}d\boldsymbol{q}\\
&\quad+i\int\hat{\phi}_p\hat{\psi}_qp_\perp^2\left(\boldsymbol{e}_\parallel\cdot(\boldsymbol{p}_\perp\times\boldsymbol{q}_\perp)\right)(\boldsymbol{q}_\perp\times\boldsymbol{e}_\parallel)\delta_{k,pq}d\boldsymbol{p}d\boldsymbol{q}\\
&\quad-i\int\hat{\psi}_p\hat{\phi}_qq_\perp^2\left(\boldsymbol{e}_\parallel\cdot(\boldsymbol{p}_\perp\times\boldsymbol{q}_\perp)\right)(\boldsymbol{q}_\perp\times\boldsymbol{e}_\parallel)\delta_{k,pq}d\boldsymbol{p}d\boldsymbol{q}\\
&\quad-\int\hat{\psi}_p\hat{\psi}_qq_\perp^2\left(\boldsymbol{e}_\parallel\cdot(\boldsymbol{p}_\perp\times\boldsymbol{q}_\perp)\right)\boldsymbol{e}_\parallel\delta_{k,pq}d\boldsymbol{p}d\boldsymbol{q}.
\end{aligned}\tag{S18}$$

The introduction of the canonical variables

$$\hat{\psi}_k = \frac{1}{2k_\perp}\sum_s A_k^s,\tag{S19a}$$

$$\hat{\phi}_k = -\frac{1}{2k_\perp^2}\sum_s sA_k^s,\tag{S19b}$$

gives

$$\begin{aligned}
\widehat{NL}(\boldsymbol{k}) &= \frac{1}{4}\sum_{s_ps_q}\int s_ps_qA_p^{s_p}A_q^{s_q}\left(\boldsymbol{e}_\parallel\cdot(\boldsymbol{p}_\perp\times\boldsymbol{q}_\perp)\right)\boldsymbol{e}_\parallel\delta_{k,pq}d\boldsymbol{p}d\boldsymbol{q}\\
&\quad-\frac{i}{4}\sum_{s_ps_q}\int A_p^{s_p}A_q^{s_q}\frac{s_p}{q_\perp}\left(\boldsymbol{e}_\parallel\cdot(\boldsymbol{p}_\perp\times\boldsymbol{q}_\perp)\right)(\boldsymbol{q}_\perp\times\boldsymbol{e}_\parallel)\delta_{k,pq}d\boldsymbol{p}d\boldsymbol{q}\\
&\quad+\frac{i}{4}\sum_{s_ps_q}\int A_p^{s_p}A_q^{s_q}\frac{s_q}{p_\perp}\left(\boldsymbol{e}_\parallel\cdot(\boldsymbol{p}_\perp\times\boldsymbol{q}_\perp)\right)(\boldsymbol{q}_\perp\times\boldsymbol{e}_\parallel)\delta_{k,pq}d\boldsymbol{p}d\boldsymbol{q}\\
&\quad-\frac{1}{4}\sum_{s_ps_q}\int A_p^{s_p}A_q^{s_q}\frac{q_\perp}{p_\perp}\left(\boldsymbol{e}_\parallel\cdot(\boldsymbol{p}_\perp\times\boldsymbol{q}_\perp)\right)\boldsymbol{e}_\parallel\delta_{k,pq}d\boldsymbol{p}d\boldsymbol{q}.
\end{aligned}\tag{S20}$$

The dummy variables $\boldsymbol{p}$, $\boldsymbol{q}$ and s_p, s_q, can be exchanged to symmetrize the equation; we find

$$\begin{aligned}
\widehat{NL}(\boldsymbol{k}) &= \frac{1}{8}\sum_{s_ps_q}\int A_p^{s_p}A_q^{s_q}\frac{\boldsymbol{e}_\parallel\cdot(\boldsymbol{p}_\perp\times\boldsymbol{q}_\perp)}{p_\perp q_\perp}(p_\perp^2-q_\perp^2)\boldsymbol{e}_\parallel\delta_{k,pq}d\boldsymbol{p}d\boldsymbol{q}\\
&\quad+\frac{i}{8}\sum_{s_ps_q}\int A_p^{s_p}A_q^{s_q}\frac{\boldsymbol{e}_\parallel\cdot(\boldsymbol{p}_\perp\times\boldsymbol{q}_\perp)}{p_\perp q_\perp}\left(s_qq_\perp-s_pp_\perp\right)(\boldsymbol{k}_\perp\times\boldsymbol{e}_\parallel)\delta_{k,pq}d\boldsymbol{p}d\boldsymbol{q}.
\end{aligned}\tag{S21}$$

Coming back to the wave amplitude equation, we can write in the anisotropic limit

$$\left(\frac{\partial\hat{\psi}_k}{\partial t} - i\nu_{\text{odd}}k_\parallel k_\perp^2\hat{\phi}_k\right)k_\perp^2\boldsymbol{e}_\parallel + \left(i\frac{\partial\hat{\phi}_k}{\partial t} + \nu_{\text{odd}}k_\parallel\hat{\psi}_k\right)k_\perp^2\boldsymbol{k}_\perp\times\boldsymbol{e}_\parallel = \widehat{NL}(\boldsymbol{k}), \tag{S22}$$

therefore, after projection and use of the dispersion relation, we obtain

$$\frac{\partial\hat{\psi}_k}{\partial t} - i\omega_k k_\perp\hat{\phi}_k = \sum_{s_p s_q}\int A_p^{s_p}A_q^{s_q}\frac{\boldsymbol{e}_\parallel\cdot(\boldsymbol{p}_\perp\times\boldsymbol{q}_\perp)}{8k_\perp^2 p_\perp q_\perp}(p_\perp^2 - q_\perp^2)\delta_{k,pq}d\boldsymbol{p}d\boldsymbol{q}, \tag{S23a}$$

$$\frac{\partial\hat{\phi}_k}{\partial t} - i\omega_k\frac{\hat{\psi}_k}{k_\perp} = \sum_{s_p s_q}\int A_p^{s_p}A_q^{s_q}\frac{\boldsymbol{e}_\parallel\cdot(\boldsymbol{p}_\perp\times\boldsymbol{q}_\perp)}{8k_\perp^2 p_\perp q_\perp}\left(s_q q_\perp - s_p p_\perp\right)\delta_{k,pq}d\boldsymbol{p}d\boldsymbol{q}. \tag{S23b}$$

With the introduction of the canonical variables (S8), the weighted addition of the previous expressions gives

$$\begin{aligned}\frac{\partial A_k^s}{\partial t} + is\omega_k A_k^s = &\sum_{s_p s_q}\int\frac{\boldsymbol{e}_\parallel\cdot(\boldsymbol{p}_\perp\times\boldsymbol{q}_\perp)}{8k_\perp p_\perp q_\perp}\left(p_\perp^2 - q_\perp^2 - sk_\perp\left(s_q q_\perp - s_p p_\perp\right)\right)\\ &\times A_p^{s_p}A_q^{s_q}\delta_{k,pq}d\boldsymbol{p}d\boldsymbol{q}.\end{aligned} \tag{S24}$$

Remaking that

$$p_\perp^2 - q_\perp^2 = (s_p p_\perp - s_q q_\perp)(s_p p_\perp + s_q q_\perp), \tag{S25}$$

we can rearrange the expression in the following manner

$$\begin{aligned}\frac{\partial A_k^s}{\partial t} + is\omega_k A_k^s = &\sum_{s_p s_q}\int\frac{\boldsymbol{e}_\parallel\cdot(\boldsymbol{p}_\perp\times\boldsymbol{q}_\perp)}{8k_\perp p_\perp q_\perp}(s_p p_\perp - s_q q_\perp)(sk_\perp + s_p p_\perp + s_q q_\perp)\\ &\times A_p^{s_p}A_q^{s_q}\delta_{k,pq}d\boldsymbol{p}d\boldsymbol{q}.\end{aligned} \tag{S26}$$

We introduce the interaction representation for waves of weak amplitude ($0 < \epsilon \ll 1$)

$$A_k^s = \epsilon a_k^s e^{-is\omega_k t}, \tag{S27}$$

and get eventually the wave amplitude equation after a few last manipulations

$$\frac{\partial a_k^s}{\partial t} = \epsilon\sum_{s_p s_q}\int L_{kpq}^{ss_p s_q}a_p^{s_p}a_q^{s_q}e^{i\Omega_{k,pq}t}\delta_{k,pq}d\boldsymbol{p}d\boldsymbol{q}, \tag{S28}$$

with $\Omega_{k,pq} \equiv s\omega_k - s_p\omega_p - s_q\omega_q$ and

$$L_{kpq}^{ss_p s_q} \equiv \frac{\boldsymbol{e}_\parallel\cdot(\boldsymbol{p}_\perp\times\boldsymbol{q}_\perp)}{8k_\perp p_\perp q_\perp}(s_p p_\perp - s_q q_\perp)(sk_\perp + s_p p_\perp + s_q q_\perp). \tag{S29}$$

Expression (S29) satisfies the following properties (relation (S12) is used)

$$L_{0pq}^{ss_p s_q} = 0, \tag{S30a}$$

$$L_{kqp}^{ss_q s_p} = L_{kpq}^{ss_p s_q}, \tag{S30b}$$

$$L_{pkq}^{s_p s s_q} = \frac{p_\parallel}{k_\parallel}L_{kpq}^{ss_p s_q}, \tag{S30c}$$

$$L_{kpq}^{-s-s_p-s_q} = L_{kpq}^{ss_p s_q}, \tag{S30d}$$

$$L_{-k-p-q}^{ss_p s_q} = L_{kpq}^{ss_p s_q}. \tag{S30e}$$

The wave amplitude equation (S28) governs the slow evolution of odd waves of weak amplitude in the anisotropic limit. It is a quadratic non-linear equation which corresponds to the interactions between waves propagating along $\boldsymbol{p}$ and $\boldsymbol{q}$, in the positive ($s_p, s_q > 0$) or negative ($s_p, s_q < 0$) direction. The symmetries listed above can be used to simplify the derivation of the kinetic equation [3]. The wave amplitude equation tells us that the non-linear coupling between the states associated with the wavevectors $\boldsymbol{p}_\perp$ and $\boldsymbol{q}_\perp$ vanishes when these wavevectors are collinear. Moreover, we note that the non-linear coupling disappears when the wavenumbers $p_\perp$ and $q_\perp$ are equal if their associated directional polarities, s_p and s_q, are also equal. These are general properties for helical waves [4–9].

Kinetic equation and solutions

Kinetic equation

The derivation of the kinetic equation for odd wave turbulence is classical. The method based on a multiple time scale was recently reviewed for inertial wave turbulence, a similar problem where waves are helical and turbulence anisotropic [3]. Note that this technique was first proposed by Benney and Saffman [2] for three-wave interactions and then extended to four-wave interactions [10] with an application to surface gravity waves.

Assuming a statistically homogeneous turbulence, we introduce the energy density spectrum $e_k^s \delta(\boldsymbol{k}+\boldsymbol{k'}) \equiv \langle a_k^s a_{k'}^s \rangle$, with $e_k^s \equiv e^s(\boldsymbol{k})$. Then, the multiple time scale method leads to the following kinetic equation

$$\begin{aligned}\frac{\partial e_k^s}{\partial t} &= \frac{\pi \epsilon^2 k_\parallel}{16} \sum_{s_p s_q} \int \left(\frac{\sin\theta_k}{k_\perp}\right)^2 \left(\frac{s_p p_\perp - s_q q_\perp}{k_\parallel}\right)^2 (s k_\perp + s_p p_\perp + s_q q_\perp)^2 \\ &\times \left[k_\parallel e_p^{s_p} e_q^{s_q} + p_\parallel e_k^s e_q^{s_q} + q_\parallel e_k^s e_p^{s_p}\right] \delta(\Omega_{kpq}) \delta_{kpq} d\boldsymbol{p} d\boldsymbol{q},\end{aligned} \tag{S31}$$

with θ_k the opposite angle to $\boldsymbol{k}_\perp$ in the triangle $\boldsymbol{k}_\perp + \boldsymbol{p}_\perp + \boldsymbol{q}_\perp = \boldsymbol{0}$. Expression (S31) is the kinetic equation for odd wave turbulence in the anisotropic limit ($k_\parallel \ll k_\perp$). Note that the kinetic equation for odd wave turbulence does not describe the slow mode ($k_\parallel = 0$) which involves strong turbulence.

Conservation laws

The kinetic equation satisfies the conservation of energy and helicity. To prove this property, we introduce the energy spectrum

$$E(\boldsymbol{k}) = e^+(\boldsymbol{k}) + e^-(\boldsymbol{k}) = \sum_s e^s(\boldsymbol{k}) \tag{S32}$$

and the helicity spectrum

$$H(\boldsymbol{k}) = k_\perp (e^+(\boldsymbol{k}) - e^-(\boldsymbol{k})) = k_\perp \sum_s s e^s(\boldsymbol{k}). \tag{S33}$$

For the energy, we find

$$\begin{aligned}\frac{\partial \int E(\boldsymbol{k}) d\boldsymbol{k}}{\partial t} &= \frac{\pi \epsilon^2}{16} \sum_{s s_p s_q} \int k_\parallel \left(\frac{\sin\theta_k}{k_\perp}\right)^2 \left(\frac{s_p p_\perp - s_q q_\perp}{k_\parallel}\right)^2 \\ &\times (s k_\perp + s_p p_\perp + s_q q_\perp)^2 \left[k_\parallel e_p^{s_p} e_q^{s_q} + p_\parallel e_k^s e_q^{s_q} + q_\parallel e_k^s e_p^{s_p}\right] \delta(\Omega_{kpq}) \delta_{kpq} d\boldsymbol{k} d\boldsymbol{p} d\boldsymbol{q}.\end{aligned} \tag{S34}$$

After a circular permutation we obtain

$$\begin{aligned}\frac{\partial \int E(\boldsymbol{k}) d\boldsymbol{k}}{\partial t} &= \frac{\pi \epsilon^2}{48} \sum_{s s_p s_q} \int (k_\parallel + p_\parallel + q_\parallel) \left(\frac{\sin\theta_k}{k_\perp}\right)^2 \left(\frac{s_p p_\perp - s_q q_\perp}{k_\parallel}\right)^2 \\ &\times (s k_\perp + s_p p_\perp + s_q q_\perp)^2 \left[k_\parallel e_p^{s_p} e_q^{s_q} + p_\parallel e_k^s e_q^{s_q} + q_\parallel e_k^s e_p^{s_p}\right] \delta(\Omega_{kpq}) \delta_{kpq} d\boldsymbol{k} d\boldsymbol{p} d\boldsymbol{q},\end{aligned} \tag{S35}$$

which is null on the resonant manifold.

Likewise, for the helicity we find

$$\begin{aligned}\frac{\partial \int H(\boldsymbol{k}) d\boldsymbol{k}}{\partial t} &= \frac{\pi \epsilon^2}{16} \sum_{s s_p s_q} \int s k_\parallel k_\perp \left(\frac{\sin\theta_k}{k_\perp}\right)^2 \left(\frac{s_p p_\perp - s_q q_\perp}{k_\parallel}\right)^2 \\ &\times (s k_\perp + s_p p_\perp + s_q q_\perp)^2 \left[k_\parallel e_p^{s_p} e_q^{s_q} + p_\parallel e_k^s e_q^{s_q} + q_\parallel e_k^s e_p^{s_p}\right] \delta(\Omega_{kpq}) \delta_{kpq} d\boldsymbol{k} d\boldsymbol{p} d\boldsymbol{q}.\end{aligned} \tag{S36}$$

After a circular permutation we obtain

$$\begin{aligned}\frac{\partial \int H(\boldsymbol{k}) d\boldsymbol{k}}{\partial t} &= \frac{\pi \epsilon^2}{48} \sum_{s s_p s_q} \int (s\omega_k + s_p \omega_p + s_q \omega_q) \left(\frac{\sin\theta_k}{k_\perp}\right)^2 \left(\frac{s_p p_\perp - s_q q_\perp}{k_\parallel}\right)^2 \\ &\times (s k_\perp + s_p p_\perp + s_q q_\perp)^2 \left[k_\parallel e_p^{s_p} e_q^{s_q} + p_\parallel e_k^s e_q^{s_q} + q_\parallel e_k^s e_p^{s_p}\right] \delta(\Omega_{kpq}) \delta_{kpq} d\boldsymbol{k} d\boldsymbol{p} d\boldsymbol{q},\end{aligned} \tag{S37}$$

which is null on the resonant manifold. Therefore, energy and helicity are conserved by the kinetic equation. Note that usually the conservation of energy is obtained using the dispersion relation while the conservation of the second invariant is obtained using the wave vector relation. This is really an odd turbulence.

Kolmogorov-Zakharov spectrum

The derivation of the stationary solutions requires a long but classical calculation. First, we assume axisymmetry and introduce the reduced spectrum

$$E_k^s \equiv E^s(k_\perp, k_\parallel) = 2\pi k_\perp e^s(\boldsymbol{k}). \tag{S38}$$

We obtain the kinetic equation

$$\begin{aligned}\frac{\partial E_k^s}{\partial t} &= \frac{\epsilon^2 k_\parallel}{32\nu_{\text{odd}}} \sum_{s_p s_q} \int_{\Delta_\perp} \left(\frac{\sin\theta_k}{k_\perp}\right) \frac{1}{k_\perp p_\perp q_\perp} \left(\frac{s_p p_\perp - s_q q_\perp}{k_\parallel}\right)^2 (s k_\perp + s_p p_\perp + s_q q_\perp)^2 \\ &\times \left[\omega_k E_p^{s_p} E_q^{s_q} + \omega_p E_k^s E_q^{s_q} + \omega_q E_k^s E_p^{s_p}\right] \delta(\Omega_{kpq}) \delta(k_\parallel + p_\parallel + q_\parallel) dp_\perp dq_\perp dp_\parallel dq_\parallel.\end{aligned} \tag{S39}$$

To simplify our analysis, we shall consider the case of zero-helicity. Then, we obtain

$$\begin{aligned}\frac{\partial E_k}{\partial t} &= \frac{\epsilon^2 k_\parallel}{128\nu_{\text{odd}}} \sum_{s s_p s_q} \int_{\Delta_\perp} \left(\frac{\sin\theta_k}{k_\perp}\right) \frac{1}{k_\perp p_\perp q_\perp} \left(\frac{s_p p_\perp - s_q q_\perp}{k_\parallel}\right)^2 (s k_\perp + s_p p_\perp + s_q q_\perp)^2 \\ &\times \left[\omega_k E_p E_q + \omega_p E_k E_q + \omega_q E_k E_p\right] \delta(\Omega_{kpq}) \delta(k_\parallel + p_\parallel + q_\parallel) dp_\perp dq_\perp dp_\parallel dq_\parallel.\end{aligned} \tag{S40}$$

We introduce the adimensionalize wave numbers $\tilde{p}_i \equiv p_i/k_i$ and $\tilde{q}_i \equiv q_i/k_i$ with $i = \perp, \parallel$. With a spectrum of the type $E_k = A k_\perp^n k_\parallel^m$, we finally obtain

$$\begin{aligned}\frac{\partial E_k}{\partial t} &= \frac{\epsilon^2 A^2}{128\nu_{\text{odd}}} k_\perp^{2n+2} k_\parallel^{2m} \sum_{s s_p s_q} \int_{\Delta_\perp} \frac{\sin\theta_k}{\tilde{p}_\perp \tilde{q}_\perp} (s_p \tilde{p}_\perp - s_q \tilde{q}_\perp)^2 (s + s_p \tilde{p}_\perp + s_q \tilde{q}_\perp)^2 \\ &\times \left[\tilde{p}_\perp^n \tilde{q}_\perp^n \tilde{p}_\parallel^m \tilde{q}_\parallel^m + \tilde{p}_\perp \tilde{q}_\perp^n \tilde{p}_\parallel \tilde{q}_\parallel^m + \tilde{p}_\perp^n \tilde{q}_\perp \tilde{p}_\parallel^m \tilde{q}_\parallel\right] \\ &\times \delta(s + s_p \tilde{p}_\perp \tilde{p}_\parallel + s_q \tilde{q}_\perp \tilde{q}_\parallel) \delta(1 + \tilde{p}_\parallel + \tilde{q}_\parallel) d\tilde{p}_\perp d\tilde{q}_\perp d\tilde{p}_\parallel d\tilde{q}_\parallel.\end{aligned} \tag{S41}$$

We apply the Kuznetsov-Zakharov transformation and find after some manipulations

$$\begin{aligned}\frac{\partial E_k}{\partial t} &= \frac{\epsilon^2 A^2}{384\nu_{\text{odd}}} k_\perp^{2n+2} k_\parallel^{2m} \sum_{s s_p s_q} \int_{\Delta_\perp} \sin\theta_k (s_p \tilde{p}_\perp - s_q \tilde{q}_\perp)^2 (s + s_p \tilde{p}_\perp + s_q \tilde{q}_\perp)^2 \\ &\times \tilde{p}_\perp^{n-1} \tilde{q}_\perp^{n-1} \tilde{p}_\parallel^m \tilde{q}_\parallel^m \left(1 + \tilde{p}_\perp^{-3-2n} \tilde{p}_\parallel^{-2m} + \tilde{q}_\perp^{-3-2n} \tilde{q}_\parallel^{-2m}\right) \left(1 + \tilde{p}_\perp^{1-n} \tilde{p}_\parallel^{1-m} + \tilde{q}_\perp^{1-n} \tilde{q}_\parallel^{1-m}\right) \\ &\times \delta(s + s_p \tilde{p}_\perp \tilde{p}_\parallel + s_q \tilde{q}_\perp \tilde{q}_\parallel) \delta(1 + \tilde{p}_\parallel + \tilde{q}_\parallel) d\tilde{p}_\perp d\tilde{q}_\perp d\tilde{p}_\parallel d\tilde{q}_\parallel.\end{aligned} \tag{S42}$$

Two stationary solutions emerge, namely

$$n = 1 \quad \text{and} \quad m = 0, \tag{S43}$$

and

$$n = -3/2 \quad \text{and} \quad m = -1/2. \tag{S44}$$

The first corresponds to the thermodynamic (zero-flux) solution, while the second is the Kolmogorov-Zakharov spectrum for which the energy flux is finite.

Properties of odd wave turbulence

Cascade direction

We introduce the axisymmetric energy flux

$$\partial_t E_k = -\frac{\partial \Pi_\perp(k_\perp, k_\parallel)}{\partial k_\perp} - \frac{\partial \Pi_\parallel(k_\perp, k_\parallel)}{\partial k_\parallel} \tag{S45}$$

and from the kinetic equation (S42) we write

$$\partial_t E_k = \frac{\epsilon^2 A^2}{384\nu_{\text{odd}}} k_\perp^{2n+2} k_\parallel^{2m} I(n,m), \tag{S46}$$

where $I(n,m)$ is the normalized collisional integral.

After integration, and taking the limit $(n,m) \to (-3/2,-1/2)$, the two components of the flux become constant and equal to $\Pi_\perp^{\text{KZ}}$ and $\Pi_\parallel^{\text{KZ}}$. Thanks to L'Hospital's rule, we obtain

$$\Pi_\perp^{KZ} = -\frac{\epsilon^2 A^2}{384\nu_{\text{odd}}} \frac{1}{2k_\parallel} \frac{\partial I(n,-1/2)}{\partial n}\Big|_{n=-3/2} \equiv \frac{\epsilon^2 A^2}{384\nu_{\text{odd}}} \frac{1}{k_\parallel} I_\perp, \tag{S47}$$

$$\Pi_\parallel^{KZ} = -\frac{\epsilon^2 A^2}{384\nu_{\text{odd}}} \frac{1}{2k_\perp} \frac{\partial I(-3/2,m)}{\partial m}\Big|_{m=-1/2} \equiv \frac{\epsilon^2 A^2}{384\nu_{\text{odd}}} \frac{1}{k_\perp} I_\parallel, \tag{S48}$$

where

$$\begin{pmatrix} I_\perp \\ I_\parallel \end{pmatrix} \equiv \sum_{s s_p s_q} \int_{\Delta_\perp} \sin\theta_k \left(s_q \tilde{q}_\perp - s_p \tilde{p}_\perp\right)^2 \left(s + s_p \tilde{p}_\perp + s_q \tilde{q}_\perp\right)^2$$
$$\tilde{p}_\perp^{-5/2} \tilde{q}_\perp^{-5/2} \tilde{p}_\parallel^{-1/2} \tilde{q}_\parallel^{-1/2} \begin{pmatrix} \tilde{p}_\parallel \ln \tilde{p}_\perp + \tilde{q}_\parallel \ln \tilde{q}_\perp \\ \tilde{p}_\parallel \ln \tilde{p}_\parallel + \tilde{q}_\parallel \ln \tilde{q}_\parallel \end{pmatrix} \left(1 + \tilde{p}_\perp^{5/2} \tilde{p}_\parallel^{3/2} + \tilde{q}_\perp^{5/2} \tilde{q}_\parallel^{3/2}\right)$$
$$\delta\left(s + s_p \tilde{p}_\perp \tilde{p}_\parallel + s_q \tilde{q}_\perp \tilde{q}_\parallel\right) \delta\left(1 + \tilde{p}_\parallel + \tilde{q}_\parallel\right) d\tilde{p}_\perp d\tilde{q}_\perp d\tilde{p}_\parallel d\tilde{q}_\parallel. \tag{S49}$$

With this notation, we find the simple relationship for the energy flux ratio

$$\frac{\Pi_\parallel^{KZ}}{\Pi_\perp^{KZ}} = \frac{k_\parallel}{k_\perp} \frac{I_\parallel}{I_\perp}, \tag{S50}$$

which can be small if $I_\parallel \sim I_\perp$ since by assumption $k_\perp \gg k_\parallel$.

ISOTROPIC KINETIC ENERGY SPECTRA

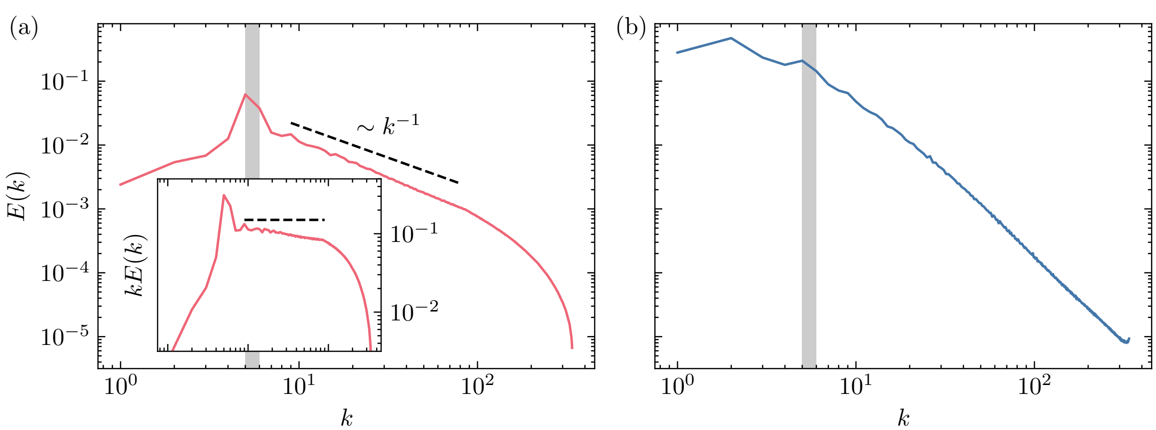

FIG. S1. The istropic kinetic energy spectra $E(k)$ for strong (a) and weak turbulence (b) as treated in the main text. Inset in (a) shows the spectrum compensated by its respective scaling prediction $E(k) \sim k^{-1}$ as suggested in Ref. [11]. For the weak turbulence in (b), no scaling prediction exists for the isotropic energy spectrum. The shaded area depicts the forcing range.

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