Canonical ensemble of a d-dimensional Reissner-Nordström black hole in a cavity

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We construct the canonical ensemble of a d-dimensional Reissner-Nordström black hole spacetime in a cavity surrounded by a heat reservoir through the Euclidean path integral formalism. The cavity radius R is fixed, and the heat reservoir is at a fixed temperature T and fixed electric charge Q. We use York's approach to find the reduced action by imposing the Hamiltonian and Gauss constraints and the appropriate conditions to the Euclideanized Einstein-Maxwell action with boundary terms, and then perform a zero loop approximation so that the paths that minimize the action contribute to the partition function. We find that, for an electric charge smaller or equal than a critical saddle electric charge Q_s, there are three solutions r_{+1}, r_{+2}, and r_{+3}, such that $r_{+1} < r_{+2} < r_{+3}$. The solutions r_{+1} and r_{+3} are stable within the ensemble, while r_{+2} is unstable. For an electric charge equal to Q_s, the solution r_{+2} merges with r_{+1} and r_{+3} at a given specific temperature. For an electric charge larger than Q_s, there is only one solution r_{+4}, which can be seen as the merging of the r_{+1} and r_{+3} solutions, with r_{+4} being stable. Since the partition function is directly related to the free energy in the canonical ensemble, we read off the free energy and calculate the thermodynamic variables, namely the entropy, the thermodynamic electric potential, the thermodynamic pressure, and the mean energy. We investigate thermodynamic stability, which is controlled by the positivity of the heat capacity at constant area and electric charge, and show that the heat capacity is discontinuous at the electric charge Q_s, signaling a turning point. We analyze the favorable states, examining the free energies of the stable black hole solutions and the free energy of electrically charged hot flat space, in order to check for possible first and second order phase transitions between the possible states. For instance, the two stable black hole solutions r_{+1} and r_{+3} are in competition between themselves, more specifically, for certain ensemble parameters there exists a first order phase transition from one solution to the other, and at the critical charge Q_s this transition turns into a second order phase transition. We also compare the thermodynamic radius of zero free energy with the generalized Buchdahl bound radius, which do not match, and comment on the physical implications, such as the possibility of total gravitational collapse of the thermodynamic system. We study the limit of infinite cavity radius and find two possibilities, the Davies and the Rindler solutions. The Davies thermodynamic solution of electrically charged black holes in $d = 4$ dimensions is recovered from the general d-dimensional canonical ensemble analysis. We obtain, in particular, the heat capacity given by Davies and the Davies point. The Rindler solution describes the black hole horizon as a Rindler horizon, and the boundary, which is at fixed temperature T provided by the reservoir, must have the necessary acceleration to reproduce the corresponding Unruh temperature. Going back to a cavity with finite radius we find that the three solutions mentioned above are related to the original York two Schwarzschild black hole solutions and to the two Davies solutions, with the middle unstable solution r_{+2} belonging simultaneously to the two sets of solutions. In this sense, York's and Davies' formalisms have been unified in our approach. In all instances we mention carefully the four-dimensional case, for which we accomplish new results, and study in detail all aspects of the five dimensional case.

I. INTRODUCTION

A. Background

The hypothesis that black holes have a thermodynamic character emerged through a series of notable developments. Bekenstein [1] introduced the idea that a black hole has an entropy proportional to the surface area of its event horizon and formulated a generalized second law of thermodynamics. Smarr found a mass formula involving all the black hole parameters [2], which was extended in a formal basis to the four laws of black hole mechanics [3]. These laws were strikingly similar to the four laws of thermodynamics. The complete description came with the discovery by Hawking [4] that black holes radiate quanta with a thermal spectrum at temperature $T_{\rm H} = \frac{\kappa}{2\pi}$, the Hawking temperature in Planck units, where κ is black hole's surface gravity, and for instance, for the simplest nonrotating black hole one has $\kappa = \frac{1}{2\pi r_+}$, so that $T_{\rm H} = \frac{1}{4\pi r_+}$, r_+ being the event horizon radius. Furthermore, the vacuum state sitting at the horizon that enables the radiation to be produced was shown to be described by the Hartle-Hawking vacuum state [5]. By assuming that the black hole is in thermal equilibrium with the radiation emitted, it was argued that black holes must indeed be thermodynamic objects, and it was found that the entropy S of a black hole has the expression $S = \frac{A_+}{4}$, the Bekenstein-Hawking entropy,

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where A_+ is the surface area of the event horizon. The thermodynamics of black holes was expanded for black holes with rotation and electric charge by Davies [6], by assigning the first law of thermodynamics together with the Hawking temperature and the Bekenstein-Hawking entropy to those black holes. In this case, it was noticed an abrupt change of the heat capacity of the system, which was presumed to yield a phase transition.

The rationale for the black hole entropy and thermodynamics has also been obtained through statistical methods, in juxtaposition to the results obtained through the analysis of quantum fields around black holes. The focus on black hole statistical methods implies one has to set up an ensemble of a physical spacetime. To build the ensemble, one needs to know a priori the microscopic description of the system. Such an accepted description is still unknown for gravity. Nevertheless, the partition function of spacetime can be computed through the Euclidean path integral approach to quantum gravity. In this approach, the partition function is given by a path integral of the exponential of the Euclidean action $I[g,\phi]$, over Euclidean metrics g and fields ϕ that permeate the space, where the integral is restricted to metrics that are periodic in the imaginary time length, i.e., $Z = \int DgD\phi \mathrm{e}^{-I[g,\phi]}$. Depending on the ensemble considered, there are always quantities that are fixed at some boundary, e.g., in the canonical ensemble the temperature given by the inverse of the imaginary time length at the boundary is fixed, with this boundary then being a heat reservoir. This method of computing the partition function inherits the difficulties of the Euclidean path integral approach to quantum gravity. For example, the map between the physical Lorentzian spacetime and the Euclidean space, that is performed through a Wick transformation on a time coordinate, is not in general well-defined or unique, covering only some sections of the Lorentzian spacetime. Moreover, there are difficulties on the convergence of the path integral. To put these difficulties aside, a zero loop approximation of the path integral is considered, where only the paths that minimize the Euclidean action are taken into account. The partition function is then given by $Z = \mathrm{e}^{-I_0}$, where now I_0 is the classical Euclidean action evaluated at one of these paths, yielding a partition function in the semiclassical approximation. One can then relate the partition function to a thermodynamic potential depending on the ensemble chosen, and the thermodynamics of the system can be worked out through the derivatives of the thermodynamic potential.

The application of the Euclidean path integral approach to known spaces, such as the Schwarzschild black hole in the canonical ensemble, and the Reissner-Nordström black in the grand canonical ensemble with fixed electric potential at the boundary, was done with a heat reservoir at infinite proper distance [7], recovering the Hawking temperature and the Bekenstein-Hawking entropy of a black hole. Yet, for those configurations in the respective ensembles, the obtained heat capacity of the black hole is negative, which means the configuration is thermodynamically unstable. It was later found that the configurations correspond to a saddle point of the action [8]. And so, the zero loop approximation is not valid for these configurations, although one can still treat them as instantons. A perturbation of the instanton yields a negative mode that makes the one-loop contribution of the path integral formally divergent, but the path integral can still be continued to the complex numbers resulting in a nonzero imaginary part. However, when applied to a Schwarzschild-anti de Sitter black hole, which can be considered a configuration of a black hole in a box, the formalism produced consistent results and stable black hole solutions [9]. It was then found that the negative mode of the pure Schwarzschild black hole ceases to exist if the heat reservoir sits at a radius equal or smaller than the photon sphere radius [10].

Soon after, York realized that the construction of canonical or grand canonical ensembles should be performed by putting a black hole space inside a cavity with a heat reservoir at finite radius [11]. Using the path integral approach and the zero loop approximation for a Schwarzschild black hole inside a cavity, he found that there are two stationary points for the Euclidean action I_0. From these two, the one with the least mass is unstable and corresponds to the Hawking-Gibbons black hole in the limit of infinite radius of the heat reservoir. The other one has the largest mass and is stable, therefore the zero loop approximation is valid for this stationary point. This motivated a series of developments, namely, the study of the canonical ensemble of a Schwarzschild black hole inside a cavity by considering a class of paths in the context of York's formalism [12], and the application of York's formalism to a system including matter was sketched in [13]. Moreover, the grand canonical ensemble of a Reissner-Nordström black hole inside a cavity was considered within York's formalism in [14], by fixing the temperature and the electric potential at the boundary of the cavity. The canonical ensemble for arbitrary configurations of self-gravitating systems was studied in [15].

There have been applications of the Euclidean path integral approach, both in Gibbons-Hawking form and in York's extended formalism, to asymptotically anti-de Sitter and de Sitter black hole spaces as well as to higher dimensions, which we now mention. Black hole spaces with negative cosmological constant within general relativity were discussed in three and four dimensions [16]. The two dimensional black hole space in Teitelboim-Jackiw theory which is asymptotically de Sitter was studied in [17]. The grand canonical ensemble of the Reissner-Nordström-anti-de Sitter space in four dimensions was constructed and analyzed in [18], and of the electric charged toroidal anti-de Sitter black hole in [19]. The stability and the negative mode for Schwarzschild-Tangherlini vacuum spacetime was described in [20]. The construction of the canonical ensemble of a four-dimensional Reissner-Nordström black hole inside a cavity was obtained in [21, 22] by adding a boundary term on the action and fixing the electric charge instead of the electric potential at the boundary of the cavity. A study of the canonical ensemble of four and higher dimensional Schwarzschild-anti de Sitter black holes was done in [23]. The canonical ensemble of black branes in arbitrary dimensions along with their phase structure was developed in [24]. The canonical ensemble has been applied to Schwarzschild black holes in a cavity in d dimensions in [25, 26]. The canonical ensemble of gravastars was analyzed in [27]. The first law of de Sitter spaces with black holes in an ensemble context was identified in [28]. A detailed analysis of the inclusion of matter is given in [29], with surprising results concerning the equilibrium and the equation of state of the matter. The grand canonical ensemble of Reissner-Nordström black holes inside cavity in dimensions $d \geq 4$ was considered in [30]. The canonical ensemble of black hole in a de Sitter background has been constructed and explored in [31]. Using the Gibbons -Hawking action formalism for electrically charged black holes in the canonical ensemble, the Davies' thermodynamic theory of black holes has been recovered in [32].

It should be mentioned that motivated by supergravity theories, the analysis of ensembles and the Euclidean path integral approach were extended to black brane solutions. It was found that the mechanical stability of black branes is related to their local thermodynamic stability [33]. This relation was further studied and proven in some cases, see [34–36].

We also note that York's path integral formalism and the thermodynamics of a hot thin shell of matter analyzed from a first law of thermodynamics basis share some similarities. This was found in [37] for thin shells with an outer Schwarzschild spacetime and in [38] for thin shells with an outer Reissner-Nordström spacetime. The analysis of hot thin shells have been extended to higher dimensions in [39] for Schwarzschild spacetimes, and in [40] for Reissner-Nordström spacetimes. In [41], the Reissner-Nordström case was revisited. A radius that will appear somewhat naturally in the analysis is the generalized Buchdahl bound radius, also called Buchdahl-Andréasson-Wright bound radius [42]. It is a dynamical radius rather than a thermodynamic one.

B. Motivation

1. *Scales*

It is important to know in which physical settings and at which scales the situations we are studying here prevail and are of interest. Black holes can exist in all scales, from Planck scales, through micro scales, up to astrophysical and cosmic scales. Planck scale and micro scale black holes with very small radii can appear through pair creation in strong field settings, or produced by head-on collisions of elementary type particles of enormous high energy. On the other hand, astrophysical and cosmic black holes arise through the gravitational collapse of huge quantities of matter. Different physical effects turn up for each range of scales and one should pick the appropriate ones that have the most impact for the black holes under study [43, 44].

The scales of interest here are scales where quantum effects determined by the Hawking radiation in a black hole environment become important. The quantity that can be taken to set the scales is then the Hawking temperature $T_{\rm H}$. In $d = 4$, one has $T_{\rm H} = \frac{1}{4\pi r_+}$ for the simplest black hole, r_+ being the horizon radius of the black hole. This is a temperature measured at infinity. Surrounding the black hole there is thus a cloud of radiation created via quantum processes. The Hawking temperature can be written as $T_{\rm H} = \frac{l_{\rm pl}}{r_+} T_{\rm pl}$, where the subscript pl means Planck quantities. Thus, we can write $T_{\rm H} = 10^{32} \frac{1}{r_+}\,{\rm K}$ with r_+ given in Planck units. We want to study quantum effects that are far from the full quantum gravity regime. If we put $r_+ = 10^{20} l_{\rm pl}$, then the Hawking temperature is $T_{\rm H} = 10^{-20} T_{\rm pl}$. In usual units one has that in this case the horizon radius has value $r_+ = 10^{-13}\,{\rm cm}$, the temperature is $T_{\rm H} = 10^{12}\,{\rm K}$, and the black hole mass m is $m = 10^{15}\,{\rm g}$. Such a black hole has the size of a neutron or proton, and the mass of a large Earth mountain, a really interesting microscopic black hole in our context, for which semiclassical effects have to be taken into account. This is the kind of system we are interested, it is a microscopic system with high temperatures, where quantum effects are important, but not full quantum gravity [45].

Now, the continual emitting of Hawking radiation depletes the black hole of its energy, the horizon radius shrinks and eventually the black hole disappears. One can think of ways to stabilize the black hole. One way is to enclose the black hole in a cavity and surround it by a heat reservoir. Another way is to give the black hole a charge, e.g., an electric or magnetic charge. It is of interest to implement both ways.

2. *Geometric and physical structure*

By enclosing the black hole in a cavity surrounded by a heat reservoir with a definite radius and at a given temperature, one is able to maintain a thermodynamic equilibrium between both temperatures, the black hole and the reservoir temperatures, and treat the system in a time independent way.

This system, black hole plus reservoir, is relevant when the temperatures of its components are sufficiently high. This means that the black hole and heat reservoir have to be of microscopic size, where the curvature of space is sufficiently big to generate a significant emission of radiation from the black holes permitting the emergence of non-negligible quantum effects within the system.

A heat reservoir at constant temperature is a physical situation that points to the building of a statistical mechanics canonical ensemble. Since the scales of the regime one is interested in are far from quantum gravity scales, but nevertheless quantum effects are important, these gravitational systems involving black holes at these micro scales can be treated semiclassically.

Due to the time independence of the system, one can use an Euclidean path integral approach to calculate the partition function of the canonical ensemble [46] by Euclideanizing the chosen time coordinate of the solution. This allows one to analyze important properties of the system, like its full thermodynamics, its thermodynamic phases, and the possible first and second order phase transitions between black holes and hot spaces. It also clarifies the reciprocal thermodynamic responses operating between event horizons and cavity walls kept at finite temperature. One could think of producing these tiny black holes in the laboratory and test important features some of them found here, notably the stability behavior and the role of the thermodynamic phases.

3. *Electric charge*

By giving some charge to the black hole, e.g., electric or magnetic charge, it is possible to stabilize it thermodynamically. In general relativity, black holes can have mass, electric charge, and angular momentum. Black holes with mass and angular momentum are used to describe astrophysical phenomena, whereas black holes with electric charge are dismissed for such phenomena since they are quickly discharged by the plasma in the surrounding medium. Notwithstanding, electrically charged black holes can be of importance when one is dealing with micro objects. In these black holes, quantum effects come into play. Vacuum polarization at the black hole event horizon can discharge the black hole, as particles with opposite charge in the polarized domains are more probable to be absorbed [47]. This happens when the temperature is sufficiently high to allow particle production of massive particles, since electric charge of contrary sign is superradiantly emitted. However, when the temperature is sufficiently low, there is not enough energy to produce charged massive particles and the black hole does not discharge. One can find different ways to stabilize the charge. For instance, if the charge is purely topological, there are no particles to radiate. Another instance is when the only particles of the theory are sufficiently massive that their creation is highly suppressed, such as a very massive magnetic monopole in a magnetically charged black hole background [48, 49]. Central charges that appear in the algebra of supergravity theories also do not suffer from pair creation instability. One can also fix the electric charge in the cavity with the black hole inside. This allows us to find stable and unstable electrically charged black holes. Indeed, for small enough black holes and relatively small temperatures all the packets of energy with positive or negative electric charge are trapped within the gravitational field of the black hole which is then electrically stable, i.e., it does not discharge. Thus electrically charged black holes, in particular Reissner-Nordström black holes, have interest in practice.

Now, Reissner-Nordström black hole spacetimes can be asymptotically anti-de Sitter, asymptotically flat, and asymptotically de Sitter [50]. Certain particle theories, notably, supergravity theories, work with a negative cosmological constant, and their black hole solutions have anti-de Sitter asymptotics. Pure general relativity has black holes which are asymptotically flat, with these spacetimes yielding the appropriate environment in the study of a sufficiently large neighborhood surrounding a black hole. In a cosmological setting and in other settings, one might want to use black holes in asymptotically de Sitter spacetimes.

4. *Higher dimensions*

The world seems to have $d = 4$ spacetime dimensions, but speculations on higher dimensions has always been in the forefront of gravitational theories. For instance, Schwarzschild and Reissner-Nordström black holes in higher dimensions, $d \geq 4$, were first conceived in a discussion connected to the problem of the dimensionality of space [51]. Properties of the spaces might differ, the Hawking temperature is now $T_{\rm H} = \frac{d-3}{4\pi r_+}$ for the Schwarzschild-Tangherlini d-dimensional black hole, r_+ being the horizon radius of the d-dimensional black hole [52].

Moreover, certain theories are well formulated only in higher-dimensional spacetimes, $d \geq 4$, such is the case of several supergravity theories, and of string and superstring theories, which makes the study of black hole solutions in d-dimensions important. In connection to these theories, there is a correspondence between black hole physics in anti-de Sitter backgrounds and a conformal field theory physics in the boundary of those same backgrounds, the AdS/CFT conjecture, which is formulated in different dimensions [53, 54].

The extra dimensions can be small, normal, or large when compared to the usual $d = 4$ ones. If one conceives relatively large extra dimensions, then one can in principle produce higher dimensional black holes in future particle accelerator machines, see, e.g., [55].

As well, in studying spacetimes with d generic dimensions, $d \geq 4$, one has the possibility of understanding what is peculiar to $d = 4$ and what is generic, with some results for the particular case $d = 4$ being recovered.

C. Aim

Our aim is to understand more deeply the quantum and thermodynamic properties of microscopic gravitational systems involving black holes, notably the interaction of a black hole with a heat bath, using an electrically charged black hole in general relativity. For that, in this work, we construct the canonical ensemble of a d-dimensional Reissner-Nordström-Tangherlini, or simply Reissner-Nordström spacetime, inside a cavity. The construction is made by obtaining the partition function through the Euclidean path integral approach in zero loop approximation. The canonical ensemble is defined by adding a boundary term to the action, by fixing the inverse temperature as the Euclidean time length at the boundary of the cavity, and by fixing as well the electric flux, i.e., by fixing the electric charge. We find three black hole solutions for the ensemble for an electric charge smaller or equal than a critical charge from which two are stable, and one black hole solution for an electric charge larger than a critical charge which is stable. We study the thermodynamics of the stable solutions and also analyze the thermodynamic stability. We perform an analysis of the thermodynamic phases, namely the phases corresponding to the two stable black holes and the phase of hot flat charged space, and discuss the possible first and second order phase transitions. We compare the radius of zero free energy with the generalized Buchdahl bound radius, also called Buchdahl-Andréasson-Wright bound radius [42]. We make the analysis of the system in the limit of infinite radius of the cavity for generic d, and find two possible limits, the Davies and the Rindler solutions. Applying the results to $d = 4$, we recover Davies' thermodynamic theory for electrically charged black holes from the canonical ensemble in the limit of infinite radius, and we also retrieve the Davies point showing that it signals a turning point rather than a second order phase transition as originally argued. The Rindler limit reveals that the cavity boundary is accelerated at the corresponding Unruh temperature. We note that the $d = 4$ canonical ensemble was mentioned in [14] and analyzed in [21, 22]. When we specifically put $d = 4$ in our analysis, we confirm the results obtained in [21, 22], as well as find other interesting new results, such as the recovering of the full thermodynamic analysis of Davies from the canonical ensemble when the cavity radius, i.e., the reservoir, is at infinity.

D. Organization

This paper is organized as follows. In Sec. II, we construct the canonical ensemble through the Euclidean path integral approach. In Sec. III, we apply the zero loop approximation, find the solutions to the canonical ensemble, and analyze their stability and the dimension dependence. We comment on the four-dimensional case and cover in detail the five-dimensional case, a feature that will be provided in all sections. In Sec. IV, we study the thermodynamics given by the canonical ensemble in the zero loop approximation. In Sec. V, we study the favorable states, comparing the stable black hole solutions with a configuration of an electrically charged shell with gravity turned off that emulates hot flat space with electric charge. We also find and comment on the thermodynamic black hole configurations that have horizon radii higher than the Buchdahl radius. In Sec. VI, we study the canonical ensemble in the limit of infinite radius of the cavity, recovering the Davies thermodynamic theory of black holes and finding the Rindler solution at the Unruh temperature. In Sec. VII, we conclude. There are two important appendices. In Appendix A, the Euclidean action for the canonical ensemble, the boundary conditions, the Ricci scalar, the Euler characteristic, and the reduced action are derived and explained in detail. In Appendix B, we perform the calculation of the radius where the free energy of the electrically charged black hole is zero and give the results for different ensembles and the generalized Buchdahl radius.

II. THE CANONICAL ENSEMBLE OF A CHARGED BLACK HOLE IN A CAVITY THROUGH THE EUCLIDEAN PATH INTEGRAL APPROACH

A. Partition function as a Euclidean path integral

In statistical mechanics, the canonical ensemble of a system is a statistical ensemble of possible configurations of the system in thermodynamic equilibrium with a reservoir of temperature T, with fixed particle number and unspecified energy. Through the canonical ensemble, it is possible to obtain the thermodynamic properties of the system in equilibrium with the heat reservoir. The quantity that holds all the thermodynamic information of the canonical ensemble is the partition function, $Z = \sum_i \mathrm{e}^{-\beta E_i}$, where the sum of all the possible states i is done, β is the ensemble inverse temperature, $\beta = \frac{1}{T}$, and E_i is the energy of each state i.

When the canonical ensemble is applied to a quantum system, one can calculate the partition function as $Z = \mathrm{Tr}\left(\mathrm{e}^{-\beta H}\right) = \sum_i \langle\psi_i|\, \mathrm{e}^{-\beta H}\, |\psi_i\rangle$, where $\mathrm{Tr}(\mathrm{e}^{-\beta H})$ is the trace of the quantum operator $\mathrm{e}^{-\beta H}$, H is the Hamiltonian of the system and the ψ_i are a basis of a Hilbert space, not necessarily the Hamiltonian eigenstates. Consider now a quantum system to be in a state ψ at time t_1 and in a state $\hat{\psi}$ at time t_2. Then, the amplitude of a system to evolve from the state ψ to $\hat{\psi}$ is $\langle\hat{\psi}, t_2|\psi, t_1\rangle = \langle\hat{\psi}|\, \mathrm{e}^{-iH(t_2-t_1)}\, |\psi\rangle$, which can be calculated by the Feynman path integral approach as $\langle\hat{\psi}, t_2|\psi, t_1\rangle = \int d[\psi]\mathrm{e}^{iI[\psi]}$, where the functional integration on ψ is done from $\psi(t_1) = \psi$ to $\psi(t_2) = \hat{\psi}$. If we now evaluate the amplitude of the system to evolve from a state ψ to the same state ψ in a time interval $(t_2 - t_1) = -i\beta$ and sum the amplitudes for all the basis states, we have that this sum is the partition function now written in the Euclidean path integral form $Z = \mathrm{Tr}(\mathrm{e}^{-\beta H}) = \int d\psi \mathrm{e}^{-I[\psi]}$, where I is now the Euclidean action of the system and the integration is done for every possible periodic function ψ with period $\beta = \frac{1}{T}$. This is the Euclidean path integral approach to construct the canonical ensemble.

It is reasonable to extend this approach to self-gravitating systems. Moreover, such extension provides a way to describe quantum gravity, i.e., the Euclidean path integral approach to quantum gravity. The partition function is then given by the Euclidean path integral

$$Z = \int DgD\phi \mathrm{e}^{-I[g,\phi]}\,, \tag{1}$$

where g is the Euclidean metric obtained from the Lorentzian metric by making a Wick transformation $t = -i\tau$, i.e., time is Euclideanized, ϕ represents all the matter fields that might be present in the system, and Dg and $D\phi$ mean integration measures of the path integral whose structures are not of concern here. Here, we construct the canonical ensemble by the Euclidean path integral approach to a spherically symmetric electrically charged black hole inside a cavity, in arbitrary d dimensions. The system will be in equilibrium with a heat reservoir at the boundary of the cavity with fixed inverse temperature β, which is given by the total Euclidean proper time of the boundary of the cavity, and with fixed electric flux, i.e., with the black hole electric charge Q fixed. The thermodynamics of the system can then be obtained by considering that the partition function of the canonical ensemble is tied to the Helmholtz free energy F through $Z = \mathrm{e}^{-\beta F}$, i.e., $\beta F = -\ln Z$. With the free energy determined, the other thermodynamic quantities are obtained by the derivatives of the free energy, noting that $F = E - TS$, where E is the thermodynamic energy of the system and S its entropy.

B. The Euclidean action for the canonical ensemble

The Euclidean action of the system consisting of an electrically charged black hole in a cavity in d dimensions is

$$\begin{aligned} I = &-\frac{1}{16\pi}\int_M R\sqrt{g}d^dx - \frac{1}{8\pi}\int_{\partial M}(K-K_0)\sqrt{\gamma}d^{d-1}x \\ &+ \frac{(d-3)}{4\Omega_{d-2}}\int_M F_{ab}F^{ab}\sqrt{g}d^dx \\ &+ \frac{(d-3)}{\Omega_{d-2}}\int_{\partial M} F^{ab}A_a n_b\sqrt{\gamma}d^{d-1}x\,, \end{aligned} \tag{2}$$

where R is the Ricci scalar given by derivatives and second derivatives of the Euclidean metric g_{ab}, g is the determinant of g_{ab}, K is the trace of the extrinsic curvature of the boundary of the cavity defined as K_{ab}, K_0 is the trace of the extrinsic curvature of the boundary of the cavity embedded in flat Euclidean space, γ is the determinant of the induced metric $\gamma_{\alpha\beta}$ on the boundary of the cavity, Ω_{d-2} is the surface area of a $d-2$ unit sphere and appears here basically for practical purposes, $F_{ab} = \partial_a A_b - \partial_b A_a$ is the Maxwell tensor given by derivatives of the vector potential A_a, n_b is the outward unit normal vector to the boundary of the cavity, a, b are spacetime indices that run from 0 to $d-1$ in the usual manner, and α, β are indices on the boundary that run from 0 to $d-2$. The boundary term depending on the Maxwell tensor must be present so that the canonical ensemble may be prescribed, see [14]. This term allows us to fix the electric flux given by the integral of the Maxwell tensor on a $(d-2)$-surface, which has the meaning of electric charge. Otherwise, the potential vector A_a must be fixed, which means the grand canonical ensemble should be prescribed, see [30] for this case.

C. Geometry, electromagnetic field, and boundary conditions

1. Geometry and boundary conditions

We assume that the Euclidean path integral is done along metrics which are spherically symmetric. Therefore, the Euclidean space is given by the warped space product $\mathbb{R}^2 \times \mathbb{S}^{d-2}$ with the warping function r^2, where $\mathbb{S}^{d-2}$ is a $(d-2)$-sphere with radius r. The Euclidean metric of such space is given by

$$ds^2 = b^2(y)d\tau^2 + \alpha^2(y)dy^2 + r^2(y)d\Omega^2_{d-2}\,, \tag{3}$$

where τ is the periodic Euclidean time with range $0 \leq \tau < 2\pi$, y is a radial coordinate with range $0 \leq y \leq 1$, $d\Omega^2_{d-2}$ is the line element of the unit $(d-2)$-sphere with total area $\Omega_{d-2} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}$, Γ being the gamma function, $b(y)$ and $\alpha(y)$ are functions of y, and $r(y)$ represents the radius that gives the area of the $(d-2)$-sphere. The functions $b(y)$, $\alpha(y)$, and $r(y)$ are unspecified for now and are to be integrated in the path integral.

The hypersurface $y = 0$ is assumed to correspond to the bifurcation two-surface of the event horizon of the charged black hole, so we must impose the conditions

$$b(0) = 0\,, \tag{4}$$

$$r(0) = r_+\,, \tag{5}$$

where r_+ is the horizon radius. The conditions given in Eqs. (4) and (5) impose that the $y = 0$ hypersurface corresponds to $\{y = 0\} \times \mathbb{S}^{d-2}$, i.e., a point times a $(d-2)$-sphere. The $y = 0$ point in the (τ, y) sector coincides with the central point of the $\mathbb{R}^2$ plane in polar coordinates, where τ is in fact the angular coordinate and y is the radial coordinate of the plane. The $y = 0$ hypersurface can be seen as the limit $y \to 0$ of y constant hypersurfaces, with these latter having an $\mathbb{S}^1 \times \mathbb{S}^{d-2}$ topology. For the metric to be smooth, as y goes to zero, the constant y hypersurfaces $\mathbb{S}^1 \times \mathbb{S}^{d-2}$ must go smoothly to $\{y = 0\} \times \mathbb{S}^{d-2}$. There are more conditions other than the two above, see Appendix A for a detailed derivation of these conditions. One of the conditions for smoothness is

$$\left(\frac{b'}{\alpha}\right)(0) = 1\,, \tag{6}$$

where $\left(\frac{b'}{\alpha}\right)(0) \equiv \left(\frac{b'}{\alpha}\right)_{y=0}$. This is a third condition and means there are no conical singularities in the Euclidean manifold. With Eq. (6) considered, one can compute the Ricci scalar R of the metric of Eq. (3) when the metric is expanded around $y = 0$ with the conditions given in Eqs. (4) and (5). One obtains that to have a well-defined Ricci scalar R and for the space to be smooth, one must impose the fourth and fifth conditions

$$\left(\frac{r'}{\alpha}\right)(0) = 0\,, \tag{7}$$

$$\left(\frac{1}{\alpha}\left(\frac{b'}{\alpha}\right)'\right)(0) = 0\,, \tag{8}$$

with $\left(\frac{r'}{\alpha}\right)(0) \equiv \left(\frac{r'}{\alpha}\right)_{y=0}$ and $\left(\frac{1}{\alpha}\left(\frac{b'}{\alpha}\right)'\right)(0) \equiv \left(\frac{1}{\alpha}\left(\frac{b'}{\alpha}\right)'\right)_{y=0}$, see Appendix A. The condition given in Eq. (7) is equivalent, in even dimensions, to requiring that the Euclidean space considered has an Euler characteristic $\chi = 2$ by the Chern-Gauss-Bonnet formula. On the other hand, in odd dimensions, the Euler characteristic vanishes, and so this requirement is not satisfactory. Nevertheless, the requirement that the Ricci scalar is well-defined suffices. One can also see that the condition given in Eq. (7) is equivalent to requiring that the event horizon of the black hole is a null hypersurface, if one performs a Wick transformation back to the Lorentzian signature. The condition given in Eq. (8) means for some coordinate y, that if $(b'\alpha^{-1})'|_{y=0}$ is nonzero finite, then $\alpha|_{y=0}$ must diverge. Indeed, this is satisfied by the Reissner-Nordström line element with coordinate choice $y = r$ found by solving the Einstein equation, as it is seen below. We note that the condition given in Eq. (8) is not referred to elsewhere, in particular it is not mentioned in [14, 30].

The hypersurface $y = 1$ corresponds to the boundary of the cavity, where two conditions are imposed

$$b(1) = \frac{\beta}{2\pi}\,, \tag{9}$$

$$r(1) = R\,. \tag{10}$$

The condition given by Eq. (9) fixes β at the boundary, with β being the inverse temperature of the cavity, $\beta = \frac{1}{T}$. This condition enforces that the total Euclidean proper time of the boundary of the cavity is fixed to be equal to the inverse temperature of the cavity, and the condition comes from the definition of the path integral as stated in Sec. II A. The condition given by Eq. (10) states that the boundary is at radius R.

2. *Electromagnetic field and boundary conditions*

For the electromagnetic Maxwell field, due to spherical symmetry and admitting the nonexistence of magnetic monopoles, the only nonvanishing components of the Maxwell tensor F_{ab} are $F_{y\tau} = -F_{\tau y}$. Moreover, we choose a gauge where the only nonvanishing component of the vector potential is $A_\tau(y)$. Therefore, the Maxwell tensor F_{ab} is described only by the term

$$F_{y\tau}(y) = \frac{dA_\tau(y)}{dy}\,. \tag{11}$$

The boundary conditions can now be imposed.

At $y = 0$, we require that

$$A_\tau(0) = 0\,, \tag{12}$$

to have regularity. This condition also fixes completely the gauge of the Maxwell field.

At $y = 1$, we fix the electric charge by specifying the electric flux given by $\int_{\substack{y=1\\ \tau=c}} F^{ab} dS_{ab} = 2i\Omega_{d-2}Q$, where c is a constant, Q is the electric charge in the cavity, $dS_{ab} = 2u_{[a}n_{b]}dS$ is the surface element of the $y = 1$ and $\tau = 0$ surface, $u_a dx^a = b d\tau$, $n_a dx^a = \alpha dy$, and dS is the surface volume, i.e.,

$$\left(b\alpha r^{d-2}\, F^{y\tau}\right)(1) = -iQ\,. \tag{13}$$

D. The action with boundary conditions

Putting together the conditions just found into the action Eq. (2), one finds that it is a function of the radius of the cavity R, the inverse temperature β, and the charge Q, which are the fixed quantities of the system, and a functional of b, α, r and A_τ. The partition function is then given at this stage by the action appropriately integrated in all paths in the path integral. We now evaluate the action Eq. (2) with the considered boundary conditions. Here we sketch the calculation, see Appendix A for full details.

We start by looking at the Ricci scalar R. The Ricci scalar R is the contraction of the Ricci tensor R_{ab} which itself is the contraction of the Riemann tensor, and moreover, one can form the Einstein tensor G_{ab} from R_{ab} and R, $G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$. The Ricci scalar for the metric in Eq. (3) is given by $-\frac{1}{16\pi}R = \frac{1}{8\pi\alpha b r^{d-2}}\left(\frac{r^{d-2}b'}{\alpha}\right)' + \frac{1}{8\pi}G^\tau{}_\tau$, where $G^\tau{}_\tau$ is the time-time component of the Einstein tensor and is given by $G^\tau{}_\tau = \frac{(d-2)}{2r'r^{d-2}}\left[r^{d-3}\left(\frac{r'^2}{\alpha^2} - 1\right)\right]'$, and the prime means derivative with respect to y. By putting the expression of the Ricci scalar into Eq. (2), we observe that the first term in the volume integration of the Ricci scalar yields $\frac{\Omega_{d-2}}{4}\left(\frac{r^{d-2}b'}{\alpha}\right)_{y=1} - \frac{\Omega_{d-2}}{4}\left(\frac{r^{d-2}b'}{\alpha}\right)_{y=0}$, i.e., a boundary term at $y = 1$ and a boundary term at $y = 0$. The term $-\int_{y=1}\frac{\sqrt{\gamma}}{8\pi}(K - K_0)d^{d-1}x$ is called the Gibbons-Hawking-York boundary term, and is given by $-\int_{y=1}\frac{\sqrt{\gamma}}{8\pi}(K - K_0)d^{d-1}x = \left(\frac{2\pi b r^{d-3}}{\mu}\left(1 - \frac{r'}{\alpha}\right)\right)_{y=1} - \frac{\Omega_{d-2}}{4}\left(\frac{r^{d-2}b'}{\alpha}\right)_{y=1}$, where it was used that the extrinsic curvature of a constant y hypersurface is $\mathbf{K} = \frac{bb'}{\alpha}d\tau + \frac{rr'}{\alpha}d\Omega^2_{d-2}$, and $\mathbf{K}_0 = rd\Omega^2_{d-2}$ is the extrinsic curvature of the hypersurface embedded in flat space and $\mu = \frac{8\pi}{(d-2)\Omega_{d-2}}$. The last term of the Gibbons-Hawking-York boundary term cancels with the boundary term at $y = 1$ of the Ricci scalar. Moreover, by using the boundary condition Eq. (6), the remaining boundary term of the volume integral of the Ricci scalar becomes $-\frac{\Omega_{d-2}}{4}\left(\frac{r^{d-2}b'}{\alpha}\right)_{y=0} = -\frac{\Omega_{d-2}}{4}r_+^{d-2}$. It is useful to rewrite the Maxwell boundary term in the action Eq. (2). Using the divergence theorem and that $\nabla_b(F^{ab}A_a) = -\frac{1}{2}F_{ab}F^{ab} + A_a\nabla_bF^{ab}$, one transforms the boundary Maxwell term into bulk terms, obtaining that the Maxwell part of the action is $-\frac{(d-3)}{4\Omega_{d-2}}\int_M F_{ab}F^{ab}\sqrt{g}d^dx + \frac{(d-3)}{\Omega_{d-2}}\int_M A_a\nabla_bF^{ab}\sqrt{g}d^dx$. Now, $-\frac{(d-3)}{4\Omega_{d-2}}F_{ab}F^{ab} = -\frac{(d-3)}{2\Omega_{d-2}}\frac{{A'_\tau}^2}{\alpha^2b^2}$, where $F_{y\tau} = A'_\tau$ was used, and $\frac{(d-3)}{\Omega_{d-2}}\nabla_bF^{ab}A_a = -\frac{(d-3)}{\Omega_{d-2}\alpha br^{d-2}}\left(\frac{r^{d-2}A'_\tau}{b\alpha}\right)'A_\tau$, where $\nabla_aF^{\tau a} = -\frac{1}{\alpha br^{d-2}}\left(\frac{r^{d-2}A'_\tau}{\alpha b}\right)'$ was used.

One can proceed with the integrations at the cavity, since the integrands do not depend on time or on the angles, and one obtains from Eq. (2) the full action

$$\begin{aligned} I[\beta, Q, R; b, \alpha, r, A_\tau] &= \frac{\beta R^{d-3}}{\mu}\left(1 - \left(\frac{r'}{\alpha}\right)(1)\right) \\ &- \frac{\Omega_{d-2}}{4}r_+^{d-2} - \frac{(d-3)}{\Omega_{d-2}}\int_M\left(\frac{r^{d-2}A'_\tau}{b\alpha}\right)'A_\tau d\tau dy d\Omega_{d-2} \\ &+ \int_M\frac{\alpha br^{d-2}}{8\pi}\left(G^\tau{}_\tau - \frac{4\pi(d-3)}{\Omega_{d-2}}\frac{{A'_\tau}^2}{\alpha^2b^2}\right)d\tau dy d\Omega_{d-2}\,, \end{aligned} \tag{14}$$

where it was used that the time length at the cavity is given by Eq. (9), i.e., $\beta = 2\pi b(1)$, and that from Eq. (10) one has $r(1) = R$. We have then the action as a functional of b, α, r and A_τ to be integrated in all paths, in the path integral.

III. THE ZERO LOOP APPROXIMATION: REDUCED ACTION, SOLUTIONS, AND STABILITY

A. Constraints and the reduced action

Due to the aforementioned difficulties in dealing with the path integral, we perform the zero loop approximation. We do this in steps. First, we find the reduced action by imposing the Hamiltonian and momentum constraints to the metric and the Gauss constraint to the Maxwell field. Second, we implement the zero loop approximation, i.e., we only consider the path that minimizes the reduced action.

The Hamiltonian constraint is $G^\tau{}_\tau = 8\pi T^\tau{}_\tau$, with $G^\tau{}_\tau$ given by $G^\tau{}_\tau = \frac{d-2}{2r'r^{d-2}}\left[r^{d-3}\left(\frac{r'^2}{\alpha^2} - 1\right)\right]'$, and $T^\tau{}_\tau = \frac{(d-3)}{\Omega_{d-2}}\frac{A'^2_\tau}{2\alpha^2b^2}$, where $T^\tau{}_\tau$ is the time-time component of the stress-energy tensor $T^a{}_b$. Thus, the Hamiltonian constraint is

$$\frac{d-2}{2r'r^{d-2}}\left[r^{d-3}\left(\frac{r'^2}{\alpha^2} - 1\right)\right]' = \frac{4\pi(d-3)A'^2_\tau}{\Omega_{d-2}\alpha^2b^2}\,. \tag{15}$$

The momentum constraint is trivially satisfied since the metric Eq. (3) is diagonal and does not depend on the Euclidean time. The Gauss constraint is $\nabla_yF^{\tau y} = 0$, which explicitly is

$$\left(\frac{r^{d-2}A'_\tau}{b\alpha}\right)' = 0\,, \tag{16}$$

The two constraint equations, Eqs. (15) and (16), are coupled, nevertheless they can be integrated. It is better to start first by integrating Eq. (16). Its integration yields

$$A'_\tau = -i\frac{q}{r^{d-2}}b\alpha\,, \tag{17}$$

where q is an integration constant. If one evaluates Eq. (17) at $y = 1$ and uses the boundary condition Eq. (13), then one obtains that

$$q = Q\ , \tag{18}$$

and so the integration constant q of the Gauss constraint is precisely the fixed electric charge Q of the ensemble. From this point onward we work with Q. By using Eq. (17) and Eq. (18), the Hamiltonian constraint becomes $\frac{d-2}{2r'r^{d-2}}\left[r^{d-3}\left(\frac{r'^2}{\alpha^2} - 1\right)\right]' = -\frac{4\pi(d-3)Q^2}{\Omega_{d-2}r^{2d-4}}$, which can be integrated to obtain

$$\frac{r'^2}{\alpha^2} \equiv f(r, Q, r_+)\,, \tag{19}$$

where

$$f(r, Q, r_+) \equiv 1 - \frac{r_+^{d-3} + \frac{\mu Q^2}{r_+^{d-3}}}{r^{d-3}} + \frac{\mu Q^2}{r^{2d-6}}\,, \tag{20}$$

with

$$\mu = \frac{8\pi}{(d-2)\Omega_{d-2}}\,. \tag{21}$$

The function f in Eq. (20) is defined for convenience, and the regularity conditions Eqs. (5) and (7) were used to determine the integration constant r_+. Although the condition Eq. (8) is not used anywhere, notice for bookkeeping that, if $y = r$ is chosen, $r' = 1$ and α diverges at $r = r_+$, therefore the condition Eq. (8) should be satisfied if $\left(\frac{b'}{\alpha}\right)'_{y=0}$ is finite. The function A'_τ in Eq. (17) is related to the Coulomb electric field in Lorentzian curved spacetime as $n_aE^a = \frac{iA'_\tau}{b\alpha} = \frac{Q}{r^{d-2}}$, where E^a is the electric field. It is important to write explicitly the extremal case, i.e., when $r_+^{2d-6} = \mu Q^2$, and we write this special radius as r_{+_e}, which is thus given by

$$r_{+_e} = \left(\mu Q^2\right)^{\frac{1}{2d-6}} . \qquad (22)$$

The function $f(r,Q,r_+)$ in Eq. (20) in the extremal case is then $f(r,Q,r_{+_e}) = \left(1 - \frac{\sqrt{\mu}Q}{r^{d-3}}\right)^2$.

The Hamiltonian, momentum, and Gauss constraints simplify the action of Eq. (14) considerably. One can see that the third term in Eq. (14) has an integrand proportional to $G^\tau{}_\tau - 8\pi T^\tau{}_\tau$ and so, applying the Hamiltonian constraint given in Eq. (15), this term vanishes. Moreover, the last term in Eq. (14) is proportional to $\left(\frac{r^{d-2}A'_\tau}{b\alpha}\right)'$ which vanishes also if the Gauss constraint given in Eq. (16) is applied. Therefore, the action Eq. (14) becomes

$$I_*[\beta,Q,R;r_+] = \frac{\beta R^{d-3}}{\mu}(1-\sqrt{f(R,Q,r_+)}) - \frac{\Omega_{d-2} r_+^{d-2}}{4} , \qquad (23)$$

where I_* is the reduced action, which is the Euclidean action evaluated on the paths that obey the Hamiltonian and Gauss constraints, and $(r'\alpha^{-1})_{y=1}$ was substituted by the solution to the Hamiltonian constraint given in Eq. (19). From Eq. (20) we have that $f(r,Q,r_+)$ appearing in Eq. (23) evaluated at the cavity radius R is given by

$$f(R,Q,r_+) \equiv 1 - \frac{r_+^{d-3} + \frac{\mu Q^2}{r_+^{d-3}}}{R^{d-3}} + \frac{\mu Q^2}{R^{2d-6}} . \qquad (24)$$

The extremal case characterized by Eq. (22) has this function at R given by $f(R,Q,r_{+_e}) = \left(1 - \frac{\sqrt{\mu}Q}{R^{d-3}}\right)^2$.

The Hamiltonian, momentum, and Gauss constraints, together with the boundary conditions and the requirement of spherical symmetry, restrict the path integral substantially. The Euclidean space is determined by the functional r_+ and so the path integral is the sum of spaces with all possible r_+. Indeed, the partition function is given by the path integral, i.e.,

$$Z = \int D r_+ \mathrm{e}^{-I_*[\beta,Q,R;r_+]} , \qquad (25)$$

with $I_*[\beta,Q,R;r_+]$ being the reduced action described in Eq. (23). There is formally another functional which is A_τ, i.e., the Maxwell field, but the action does not depend explicitly on A_τ, it only depends on the electric charge which is fixed at the cavity. This means the integration over paths of A_τ can be absorbed by a normalization and does not yield any contribution to the constrained path integral.

B. Reduced action evaluated at stationary points, I_0: Analytic investigation of the existence of stationary points in generic d dimensions

1. *Reduced action evaluated at stationary points, I_0*

To further simplify the path integral in the partition function of Eq. (25), we perform the zero loop approximation, i.e., we only consider the path that minimizes the action given in Eq. (23). The partition function in this approximation is given by

$$Z[\beta,R,Q] = \mathrm{e}^{-I_0[\beta,R,Q]} , \qquad (26)$$

where

$$I_0[\beta,R,Q] = I_*[\beta,R,Q;r_+[\beta,R,Q]] , \qquad (27)$$

is the action in Eq. (23) evaluated at its minimum with respect to r_+. The function $r_+[\beta,R,Q]$ corresponds to a black hole solution that is in equilibrium with the cavity and it is determined by a stationary point of the action, i.e., $\left(\frac{\partial I_*}{\partial r_+}\right)_{r_+=r_+[\beta,R,Q]} = 0$.

2. *Equations for the d-dimensional stationary points*

Thus, the stationary points of the reduced action I_*, given by $\left(\frac{\partial I_*}{\partial r_+}\right)_{r_+=r_+[\beta,R,Q]} = 0$, can be found through Eq. (23) to give

$$\beta = \iota(r_+) \, , \; \iota(r_+) \equiv \frac{4\pi}{(d-3)} \frac{r_+^{d-2}}{r_+^{d-3} - \frac{\mu Q^2}{r_+^{d-3}}} \sqrt{f(R,Q,r_+)} , \qquad (28)$$

where $\iota(r_+)$ is the inverse temperature function, defined here for convenience. Note that for fixed R and Q, ι is indeed a function of r_+ alone. The solutions $r_+[\beta,R,Q]$ of Eq. (28) are stationary points of the action in Eq. (23) and are obtained from inverting Eq. (28). To help in the analysis, we define in this section a horizon radius parameter x and an electric charge parameter y as

$$x = \frac{r_+}{R} , \qquad y = \frac{\mu Q^2}{R^{2d-6}} . \qquad (29)$$

Rearranging Eq. (28) we obtain

$$(x^{2d-6} - y)^2 \left(\frac{(d-3)\beta}{4\pi R}\right)^2 - x^{3d-7}(1-x^{d-3})(x^{d-3}-y) = 0. \qquad (30)$$

The equation above, Eq. (30), can be reduced at most to sixth polynomial order for $d=5$, while for other dimensions the polynomial order is higher. Therefore, we did not find any analytical solution to this equation for any specific value of d.

It should be noted that the nonextremal condition for the black hole is best seen by putting in the form

$$x_e \leq x \leq 1\,, \tag{31}$$

where x_e is the extremal x related to the extremal y, denoted as y_e, by

$$y_e = x_e^{2d-6}\,, \tag{32}$$

see Eq. (22).

3. Saddle points of the action in d dimensions

Although it is not possible to find exact solutions for x and y, nevertheless, we are able to obtain analytically the limiting values for the solutions. These are determined by the saddle points of the action I_* described as $\left(\frac{\partial^2 I_*}{\partial r_+^2}\right)_0 = 0$, where the subscript 0 means that the quantity inside parenthesis is evaluated at the stationary point. Now, $\left(\frac{\partial^2 I_*}{\partial r_+^2}\right)_0 = -\frac{\Omega_{d-2}(d-2)r_+^{d-3}}{4}\beta^{-1}\frac{\partial \iota}{\partial r_+}$, so the saddle points of the action are given by the equation $\frac{\partial \iota}{\partial r_+} = 0$ together with Eq. (28). This condition can be put as a function of the variables x and y and it simplifies to

$$\frac{d-1}{2}x^{4d-12} - (1+y)x^{3d-9} - 3(d-3)yx^{2d-6} + (2d-5)y(1+y)x^{d-3} - \frac{3d-7}{2}y^2 = 0\,. \tag{33}$$

Equation (33) is a polynomial equation of order four in x^{d-3} and it can be solved analytically. We enumerate the solutions.

For $y = 0$, one has the electrically uncharged case and it was discussed in [11] for $d = 4$, [25] for $d = 5$, and [26] for generic d.

For $0 < y < y_s$, there are four real roots of Eq. (33), from which only two obey the condition $0 < y < x^{2d-6}$ which is the nonextremal condition, see Eq. (31), and where y_s is a saddle or critical electric charge parameter to be determined. We designate the two saddle points of the action, i.e., the two solutions of interest of Eq. (33), as $x_{s1} = x_{s1}(y)$ and $x_{s2} = x_{s2}(y)$, where $x_{s1} \leq x_{s2}$. Now, we find explicitly the saddle points of the action. They are

$$x_{s1}^{d-3} = \frac{1+y}{2(d-1)} + \xi - \frac{1}{2}\sqrt{2\eta + \frac{\zeta}{\xi} - 4\xi^2}\,, \tag{34}$$

$$x_{s2}^{d-3} = \frac{1+y}{2(d-1)} + \xi + \frac{1}{2}\sqrt{2\eta + \frac{\zeta}{\xi} - 4\xi^2}\,, \tag{35}$$

where

$$\begin{aligned}
\eta &= \frac{3(1+y)^2 + 12(d-1)(d-3)y}{2(d-1)^2}\,,\\
\zeta &= \frac{(1+y)}{(d-1)^3}\left(y^2 - (4d^3 - 24d^2 + 48d - 30)y + 1\right)\,,\\
\xi &= \frac{1}{2}\sqrt{\frac{2}{3}\eta + \frac{2}{3(d-1)}\frac{\sigma^2+\sigma_0}{\sigma}}\,,\\
\sigma &= \left(\frac{\sigma_1 + \sqrt{\sigma_1^2 - 4\sigma_0^3}}{2}\right)^{\frac{1}{3}}\,,\\
\sigma_0 &= 3(2d-5)y(1-y)^2\,,\\
\sigma_1 &= 54(d-3)(d-2)^2(1-y)^2y^2\,.
\end{aligned} \tag{36}$$

For $y = y_s$, both saddle points merge into a single one. We write the value of the saddle point $x_s \equiv x_{s1} = x_{s2}$ at $y = y_s$, which is a saddle point with the feature that the third derivative of the action also vanishes. The saddle point $x_s \equiv x_{s1} = x_{s2}$ is given by

$$x_s^{d-3} = \frac{1}{2(d-1)(2d-5)} \times \left[(d-1)(3d-7)(3d^2-16d+22) - 3\sqrt{3}(d-2)^2(d-3)\sqrt{(d-1)(3d-7)}\right]\,, \tag{37}$$

which occurs at $y = y_s$ given by

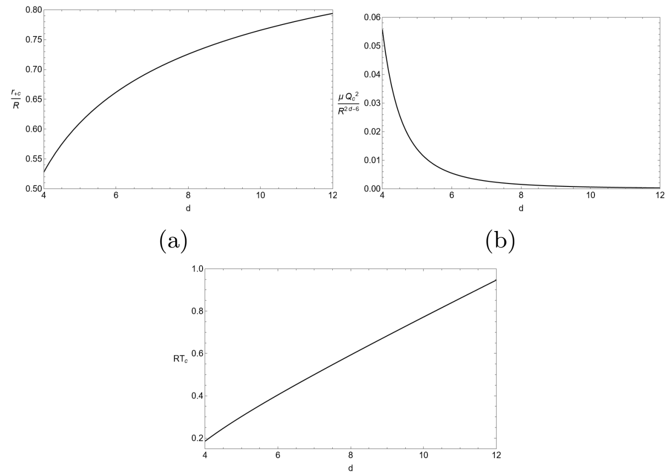
$$y_s = \frac{1}{4(d-1)(2d-5)^3(3d-7)} \times \left[(d-1)(3d-7)(3d^2-16d+22) - 3\sqrt{3}(d-3)(d-2)^2\sqrt{(d-1)(3d-7)}\right]^2\,. \tag{38}$$

Of course, to x_s corresponds an r_{+s} through $r_{+s} = x_s R$, and to y_s corresponds a Q_s through $Q_s = \frac{y_s R^{2d-6}}{\mu}$, where we have not put the subscript s in R in these formulas because, for finite R, one can always assume R fixed. Putting the values given in Eqs. (37) and (38) into Eq. (30), one finds RT_s,

$$RT_s = RT_s(x_s, y_s) \tag{39}$$

the temperature parameter at which x_s is a solution of the black hole for $y = y_s$. The values of x_s, y_s, and RT_s are displayed for different values of d in Fig. 1. It can be seen that both x_s and RT_s increase as d increases, and y_s decreases as d increases.

For $y_s < y < 1$, there are no roots of Eq. (33) that obey the condition $0 < y < x^{2d-6}$ which is the nonextremal condition, see Eq. (31), i.e., there are no saddle points of the action.



(a)

(b)

(c)

FIG. 1: Plots of the saddle point (x_s, y_s, T_s) of the action, where the third derivative of the action also vanishes, as functions of the number of dimensions d. (a) Plot of $x_s = \frac{r_{+s}}{R}$ as a function of d; (b) plot of $y_s = \frac{\mu Q_s^2}{R^{2d-6}}$ as a function of d; (c) plot of RT_s as a function of d.

4. The solutions in d dimensions: Qualitative analysis

For $y = 0$, one has the electrically uncharged case and it has been discussed in [11] for $d = 4$, [25] for $d = 5$, and [26] for generic d.

For $0 < y < y_s$, from the results for the saddle points of the action, one can make a qualitative analysis and find that there are three solutions $x(\beta, y)$, or if one prefers $x(T, y)$, of Eq. (30). These three solutions we designate by x_1, x_2, and x_3. The solution x_1 exists in the interval of temperatures $0 < T < T_1$ and is bounded by $x_e < x_1(T, y) < x_{s1}(y)$, where the values of the solution at the bounds are $x_1(0, y) = x_e$, with x_e defined in Eq. (32), and $x_1(T_1, y) = x_{s1}(y)$, with T_1 being defined by the latter relation. The solution x_2 exists in the interval of temperatures $T_1 > T > T_2$ and is bounded by $x_{s1}(y) < x_2(T, y) < x_{s_2}(y)$, where the values of the solution at the bounds are $x_2(T_1, y) = x_{s1}(y)$ and $x_2(T_2, y) = x_{s2}(y)$, with T_2 being defined by the former relation. The solution x_3 exists in the interval of temperatures $T_2 < T < \infty$, and is bounded by $x_{s2}(y) < x_3(T, y) < 1$, where the values of the solution at the bounds are $x_3(T_2, y) = x_{s2}(y)$ and $x_3(T \to \infty, y) = 1$. As y_s decreases with the increase of d, this means that the area of the region of existence of these solutions decreases with the increase of d, as it is squeezed toward lower values of the electric charge.

For $y = y_s$, there are still three solutions x_1, x_2, and x_3, with the solution x_2 having been reduced to a point, more precisely to the saddle point of $\iota(r_+)$ given as $x_2(T_s, y_s) = x_s$, with T_s being defined by the latter relation. The bounds of x_1 and x_3 are the same as the case $0 < y < y_s$, except that $x_{s1}(y_s) = x_{s2}(y_s) = x_s$ and $T_s = T_1 = T_2$.

For $y_s < y < 1$, there is only one solution x_4 that exists for all T and is bounded by $x_e < x_4(T, y) < 1$, where $x_4(0, y) = x_e$ and $x_4(T \to \infty, y) = 1$.

C. Stability condition

To determine if the solutions are minima of the action and thus stable, we must go beyond the zero loop approximation and expand the action and the path integral around the stationary point. Thus, we write $I_* = I_0 + \left(\frac{\partial I_*}{\partial r_+}\right)_0 \delta r_+ + \left(\frac{\partial^2 I_*}{\partial r_+^2}\right)_0 \delta r_+^2$, where the subscript 0 means that the quantity inside parenthesis is evaluated at the stationary point, $I_0 = I_*(\beta, Q, R; (r_+)_0)$, and $\delta r_+ = r_+ - (r_+)_0$. Then, the partition function can be expanded as

$$Z = \mathrm{e}^{-I_0} \int D\delta r_+ \mathrm{e}^{-\left(\frac{\partial^2 I_*}{\partial r_+^2}\right)_0 \delta r_+^2} . \tag{40}$$

The partition function in Eq. (40) contains one loop contributions that obey the spherical symmetry of the geometry, the boundary conditions, and the Hamiltonian and Gauss constraints. For the path integral to be well defined, we must have

$$\left(\frac{\partial^2 I_*}{\partial r_+^2}\right)_0 > 0 , \tag{41}$$

so that the stationary point is a minimum and stable, otherwise the integral may blow up or be continued to a complex function, indicating that the stationary point is not a minimum and is therefore an instanton. The second derivative of the action Eq. (23) can be simplified into $\left(\frac{\partial^2 I_*}{\partial r_+^2}\right)_0 = -\frac{\Omega_{d-2}(d-2)r_+^{d-3}}{4\beta}\frac{\partial\iota}{\partial r_+}$. Thus, the stability condition reduces to $\frac{\partial \iota}{\partial r_+} < 0$, meaning that the solution is stable when $\frac{r_+}{R}$ increases with a decrease in the inverse temperature, and so with an increase in the temperature. Now, $\frac{\partial\iota}{\partial r_+} = \frac{\beta}{r_+}[(d-2) - (d-3)\frac{r_+^{2d-6}+\mu Q^2}{r_+^{2d-6}-\mu Q^2} + \frac{d-3}{2f}(\frac{\mu Q^2}{r_+^{d-3}R^{d-3}} - \frac{r_+^{d-3}}{R^{d-3}})]$, where the expression of ι in Eq. (28) was used, and so stability means $(d-2) - (d-3)\frac{r_+^{2d-6}+\mu Q^2}{r_+^{2d-6}-\mu Q^2} + \frac{d-3}{2f}(\frac{\mu Q^2}{r_+^{d-3}R^{d-3}} - \frac{r_+^{d-3}}{R^{d-3}}) < 0$. In terms of the variables x and y defined as $x = \frac{r_+}{R}$ and $y = \frac{\mu Q^2}{R^{2d-6}}$, see Eq. (29), the stability condition is

$$\begin{aligned} &\frac{d-1}{2}x^{4d-12} - (1+y)x^{3d-9} - 3(d-3)yx^{2d-6} \\ &+ (2d-5)y(1+y)x^{d-3} - \frac{3d-7}{2}y^2 > 0 . \end{aligned} \tag{42}$$

The range of x is $x_e < x < 1$, where x_e is a function of y_e, see Eq. (32). In the case of $0 \le y < y_s$, the condition of stability reduces to two intervals in x, one is $0 < x < x_{s1}(y)$ and the other is $x_{s2}(y) < x < 1$.

Therefore, the solutions x_1 and x_3 are stable, while the solution x_2 is unstable. Moreover, the points $x = x_{s1}$ and $x = x_{s2}$ are saddle points of the action as previously stated, and so they are neutrally stable. In the case of $y = y_s$, the same applies as the previous case. In the case of $y_s < y < 1$, the stability condition is satisfied in the interval $x_e < x < 1$ and so the solution x_4 is stable.

It is of interest to pick specific dimensions d. Due to its real importance we study $d = 4$, and as a typical case of higher dimension we analyze carefully $d = 5$.

D. $d = 4$: Stationary points and stability in four dimensions

We now comment on the particular case of four dimensions, $d = 4$. The original results were presented in [21, 22], we show here that they are in agreement with ours and we also display new and interesting features for this case.

First, we want to understand qualitatively $x \equiv \frac{r_+}{R}$ as a function of the temperature parameter RT, i.e., $x(RT)$, for the several distinct electric charge parameter y regions. Recall that the value of y_s is important since it separates the behavior of the solutions. From Eq. (38), in $d = 4$ it is $y_s = (\sqrt{5} - 2)^2 = 0.056$, the latter equality being approximate. The solutions can then be divided using the electric charge parameter y in the solution for the no charge case $y = 0$, solutions for the charge parameter in the region $0 < y < (\sqrt{5} - 2)^2$, the solution for $y = y_s = (\sqrt{5}-2)^2$, and solutions for the charge parameter in the region $(\sqrt{5}-2)^2 < y < 1$. We can comment now on $x(RT)$ within each y division. For $y = 0$, it describes the uncharged case and the solution is known, it is the original York solution [11], and consists of two solutions, here represented as x_2 and x_3. The solution x_{s2} happens when x_2 and x_3 meet at temperature $RT = \frac{3\sqrt{3}}{8\pi} = 0.207$, the latter equality being approximate. For the electric charge in the range $0 < y < (\sqrt{5} - 2)^2$, there are three solutions x_1, x_2 and x_3, where x_1 is stable, x_2 is unstable, and x_3 is stable. For very small charges, the temperature T_1, which is the temperature at which x_{s1} is a solution for the black hole at the given charge, is very high, tending to infinite when the charge tends to zero. For very small charges, the temperature T_2, which is the temperature at which x_{s2} is a solution for the black hole at the given charge, is very near the minimum temperature of the solutions of the canonical ensemble of the Schwarzschild black hole in four dimensions, i.e., $RT = \frac{3\sqrt{3}}{8\pi}$, mentioned above. Increasing the electric charge from small values, one has that the saddle points x_{s1} and x_{s2} approach each other. For the electric charge parameter given by $y = (\sqrt{5} - 2)^2 = y_s$, the saddle points x_{s1} and x_{s2} meet, and at this electric charge, the solution x_1 is described by a curve, the solution x_2 is now reduced to a point that coincides with $x_s = x_{s1} = x_{s2}$, and the solution x_3 is described by another curve. All solutions are stable, more precisely, x_1 is stable, x_2 is neutrally stable, and x_3 is stable. For electric charge in the range $(\sqrt{5} - 2)^2 < y < 1$, there is only one solution x_4 which represents the union of x_1 and x_3, with x_2 having disappeared. Also, the solution x_4 is stable.

Second, we want to understand qualitatively $x \equiv \frac{r_+}{R}$ as a function of the electric charge parameter $y \equiv \frac{\mu Q^2}{R^2}$, with $\mu = 1$ here, i.e., $x(y)$, for the several distinct temperature parameter RT regions. Recall that the value of RT_s and the value of minimum temperature in the uncharged case $RT = \frac{3\sqrt{3}}{8\pi}$ are important since they separate the behavior of the solutions. In $d = 4$, the value of the temperature corresponding to y_s and x_s is $RT_s = 0.185$, this equality being approximate. Thus, the temperature parameter regions are $0 < RT < 0.185$, $RT_s = 0.185$, $0.185 < RT \leq \frac{3\sqrt{3}}{8\pi} = 0.207$, and $\frac{3\sqrt{3}}{8\pi} < RT < \infty$. We can comment now on $x(y)$ within each RT division. For $0 < RT < 0.185$, there are only two solutions, which are x_1 in the interval $0 < y < y_s$ and x_4 in the interval $y_s \leq y < 1$, with $y_s = (\sqrt{5} - 2)^2$. For $RT_s = 0.185$ corresponding to y_s and x_s, with this equality being approximate, there are four solutions, but two of them are degenerate. Indeed, there is the x_1 solution, there are the x_2 and x_3 solutions that degenerate into a point $x_2 = x_3$ at $y = y_s$, and there is the x_4 solution. For $0.185 < RT \leq \frac{3\sqrt{3}}{8\pi} = 0.207$, the latter equality being approximate, there are the four solutions x_1, x_2, x_3 and x_4. The solutions x_1, x_2, and x_3 lie in the range $0 < y < y_s$, and the solution x_4 exists only for $y_s < y < 1$. We can note again that the solution x_4 is a continuation in y, i.e., in Q, of the solutions x_1 and x_3, and so in a sense x_4 is the union of x_1 and x_3. For $\frac{3\sqrt{3}}{8\pi} < RT < \infty$, there are also the four solutions but x_2 and x_3 are discontinuous.

E. $d = 5$: Stationary points and stability in five dimensions

1. *Behavior of the solutions and plots*

We now present in some detail the particular case of five dimensions, $d = 5$, as a typical higher dimensional case. The behavior of the solutions will be developed for this case with explanations and plots.

First, we analyze $x \equiv \frac{r_+}{R}$ as a function of the temperature parameter RT, for the several regions of the electric charge parameter y. Once more, the value of y_s is important for the analysis since it separates the regions of different behavior for the solutions. From Eq. (38), in $d = 5$ it is $y_s = \frac{(68-27\sqrt{6})^2}{250} = 0.014$, the latter equality being approximate. We can divide the analysis into the following regions of the electric charge parameter y: the no charge case $y = 0$, the electric charge parameter in the region $0 < y < \frac{(68-27\sqrt{6})^2}{250}$, the specific case of the critical charge $y = y_s = \frac{(68-27\sqrt{6})^2}{250}$, and the electric charge parameter in the region $\frac{(68-27\sqrt{6})^2}{250} < y < 1$. We now describe the solutions $x(RT)$ for each region of y, and for that we display in Fig. 2 the plots of the solutions $x \equiv \frac{r_+}{R}$ as a function of RT of the canonical ensemble in five dimensions, $d = 5$. An important line in such

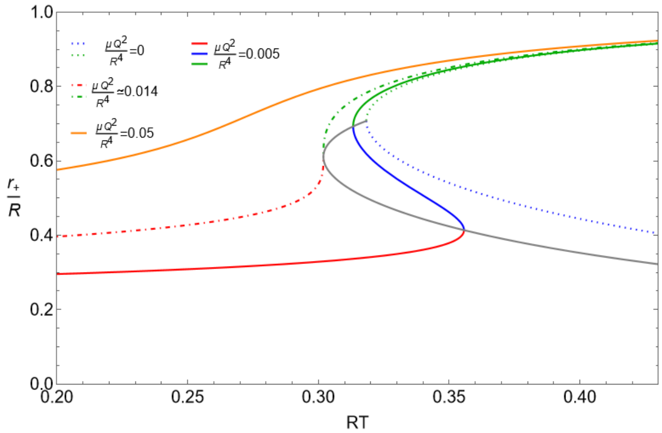


FIG. 2: Plots of the solutions $x \equiv \frac{r_+}{R}$ as a function of RT of the canonical ensemble in five dimensions, $d = 5$, for four values of the electric charge parameter $y \equiv \frac{\mu Q^2}{R^4}$, with $\mu = \frac{4}{3\pi}$ here. The four values of the electric charge parameter y are $y = 0$ in dotted lines, $y = 0.005$ in full lines, $y = \frac{(68-27\sqrt{6})^2}{250} = 0.014$ in dot dashed lines, the latter equality being approximate, and $y = 0.05$ in an orange full line. The solution $x_1 = \frac{r_{+1}}{R}$ is represented in red, $x_2 = \frac{r_{+2}}{R}$ is represented in blue, $x_3 = \frac{r_{+3}}{R}$ is represented in green, and $x_4 = \frac{r_{+4}}{R}$ is represented in orange. The gray curve describes the trajectory of the saddle points of the action $x_{s1} = \frac{r_{+s1}}{R}$ and $x_{s2} = \frac{r_{+s2}}{R}$ by changing the electric charge parameter, and it separates the regions of existence of the solutions $x_1 = \frac{r_{+1}}{R}$, $x_2 = \frac{r_{+2}}{R}$, and $x_3 = \frac{r_{+3}}{R}$.

plots is the line in gray in the figure, that represents the trajectory of the saddle points x_{s1} and x_{s2} of the action by varying the electric charge. This gray line separates the regions where the solutions x_1, x_2, and x_3 can be found. More precisely, the two saddle points x_{s1} and x_{s2} are the bounds of the solution x_2. For $y = 0$, we have the uncharged case. The solution has been analyzed in [25], and consists of two solutions, here represented as x_2 and x_3. At the saddle point x_{s2}, the solutions x_2 and x_3 meet at temperature $RT = \frac{1}{\pi}$. For the electric charge parameter y in the region $0 < y < \frac{(68-27\sqrt{6})^2}{250}$, which can be visualized by the $y = 0.005$ case in the plot, there are three solutions x_1, x_2, and x_3, where again x_1 is stable, x_2 is unstable, and x_3 is stable, see below for the discussion of thermodynamic stability. This case is representative of small electric charges. For very small charges, the temperature T_1 corresponding to the saddle point x_{s1} assumes very large values and tends to infinity when the charge tends to zero. Moreover, the temperature T_2, corresponding to the saddle point x_{s2} is close to the minimum temperature of the solutions of the canonical ensemble of the Schwarzschild black hole in five dimensions $RT = \frac{1}{\pi}$. Note that the figure with the plots for small electric charge parameter yields a unification of York and Davies, as the two solutions are here represented. More precisely, the blue and green lines correspond to the unstable and stable black holes of York [11], respectively, and the red and blue lines correspond to the stable and unstable black holes of Davies [6], respectively, see below for these latter black holes. Increasing the electric charge from small values, one sees that the saddle points x_{s1} and x_{s2} approach each other along the gray curve. For the saddle electric charge $y = y_s \frac{(68-27\sqrt{6})^2}{250} = 0.014$, with the latter equality being approximate, the saddle points x_{s1} and x_{s2} are equal as $x_{s1} = x_{s2} = x_s$. While x_1 and x_3 are described by a curve, the solution x_2 reduces to a point $x_2 = x_s$ that connects both solutions x_1 and x_3. Regarding stability, x_1 is stable, x_2 is neutrally stable, and x_3 is stable. For the electric charge parameter y in the region $\frac{(68-27\sqrt{6})^2}{250} < y < 1$, which is represented in the plot by the case $y = 0.05$, there is only one solution x_4, that is in a sense the continuation of x_1 and x_3, with x_2 having disappeared. We note that x_4 is a stable solution.

Second, we want to describe $x \equiv \frac{r_+}{R}$ as a function of the electric charge parameter $y \equiv \frac{\mu Q^2}{R^4}$, with $\mu = \frac{4}{3\pi}$ here, for the several regions of the temperature parameter RT. Here, the value of RT_s and the value of the minimum temperature of the uncharged case $RT = \frac{1}{\pi}$ are important since they separate the regions of different behavior for the solutions. In $d = 5$, the temperature corresponding to $x_s(y_s)$ is $RT_s = 0.302$, with this equality being approximate. We then consider the temperature parameter regions $0 < RT < 0.302$, $RT_s = 0.302$, $0.302 < RT \leq \frac{1}{\pi} = 0.318$, the latter equality being approximate, and $\frac{1}{\pi} < RT < \infty$. We now describe $x(y)$ within each RT region, and for that we display in Fig. 3 plots of the solutions $x \equiv \frac{r_+}{R}$ as a function of $y \equiv \frac{\mu Q^2}{R^4}$, $\mu = \frac{4}{3\pi}$, of the canonical ensemble in five dimensions, $d = 5$. For the temperature parameter RT in the range $0 < RT < 0.302$, of which $RT = 0.15$ is represented in the figure, there are only two solutions to display, which are x_1 in the interval $0 < y < y_s$, and x_4 in the interval $y_s \leq y < 1$, with $y_s = \frac{(68-27\sqrt{6})^2}{250}$. For the temperature parameter RT given by $RT = RT_s = 0.302$, this equality being approximate, one has the curves of the x_1 solution and the x_4 solution, while the x_2 and x_3 solutions degenerate into a point $x_2 = x_3$ at $y = y_s$. For the temperatures $0.302 < RT \leq \frac{1}{\pi} = 0.318$, of which $RT = 0.31$ and $RT = \frac{1}{\pi}$ are represented in the figure, one has the solutions x_1, x_2 and x_3 lying in the range $0 < y < y_s$, while the solution x_4 lies in the range $y_s < y < 1$. The figure shows explicitly that the solution x_4 is a continuation in the electric charge parameter y of the solutions x_1 and x_3. Note also that the gray curve in the figure bounds the solution x_2. For $\frac{1}{\pi} < RT < \infty$, which is represented by $RT = 0.4$ in the figure, one has also the four solutions but the segments of x_2 and x_3 are discontinuous.

2. Interpretation of the results

These results merit some underlying understanding of the physics, which we now give in terms of the thermal wavelength λ which is proportional to the inverse of the temperature, $\lambda = \frac{1}{T}$. We give the reasoning for the plots of the solutions $x \equiv \frac{r_+}{R}$ as a function of RT of the canonical ensemble shown in Fig. 2. We analyze the solutions from small electric charge to large electric charge, and from low to high temperature T with R fixed. We note that small RT corresponds to low T here.

To start with, we analyze the case for a given small electric charge. For small T, the associated thermal wavelength λ is large and is stuck to the cavity walls, which means that if there were no electric charge, there would be no black hole. But since there is a fixed electric charge, there is a small black hole with radius r_+ of the order of the length scale set by the charge itself. This black hole does not form by collapse, its presence comes from topological constraints. The black hole is stable, small perturbations cannot evaporate it. For the smallest possible T, $T = 0$, the black hole is an extremal black hole. For small temperature, there is only one black hole solution which is this one. For an intermediate T, as the temperature increases, one has that the associated thermal wavelength λ decreases. The black hole with small r_+ is still there, but there is now the possibility of forming black holes via collapse, indeed the thermal wavelengths are no more stuck to the cavity walls and the existent thermal energy can collapse. One black hole that can form in this way has radius r_+ of the order of λ and is thermodynamically unstable since clearly it can evaporate. The other black hole that can form in this way has radius r_+ large such that $R - r_+$ is of the order of λ, and is thermodynamically stable, the reservoir and the black hole exchange quanta of λ in a stabilizing way. For intermediate temperatures, there are thus three black hole solutions for each temperature. For high T, as the temperature increases and the associated wavelength λ gets even smaller. The smallest black hole r_+ ceases to exist because, due to the turbulence created by the high temperature, there is no way to maintain the electric charge coherently at the center of the cavity. The intermediate black hole r_+ ceases to exist because the electric charge repulsion is sufficient to halt gravitational collapse of this black hole with intermediate r_+. The large black hole r_+ still exists, as it has sufficient mass to overcome the electric repulsion and still collapses. For high T, therefore only the large black hole exists. This is for a typical reasonably low electric charge Q, and we see there is an interplay between the two quantities that characterize the ensemble, namely, the temperature T and the electric charge Q.

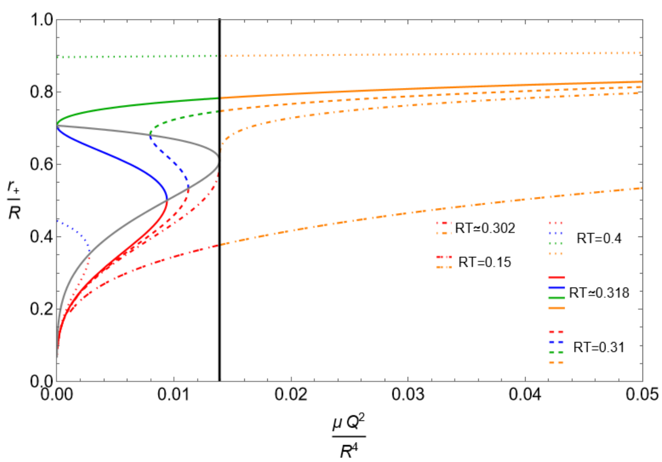


FIG. 3: Plots of the solutions $x \equiv \frac{r_+}{R}$ as a function of $y \equiv \frac{\mu Q^2}{R^4}$ of the canonical ensemble in five dimensions, $d = 5$, for five values of the temperature parameter RT, with $\mu = \frac{4}{3\pi}$. The five values of RT are $RT = 0.15$ in double dashed lines, $RT = RT_s = 0.302$ in dot dashed lines, $RT = 0.31$ in dashed lines, $RT = \frac{1}{\pi} = 0.318$, in full lines, the latter equality being approximate, and $RT = 0.4$ in dotted lines. The solution $x_1 = \frac{r_{+1}}{R}$ is represented in red, $x_2 = \frac{r_{+2}}{R}$ is represented in blue, $x_3 = \frac{r_{+3}}{R}$ is represented in green, and $x_4 = \frac{r_{+4}}{R}$ is represented in orange. The black line, corresponding to $y = y_s = \frac{(68-27\sqrt{6})^2}{250}$, separates the solution $x_4 = \frac{r_{+4}}{R}$ from the remaining solutions. The gray line corresponds to the trajectory of the saddle points of the action $x_{s1} = \frac{r_{+s1}}{R}$ and $x_{s2} = \frac{r_{+s2}}{R}$, which bounds the region where $x_2 = \frac{r_{+2}}{R}$ exists.

We now analyze the case of high electric charge. Again here, for small T, the associated thermal wavelength λ is large and is stuck and cannot collapse. But since there is a fixed electric charge, there is a small black hole with radius r_+ of the order of the length set by the charge itself, its presence comes from topological constraints, is stable, i.e., small perturbations cannot evaporate it. $T = 0$ yields an extremal black hole. At intermediate T, there is turbulence to disperse the black hole with topological features but it is possible to have sufficient mass to collapse the existent thermal energy into the large black hole, with $R - r_+$ starting to be comparable to λ. Note that the intermediate black hole does not exist because the electric charge is large enough to counter its collapse. For high T, as the temperature increases and the associated wavelength λ gets smaller, the large black hole r_+ has sufficient mass to overcome the electric repulsion and the thermal energy collapses, being stable. For all temperatures, there is thus one black hole solution only for each temperature. It is in a sense the union of the topological black hole with the large collapsed black hole as the temperature T increases, the intermediate one having disappeared. Following this reasoning, one can also extend this interpretation to the plots of the solutions $x \equiv \frac{r_+}{R}$ as a function of $\frac{\mu Q^2}{R^4}$ in Fig. 3.

IV. THERMODYNAMICS FROM THE CANONICAL ENSEMBLE OF A CHARGED BLACK HOLE INSIDE A CAVITY IN d DIMENSIONS

A. Free energy, entropy, pressure, electric potential, and thermodynamic energy

With the zero loop approximation performed, the partition function is $Z = \mathrm{e}^{-I_0[\beta,R,Q]}$, and simultaneously, in the canonical ensemble, it is also given by $Z = \mathrm{e}^{-\beta F}$, where F is the Helmholtz free energy. Therefore, we have the relation $I_0[\beta,R,Q] = \beta F$, i.e., the action is related to the free energy of the charged black hole in the cavity by

$$F = T\, I_0[\beta,R,Q]\,. \tag{43}$$

The free energy of the system containing the charged black hole is then

$$F = \frac{R^{d-3}}{\mu}\left(1-\sqrt{f\left(R,Q,r_+\right)}\right) - T\frac{\Omega_{d-2}r_+^{d-2}}{4}\,, \tag{44}$$

with $f(R,Q,r_+) \equiv 1 - \frac{r_+^{d-3}+\frac{\mu Q^2}{r_+^{d-3}}}{R^{d-3}} + \frac{\mu Q^2}{R^{2d-6}}$, see Eq. (24).

The Helmholtz free energy is given in terms of the internal energy E, the temperature T, and the entropy S by the relation

$$F = E - TS\,. \tag{45}$$

By definition F has the differential

$$dF = -SdT - pdA + \phi dQ\,, \tag{46}$$

where, in addition to the entropy S, the area A, and the electric charge Q, there is the thermodynamic pressure p, and the thermodynamic electric potential ϕ. And so, we can obtain the thermodynamic quantities from the derivatives of the free energy F, more precisely, the entropy is $S = -\left(\frac{\partial F}{\partial T}\right)_{A,Q}$, the pressure is $p = -\left(\frac{\partial F}{\partial A}\right)_{T,Q}$, and the electric potential is $\phi = \left(\frac{\partial F}{\partial Q}\right)_{T,A}$, where here the subscript indicates the quantities that are fixed while performing the derivative. In Eq. (46), some of the dependence on T, A, and Q is implicit on the solution for the horizon radius $r_+ = r_+(T,A,Q)$, as it is evaluated at the minima of the action. To simplify the calculation of the derivatives, we can perform the chain rule and the fact that, since $r_+ = r_+(T,A,Q)$, the derivative of the reduced action obeys $\left(\frac{\partial I_*}{\partial r_+}\right)_{T,R,Q} = \left(\frac{\partial F}{\partial r_+}\right)_{T,R,Q} = 0$, to get for example $S = -\left(\frac{\partial F}{\partial T}\right)_{A,Q} = -\left(\frac{\partial F}{\partial T}\right)_{R,Q,r_+} - \left(\frac{\partial F}{\partial r_+}\right)_{T,R,Q}\frac{\partial r_+}{\partial T} = -\left(\frac{\partial F}{\partial T}\right)_{R,Q,r_+}$, and this also holds similarly for the computation of the pressure and the electric potential. Therefore, the thermodynamic quantities can be computed to be $S = -\left(\frac{\partial F}{\partial T}\right)_{R,Q,r_+}$, $p = -\frac{1}{(d-2)\Omega_{d-2}R^{d-3}}\left(\frac{\partial F}{\partial R}\right)_{T,Q,r_+}$, $\phi = \left(\frac{\partial F}{\partial Q}\right)_{T,R,r_+}$ and $E = F - TS$. The entropy is then given as

$$S = \frac{A_+}{4}\,, \tag{47}$$

where $A_+ \equiv \Omega_{d-2}r_+^{d-2}$ is the area of the event horizon, and so this is the usual Hawking-Bekenstein expression for the entropy of a black hole. The thermodynamic pressure is

$$p = \frac{d-3}{16\pi R\sqrt{f}}\left((1-\sqrt{f})^2 - \frac{\mu Q^2}{R^{2d-6}}\right)\,, \tag{48}$$

the thermodynamic electric potential is

$$\phi = \frac{Q}{\sqrt{f}}\left(\frac{1}{r_+^{d-3}} - \frac{1}{R^{d-3}}\right)\,, \tag{49}$$

and finally, from Eq. (45), the thermodynamic energy is given by

$$E = \frac{R^{d-3}}{\mu}\left[1 - \sqrt{\left(1-\frac{r_+^{d-3}}{R^{d-3}}\right)\left(1-\frac{\mu Q^2}{r_+^{d-3}R^{d-3}}\right)}\right]\,, \tag{50}$$

Collecting Eqs. (47)-(50), one finds that the first law of thermodynamics in the form

$$dE = TdS - pdA + \phi dQ\,, \tag{51}$$

holds. It is interesting to note, and surely not a coincidence, that these thermodynamic quantities are identical to the ones calculated for a self-gravitating charged shell, where the first law of thermodynamics is imposed, and, the charged shell assumes the temperature equation of state of a black hole and the thermodynamic pressure equation of state of the cavity, see [40].

B. Euler relation equation and Gibbs-Duhem relation

With the thermodynamic quantities obtained in Eqs. (47)-(50), one can get an integrated first law of thermodynamics known as Euler equation. For that, one rewrites the energy in Eq. (50) in terms of the entropy in Eq. (47), the area $A = \Omega_{d-2}R^{d-2}$, and the electric charge Q as

$$E = \frac{(d-2)A^{\frac{d-3}{d-2}}\Omega_{d-2}^{\frac{1}{d-2}}}{8\pi}\times \left(1 - \sqrt{\left(1-\left(\frac{4S}{A}\right)^{\frac{d-3}{d-2}}\right)\left(1-\frac{\mu Q^2\Omega_{d-2}^{2\frac{d-3}{d-2}}}{(4SA)^{\frac{d-3}{d-2}}}\right)}\right)\,. \tag{52}$$

We have that the energy is a function $E = E(S, A, Q)$, and if a scaling is performed on the thermodynamic quantities $S \to \nu S$, $A \to \nu A$ and $Q \to \nu^{\frac{d-3}{d-2}} Q$, then it can be verified that $E(\nu S, \nu A, \nu^{\frac{d-3}{d-2}} Q) = \nu^{\frac{d-3}{d-2}} E(S, A, Q)$. According to the Euler relation theorem, and considering that the differential of the energy is given by the first law of thermodynamics Eq. (51), we have the Euler equation

$$E = \frac{d-2}{d-3}(TS - pA) + \phi Q\,. \tag{53}$$

One can furthermore differentiate Eq. (53) and use the first law of thermodynamics to obtain

$$\frac{1}{d-3}\left(TdS - pdA\right) + \frac{d-2}{d-3}(SdT - Adp) + Qd\phi = 0\,, \tag{54}$$

which is the Gibbs-Duhem relation.

C. Heat capacity

A system to be thermodynamically stable must have positive heat capacity at constant area and constant electric charge $C_{A,Q}$, i.e.,

$$C_{A,Q} \geq 0\,, \tag{55}$$

where $C_{A,Q} \equiv T\left(\frac{\partial S}{\partial T}\right)$. In section III B, we have shown that the stability condition in the ensemble formalism was reduced to the condition $\frac{\partial \iota}{\partial r_+} < 0$. The derivative above can be put in terms of thermodynamic variables, and then in terms of the heat capacity. The inverse temperature function $\iota(r_+)$ is a function of r_+, R and Q. The variables Q and R are already thermodynamic variables. The quantity r_+ is also a thermodynamic variable since we have obtained that $S = \frac{\Omega_{d-2} r_+^{d-2}}{4}$. Therefore, since $\beta = \iota(r_+)$, we have that $\frac{\partial \iota}{\partial r_+} = -\frac{1}{T}\left(\frac{\partial S}{\partial r_+}\right)\frac{1}{C_{A,Q}}$, where we have used the definition of the heat capacity at constant area and constant electric charge.

The heat capacity is then

$$C_{A,Q} = \frac{(d-2)R^{d-2} f\left(\frac{r_+^{d-3}}{R^{d-3}} - \frac{\mu Q^2}{R^{d-3} r_+^{d-3}}\right)\frac{\Omega_{d-2} r_+^{d-2}}{4R^{d-2}}}{\frac{d-3}{2}\left(\frac{r_+^{d-3}}{R^{d-3}} - \frac{\mu Q^2}{r_+^{d-3} R^{d-3}}\right)^2 - f\left(\frac{r_+^{d-3}}{R^{d-3}} - (2d-5)\frac{\mu Q^2}{r_+^{d-3} R^{d-3}}\right)}\,. \tag{56}$$

Since to be thermodynamically stable one has that $C_{A,Q} \geq 0$, thermodynamic stability reduces to Eq. (42) after rearrangements and definitions. Thus, the physical interpretation is that the stability of the solutions is controlled by the heat capacity at constant area and charge, as it should be in the canonical ensemble. This quantity is tied to the derivative of the inverse temperature given by Eq. (28) and so the condition reduces to the intervals given by the stationary points of $\iota(r_+, R, Q)$, or the saddle points of the action. Moreover, solutions where r_+ increases as T increases are stable and solutions where r_+ decreases as T increases are unstable.

It is interesting to see what happens when one fixes $\frac{r_+}{R}$ and change the electric charge parameter $\frac{\mu Q^2}{R^{2d-6}}$. For $\frac{r_+}{R} > \left(\frac{2}{d-1}\right)^{\frac{1}{d-3}}$, the heat capacity is always positive. The limit of the bound happens for the uncharged black hole, black holes that obey this inequality and have any finite electric charge have positive heat capacity. For $0 \leq \frac{r_+}{R} \leq \left(\frac{2}{d-1}\right)^{\frac{1}{d-3}}$, the sign of the heat capacity $C_{A,Q}$ changes according to the electric charge. $C_{A,Q}$ is positive for sufficiently high electric charge parameter $\frac{\mu Q^2}{R^{2d-6}}$, and is negative for sufficiently low electric charge parameter $\frac{\mu Q^2}{R^{2d-6}}$, the change in sign happening at the definite value of the charge satisfying Eq. (33) with fixed $\frac{r_+}{R}$. We note that this does not indicate a phase transition since $\frac{r_+}{R}$ is not a thermodynamic variable controlled in the ensemble. At that definite value of the charge parameter, there is rather a turning point describing the ratio of scales at which there is stability.

The thermodynamic variables are the temperature and the electric charge, and therefore the heat capacity must be analyzed in terms of these quantities, instead of $\frac{r_+}{R}$ and the electric charge. For the range of electric charges $0 < \frac{\mu Q^2}{R^{2d-6}} < \frac{\mu Q_s}{R^{2d-6}}$, one has three curves for the heat capacity as a function of the temperature, one for each solution. The heat capacity is positive for the solutions r_{+1} and r_{+3}, while it is negative for r_{+2}. The heat capacity diverges when the solutions reach the temperatures of the saddle points of the action, which are the turning points. For the critical charge parameter $\frac{\mu Q_s^2}{R^{2d-6}}$, one has two curves for the heat capacity as a function of the temperature. In this particular case, the two curves are described by the solutions r_{+1} and r_{+3} and it is positive for both. Moreover, there is a discontinuity between the two curves at RT_s, where the heat capacity diverges. This point indeed does mark a second order phase transition between r_{+1} and r_{+3}, as both solutions are stable and it can be seen that the free energy is continuous at RT_s for $\frac{\mu Q_s^2}{R^{2d-6}}$. For the range $\frac{\mu Q^2}{R^{2d-6}} > \frac{\mu Q_s^2}{R^{2d-6}}$, there is only one curve for the heat capacity as a function of the temperature, corresponding to the solution r_{+4} and it is always positive.

We can now give the thermodynamic expressions with commentaries for the particular dimensions $d = 4$ and $d = 5$.

D. $d = 4$: Thermodynamics in four dimensions

For $d = 4$, we write the results explicitly. The entropy is given as $S = \pi r_+^2$, which is the usual Hawking-Bekenstein formula $S = \frac{A_+}{4}$, with $A_+ = 4\pi r_+^2$ being the area of the event horizon. The pressure is $p =$

$\frac{1}{16\pi R\sqrt{f}}\left((1-\sqrt{f})^2-\frac{Q^2}{R^2}\right)$ where we have used $\mu = 1$ and $f = 1 - \frac{r_+ + \frac{Q^2}{r_+}}{R} + \frac{Q^2}{R^2}$. The electric potential is $\phi = \frac{Q}{\sqrt{f}}\left(\frac{1}{r_+}-\frac{1}{R}\right)$. Finally, the mean energy is given by $E = R\left[1-\sqrt{\left(1-\frac{r_+}{R}\right)\left(1-\frac{Q^2}{r_+ R}\right)}\right]$. One can then write the energy in terms of S, $A = 4\pi R^2$, and Q, i.e., $E = E(S, A, Q)$ to obtain the Euler relation $E = 2(TS - pA) + \phi Q$. The Gibbs-Duhem relation is $TdS - pdA + 2(SdT - Adp) + Qd\phi = 0$.

The heat capacity, the quantity that controls thermodynamic stability, is

$$C_{A,Q} = \frac{2R^2 f\left(\frac{r_+}{R}-\frac{Q^2}{R^2}\frac{R}{r_+}\right)\frac{\pi r_+^2}{R^2}}{\frac{1}{2}\left(\frac{r_+}{R}-\frac{Q^2}{R^2}\frac{R}{r_+}\right)^2 - f\left(\frac{r_+}{R}-\frac{3Q^2}{R^2}\frac{R}{r_+}\right)}. \quad (57)$$

One could fix $\frac{r_+}{R}$ and change the electric charge parameter $\frac{Q^2}{R^2}$ in Eq. (57). As seen in the general d case, we find that for $\frac{r_+}{R} > \frac{2}{3}$, the heat capacity is always positive, and for $0 \le \frac{r_+}{R} \le \frac{2}{3}$, the sign of the heat capacity $C_{A,Q}$ changes depending on the electric charge, being positive for a region of high electric charge parameter $\frac{Q^2}{R^2}$, and being negative for a region of low electric charge parameter $\frac{Q^2}{R^2}$. This does not indicate a phase transition but rather a turning point. To see this fact and verify the true phase transitions, one must analyze the heat capacity in terms of the fixed quantities of the ensemble, i.e., the temperature and the electric charge. For the range of charge parameters $0 < \frac{\mu Q^2}{R^2} < (\sqrt{5}-2)^2$, where in $d = 4$ one has $\frac{\mu Q_s^2}{R^2} = (\sqrt{5}-2)^2$, the heat capacity has a curve for each solution r_{+1}, r_{+2}, and r_{+3}, being positive for r_{+1} and r_{+3}, and being negative for r_{+2}. When the solutions reach the temperatures of the saddle points of the action, i.e., the turning points, the heat capacity diverges but this only indicates conditions for stability of the ensemble, there are no phase transitions at these points. For the critical charge $\frac{\mu Q_s^2}{R^2} = (\sqrt{5}-2)^2$, the heat capacity has two curves as a function of the temperature, r_{+1} and r_{+3}, being positive for both solutions. For this case, there is a discontinuity between the two curves at $RT_s = 0.185$, where the heat capacity diverges. This point indeed signals a second order phase transition between r_{+1} and r_{+3}, as both solutions are stable and it can be seen that the free energy is continuous at $RT_s = 0.185$ for $\frac{\mu Q^2}{R^2} = (\sqrt{5}-2)^2$. For the range of charge parameters $\frac{\mu Q^2}{R^2} > (\sqrt{5}-2)^2$, one only has that the heat capacity of r_{+4} as a function of the temperature is always positive. In [21, 22] some of these results for $d = 4$ are presented.

E. $d = 5$: Thermodynamics in five dimensions

Here, we make the results explicit for the case $d = 5$. The entropy is given as $S = \frac{\pi^2 r_+^3}{2}$, matching the usual Hawking-Bekenstein formula $S = \frac{A_+}{4}$, with $A_+ = 2\pi^2 r_+^3$ being the area of the event horizon. The pressure yields $p = \frac{2}{16\pi R\sqrt{f}}\left((1-\sqrt{f})^2 - \frac{4Q^2}{3\pi R^4}\right)$, where we have used $\mu = \frac{4}{3\pi}$ and $f = 1 - \frac{r_+^2 + \frac{4Q^2}{3\pi r_+^2}}{R^2} + \frac{4Q^2}{3\pi R^4}$. The electric potential yields $\phi = \frac{Q}{\sqrt{f}}\left(\frac{1}{r_+^2}-\frac{1}{R^2}\right)$. And the energy has the expression $E = \frac{3\pi R^2}{4}\left[1-\sqrt{\left(1-\frac{r_+^2}{R^2}\right)\left(1-\frac{4Q^2}{3\pi r_+^2 R^2}\right)}\right]$. These thermodynamic quantities are identical to the ones calculated for a self-gravitating charged shell, where the first law of thermodynamics is imposed, and the charged shell assumes the equation of state of the black hole, see [40]. The energy can be written in terms of S, $A = 2\pi^2 R^3$, and the electric charge Q, as $E = E(S, A, Q)$ to obtain the Euler relation $E = \frac{3}{2}(TS - pA) + \phi Q$. The Gibbs-Duhem relation yields $\frac{1}{2}(TdS - pdA) + \frac{3}{2}(SdT - Adp) + Qd\phi = 0$.

The heat capacity is

$$C_{A,Q} = \frac{3R^3 f\left(\frac{r_+^2}{R^2}-\frac{4Q^2}{3\pi R^2 r_+^2}\right)\frac{\pi^2 r_+^3}{2R^3}}{\left(\frac{r_+^2}{R^2}-\frac{4Q^2}{3\pi R^4}\frac{R^2}{r_+^2}\right)^2 - f\left(\frac{r_+^2}{R^2}-\frac{20Q^2}{3\pi R^4}\frac{R^2}{r_+^2}\right)}. \quad (58)$$

Regarding the behavior of the heat capacity with fixed $\frac{r_+}{R}$ as a function of the electric charge parameter $\frac{Q^2}{R^4}$, one has that the heat capacity is always positive for $\frac{r_+}{R} > \frac{\sqrt{2}}{2}$, and the heat capacity changes signs for $0 \le \frac{r_+}{R} \le \frac{\sqrt{2}}{2}$, being positive for high electric charge parameter $\frac{Q^2}{R^4}$, and being negative for low electric charge parameter $\frac{Q^2}{R^4}$. As already noted, to understand the turning points and the possible phase transitions of the solutions, one must analyze the behavior of the heat capacity through its dependence in the temperature and the electric charge, see Fig. 4. For a fixed electric charge parameter in the range $0 < \frac{\mu Q^2}{R^4} < \frac{(68-27\sqrt{6})^2}{250}$, where in $d = 5$ one has $\frac{\mu Q_s^2}{R^4} = \frac{(68-27\sqrt{6})^2}{250}$, the heat capacity is described by three curves, one for each solution r_{+1}, r_{+2}, and r_{+3}, being positive for r_{+1} and r_{+3}, and being negative for r_{+2}, see Fig. 4 for the case $\frac{\mu Q^2}{R^4} = 0.005$. The heat capacity in this range of charges diverges at the turning points of the solutions, as seen by the dashed black lines, indicating the conditions for stability of the solutions and not signaling any phase transition. For the electric charge $\frac{\mu Q_s^2}{R^4} = \frac{(68-27\sqrt{6})^2}{250}$, the heat capacity is positive, as it is described by the curves of the solution r_{+1} and r_{+3}. The heat capacity diverges at $RT_s = 0.302$, the solid black line, and here one in fact has a second order transition, from r_{+1} to r_{+3}, as these are both stable solutions, and the free energy is continuous there. For $\frac{\mu Q^2}{R^4} > \frac{(68-27\sqrt{6})^2}{250}$, the heat capacity is always positive, as it is described only by the solution r_{+4}.

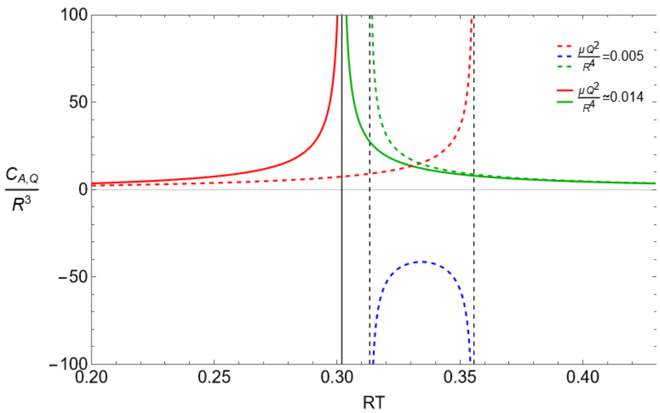


FIG. 4: The heat capacity $C_{A,Q}$ in R^3 units, $\frac{C_{A,Q}}{R^3}$, in $d = 5$, as a function of the temperature for two values of the electric charge, $\frac{\mu Q^2}{R^4} = 0.005$ and $\frac{\mu Q^2}{R^4} = \frac{\mu Q_s^2}{R^4} = 0.014$ approximately, for solutions r_{+1} in red, r_{+2} in blue, and r_{+3} in green. The dashed black lines mark the turning points of the solutions and the solid black line marks the second order phase transition between the stable solutions r_{+1} and r_{+3}.

V. FAVORABLE PHASES OF THE d DIMENSIONAL CANONICAL ENSEMBLE OF AN ELECTRICALLY CHARGED BLACK HOLE IN A CAVITY AND PHASE TRANSITIONS

A. Black hole sector of the canonical ensemble and favorable phases

Consider the black hole sector of the canonical ensemble and the corresponding free energy. Since free energy F and action I_0 are related by $F = \frac{I}{\beta} = T\, I_0$, the black hole free energy $F_{\rm bh}$ can be taken directly from Eq. (44) to be rewritten as

$$F_{\rm bh} = \frac{R^{d-3}}{\mu}\left(1 - \sqrt{f(R,Q,r_+)}\right) - T\frac{A_+}{4}\,, \qquad (59)$$

where in this section we put a bh subscript in F to denote that it is a black hole free energy to distinguish from other possible free energies. Since $A_+ \equiv \Omega_{d-2} r_+^{d-2}$ and $r_+ = r_+(T,R,Q)$, the black hole solutions have their free energies of the form $F_{\rm bh}(T,R,Q)$. For a system characterized by the free energy, the one that has the lower free energy $F_{\rm bh}$, for given R, T, and Q, is the one that is thermodynamically favored. Thus, we can find the black hole that is favored.

We have shown that in the zero loop approximation, there are different black hole solutions depending on the electric charge and temperature of the reservoir, see Sec. II. For sufficiently low electric charge parameter, i.e., for $0 \le \frac{\mu Q^2}{R^{2d-6}} < \frac{\mu Q_s^2}{R^{2d-6}}$, where Q_s is the saddle electric charge value, corresponding to the saddle electric charge parameter $y_s = \frac{\mu Q_s^2}{R^{2d-6}}$, we have seen that there can be up to three solutions $\frac{r_{+1}}{R}$, $\frac{r_{+2}}{R}$, and $\frac{r_{+3}}{R}$. Now we comment on the free energies $F_{\rm bh}$ of these three solutions. The solution $\frac{r_{+1}}{R}$ has positive free energy for all the temperatures in which the solution exists. The solution $\frac{r_{+2}}{R}$ has also positive free energy always, but it is unstable, so we are not interested in it here. The solution $\frac{r_{+3}}{R}$, has a temperature for each electric charge parameter $\frac{\mu Q^2}{R^{2d-6}}$ at which the free energy becomes zero, which we define as $T_{F_{\rm bh}=0}(Q)$ or $T_{F_{\rm bh}=0}(\frac{\mu Q^2}{R^{2d-6}})$, thus $\frac{r_{+3}}{R}$ can have positive or negative free energy. For the saddle charge parameter $\frac{\mu Q^2}{R^{2d-6}} = \frac{\mu Q_s^2}{R^{2d-6}}$, the solution $\frac{r_{+1}}{R}$ has positive free energy, there is a solution where $\frac{r_{+1}}{R} = \frac{r_{+2}}{R} = \frac{r_{+3}}{R}$ which has positive free energy, and the solution $\frac{r_{+3}}{R}$ has again a temperature $T_{F_{\rm bh}=0}(Q_s)$ or $T_{F_{\rm bh}=0}(\frac{\mu Q_s^2}{R^{2d-6}})$, at which the free energy becomes zero, thus $\frac{r_{+3}}{R}$ can have positive or negative free energy. For higher values of the electric charge parameter, i.e., for $\frac{\mu Q_s^2}{R^{2d-6}} < \frac{\mu Q^2}{R^{2d-6}} < 1$, the solution $\frac{r_{+4}}{R}$ has also a temperature $T_{F_{\rm bh}=0}(Q)$, or $T_{F_{\rm bh}=0}(\frac{\mu Q_s^2}{R^{2d-6}})$, at which the free energy becomes zero, thus $\frac{r_{+4}}{R}$ can have positive or negative free energy. The temperature $T_{F_{\rm bh}=0}(Q)$ can be calculated by solving $F_{\rm bh} = 0$, with $F_{\rm bh}$ given in Eq. (59) for either the solution $\frac{r_{+3}}{R}$ or $\frac{r_{+4}}{R}$. One can instead put the free energy in terms of the mass m and electric charge Q through Eq. (28) and through the relation $2\mu m = r_+^{d-3} + \frac{\mu Q^2}{r_+^{d-3}}$, so that $F_{\rm bh} = 0$ reduces to a quartic equation for the mass m as a function of the electric charge, see Appendix B. After solving it, one can then recover the value of r_+ and consequently the value $T_{F_{\rm bh}=0}(\frac{\mu Q^2}{R^{2d-6}})$. For temperatures lower than $T_{F_{\rm bh}=0}(\frac{\mu Q^2}{R^{2d-6}})$, the solutions have positive free energy and for temperatures higher than $T_{F_{\rm bh}=0}(\frac{\mu Q^2}{R^{2d-6}})$, the solutions have negative free energy.

There is another important temperature, T_f, which depends on the electric charge Q, i.e., on the electric charge parameter $\frac{\mu Q^2}{R^{2d-6}}$, and at which the favorability of one phase over the other changes. For the electric charge parameter within the region $0 \le \frac{\mu Q^2}{R^{2d-6}} < \frac{\mu Q_s^2}{R^{2d-6}}$, there is a phase favorability temperature T_f at which the solutions $\frac{r_{+1}}{R}$ and $\frac{r_{+3}}{R}$ have the same free energy. In other words, the solutions $\frac{r_{+1}}{R}$ and $\frac{r_{+3}}{R}$ are stable, and thus within the black hole sector they compete between themselves to be the most favored phase. Specifically, for temperatures lower than T_f, the solution $\frac{r_{+1}}{R}$ is either more favorable than $\frac{r_{+3}}{R}$, or is the only existing solution if the temperature is low enough. For a temperature equal to T_f, the solutions $\frac{r_{+1}}{R}$ and $\frac{r_{+3}}{R}$ are equally favorable, i.e., they coexist equally. For temperatures higher than T_f, either the solution $\frac{r_{+3}}{R}$ is more favorable than $\frac{r_{+1}}{R}$, or is the only existing solution if the temperature is high enough. For the electric charge parameter given by $\frac{\mu Q^2}{R^{2d-6}} = \frac{\mu Q_s^2}{R^{2d-6}}$, the temperature T_f is the temperature at which $\frac{r_{+1}}{R} = \frac{r_{+2}}{R} = \frac{r_{+3}}{R}$ and all have the same free energy, i.e., $\frac{r_{+1}}{R}$ and $\frac{r_{+3}}{R}$ coexist. For temperatures lower than T_f, the solution $\frac{r_{+1}}{R}$ is the only existing solution. For temperatures higher than T_f, the solution $\frac{r_{+3}}{R}$ is the only existing solution. For the electric charge parameter within the region $\frac{\mu Q_s^2}{R^{2d-6}} < \frac{\mu Q^2}{R^{2d-6}} < 1$, there is only one black hole solution, it is $\frac{r_{+4}}{R}$. Within the black hole sector it is surely the most favored state since it is stable and there is no other solution. It can have positive or negative free energy.

B. Hot flat space sector of the electrically charged canonical ensemble

Let us consider a possible electrically charged hot flat space sector, i.e., a cavity with nothing in it with its boundaries defined by R, T, and Q, the settings of the canonical ensemble.

To have such a solution one can think in trying to decrease r_+ up to zero, to a point where there is no more a black hole and thus obtain flat space. However, this is not possible, since there is a minimum limit for r_+ given by $r_+ = r_{+e}$ corresponding to the extremal black hole. At r_{+e}, the free energy tends to $F_{\rm bh} = \frac{Q}{\sqrt{\mu}}$, and it is then impossible to decrease r_+ further. Regarding extremal black holes, the only temperature that such solutions exist is at $T = 0$ and we do not consider them here as it is only one point of the ensemble, although it is a very interesting one. We simply note, that there is no other immediate solution of the action that can be a candidate for a stationary point of the reduced action. Thus, to emulate electrically charged hot flat space one has to go beyond the black hole sector. One can consider, for example, a shell with radius $r_{\rm shell}$, coated with the required electric charge Q, and with gravity turned off, i.e., the constant of gravitation is set to zero. The action of the system if one considers terms depending only on the Maxwell field can be calculated to give the free energy as $F_{\rm shell} = \frac{Q^2}{2}\left(\frac{1}{r_{\rm shell}^{d-3}} - \frac{1}{R^{d-3}}\right)$, i.e.,

$$F_{\rm shell} = \frac{Q^2}{2r_{\rm shell}^{d-3}}\left(1 - \frac{r_{\rm shell}^{d-3}}{R^{d-3}}\right). \tag{60}$$

Thus, for a given $r_{\rm shell}$, one has that $F_{\rm shell}$ has a given constant fixed value. There are two limits that one can mention. One limit is when $r_{\rm shell}$ is very small. One could see this limit as an electrically charged central point surrounded by hot flat space, where quantum fluctuations of the hot flat space generate electric charge. But this seems to lead to a divergent free energy. Note that the behavior mentioned for $r_{\rm shell}$ very small contrasts with the grand canonical ensemble case [30], where $r_{\rm shell} = 0$ corresponds to a zero grand potential. The other limit is when $r_{\rm shell} = R$ and so the free energy is zero. This means that all the charge is infinitesimally near the boundary of the cavity, i.e., it is at the boundary of the cavity itself and there is hot flat space inside the cavity. Thus, the more interesting limit is the latter one, when $r_{\rm shell} = R$, and the charge is gathered near the boundary of the cavity giving $F_{\rm shell} = 0$. Since in this case the shell emulates hot flat space with electric charge at the boundary, one has $F_{\rm shell} = F_{\rm hfs} = 0$. Nevertheless, it is interesting to compare the toy model of a shell with free energy $F_{\rm shell}$ given in Eq. (60) for several $\frac{r_{\rm shell}}{R}$, and in particular for $\frac{r_{\rm shell}}{R} = 1$, with the black hole free energy $F_{\rm bh}$ given in Eq. (59).

One could further think in building an equivalent system with the constant of gravitation turned on, such as an electrically charged self-gravitating shell close to the boundary of the cavity. Still, it is unclear if there is a possible conversion of this system to a charged black hole, and vice versa, since the two systems correspond to different topologies and also to a different action, as here we do not consider the matter sector. So we stick to the electric shell with gravitation turned off.

C. Favorable phases: First and second order phase transitions

It is thus of interest to understand what are the favorable states of the ensemble, i.e., of an ensemble of a cavity with fixed radius R, fixed temperature T, and fixed electric charge Q, all values of these quantities set by the reservoir.

A thermodynamic system tends to be in a state in which its thermodynamic potential, associated to the ensemble considered, has the lowest value. In our case, the thermodynamic potential is the Helmholtz free energy F, and so a state is favored relatively to another if it has lower F for given R, T, and Q. If a system is in a stable state but with a higher free energy F than another stable state, it is probable that the system undergoes a conversion, i.e., a phase transition, to the stable state with the lowest free energy. Indeed, in the calculation of the partition function by the path integral approach, if there are two stable configurations, i.e., two states that minimize the action, then the largest contribution to the partition function is given by the configuration with the lowest action or, in thermodynamic language, with the lowest free energy. This type of phase transitions are first order since the free energy is continuous, but the first derivatives are discontinuous.

In the case of the canonical ensemble of an electrically charged black hole inside a cavity in d dimensions, we must compare the free energy between all the stable black hole solutions of the ensemble, i.e., one has to compute $F_{\rm bh}$ given in Eq. (59), for the possible solution $r_+(R,T,Q)$. For any d we note that in this ensemble one can have three solutions for the same temperature, two of them are stable. The stable black hole with lowest $F_{\rm bh}$ is the one that is favored. This means that considering only the two stable black hole solutions, one would then have a first order phase transition from r_{+1} to r_{+3}, for the electric charge parameter in the range $0 < \frac{\mu Q^2}{R^{2d-6}} < \frac{\mu Q_s^2}{R^{2d-6}}$, and in the limit of the charge parameter with value $\frac{\mu Q^2}{R^{2d-6}} = \frac{\mu Q_s^2}{R^{2d-6}}$, this first order phase transition becomes a second order phase transition. It is also interesting to compare the black hole solutions with the nongravitating electrically charged shell case for the same boundary data, which has free energy given in Eq. (60). As we argued above, this shell is useful in mimicking charged hot flat space inside the cavity. Depending on the value of the radius of the shell $\frac{r_{\rm shell}}{R}$, this free energy can go from infinity, when $\frac{r_{\rm shell}}{R}=0$, to zero, when $\frac{r_{\rm shell}}{R}=1$. In the case of $\frac{r_{\rm shell}}{R}=0$, the shell is never favored, while for $\frac{r_{\rm shell}}{R}=1$, i.e., the case of hot flat space with the electric charge at the boundary, there is a region in which it is favored. We proceed, by essentially assuming a shell with $\frac{r_{\rm shell}}{R}=1$, so that $F_{\rm shell}=F_{\rm hfs}=0$.

Another issue that should be raised in the connection to favorable states, although it does not come directly from the ensemble formalism and its thermodynamics, is that there is a black hole radius r_+, more precisely, there is a ratio $\frac{r_+}{R}$, for which the thermodynamic energy contained within R is higher than the Buchdahl bound or, in our context, the generalized Buchdahl bound [42]. When this happens, that energy content should collapse into a black hole. In this situation there is no more favorable phase considerations, the unique phase is a black hole. Indeed, the generalized Buchdahl bound yields the maximum mass, or maximum energy, that can be enclosed in a d-dimensional cavity with electric charge Q, before the system shows up some kind of singularity. At the bound or above, the system most likely tends to gravitational collapse. Since the mass of a system is related to the gravitational radius, it also sets a bound on the ratio $\frac{r_+}{R}$. In our context, one should consider this bound as yielding, for a fixed R, the mass m, or the gravitational radius r_+, above which the energy within the system is sufficiently large that the system cannot support itself gravitationally and collapses. We can now apply this concept to the case that interest us here.

In the Schwarzschild black hole case in d dimensions it was found in [26], that the canonical ensemble yields $F_{\rm bh}=0$ when $\frac{r_+}{R}$ has the Buchdahl bound value, $\left(\frac{r_+}{R}\right)_{\rm Buch}$. Since we are envisaging R as fixed, we write $\left(\frac{r_+}{R}\right)_{\rm Buch}\equiv\frac{r_{+\rm Buch}}{R}$ to simplify the notation. In a d-dimensional Schwarzschild spacetime one has $\frac{r_{+\rm Buch}}{R}=\left(\frac{4(d-2)}{(d-1)^2}\right)^{\frac{1}{d-3}}$. One can infer that black hole solutions with higher $\frac{r_+}{R}$, i.e., higher temperatures RT, yield gravitational collapse. Since zero free energy in this electrically uncharged case, is also the free energy of hot flat space, $F_{\rm hfs}=0$, one sees that in the uncharged case one passes directly from a situation where a hot flat space phase is favored relatively to a black hole phase, to a situation where the phase is a phase where surely there is a black hole, not merely a phase in which the black hole is favored.

Now, in our setting, i.e., in the canonical ensemble for a black hole with electric charge, one finds that for $F_{\rm bh}=0$ only the bigger black hole exists, and it gives a value for $\frac{r_+}{R}$ that is higher than the Buchdahl bound value. Thus, there is a definite $F_{\rm bh}$ value greater than zero where the Buchdahl value $\frac{r_{+\rm Buch}}{R}$ is met, as we have found by numerical means up to $d=16$, but did not prove for all d. For this definite value of $F_{\rm bh}$ or lower values of it, the system has high enough temperature and high enough self-thermodynamic energy to undergo gravitational collapse. When this happens there is no more coexistence of phases, there is only the black hole phase. Below the saddle, or critical, charge, i.e., below the electric charge parameter given by $\frac{\mu Q_s^2}{R^{2d-6}}$, it is the black hole solution $\frac{r_{+3}}{R}$ that achieves $\frac{r_{+\rm Buch}}{R}$. Above the saddle charge, i.e., above $\frac{\mu Q_s^2}{R^{2d-6}}$, it is the black hole solution $\frac{r_{+4}}{R}$ that achieves $\frac{r_{+\rm Buch}}{R}$. In contrast, if we consider the grand canonical ensemble with electric charge, rather than the canonical ensemble we are studying here, it was found [30] that $W_{\rm bh}=0$, where $W_{\rm bh}$ is the grand potential free energy related to the grand canonical ensemble, gives a value for $\frac{r_+}{R}$ which is lower than the Buchdahl bound value. In the grand canonical ensemble, there is only one stable black hole. So, this means that for $W_{\rm bh}=0$, the two phases black hole and hot flat space coexist equally. For $W_{\rm bh}<0$ up to some definite negative value, then the two phases, black hole and hot flat space, coexist but the black hole dominates. For the definite negative value of $W_{\rm bh}$, the radius $\frac{r_+}{R}$ is the Buchdahl bound value $\frac{r_{+\rm Buch}}{R}$. For even lower $W_{\rm bh}$, i.e., for higher temperature parameter RT, one has $\frac{r_+}{R}$ larger than $\frac{r_{+\rm Buch}}{R}$ and the system collapses, or is collapsed, there is thus no coexistence, only the black hole phase remains. Although numerically all three radii $\frac{r_+}{R}$, namely, the canonical zero free energy, the Buchdahl, and the grand canonical zero grand potential, are very close, see Appendix B, it seems that a connection between the ensemble stability and the mechanical stability of matter is elusive here. A comment is in order. The Buchdahl bound applies to a self-gravitating mechanical system consisting of a ball of matter of radius R. Our system is a thermodynamic system, with boundary data, namely R, T, and Q, and contains no matter. One can argue that in higher orders of approximation, the system contains packets of energy and one can plausibly deduce that the system must collapse once the Buchdahl bound is surpassed. Be as it may, the inference we have made comes from dynamics, not thermodynamics, and therefore is strictly outside our approach.

To better understand the issues and make progress one has to pick up definite dimensions. We now specify our generic d-dimensional results to the dimensions $d=4$ and $d=5$. We comment on the dimension $d=4$, and will do a thorough analysis for the dimension $d=5$.

D. $d=4$: Analysis

For $d=4$, as for any d, this ensemble can have either one or three black hole solutions for a given temperature. When there are three, two of them are stable and are of interest in the consideration of the most favorable phase, while the remaining solution is unstable and is of no interest in the consideration of the most favorable phase. The two that are stable have to be compared against one another to see which is the most favorable phase.

We start by comparing the free energy of the several black hole solutions that exist in this ensemble between themselves. From Eq. (59), in $d = 4$, the black hole free energy is

$$F_{\rm bh} = R\left(1 - \sqrt{f(R,Q,r_+)}\right) - T\frac{A_+}{4}\,, \qquad (61)$$

where here $\frac{A_+}{4} = \pi r_+^2$, $f(R,Q,r_+) \equiv 1 - \frac{r_+ + \frac{Q^2}{r_+}}{R} + \frac{Q^2}{R^2}$, we have used $\mu = 1$, and $r_+ = r_+(T,R,Q)$. In $d = 4$, the saddle electric charge parameter value $\frac{Q_s^2}{R^2} = (\sqrt{5}-2)^2 = 0.056$, the last equality being approximate, separates the region with only one solution from the region with three solutions.

A first set of general and specific comments can be made, namely about the positivity of the free energy for each solution. For $0 \le \frac{Q^2}{R^2} < \frac{Q_s^2}{R^2}$, the stable black hole solution $\frac{r_{+1}}{R}$ has positive $F_{\rm bh}$ for all the temperatures in which the solution exists. The same happens for the solution $\frac{r_{+2}}{R}$, but this solution is of not interest here since it is unstable. The other stable black hole solution $\frac{r_{+3}}{R}$ has a temperature $T_{F_{\rm bh}=0}$ depending on the electric charge, at which the free energy becomes zero, and so the black hole solution $\frac{r_{+3}}{R}$ can have $F_{\rm bh}$ positive or negative. For the critical charge $\frac{Q^2}{R^2} = \frac{Q_s^2}{R^2}$, with $\frac{Q_s^2}{R^2} = 0.056$ approximately, the stable black hole solution $\frac{r_{+1}}{R}$ has positive free energy, the point $\frac{r_{+1}}{R} = \frac{r_{+2}}{R} = \frac{r_{+3}}{R}$ has positive free energy, and the stable black hole solution $\frac{r_{+3}}{R}$ has a temperature $T_{F_{\rm bh}=0}$ at which the free energy becomes zero. For $\frac{Q_s^2}{R^2} < \frac{Q^2}{R^2} < 1$, the only black hole solution is $\frac{r_{+4}}{R}$, which is stable, and it has a temperature $T_{F_{\rm bh}=0}$ depending on the electric charge, at which the free energy becomes zero. So, the free energy of $\frac{r_{+4}}{R}$ can be positive or negative. Quite generally one can calculate $T_{F_{\rm bh}=0}$ by solving $F_{\rm bh} = 0$, with $F_{\rm bh}$ given in Eq. (61), for either the solution $\frac{r_{+3}}{R}$ or $\frac{r_{+4}}{R}$. The free energy can be written in terms of m and Q through Eq. (28) in $d = 4$ and through $2m = r_+ + \frac{Q^2}{r_+}$, allowing us to reduce $F_{\rm bh} = 0$ into a quartic equation for the mass, see Appendix B. The solutions have positive free energy for temperatures lower than $T_{F_{\rm bh}=0}$, and the solutions have negative free energy for temperatures higher than $T_{F_{\rm bh}=0}$.

A second set of general and specific comments can be made, namely about the favorability between black hole solutions. For $0 \le \frac{Q^2}{R^2} < \frac{Q_s^2}{R^2}$, there is a favorability temperature T_f which depends on the electric charge, and at which the solutions $\frac{r_{+1}}{R}$ and $\frac{r_{+3}}{R}$ have the same free energy. For temperatures lower than T_f, the solution r_{+1} is more favorable than $\frac{r_{+3}}{R}$, or it is the only existing solution. For temperatures higher than T_f, the solution $\frac{r_{+3}}{R}$ is more favorable than $\frac{r_{+1}}{R}$, or it is the only existing solution. For the critical charge $\frac{Q^2}{R^2} = \frac{Q_s^2}{R^2}$, the temperature T_f is the temperature at which $\frac{r_{+1}}{R} = \frac{r_{+2}}{R} = \frac{r_{+3}}{R}$ and all have the same free energy, i.e., the stable solutions $\frac{r_{+1}}{R}$ and $\frac{r_{+3}}{R}$ coexist. For $\frac{Q_s^2}{R^2} < \frac{Q^2}{R^2} < 1$, there is only one black hole solution, it is $\frac{r_{+4}}{R}$, and, since it is stable, it is favored. We can now consider phase transitions between the two stable black hole solutions. There is a first order phase transition from r_{+1} to r_{+3}, for the electric charge parameter in the range $0 < \frac{\mu Q^2}{R^2} < \frac{\mu Q_s^2}{R^2}$ and, additionally, in the limit of the electric charge parameter with value $\frac{\mu Q^2}{R^2} = \frac{\mu Q_s^2}{R^2}$, this first order phase transition turns into a second order phase transition.

We now comment on the comparison in $d = 4$ between the black hole phases just discussed above with hot flat space phase, which we have emulated by a nonself-gravitating shell. In $d = 4$, the free energy of the shell is

$$F_{\rm shell} = \frac{Q^2}{2r_{\rm shell}}\left(1 - \frac{r_{\rm shell}}{R}\right)\,, \qquad (62)$$

where $r_{\rm shell}$ is the radius of the shell, see Eq. (60). So $F_{\rm shell}$ depends on the electric charge Q, on $r_{\rm shell}$, and on R, but is a constant as a function of the temperature T. The case of a very small shell will lead to a very high free energy due to the dependence on $\frac{Q}{r_{\rm shell}}$, and therefore, for this case the region of favorability for the shell lies in very small values of the charge. There are also the cases of intermediate shell radius which would have to be analyzed specifically. The other limiting case is when the charge is near the boundary of the cavity, with the free energy of this case tending to zero. Ultimately, the black hole is favored when $F_{\rm bh} < F_{\rm shell}$, both coexist equally when $F_{\rm bh} = F_{\rm shell}$, and the black hole is not favored when $F_{\rm bh} > F_{\rm shell}$. When the radius of the shell is at the cavity radius, $\frac{r_{\rm shell}}{R} = 1$, then the shell has zero free energy and emulates hot flat space with electric charge at the boundary. Then, the free energy of hot flat space is $F_{\rm shell} = F_{\rm hfs} = 0$. The black hole is not favored when $F_{\rm bh} > 0$, both the black hole and hot flat space coexist equally when $F_{\rm bh} = 0$, and the black hole is favored when $F_{\rm bh} < 0$. When the system finds itself in a phase that is not favored, it will make a first order phase transition to the favored phase.

The problem of the thermodynamic phases is even more complicated as we have mentioned already. When there is no electric charge, i.e., for the Schwarzschild space in $d = 4$, it was found in [26] that, in the canonical ensemble, the condition $F_{\rm bh} = 0$ yields a value for $\frac{r_+}{R}$ that is equal to the generalized Buchdahl bound [42], i.e., the limiting value $\left(\frac{r_+}{R}\right)_{\rm Buch}$ for gravitational collapse of a self-gravitating system of energy E and radius R. Since we are envisaging R as fixed, we write $\left(\frac{r_+}{R}\right)_{\rm Buch} \equiv \frac{r_{+\rm Buch}}{R}$ to simplify the notation, and in $d = 4$ one has $\frac{r_{+\rm Buch}}{R} = \frac{8}{9} = 0.89$, the latter equality being approximate. This result means that, in the uncharged case, as soon as the black hole phase is favored, there is no further coexistence with hot flat space, and the system collapses. For nonzero electric charge there is no more coincidence. Here, to discuss this issue of favorability between black hole and hot flat space, we are going to consider the case for which the free energy of the shell is zero, $F_{\rm shell}=0$, i.e., the case of hot flat space with electric charge at the boundary, $\frac{r_{\rm shell}}{R}=1$. In this case, the shell is situated at the cavity, and so $F_{\rm shell}$ is the free energy of hot flat space, $F_{\rm hfs}$, which is zero. For nonzero electric charge Q, i.e., nonzero charge parameter $\frac{Q^2}{R^2}$, we find that in the canonical ensemble, the condition $F_{\rm bh}=0$ yields a $\frac{r_+}{R}$ value, both for $\frac{r_{+3}}{R}$ and $\frac{r_{+4}}{R}$, that is higher than the generalized Buchdahl bound. Notice that the generalized Buchdahl bound here is the limiting value of $\frac{r_+}{R}$ for gravitational collapse of a self-gravitating system of energy E, electric charge Q, and radius R. For an electric charge parameter lower or equal than the saddle value $\frac{Q_s^2}{R^2}$, only the solution $\frac{r_{+3}}{R}$ can take the value of the Buchdahl bound, corresponding to a positive free energy and some temperature value RT. For a system with this RT or higher, then the system collapses gravitationally into a black hole with the corresponding $\frac{r_{+3}}{R}$. For an electric charge higher or equal than the saddle value $\frac{Q_s^2}{R^2}$, the solution $\frac{r_{+4}}{R}$ can take the value of the Buchdahl bound, having a definite positive value of $F_{\rm bh}$, at some temperature parameter RT. For a system with this RT or higher, the system again collapses gravitationally into a black hole with the corresponding $\frac{r_{+4}}{R}$. Interesting to note that in the grand canonical ensemble, where there is only one stable black hole solution, it was found [30] that the equation $W_{\rm bh}=0$, $W_{\rm bh}$ denoting the grand potential, yields a $\frac{r_+}{R}$ value that is lower than the Buchdahl bound. Thus, in this case, when $W_{\rm bh}=0$ for the system, the two phases coexist, black hole and hot flat space. For $W_{\rm bh}<0$, the black hole phase dominates in relation to hot flat space. And for a certain definite negative value of $W_{\rm bh}$, the value of $\frac{r_+}{R}$ of the system is the same as the value of the Buchdahl bound. From then on the system collapses, the only phase being the black hole phase, and there is no coexistence of phases, see also Appendix B. Here we have given plausible arguments for the gravitational collapse of the system when there is too much energy inside the cavity, although we have not performed a thermodynamic treatment of the collapsed phase.

E. $d=5$: Analysis

For $d=5$, as for any d, this ensemble has between one and three black hole solutions for a given temperature. When there are three solutions, two of them are stable and are going to be considered here, while the remaining is unstable and is of no interest in this analysis. The two that are stable have to be compared against one another to see which is the most favorable phase.

We start by comparing the free energy of the several black hole solutions that exist in this ensemble between themselves. In $d=5$, the black hole free energy is

$$F_{\rm bh}=\frac{R^2}{\mu}\left(1-\sqrt{f(R,Q,r_+)}\right)-T\frac{A_+}{4}\,, \tag{63}$$

where here $\frac{A_+}{4}=\frac{\pi^2 r_+^3}{2}$, $f(R,Q,r_+)\equiv 1-\frac{r_+^2+\frac{\mu Q^2}{r_+^2}}{R^2}+\frac{\mu Q^2}{R^4}$, $\mu=\frac{4}{3\pi}$, and $r_+=r_+(T,R,Q)$. To help in the analysis, we plot in Fig. 5 $F_{\rm bh}$ as a function of the temperature parameter RT, for fixed electric charge parameter $\frac{\mu Q^2}{R^4}$ in $d=5$. Recall that in $d=5$, one has the saddle electric charge parameter value $\frac{\mu Q_s^2}{R^4}=\frac{(68-27\sqrt{6})^2}{250}=0.014$, the last equality being approximate.

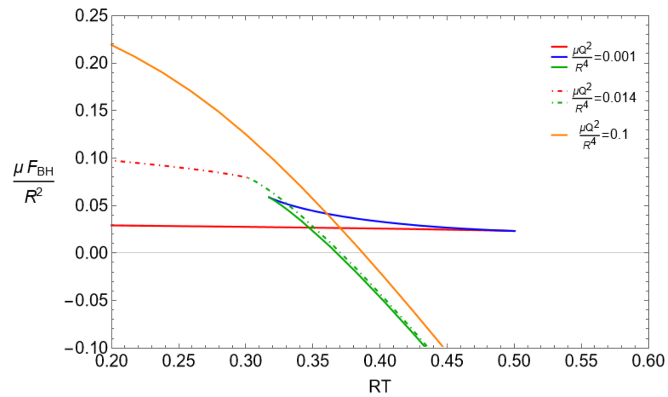


FIG. 5: Free energy $F_{\rm bh}$ of the charged black hole solutions of the canonical ensemble in $d=5$, given as a quantity with no units $\frac{\mu F_{\rm bh}}{R^2}$, as a function of the temperature parameter RT for several electric charge parameters $\frac{\mu Q^2}{R^4}$, where $\mu=\frac{4}{3\pi}$. For $\frac{\mu Q^2}{R^4}=0.001$, the solution r_{+1} is in red, the solution r_{+2} is in blue, and the solution r_{+3} is in green, all of them in solid lines. For $\frac{\mu Q^2}{R^4}=\frac{(68-27\sqrt{6})^2}{250}=0.014$, the latter equality being approximate, the solution r_{+1} is in red and the solution r_{+3} is in green, all of them in dashed lines. For $\frac{\mu Q^2}{R^4}=0.1$, the solution r_{+4} is in orange, in solid line. See text for all the details.

A first set of general and specific comments can be made directly from Fig. 5, regarding the positivity of the free energy for each solution. For relatively low electric charge parameter $0\leq\frac{\mu Q^2}{R^4}<\frac{\mu Q_s^2}{R^4}$, where $\mu=\frac{4}{3\pi}$ in $d=5$, the solution $\frac{r_{+1}}{R}$ has positive $F_{\rm bh}$ for all the temperatures in which the solution exists. The same happens for the solution $\frac{r_{+2}}{R}$, but this solution is of no interest here since it is unstable. The solution $\frac{r_{+3}}{R}$ has a temperature $T_{F_{\rm bh}=0}$ depending on the electric charge at which the free energy becomes zero, and so $\frac{r_{+3}}{R}$ can have $F_{\rm bh}$ positive or negative. In the figure, this range of the electric charge parameter is represented by the case $\frac{\mu Q^2}{R^4}=0.001$. We see that for $\frac{\mu Q^2}{R^4}=0.001$, one has for the $\frac{r_{+3}}{R}$ solution that $T_{F_{\rm bh}=0}=0.367$ approximately. For the saddle charge $\frac{\mu Q^2}{R^4}=\frac{\mu Q_s^2}{R^4}$, with $\frac{\mu Q_s^2}{R^4}=0.014$ approximately, the solution $\frac{r_{+1}}{R}$ is positive, the point $\frac{r_{+1}}{R}=\frac{r_{+2}}{R}=\frac{r_{+3}}{R}$ is positive, and the solution $\frac{r_{+3}}{R}$ has a temperature $T_{F_{\rm bh}=0}=0.37$ at which the free energy becomes zero. For relatively high electric charge parameter $\frac{\mu Q_s^2}{R^4}<\frac{\mu Q^2}{R^4}<1$, the only solution is $\frac{r_{+4}}{R}$ and it has a temperature $T_{F_{\rm bh}=0}$ depending on the electric charge. So $F_{\rm bh}$ of the black hole $\frac{r_{+4}}{R}$ can be positive or negative. In the figure, this range of $\frac{\mu Q^2}{R^4}$ is represented by the case $\frac{\mu Q^2}{R^4} = 0.1$. We see that for $\frac{\mu Q^2}{R^4} = 0.1$, one has that the solution $\frac{r_{+4}}{R}$ has $T_{F_{\rm bh}=0} = 0.387$ approximately. Quite generally, one can calculate $T_{F_{\rm bh}=0}$ by solving $F_{\rm bh} = 0$, with $F_{\rm bh}$ given in Eq. (63) for either the solution $\frac{r_{+3}}{R}$ or $\frac{r_{+4}}{R}$. One obtains a quartic equation for the mass $2\mu m = r_+^2 + \frac{\mu Q^2}{r_+^2}$, with here $\mu = \frac{3}{4\pi}$, as a function of the electric charge, see Appendix B. For temperatures lower than $T_{F_{\rm bh}=0}$, the solutions have positive free energy and for temperatures higher than $T_{F_{\rm bh}=0}$, the solutions have negative free energy.

A second set of general and specific comments can be made directly from Fig. 5, regarding the favorability between black hole solutions. For a range of low electric charge parameter $0 \le \frac{\mu Q^2}{R^4} < \frac{\mu Q_s^2}{R^4}$, the solutions $\frac{r_{+1}}{R}$ and $\frac{r_{+3}}{R}$ have the same free energy at a specific temperature T_f, i.e., the phase favorability temperature which depends on $\frac{\mu Q^2}{R^4}$. For temperatures lower than T_f, the solution $\frac{r_{+1}}{R}$ either has lower free energy than $\frac{r_{+3}}{R}$ or it is the only existing solution, and so $\frac{r_{+1}}{R}$ is more favorable. For a temperature equal to T_f, the solutions $\frac{r_{+1}}{R}$ and $\frac{r_{+3}}{R}$ have the same free energy and they are equally favorable, meaning they coexist equally. For temperatures higher than T_f, the solution $\frac{r_{+3}}{R}$ either has lower free energy than $\frac{r_{+1}}{R}$ or it is the only existing solution, and so $\frac{r_{+3}}{R}$ is more favorable. This is represented for $\frac{\mu Q^2}{R^4} = 0.001$ in the figure. We see that in this case, the favorability temperature is $RT_f = 0.347$ approximately. Also, for $RT < 0.32$, there is only the $\frac{r_{+1}}{R}$ solution, whereas for $RT > 0.50$ there is only the $\frac{r_{+3}}{R}$ solution. The solution $\frac{r_{+2}}{R}$ is unstable and does not enter in this analysis, however it is plotted in the figure to show a continuity of the free energy on the three solutions. For saddle charge $\frac{\mu Q^2}{R^4} = \frac{\mu Q_s^2}{R^4} = 0.014$, the latter equality being approximate, which is shown in the figure, the temperature $T_f = 0.30$, approximately, is the temperature at which $\frac{r_{+1}}{R} = \frac{r_{+2}}{R} = \frac{r_{+3}}{R}$, and all have the same free energy, i.e., $\frac{r_{+1}}{R}$ and $\frac{r_{+3}}{R}$ coexist. For temperatures lower than T_f, the solution $\frac{r_{+1}}{R}$ is the only existing solution. For temperatures higher than T_f, the solution $\frac{r_{+3}}{R}$ is the only existing solution. For the higher values of the electric charge parameter, i.e., for $\frac{\mu Q_s^2}{R^4} < \frac{\mu Q^2}{R^4} < 1$, there is only one black hole solution $\frac{r_{+4}}{R}$ that is stable, and so it is favorable. This is represented in the figure by the case $\frac{\mu Q^2}{R^4} = 0.1$. We can now consider phase transitions between the two stable black hole solutions. One has a first order phase transition from r_{+1} to r_{+3}, for the electric charge parameter in the range $0 < \frac{\mu Q^2}{R^4} < \frac{\mu Q_s^2}{R^4}$. Moreover, in the limit of the electric charge parameter given by the value $\frac{\mu Q^2}{R^4} = \frac{\mu Q_s^2}{R^4}$, this first order phase transition becomes a second order phase transition. This can be seen from Fig. 5, since the intersection point represents a first order phase transition, and at the limit of the critical charge, this point represents a second order phase transition.

We now compare, in $d = 5$, the black hole phases discussed just above with hot flat space phase which we have emulated by a nonself-gravitating shell, see Fig. 6. The

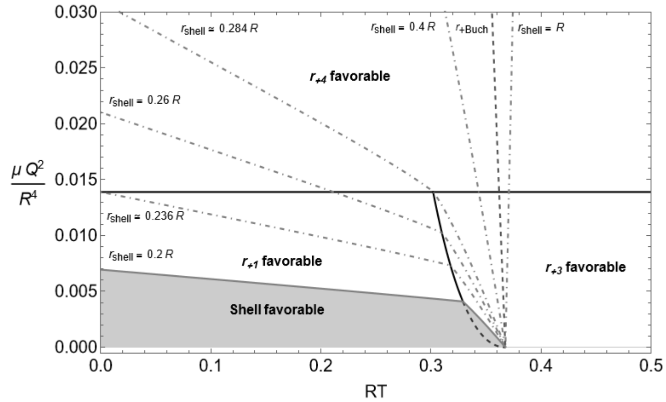


FIG. 6: Favorable states of the canonical ensemble of an electrically charged black hole inside a cavity in $d = 5$ in an electric charge Q times temperature T, more precisely, $\frac{\mu Q^2}{R^4} \times RT$ plot. It is displayed the region where the black hole r_{+1} is a favorable phase, the region where the black hole r_{+3} is a favorable phase, and the region where the black hole r_{+4} is a favorable phase. The delimiters of the favorable regions of the black hole solutions are the black lines, including the dashed line. It is also incorporated the solution for a nongravitating electrically charged shell as a simulator for hot flat space. The electrically charged shell with $\frac{r_{\rm shell}}{R} = 0$ is never favored. The electrically charged shell with $\frac{r_{\rm shell}}{R} = 0.2$ is favored in the region in gray, this case is given as an example. The upper delimiter of the region of favorability of electrically charged shells with $\frac{r_{\rm shell}}{R} = 0.236$ approximately, $\frac{r_{\rm shell}}{R} = 0.26$, $\frac{r_{\rm shell}}{R} = 0.284$ approximately, $\frac{r_{\rm shell}}{R} = 0.4$ and $\frac{r_{\rm shell}}{R} = 1$, which better simulates hot flat space, are given by the dot-dashed lines. The Buchdahl condition line, i.e., $r_{+\rm Buch}$, above which there is presumably collapse is given by a thick black dash line. See text for details.

favorable states for each electric charge and temperature, and for various values of the shell radius can be seen in the figure. The free energy of the shell for the case $d = 5$ is

$$F_{\rm shell} = \frac{Q^2}{2r_{\rm shell}^2}\left(1 - \frac{r_{\rm shell}^2}{R^2}\right), \tag{64}$$

where $r_{\rm shell}$ is the radius of the shell, see Eq. (60). So the shell free energy $F_{\rm shell}$ has a dependence on electric charge Q, on $r_{\rm shell}$, and on R, but as a function of the temperature T, the free energy is a constant. Due to the term $\frac{Q^2}{r_{\rm shell}^2}$, the free energy becomes divergent for a very small shell and fixed electric charge. Therefore, the region of favorability for the very small shell lies in very small values of the electric charge Q. There are the cases of intermediate shell radius that are represented in the figure, namely the cases $\frac{r_{\rm shell}}{R} = 0.2, 0.236, 0.26, 0.284, 0.4$, with 0.236 and 0.284 being approximate values. The more interesting limiting case is when the electric charge is near or at the boundary of the cavity, $\frac{r_{\rm shell}}{R} = 1$. The free energy of the shell in this limit is zero. The black hole solution is favored compared to the shell when $F_{\rm bh} < F_{\rm shell}$, while both the black hole and the shell coexist equally when $F_{\rm bh} = F_{\rm shell}$, and the black hole is not favored compared to the shell when $F_{\rm bh} > F_{\rm shell}$. The gray dashed curves in the figure represent the condition $F_{\rm bh} = F_{\rm shell}$ for each shell radius, delimiting the regions where the black hole is favorable, for higher temperature, and where the shell is favorable, for lower temperature. When the radius of the shell is at the cavity radius, $\frac{r_{\rm shell}}{R} = 1$, the free energy of the shell becomes zero, emulating hot flat space with free energy $F_{\rm shell} = F_{\rm hfs} = 0$. This is the case of hot flat space with electric charge at the boundary. Again, the black hole is not favored compared to hot flat space when $F_{\rm bh} > 0$, while both the black hole and hot flat space coexist equally when $F_{\rm bh} = 0$, and the black hole is favored compared to hot flat space when $F_{\rm bh} < 0$. The gray dashed curve $r_{\rm shell} = R$ in the figure corresponds to the boundary of the regions of favorability $F_{\rm bh} = 0$, and for higher temperature, the black hole is favorable, while for lower temperature, hot flat space is favorable. If for some reason the system is in an unfavored phase, then a first order phase transition occurs to a favored phase.

The problem of the thermodynamic phases is more involved as we mentioned already. When there is no electric charge, one has Schwarzschild space in $d = 5$. It was found in [25, 26] that, in the canonical ensemble of Schwarzschild space in $d = 5$, the condition $F_{\rm bh} = 0$ corresponds to a value for $\frac{r_+}{R}$ that is equal to the generalized Buchdahl bound radius [42], which is the value $\left(\frac{r_+}{R}\right)_{\rm Buch}$ for gravitational collapse of a self-gravitating system of energy E and radius R. Since we are maintaining R fixed, we write $\left(\frac{r_+}{R}\right)_{\rm Buch} \equiv \frac{r_{+\rm Buch}}{R}$, and in $d = 5$, one has $\frac{r_{+\rm Buch}}{R} = \frac{\sqrt{3}}{2} = 0.86$, the latter equality being approximate. Since for $Q = 0$, the free energy of hot flat space is zero, $F_{\rm hfs} = 0$, meaning that there is no further coexistence with hot flat space as soon as the black hole phase is favored, because the system tends to collapse. For nonzero electric charge parameter $\frac{\mu Q^2}{R^4}$ there is no coincidence. To compare the free energies, we consider the case in which the shell has radius equal to the cavity radius, $\frac{r_{\rm shell}}{R} = 1$, and so $F_{\rm shell} = 0$, meaning that the shell is a surrogate for hot flat space, i.e., $F_{\rm shell} = F_{\rm hfs} = 0$, indeed it is hot flat space with electric charge at the boundary. For nonzero $\frac{\mu Q^2}{R^4}$, we find that in the canonical ensemble $F_{\rm bh} = 0$ results in a $\frac{r_+}{R}$ value, both for $\frac{r_{+3}}{R}$ and $\frac{r_{+4}}{R}$, that is higher than the generalized Buchdahl bound, which is the value of $\frac{r_+}{R}$ for gravitational collapse of a self-gravitating system of energy E, electric charge Q, and radius R, see Fig. 7. For an electric charge parameter lower or equal than the saddle value $\frac{\mu Q_s^2}{R^4}$, there is a temperature RT at which the solution $\frac{r_{+3}}{R}$ can assume the value of the Buchdahl bound, corresponding to a positive free energy lower than the free energy of $\frac{r_{+1}}{R}$. For a system with this RT or higher, the system must suffer gravitational collapse into a black hole with the corresponding $\frac{r_{+3}}{R}$. For an electric charge higher than the saddle value y_s, there is again a temperature RT at which $\frac{r_{+4}}{R}$ assumes the Buchdahl bound, with positive value of $F_{\rm bh}$. For a system with this RT or higher, then the system must collapse gravitationally into a black hole with the corresponding $\frac{r_{+4}}{R}$. Interesting to note that the picture in the grand canonical ensemble is different. It was found [30] that the equation $W_{\rm bh} = 0$, with $W_{\rm bh}$ denoting the grand potential, results in a $\frac{r_+}{R}$ value for the single stable black hole, that is lower than the generalized Buchdahl bound. One has thermodynamically that when the system has $W_{\rm bh} = 0$ the black hole phase and hot flat space phase coexist, for $W_{\rm bh} < 0$ the black hole phase dominates, and for a certain definite negative value of $W_{\rm bh}$ the value of $\frac{r_+}{R}$ of the system is the same as the value of the Buchdahl bound. For larger temperatures, therefore the system must collapse gravitationally. The only phase of the system is the black hole phase and so there is no more coexistence, see Fig. 7 and Appendix B. We admit we have not done a thermodynamic treatment of gravitational collapse, but the arguments given in this paragraph are plausible enough to assure us that once there is sufficient thermodynamic energy inside the cavity, collapse to a black hole sets in inevitably.

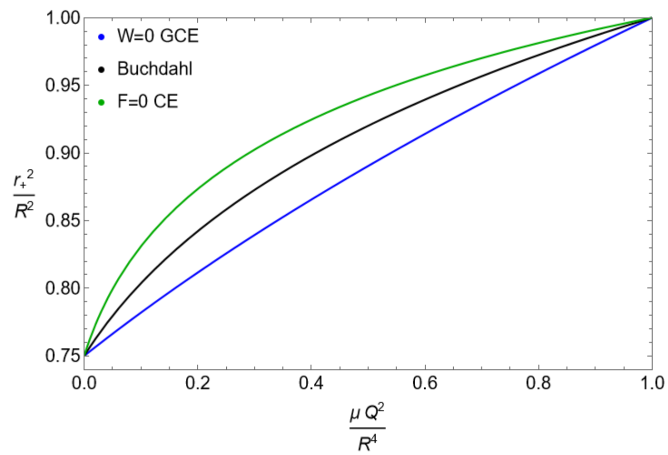


FIG. 7: Ratio $\frac{r_+^2}{R^2}$ in terms of the electric charge parameter $\frac{\mu Q^2}{R^4}$, $\mu = \frac{4}{3\pi}$, for $d = 5$ for three different cases: given by the condition $F_{\rm bh} = 0$ in the canonical ensemble in green, representing the stable solution $\frac{r_{+3}}{R}$; given by the condition $W_{\rm bh} = 0$ in the grand canonical ensemble in blue, representing the only stable solution; and given by generalized Buchdahl condition in black.

VI. INFINITE CAVITY RADIUS: THE DAVIES LIMIT AND THE RINDLER LIMIT

A. Ensemble solutions in the $R \to +\infty$ limit: Davies and Rindler

We now analyze the infinite cavity radius limit, and discuss each solution that arises from this limit. As it turns, by performing $R \to +\infty$ limit while keeping T fixed and Q fixed, three different solutions are found. One observes from Sec. III, that there are three solutions for $r_+(R,T,Q)$ if $\frac{\mu Q^2}{R^{2d-6}} < \frac{\mu Q_s^2}{R^{2d-6}}$. By performing the $R \to +\infty$ limit, the term $\frac{\mu Q^2}{R^{2d-6}}$ approaches zero, and so the solutions of the ensemble in this limit should correspond to these three solutions under the $R \to +\infty$ limit. For the smallest and intermediate solutions, the limit $R \to +\infty$ must be performed by fixing T and Q, while doing $\frac{r_+}{R} \to 0$. For the largest solution, the limit $R \to +\infty$ must be performed by fixing T and Q, while doing $\frac{r_+}{R} \to 1$. The smallest and intermediate solutions correspond to Davies thermodynamic solutions, while the largest solution limit corresponds to the Rindler solution. These solution limits occur for any d, in particular for $d = 4$ and $d = 5$ that we have been analyzing in more detail. In Fig. 8, the behavior of the three solutions in $d = 5$ can be seen for a charge $\mu Q^2 = 0.005$, $\mu = \frac{4}{3\pi}$, for two different R, $R = 5$ and $R = 100$, where the latter R gives an idea of the $R \to \infty$ limit. In this limit the scale R is lost, the scales set by the electric charge Q and temperature T at infinity are now the only two scales of the canonical ensemble. We now comment briefly on each solution.

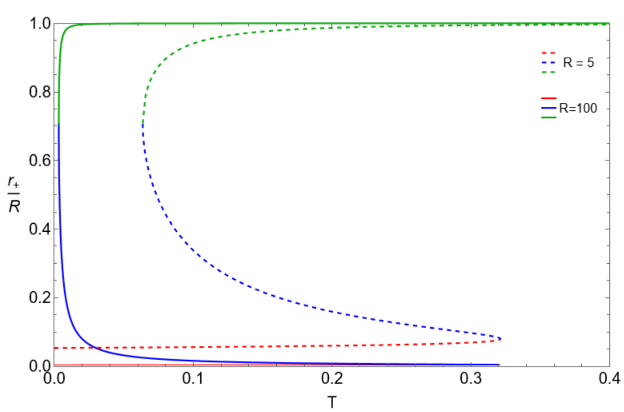


FIG. 8: Plot of the solutions r_{+1} in red, r_{+2} in blue and r_{+3} in green of the canonical ensemble in $d = 5$ as $\frac{r_+}{R}$ as a function of T in Planck units, for $\mu Q^2 = 0.005$, $\mu = \frac{4}{3\pi}$, and for two values of R, $R = 5$ in dashed lines, and $R = 100$ in filled lines. One can see the emergence of the r_{+1} and r_{+2} solution limits corresponding to the Davies limit as they get closer to the $\frac{r_+}{R} = 0$ axis, and the r_{+3} solution limit corresponding to the Rindler limit as it gets closer to the $\frac{r_+}{R} = 1$ axis.

The Davies solution corresponds to the smallest and intermediate solution limits of the canonical ensemble when taking $R \to +\infty$, with fixed T and Q. Thus, these are the solutions of the electrically charged black hole in the canonical ensemble with reservoir at infinity. This can be seen directly from the expression of the temperature in Eq. (28). Since for these solutions the behavior is $\frac{r_+}{R} \to 0$, one can maintain r_+ finite during the limit $R \to +\infty$, thus obtaining the temperature formula $T = \frac{d-3}{4\pi} \frac{r_+^{d-3} - \frac{\mu Q^2}{r_+^{d-3}}}{r_+^{d-2}}$, which is obeyed by the smallest and intermediate solutions. This is precisely the Hawking temperature for the electrically charged black hole. From Fig. 8 we see that the two solutions tend to the axis $\frac{r_+}{R} = 0$ and seem to get overlapped, which is due to the vertical axis being $\frac{r_+}{R}$. If one regularizes the solutions through multiplying by R, one obtains the two solutions in d dimensions, which for $d = 4$ are the Davies thermodynamic solutions. Moreover, one can see that the solutions do not exist for all temperatures. This is because the two solutions only exist up to a critical temperature, the generalized Davies temperature, after which there are no solutions. In the case represented in Fig. 8 which is $d = 5$, the generalized Davies temperature, i.e., the temperature when $R \to \infty$, has the expression $T_s = \frac{4}{10\pi\left(\sqrt{5\mu Q^2}\right)^{\frac{1}{2}}}$, and so for $\mu Q^2 = 0.005$ as in the figure it yields $T_s = 0.320$, with the last equality being approximate.

The Rindler solution is the largest solution limit that can be obtained from the ensemble by keeping T and Q fixed, while doing $R \to +\infty$ and $r_+ \to R$ in Eq. (28). In Fig. 8, this solution is the one that tends to $\frac{r_+}{R} = 1$. The temperature dependence on the charge goes with $\frac{\mu Q^2}{R^{d-3} r_+^{d-3}}$, therefore such dependence in the limit $r_+ \to R$ and $R \to +\infty$ disappears. This happens because the horizon radius of the black hole tends to infinity and any contributions given by the charge become negligible. The expression for the temperature is now the temperature of an electrically uncharged black hole $T = \frac{d-3}{4\pi r_+ \sqrt{1 - \frac{r_+^{d-3}}{R^{d-3}}}}$. Imposing that T is fixed and finite leads to the condition that $r_+ \sqrt{1 - \frac{r_+^{d-3}}{R^{d-3}}}$ must tend to some constant when $R \to +\infty$ and $r_+ \to R$. One can show that in this limit the event horizon of the black hole reduces to the Rindler horizon and the cavity boundary is accelerated to yield the Unruh temperature T set by the reservoir.

We now analyze in full detail the smallest and intermediate solution limits arising from $R \to +\infty$, i.e., the Davies solution. These are relevant since the formalism in this limit yields the Davies' thermodynamic theory of black holes for $d = 4$. We also analyze in full detail the largest solution limit arising from $R \to +\infty$, i.e., the Rindler solution.

B. Infinite cavity radius and Davies' thermodynamic theory of black holes: Canonical ensemble, thermodynamics, and stability of electrically charged black hole solutions in the $R \to +\infty$ limit

1. The action for the canonical ensemble in the $R \to +\infty$ limit

The limit of infinite cavity radius for the small and intermediate solutions yields that the canonical ensemble is essentially defined by the temperature T and the electric charge Q at infinity. It is this $R \to +\infty$ limit that in four dimensions gives Davies results [6]. This means that Davies' thermodynamic theory of black holes, in this case of electrically charged black holes, can be seen within the canonical ensemble formalism. Here we have results for d dimensions in the $R \to +\infty$ limit, $d = 4$ being a particular case.

In the limit of infinite radius, the analysis above needs to be taken with care, since the quantities above depend on the scale given by the cavity radius R. To proceed with this limit, one must start from the reduced action in Eq. (23) and perform the $R \to +\infty$ limit to obtain

$$I_* = \frac{\beta}{\mu}\left(\frac{r_+^{d-3}}{2} + \frac{\mu Q^2}{2r_+^{d-3}}\right) - \frac{\Omega_{d-2} r_+^{d-2}}{4} \,. \qquad (65)$$

The extrema of the action occurs when

$$\beta = \iota(r_+)\,, \qquad \iota(r_+) \equiv \frac{4\pi}{(d-3)} \frac{r_+^{d-2}}{r_+^{d-3} - \frac{\mu Q^2}{r_+^{d-3}}} \,. \qquad (66)$$

This is the inverse Hawking temperature of the Reissner-Nordström black hole measured at infinity, i.e., performing the limit of infinite radius into Eq. (28).

2. Solutions and stability of the canonical ensemble in the $R \to +\infty$ limit

To find the solutions of this canonical ensemble, we must invert Eq. (66) to get $r_+(\beta, Q)$, i.e., $r_+(T, Q)$. This can be done by solving the following equation

$$\left(\frac{(d-3)}{4\pi T}\right)(r_+^{2d-6} - \mu Q^2) - r_+^{2d-5} = 0\,, \qquad (67)$$

which generally is not solvable analytically for generic d, although one can perform some qualitative analysis. The function $\iota(r_+)$ in Eq. (66) has a minimum at $r_{+s1}^{d-3} = \sqrt{(2d-5)\mu}\,Q$, which is a saddle point of the action for the black hole and which we write as

$$r_{+s1} = \left(\sqrt{(2d-5)\mu}\,Q\right)^{\frac{1}{d-3}} \,. \qquad (68)$$

This saddle point of the action of the black hole has the temperature $T_{s1}^{-1} = \frac{2\pi}{(d-3)^2}(2d-5)(\sqrt{(2d-5)\mu}Q)^{\frac{1}{d-3}}$, i.e.,

$$T_{s1} = \frac{(d-3)^2}{2\pi(2d-5)(\sqrt{(2d-5)\mu}Q)^{\frac{1}{d-3}}} \,. \qquad (69)$$

In $d = 4$, this T_{s1} is the Davies temperature, and so Eq. (69) is the generalization of Davies temperature for higher dimensions.

By inspection, one finds that for temperatures $T \leq T_{s1}$ there are two black holes, and for $T > T_{s1}$ there are no black hole solutions. Indeed, for temperatures in the interval $0 < T \leq T_{s1}$, there are two solutions, the solution $r_{+1}(T, Q)$ and the solution $r_{+2}(T, Q)$. The solution $r_{+1}(T, Q)$ is bounded in the interval $(\mu Q^2)^{\frac{1}{2d-6}} < r_{+1}(T, Q) \leq r_{+s1}$, where $r_{+1}(T \to 0, Q) = (\mu Q^2)^{\frac{1}{2d-6}} = r_{+e}$, r_{+e} being the radius of the extremal black hole, and $r_{+1}(T_{s1}, Q) = r_{+s1}$. Moreover, $r_{+1}(T, Q)$ is an increasing monotonic function in T. The solution $r_{+2}(T, Q)$ is bounded from below, i.e., $r_{+2}(T, Q) > r_{+s1}$, where $r_{+2}(T_{s1}, Q) = r_{+s1}$, and is unbounded from above, since at $T \to 0$, the solution r_{+2} tends to infinity. Moreover, $r_{+2}(T, Q)$ is a decreasing monotonic function in T. We note that the action given in Eq. (65) with r_+ holding for $r_{+1}(T, Q)$ or $r_{+2}(T, Q)$ is the action in zero loop approximation that has been found in [32] directly from the Gibbons-Hawking approach, rather than from York's approach for a given R with subsequently taking the $R \to \infty$ limit, as we have been doing here.

Regarding stability, a solution is stable if $\frac{\partial \iota(r_+)}{\partial r_+} < 0$, as we have seen in the case of finite cavity. This gives

$$r_+ \leq r_{+s1} \,, \qquad (70)$$

with r_{+s1} given in Eq. (68). This means that the solution is stable if the radius r_+ increases as the temperature increases. Therefore, the solution r_{+1} is stable since it has this monotonic behavior, while the solution r_{+2} is unstable since it has an opposite monotonic behavior.

3. Thermodynamics in the $R \to +\infty$ limit

(i) Entropy, electric potential, and energy

With the solutions of the canonical ensemble found in the limit of infinite radius of the cavity, $R \to +\infty$, one can find I_0, i.e., the action in the zero loop approximation given in Eq. (65) evaluated at the extrema of Eq. (66). The thermodynamics for the system follows through the correspondence $F = TI_0$, where F again is the Helmholtz free energy of the system and thus it can be written for this case as

$$F = \frac{1}{\mu}\left(\frac{r_+^{d-3}}{2} + \frac{\mu Q^2}{2r_+^{d-3}}\right) - \frac{T\Omega_{d-2} r_+^{d-2}}{4} \,, \qquad (71)$$

where r_+ can be $r_{+1}(T, Q)$ or $r_{+2}(T, Q)$. Using the same calculation method from Sec. IV A, we obtain the entropy as $S = \frac{\Omega_{d-2} r_+^{d-2}}{4}$, i.e.

$$S = \frac{1}{4} A_+ \,. \tag{72}$$

The thermodynamic pressure p is zero,

$$p = 0 \,. \tag{73}$$

The thermodynamic electric potential is

$$\phi = \frac{Q}{r_+^{d-3}} \,, \tag{74}$$

which is equal to the pure electric potential. The energy, given by $E = F + TS$, can be written as $E = \frac{r_+^{d-3}}{2\mu} + \frac{Q^2}{2r_+^{d-3}}$. But the spacetime mass m is given by $m = \frac{r_+^{d-3}}{2\mu} + \frac{Q^2}{2r_+^{d-3}}$, see also Appendix B, so that the thermodynamic energy and the spacetime mass are the same in the $R \to +\infty$ limit, i.e.,

$$E = m \,. \tag{75}$$

Thus, we can write the free energy given in Eq. (71) as

$$F = m - TS \,. \tag{76}$$

We must note that the expressions for the entropy, the pressure, the thermodynamic electric potential, and the energy are consistent with the limit of infinite radius to the respective expressions in Sec. IV A. Moreover, in this limit, the pressure p vanishes, which is consistent with the absence of the variable R in the action.

(ii) Smarr formula and the first law of black holes

The energy in Eq. (75) can be rewritten in terms of the entropy and the charge as $E = \frac{1}{2\mu}\left(\frac{4S}{\Omega_{d-2}}\right)^{\frac{d-3}{d-2}} + \frac{Q^2}{2}\left(\frac{4S}{\Omega_{d-2}}\right)^{\frac{3-d}{d-2}}$. The energy function possesses the scaling property $\nu^{\frac{d-3}{d-2}} E = E(\nu S, \nu^{\frac{d-3}{d-2}} Q)$, which allows the use of the Euler relation theorem to have $E = \frac{d-3}{d-2} TS + \phi Q$, which is the formula obtained in Sec. IV B without the term pA. Indeed, the term pA in the limit of infinite reservoir radius has leading order $R^{-(d-3)}$, and so it vanishes. Since from Eq. (75) $E = m$, we obtain

$$m = \frac{d-3}{d-2} TS + \phi Q \,, \tag{77}$$

which is the Smarr formula.

In this case the law

$$dm = TdS + \phi dQ \,, \tag{78}$$

holds. This is exactly the first law of black hole mechanics. This can be obtained from Eq. (51) in the $R \to \infty$ limit. For R finite, there is a first law of thermodynamics of the cavity and does not correspond to the law of black hole mechanics. For $R \to \infty$, the first law of black hole thermodynamics and the first law of black hole mechanics coincide into one same law, which is quite remarkable. Moreover, in the electrically charged case, as opposed to the Schwarzschild case, the thermodynamics of the canonical ensemble is valid, since there is a region of the electric charge where the system is thermodynamically stable. It is from Eq. (78) that Davies has started his thermodynamic theory of black holes for $d = 4$. We have deduced it from the action Eq. (65).

(iii) Heat capacity and stability

The thermodynamic stability can be seen directly from applying the limit of infinite radius of the cavity in Eq. (56) and obtain the condition for the positivity of the heat capacity, which ensures that a solution is stable. The heat capacity in this limit is

$$\begin{aligned} C_Q &= \frac{(d-2)\Omega_{d-2} r_+^{d-2}(r_+^{2d-6} - \mu Q^2)}{4\left((2d-5)\mu Q^2 - r_+^{2d-6}\right)} \\ &= \frac{S^3 E T}{\frac{(d-3)\Omega_{d-2}^3}{4^5\pi^2}\left[\frac{(3d-8)\mu^2 Q^4}{\left(\frac{4S}{\Omega_{d-2}}\right)^{\frac{d-4}{d-2}}} + (d-4)\left(\frac{4S}{\Omega_{d-2}}\right)^{\frac{3d-8}{d-2}}\right] - T^2 S^3}, \end{aligned} \tag{79}$$

where we have dropped the subscript A in $C_{A,Q}$ since the evaluation is at infinity, and in the second equality we wrote the heat capacity in terms of the thermodynamic variables S, E, and T. So there is stability if $C_Q \geq 0$, i.e., $r_+ \leq \left[(2d-5)\mu Q^2\right]^{\frac{1}{2d-6}}$, which is Eq. (70) together with Eq. (68). This means that the solution r_{+1} is thermodynamically stable whereas the solution r_{+2} is unstable. It must be noted also that r_{+1} is an increasing monotonic function in T, which means the energy of the black hole increases of the temperature increases, as it is expected from a stable system. The opposite happens to the solution r_{+2}, since it is a decreasing monotonic function in T and so the energy of the black hole decreases as temperature increases.

4. Favorable phases

There are two stable phases. The small black hole r_{+1} and hot flat space with electric charge at infinity. Since the black hole r_{+1} has positive free energy and hot flat space with electric charge at infinity has zero free energy, and systems with lower free energy are preferred, whenever the system finds itself in the black hole r_{+1} solution it tends to transition to the hot flat space with electric charge at infinity phase.

5. $d=4$: *Analysis leading to Davies' thermodynamic theory of black holes and Davies point*

The dimension $d=4$ is specially interesting since in the $R\to\infty$ gives the results of Davies' thermodynamic theory of black holes [6]. In this setting, the reservoir of temperature T and electric charge Q is at infinity.

The reduced action in Eq. (65) in $d=4$ gives

$$I_* = \frac{\beta}{2}\left(r_+ + \frac{Q^2}{r_+}\right) - \pi r_+^2\,, \tag{80}$$

where $\mu=1$ and $\Omega_2 = 4\pi$. The stationary points in $d=4$ occur when

$$\beta = \iota(r_+)\,, \qquad \iota(r_+) \equiv \frac{4\pi r_+^2}{r_+ - \frac{Q^2}{r_+}}\,, \tag{81}$$

corresponding to the inverse Hawking temperature of a charged black hole in $d=4$.

We invert Eq. (81) to get the solutions $r_+(T,Q)$. This results in solving

$$\left(\frac{1}{4\pi T}\right)(r_+^2 - Q^2) - r_+^3 = 0\,, \tag{82}$$

although we do not present the solutions here. The minimum of function $\iota(r_+)$ in Eq. (81) occurs at $r_{+s1} = \sqrt{3}\,Q$, being a saddle point of the action of the black hole. We write the horizon radius of the saddle point as

$$r_{+\mathrm{D}} = \sqrt{3}\,Q\,, \tag{83}$$

as in $d=4$ it gives the Davies horizon radius. Since $r_+ = m + \sqrt{m^2-Q^2}$, this means $m = \frac{2}{\sqrt{3}}Q$ at the saddle point, a result that can be found in [6]. The temperature corresponding to the saddle point is Eq. (69) in $d=4$, or explicitly

$$T_{\mathrm{D}} = \frac{1}{6\sqrt{3}\pi Q}\,, \tag{84}$$

which is the Davies temperature, and it is a result that can be extracted from [6].

We present a summary of the behavior of the solutions for $d=4$. For $0<T\le T_{\mathrm{D}}$, there are two solutions, the solution $r_{+1}(T,Q)$ and the solution $r_{+2}(T,Q)$. The solution $r_{+1}(T,Q)$ increases monotonically with T and lies in the interval $r_{+_e} < r_{+1}(T,Q) \le r_{+\mathrm{D}}$, where $r_{+1}(T\to 0,Q) = r_{+_e} = Q$ and $r_{+1}(T_{\mathrm{D}},Q) = r_{+\mathrm{D}} = \sqrt{3}\,Q$. The solution $r_{+2}(T,Q)$ decreases monotonically with T and lies in the interval $r_{+\mathrm{D}} < r_{+2}(T,Q) < \infty$, where $r_{+2}(T_{\mathrm{D}},Q) = r_{+\mathrm{D}} = \sqrt{3}\,Q$. For $T_{\mathrm{D}} < T$, there are no black hole solutions. Regarding stability, a solution is stable if $\frac{\partial\iota(r_+)}{\partial r_+}\le 0$, i.e.

$$r_+ \le r_{+\mathrm{D}}\,. \tag{85}$$

With $r_{+\mathrm{D}}$ given in Eq. (83), Eq. (85) can be turned in to the region in the electric charge $\frac{1}{\sqrt{3}}r_+ \le Q \le r_+$, the latter term being simply the restriction to nonextremal case. From Eq. (85), we have that the solution r_{+1} is stable while the solution r_{+2} is unstable.

We summarize now the results for thermodynamics in $d=4$. The free energy of the system is $F = TI_0$, coming from the zero loop approximation of the path integral. From Eq. (80), the free energy is

$$F = \frac{1}{2}\left(r_+ + \frac{Q^2}{r_+}\right) - T\,\pi r_+^2\,. \tag{86}$$

From the derivatives of the free energy, we obtain the entropy $S=\pi r_+^2$, i.e., $S = \frac{1}{4}A_+$, the thermodynamic pressure $p=0$ since there is no area dependence, the electric potential $\phi = \frac{Q}{r_+}$, and the energy $E = \frac{1}{2}\left(r_+ + \frac{Q^2}{r_+}\right)$, from $E = F+TS$. Considering that this is the expression for the spacetime mass m, we have $E=m$. The free energy of Eq. (86) is then $F = m - TS$.

The Smarr formula for $d=4$ is

$$m = \frac{1}{2}TS + \phi Q\,. \tag{87}$$

Indeed, the first law of black hole mechanics $dm = TdS + \phi dQ$ coincides with the first law of thermodynamics, see above. The first law of black hole mechanics is the expression from which Davies [6] started his analysis. We have started our analysis from the action Eq. (80) and actually derived the first law from first principles. Moreover, the system is stable thermodynamically in a range of values of the electric charge. On the other hand, the electrically charged case in the grand canonical ensemble with the reservoir at infinity is unstable. Gibbons and Hawking through the action and the path integral approach [7] noticed this instability problem but did not venture into the electric canonical ensemble to cure it.

The heat capacity of Eq. (79) is for $d=4$ given by

$$C_Q = \frac{2\pi r_+^2\left(1-\frac{Q^2}{r_+^2}\right)}{3\frac{Q^2}{r_+^2}-1} = \frac{S^3 E T}{\frac{\pi Q^2}{4} - T^2 S^3}\,, \tag{88}$$

where in the second equality we wrote the heat capacity in terms of the thermodynamic variables S, E, and T. The system is thermodynamically stable if $Q \ge \frac{1}{\sqrt{3}}r_+$, i.e., $\frac{1}{\sqrt{3}}r_+ \le Q \le r_+$, the latter term being the condition for nonextremal case. The system is thermodynamically unstable if $0 \le Q < \frac{1}{\sqrt{3}}r_+$. This is the same result as given in Eq. (85) together with Eq. (83). The heat capacity C_Q is infinitely positive at the point $Q = \frac{1}{\sqrt{3}}r_+$ if one approaches it from higher Q, the heat capacity C_Q is infinitely negative if one approaches the point $Q = \frac{1}{\sqrt{3}}r_+$ from lower Q. The heat capacity goes to zero at the extremal case $Q = r_+$. Precisely at the point $Q = \frac{1}{\sqrt{3}}r_+$, this behavior of the heat capacity was found in [6], and it was classified as being similar to a second order phase transition. However, this point is a turning point rather than a second order phase transition. This turning point indicates the ratio of scales at which one has stability. Indeed, when analyzing the heat capacity in terms of the temperature and electric charge, one has two distinctive curves, one for each solution, diverging at this point. But the unstable solution cannot be considered as a phase, due to its instability. The system will always remain in the stable configuration. Note that the formula for C_Q in the second line of Eq. (88) is the same formula found in [6] by performing in Eq. (88) the redefinitions $S \to 8\pi S$, $T \to \frac{1}{8\pi}T$ and $\frac{C_Q}{8\pi} \to C_Q$.

6. $d = 5$: Analysis

The dimension $d = 5$ is a typical higher dimension that we have been analyzing. We present here the summary for this specific case in the $R \to +\infty$ limit.

The reduced action in Eq. (65) in $d = 5$ can be written simply as

$$I_* = \frac{\beta}{2}\left(\frac{3\pi r_+^2}{4} + \frac{Q^2}{r_+^2}\right) - \frac{\pi^2 r_+^3}{2}\,, \tag{89}$$

where we have used $\mu = \frac{4}{3\pi}$ and $\Omega_3 = 2\pi^2$. The stationary points are described by

$$\beta = \iota(r_+)\,, \qquad \iota(r_+) \equiv \frac{2\pi r_+^3}{r_+^2 - \frac{4Q^2}{3\pi r_+^2}}. \tag{90}$$

again corresponding to the inverse Hawking temperature of a charged black hole in $d = 5$.

The solutions are found by inverting Eq. (90) to get $r_+(\beta, Q)$, i.e., $r_+(T, Q)$. This is the same as solving

$$\left(\frac{1}{2\pi T}\right)(r_+^4 - \frac{4}{3\pi}Q^2) - r_+^5 = 0\,, \tag{91}$$

which cannot be done analytically. However, it can be analyzed qualitatively or solved numerically, see Fig. 9 for this case of five dimensions. The function $\iota(r_+)$ in Eq. (90) possesses a minimum at

$$r_{+s1} = \left(\sqrt{\frac{20}{3\pi}}\,Q\right)^{\frac{1}{2}}\,, \tag{92}$$

which corresponds to a saddle point of the action of the black hole. This generalizes the Davies radius to $d = 5$. The temperature at this saddle point is $T_{s1}^{-1} = \frac{10\pi}{4}(\sqrt{\frac{20}{3\pi}}\,Q)^{\frac{1}{2}}$, i.e.,

$$T_{s1} = \frac{4}{10\pi\left(\sqrt{\frac{20}{3\pi}}\,Q\right)^{\frac{1}{2}}}\,. \tag{93}$$

This generalizes the Davies temperature for $d = 5$.

We summarize the behavior of the solutions in $d = 5$. For temperatures $0 < T \le T_s$ there are two solutions, the solution $r_{+1}(T, Q)$ and the solution $r_{+2}(T, Q)$. The solution $r_{+1}(T, Q)$ increases monotonically with the temperature and is bounded by $r_{+e} < r_{+1}(T, Q) \le r_{+s}$, where $r_{+1}(T \to 0, Q) = r_{+e} = \left(\sqrt{\frac{4}{3\pi}}\,Q\right)^{\frac{1}{2}}$ is the extremal black hole, and $r_{+1}(T_{s1}, Q) = r_{+s} = \left(\sqrt{\frac{20}{3\pi}}\,Q\right)^{\frac{1}{2}}$. The solution $r_{+2}(T, Q)$ decreases monotonically with the temperature and assumes values in the interval $r_{+s1} < r_{+2}(T, Q) < \infty$, where $r_{+2}(T_s, Q) = r_{+s}$. See Fig. 9 for the plots of r_{+1} and r_{+2}. Regarding stability, a stable solution obeys $\frac{\partial\iota(r_+)}{\partial r_+} \le 0$. This condition becomes

$$r_+ \le r_{+s1}\,. \tag{94}$$

With r_{+s1} given in Eq. (92), Eq. (94) can be transformed to $\left(\frac{3\pi}{20}\right)^{\frac{1}{2}} r_+^2 \le Q \le \left(\frac{3\pi}{4}\right)^{\frac{1}{2}} r_+^2$, the latter term being the restriction to the nonextremal case. From Eq. (94), we obtain that r_{+1} is stable and that r_{+2} is unstable. The

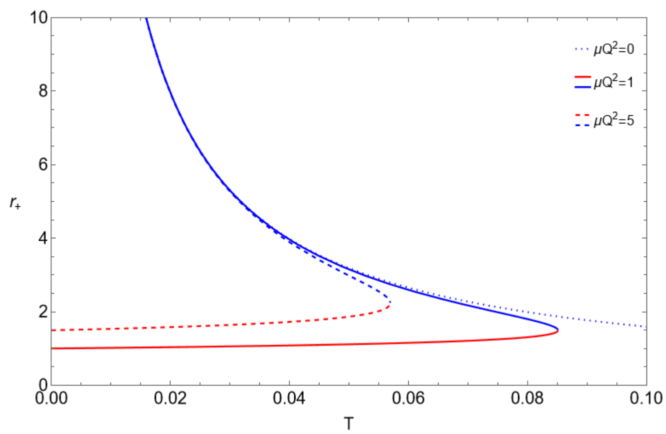


FIG. 9: Plot of the two solutions $r_{+1}(T, Q)$, in red, and $r_{+2}(T, Q)$, in blue, of the charged black hole in the canonical ensemble for infinite cavity radius, for two values of the charge, $\mu Q^2 = 1$ in filled lines, and $\mu Q^2 = 5$ in dashed lines, $\mu = \frac{4}{3\pi}$, in $d = 5$.

plots in Fig. 9 show the discussion above, namely the stable branch r_{+1} and the unstable branch r_{+2}. It is also seen clearly that the plot of Fig. 9 is the limiting $R \to \infty$ case of Fig. 2. From Fig. 2 one finds that when $R \to \infty$, the solution r_{+3} disappears, leaving r_{+1} and r_{+2}, with r_{+1} and r_{+2} meeting at a maximum temperature. Also, from Fig. 2 we see that the r_{+2} and r_{+3} branches meet at a minimum temperature, and these branches are the ones that appears in the zero charge case of York, here slightly modified due to the existence of an electric charge Q. More specifically, comparing Fig. 9 with Fig. 2, one notes that the red and blue lines of Fig. 9 are the stable and unstable black holes of Davies, here in $d = 5$, and the red and blue lines of Fig. 2 are precisely these branches of black holes for finite reservoir radius R. The blue and green branches in Fig. 2 correspond to York black holes. Thus, Fig. 2 is a unified plot of York and

Davies black holes. Note further from Fig. 9, that for the electric charge going to zero, the branch that survives in Fig. 9 is the blue branch, which corresponds to the unstable black hole r_{+2}, and the solution goes up to the point characterized by $T=\infty$ and $r_+=0$. This branch corresponds to the original unstable Hawking black hole, the black hole also found in the Gibbons-Hawking path integral approach.

We present the summary of the results for the thermodynamics in $d=5$. The free energy can be obtained from the zero loop approximation of the path integral as $F=TI_0$. From Eq. (89), the free energy takes the form

$$F=\frac{1}{2}\left(\frac{3\pi r_+^2}{4}+\frac{Q^2}{r_+^2}\right)-T\frac{\pi^2 r_+^3}{2}\,. \tag{95}$$

From its derivatives, we obtain the entropy as $S=\frac{1}{4}A_+$, $A_+=2\pi^2 r_+^3$, the thermodynamic pressure as $p=0$, the thermodynamic electric potential as $\phi=\frac{Q}{r_+^2}$, and the energy, given by $E=F-TS$, as $E=\frac{3\pi r_+^2}{8}+\frac{Q^2}{2r_+^2}$. Note that this is exactly the expression for the spacetime mass m, so the mean energy is $E=m$. The free energy of Eq. (95) becomes $F=m-TS$.

The Smarr formula in $d=5$ takes the form

$$m=\frac{2}{3}TS+\phi Q\,. \tag{96}$$

Also, one has that the law $dm=TdS+\phi dQ$ holds. And so the first law of black hole mechanics coincides with the first law of thermodynamics. Also, the system is stable thermodynamically in a small region of the charge, so this correspondence is valid.

The heat capacity of Eq. (79) is now in $d=5$ given by

$$\begin{aligned}C_Q=&\frac{3\pi^2 r_+^3\left(1-\frac{4}{3\pi}\frac{Q^2}{r_+^4}\right)}{2\left(\frac{20}{3\pi}\frac{Q^2}{r_+^4}-1\right)}\\=&\frac{S^3ET}{\frac{7\pi^2}{36}Q^4\left(\frac{2S}{\pi^2}\right)^{-\frac{1}{3}}+\frac{\pi^4}{4^3}\left(\frac{2S}{\pi^2}\right)^{\frac{7}{3}}-T^2S^3}\,,\end{aligned} \tag{97}$$

where in the second equality is in terms of the thermodynamic variables S, E, and T. One has instability if $0\le Q<\left(\frac{3\pi}{20}\right)^{\frac{1}{2}}r_+^2$, with Q meaning its absolute modulus. One has thermodynamic stability if $\left(\frac{3\pi}{20}\right)^{\frac{1}{2}}r_+^2\le Q\le\left(\frac{3\pi}{4}\right)^{\frac{1}{2}}r_+^2$, the latter term being the condition for the nonextremal case, and this can also be derived from Eq. (94) together with Eq. (92). The heat capacity C_Q is infinitely positive at the point $Q=\left(\frac{3\pi}{20}\right)^{\frac{1}{2}}r_+^2$, if this point is approached from higher Q, the heat capacity C_Q is infinitely negative, if the point is approached from lower Q. This is a turning point of the solutions, indicating the condition for stability. This is properly seen when analyzing the heat capacity with fixed temperature and electric charge, see Fig. 10. Indeed, the heat capacity is

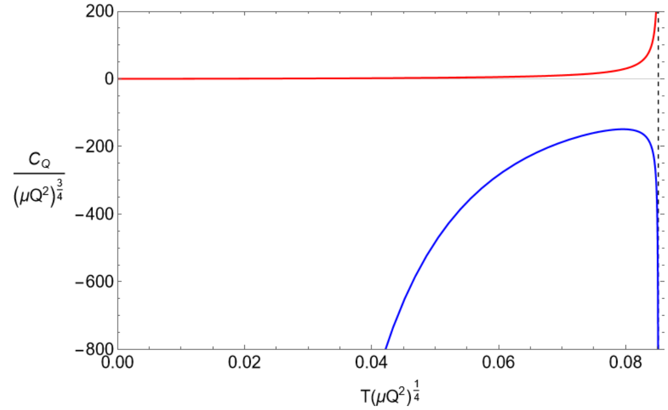


FIG. 10: The heat capacity C_Q in $(\mu Q^2)^{\frac{3}{4}}$ units, $\frac{C_Q}{(\mu Q^2)^{\frac{3}{4}}}$, is given as a function of the temperature $T(\mu Q^2)^{\frac{1}{4}}$ in $d=5$. In red, the heat capacity of r_{+1} is represented, while in blue, the heat capacity of r_{+2} is shown. There is a turning point at $T(\mu Q^2)^{\frac{1}{4}}=\frac{4}{10\pi 5^{\frac{1}{4}}}$.

described by two curves, one for each solution r_{+1} and r_{+2}, being positive for r_{+1} and negative for r_{+2}. The system cannot be sustained in the solution r_{+2} since it is unstable and so it can only be at the stable solution r_{+1}.

C. Infinite cavity radius and the Rindler limit: Cavity boundary at the Unruh temperature

The largest solution limit can be obtained by keeping T and Q fixed, while doing $R\to+\infty$ and $r_+\to R$ in Eq. (28). The temperature dependence on the charge goes with $\frac{\mu Q^2}{R^{2d-6}}$, therefore such dependence in the limit $r_+\to R$ and $R\to+\infty$ disappears. Intuitively, the black hole becomes very large such that any contributions from the charge become negligible. Then, the expression for the temperature reduces to the noncharged case, $T=(d-3)(4\pi r_+)^{-1}(1-\frac{r_+^{d-3}}{R^{d-3}})^{-\frac{1}{2}}$, but we still need to apply the limit. The requirement that T is fixed and so finite leads to the condition that $r_+\sqrt{1-\frac{r_+^{d-3}}{R^{d-3}}}$ must tend to some constant under the limit of $R\to+\infty$ and $r_+\to R$. Still, it seems unclear a priori what the system in this limit describes.

In order to understand the limit, one can first consider the Euclidean Schwarzschild metric $ds^2=R^2\frac{4r_+^2}{R^2(d-3)^2}(1-\frac{r_+^{d-3}}{r^{d-3}})\,d\tau^2+(1-\frac{r_+^{d-3}}{r^{d-3}})^{-1}R^2\,d(\frac{r}{R})^2+R^2(\frac{r^2}{R^2})d\Omega_{d-2}^2$, where we introduced the normalization by R in the line element, with $0\le\tau<2\pi$ and $r_+<r\le R$. First, we need to consider $r_+\to R$ in the limit of infinite cavity and only then perform $R\to\infty$. Therefore, we must consider the near horizon expansion of the metric. The normalized proper radial length is given by $\epsilon(r)=\frac{1}{R}\int_{r_+}^{r}(1-\frac{r_+^{d-3}}{\rho^{d-3}})^{-\frac{1}{2}}d\rho=$

$\frac{2}{d-3}(\frac{R}{r_+})^{\frac{d-5}{2}}\sqrt{(\frac{r}{R})^{d-3}-(\frac{r_+}{R})^{d-3}}$, valid at the near horizon, spanning the interval $0<\epsilon<\epsilon(R)$. One can therefore rewrite the Schwarzschild metric in this limit as $ds^2=(R^2\epsilon^2+\mathcal{O}(\epsilon^4))d\tau^2+R^2d\epsilon^2+(R^2+\mathcal{O}(\epsilon^2))d\Omega^2$. Notice however that as $r_+\to R$, the total normalized radial proper length $\epsilon(R)$ tends to zero. It is now that we perform the limit $R\to+\infty$ but such that $R\epsilon(R)$ tends to a constant, which we write as $\bar{R}$, $\bar{R}\equiv R\epsilon(R)$. Thus, we have a new proper length $\bar{r}$, defined as

$$\bar{r}\equiv R\epsilon(r)\,,\qquad 0<\bar{r}<\bar{R}\,. \tag{98}$$

The metric becomes in this limit

$$ds^2=\bar{r}^2d\tau^2+d\bar{r}^2+R^2d\Omega^2\,, \tag{99}$$

i.e., it becomes the two-dimensional Euclideanized Rindler metric times a $(d-2)$-sphere with infinite radius. The metric on the $(d-2)$-sphere can be normalized by choosing a specific point on the sphere and performing the expansion around such point, obtaining $R^2d\Omega^2=\sum_{i=1}^{d-2}(dx^i)^2$, where x^i are the new coordinates. The metric then reduces to the d dimensional Euclideanized Rindler space. The system can now be interpreted as follows. The event horizon of the black hole reduces to the Rindler horizon at $\bar{r}=0$, while the cavity boundary is located at $\bar{R}$ and it is being accelerated. The proper acceleration of the cavity is precisely $\frac{1}{\bar{R}}$ and the temperature measured at the boundary of the cavity is $T=\frac{1}{2\pi\bar{R}}$.

We now analyze what happens to the thermodynamic quantities in this Rindler solution limit. First, the temperature in Eq. (28) is finite and equals to $T=\frac{1}{2\pi\bar{R}}$. Since T is fixed by the ensemble this gives the solution for the cavity boundary, namely

$$\bar{R}=\frac{1}{2\pi T}\,. \tag{100}$$

To be in equilibrium with the temperature T of the reservoir, the boundary itself $\bar{R}$ has to have a Rindler acceleration that matches its Unruh temperature. The free energy in Eq. (44) diverges negatively, $F\to-\infty$. It diverges as $F=\frac{R^{d-3}}{\mu}-\frac{\Omega_{d-2}}{8\pi\bar{R}}R^{d-2}$, and is negative since the power R^{d-2} is always larger than R^{d-3} for $R\to+\infty$. This divergence is due to the fact that the area is divergent. Thus, it is better to work with a specific free energy, $\bar{F}$, a free energy per unit area, defined as $\bar{F}\equiv\frac{F}{\Omega_{d-2}R^{d-2}}$. Then,

$$\bar{F}=-\frac{1}{8\pi\bar{R}}\,, \tag{101}$$

so it is negative. From Eq. (47), the entropy also diverges, $S\to\infty$, it diverges as $S=\frac{\Omega_{d-2}R^{d-2}}{4}$. Defining a specific entropy $\bar{S}\equiv\frac{S}{\Omega_{d-2}R^{d-2}}$

$$\bar{S}=\frac{1}{4}\,, \tag{102}$$

so it is a constant. The thermodynamic pressure in Eq. (48) is finite, which we write as

$$\bar{p}=\frac{1}{8\pi\bar{R}}\,, \tag{103}$$

so $\bar{p}=\frac{T}{4}$. The electric potential in Eq. (49) is zero,

$$\bar{\phi}=0\,. \tag{104}$$

The thermodynamic energy from Eq. (50) obeys $E\to\infty$, it diverges as $E=\frac{R^{d-3}}{\mu}$ positively. Defining a specific energy, $\bar{E}$, as $\bar{E}\equiv\frac{E}{\Omega_{d-2}R^{d-2}}$, one obtains

$$\bar{E}=0\,. \tag{105}$$

The heat capacity in Eq. (56) goes positively as $C_A=\frac{(d-2)(d-3)\Omega_{d-2}}{2}R^{d-4}\bar{R}^2$. So $C_A=4\pi\bar{R}^2\quad\text{for}\,d=4$ and $C_A\to\infty\quad\text{for}\,d>4$, i.e., for $d=4$ is finite and depends on the temperature as $C_A=\frac{1}{\pi T^2}$, and for $d>4$ diverges. Since C_A is positive, this solution can then be considered stable. Defining a specific heat capacity, $\bar{C}_A$, as $\bar{C}_A\equiv\frac{C_A}{\Omega_{d-2}R^{d-2}}$ gives

$$\bar{C}_A=0\,. \tag{106}$$

Although this solution has divergent quantities, when one resorts to specific quantities, as one should since the system is infinite, one finds finite quantities.

For the ensemble with infinite radius one can try to define what is the most preferred phase thermodynamically. However, it seems that the two limiting solutions have different character. In the Davies solution there is still a net electrically charge Q at infinity. In the Rindler solution the electric charge has disappeared from the context, so it is in fact a zero electric charge solution. Although the starting ensembles are the same, the final ensembles in the infinite radius limit are different. From the free energies, given that the stable black hole in Davies solution has positive free energy and the Rindler one has infinite negative free energy, one would conclude that the Rindler solution is the most preferred phase. But in fact the two solutions belong to different ensembles and cannot be compared. As we have mentioned, the Davies stable solution tends to disperse to hot flat space with electric charge at infinity.

VII. CONCLUSIONS

We have analyzed the canonical ensemble of a Reissner-Nordström black hole in a cavity for arbitrary dimensions. The construction of the canonical ensemble was done through the computation of the partition function in the Euclidean path integral approach. The action is the usual Einstein-Hilbert-Maxwell action with the Gibbons-Hawking-York boundary term and an additional Maxwell boundary term so that the canonical ensemble is well defined, all terms having been Euclideanized. We assumed that the heat reservoir has a spherical boundary at finite radius R, where the temperature is fixed as the inverse of the Euclidean proper time length at the boundary, and also the electric charge is fixed by fixing the electric flux at the boundary. We then restricted to spherically symmetric spaces and assume regularity boundary conditions that avoid the presence of conical and curvature singularities.

The zero loop approximation was then performed by first imposing the Hamiltonian and the Gauss constraints, obtaining a reduced action that depends on the fixed inverse temperature β, electric charge Q, and the radius of the boundary R, and also depends on the radius of the event horizon r_+ as a variable that is integrated through the path integral. We then found the equation for the stationary points of the reduced action which are the solutions $r_+[\beta, Q, R]$, and the condition of stability of the solutions. The equation cannot be solved analytically.

The existence of the solutions of the ensemble were analyzed for arbitrary dimensions. For charges smaller than a saddle, or critical, electric charge, there are always three possible solutions where the one with the smallest radius and the one with the largest radius are stable, and the other with intermediate radius is unstable. The value of the saddle charge and the value of the radii that bound these solutions, which are saddle points of the reduced action, were found analytically. For the saddle charge, the unstable solution reduces to a point, having formally only two solutions which are stable. For charges larger than the saddle charge, there is only one solution, and this solution is stable. This analysis was then applied to the four and five dimensional cases. Regarding stability, the solutions are stable if the radius of the event horizon increases as the temperature increases. For this case, the condition is given in terms of the saddle points of the reduced action.

The thermodynamics of the electrically charged black hole was obtained using that the partition function is related to the Helmholtz free energy of the system in the canonical ensemble. Through the zero loop approximation, the free energy was obtained. The entropy, the thermodynamic electric potential, the thermodynamic pressure, and the thermodynamic energy were retrieved through the derivatives of the free energy. More precisely, the entropy is the Bekenstein-Hawking entropy, the pressure has the same expression of the pressure of a self-gravitating charged shell with radius R, and the thermodynamic electric potential is given by the usual expression. The mean thermodynamic energy, which can be identified with a quasilocal energy, was calculated through the definition of free energy. Regarding thermodynamic stability, the configurations are stable if the heat capacity with constant charge and area is positive. The integrated first law, i.e., the Euler formula, and the Gibbs-Duhem relation were also found.

We analyzed the favorable states in the canonical ensemble. A favorable state is a stable state of the ensemble that has the lowest value of the free energy. In some sense, transitions can occur between phases. Here, for an electric charge lower than the critical charge, there are two stable black hole solutions that are in competition, with an existing first order phase transition between them. For the critical charge, this first order phase transition becomes a second order phase transition. For a charge larger than the critical charge, there is only one stable black hole solution. In the uncharged case, there is a stable solution and hot flat space. Pure hot flat space does not seem to be a solution of the canonical ensemble since the charge is fixed. Instead, we compare the stable solutions with a nonself-gravitating charged sphere. This covers two limits, the case where we have flat space with a charge at the center, which is not a solution and is never favorable, and another case where the charge resides near the cavity or at the cavity. In this last case, it would act as a hot flat space with electric charge at the boundary and the corresponding free energy vanishes. Considering this latter case There is a first order phase transition between the largest black hole and hot flat space with electric charge at the boundary. The black hole solutions and the charged shell model have been compared in a phase diagram.

In this work of the canonical ensemble of a Reissner-Nordström black hole in a cavity for four and higher dimensions there are several main achievements which can be stated:

First, the construction of the canonical ensemble and the thermodynamic analysis of all generic d dimensions in a unified way was done. Moreover, significant cases were presented in all the detail, namely, the dimension $d = 4$ as the most important case, and the dimension $d = 5$ as a typical higher dimensional case.

Second, in the analysis of the specific heat $C_{A,Q}$ in terms of the temperature and the electric charge, we found the existence of a second order phase transition between the two stable solutions for a critical electric charge parameter $\frac{\mu Q_s}{R^{2d-6}}$ in arbitrary dimensions. For lower electric charge $\frac{\mu Q}{R^{2d-6}}$, we found two turning points indicating the stability of the solutions, where the heat capacity diverges and is double valued. For higher charge $\frac{\mu Q}{R^{2d-6}}$, we found that the heat capacity is always positive.

Third, since in the canonical ensemble one can have two stable black hole solutions, an analysis of the free energy has enabled us to pick the black hole solution that is most favored according to the temperature and electric charge of the ensemble and find the possible first order phase transitions. Moreover, a comparison with the free energy of hot flat space, emulated by an electric shell at the boundary, has revealed the thermodynamic phase that is favored. We have also argued that the Buchdahl bound is important in this context, and the free energies for which the bound is superseded were found, for higher free energies gravitational collapse sets in.

Fourth, the Davies thermodynamic theory of black holes has been shown to follow from the electric charged canonical ensemble in the infinite large reservoir limit when $d = 4$. The two ensemble solutions of lower radii maintain, in this limit, their black hole character. One, with smallest radius, is the stable one, and the other with intermediate radius is unstable. These two solutions meet at a saddle point. The thermodynamic quantities were found and in particular, the heat capacity at constant area and charge was found. In $d = 4$, the expression of the heat capacity reduces to the expression found by Davies. We have started from the action and the path integral approach for a reservoir at infinity and showed that the formalism gives the first law of black hole mechanics which, of course, is also the first law of thermodynamics for black holes. Davies, in the $d = 4$ formulation of the theory, started directly from the first law of black hole mechanics. These results, reached through different means, point toward the equivalence between black hole mechanics and black hole thermodynamics through the canonical ensemble.

Fifth, the limit of infinite radius of the boundary of the cavity, has revealed a surprise solution. Indeed, the largest black hole solution of the ensemble, changes character in this limit. The black hole solution turns into a Rindler solution with the ensemble fixed temperature being the Unruh temperature of the now accelerated boundary.

Sixth and last, the York path integral procedure, which was originally applied to Schwarzschild black holes, has been followed throughout this work for Reissner-Nordström black holes. We have shown that the black hole solutions found represent the unification of York electrically uncharged black holes and Davies electric charged black holes, in a remarkable way. Indeed, the two York type solutions, one larger and stable, one smaller and unstable, do appear, and the two Davies type solutions, the smaller and unstable, and the even smaller and stable also do appear, in a remarkable way. York and Davies results follow from two different limits of our work. York results follow from taking the zero electric charge limit. Davies results follow from taking the infinite cavity radius limit, i.e., by putting the heat reservoir at infinity. This latter case can also be seen to stem from York's generic reduced action approach with the boundary at infinity, which in turn yields the Gibbons-Hawking path integral formulation to black hole thermodynamics. The Gibbons-Hawking approach was originally applied to electrically uncharged black holes and it was found that there was an unstable black hole solution, the Hawking black hole, and thus no consistent thermodynamics. It was also applied to an electrically charged black hole in the grand canonical ensemble, and it was found a solution that was unstable. Had it been applied directly to electrically charged black holes in the canonical ensemble, one would have found that thermodynamic stable solutions exist to vindicate the approach. We have filled this gap here.

What does remain to be understood? Here the interest has been in the thermodynamic interaction of a black hole in a cavity with a boundary of finite size and fixed temperature, as well as in the interaction of the gravitational field with the electromagnetic field in such a system. The formalism by its very distinctive features, i.e., its Euclidean character, applies only to the outside of a black hole event horizon. The black hole interior and its singularity are not considered in the analysis. Thus, the question about the nature of the singularity remains. It is expected that the singularity is described by a Planck scale object, however intricate the description might be. A canonical formalism for micro black holes, say of the order of ten Planck radii, seems valid, after all Hawking radiation, a tamed radiation at most of the scales, if left by itself, slowly peels the singularity away. If that radiation interacts harmoniously with the boundary of a cavity, a thermodynamic procedure might be valid and show how the black hole horizon and the singularity fuse into one single describable object.

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Appendix A: The Euclidean action for the canonical ensemble, boundary conditions, Ricci scalar, Euler characteristic, and the action with boundary conditions

In this appendix, we derive the conditions that were set in Sec. II to find the reduced action from the general Euclidean action. Some of the equations appearing in that section are repeated here for the sake of completeness and self-containment.

The system consisting of an electrically charged black hole inside a cavity in d dimensions has an Euclidean action

$$I = -\frac{1}{16\pi}\int_M R\sqrt{g}d^dx - \frac{1}{8\pi}\int_{\partial M}(K-K_0)\sqrt{\gamma}d^{d-1}x + \frac{(d-3)}{4\Omega_{d-2}}\int_M F_{ab}F^{ab}\sqrt{g}d^dx + \frac{(d-3)}{\Omega_{d-2}}\int_{\partial M}F^{ab}A_a n_b\sqrt{\gamma}d^{d-1}x\,. \tag{A1}$$

R is the Ricci scalar of the space, g is the determinant of the metric g_{ab}, the extrinsic curvature of the boundary of the cavity is K_{ab}, K is its trace, K_0 denominates the trace of the extrinsic curvature of the boundary of the cavity embedded in flat Euclidean space, γ is the determinant of the induced metric $\gamma_{\alpha\beta}$ on the boundary of the cavity, Ω_{d-2} is the surface area of a $d-2$ unit sphere and appears for practical purposes, $F_{ab} = \partial_a A_b - \partial_b A_a$ is the Maxwell tensor, A_a is the electromagnetic vector potential, and n_b is the outward unit normal vector to the boundary of the cavity. The indices a, b label the spacetime indices running from 0 to $d-1$, and α, β are indices on the boundary running from 0 to $d-2$. To prescribe the canonical ensemble, one has to set a boundary term related to the Maxwell tensor [14]. This term fixes the electric flux given by the integral of the Maxwell tensor on a $(d-2)$-surface, i.e., it fixes the electric charge. If instead, the potential vector is fixed, one is in the presence of the grand canonical ensemble, see [30] for this case. Note that Eq. (A1) corresponds to Eq. (2) in the main text. It is useful to rewrite the Maxwell boundary term in the action Eq. (A1). Using the divergence theorem and that $\nabla_b(F^{ab}A_a) = -\frac{1}{2}F_{ab}F^{ab} + \nabla_b F^{ab} A_a$, one transforms the boundary Maxwell term into a bulk term, obtaining the action

$$\begin{aligned} I = &-\frac{1}{16\pi}\int_M R\sqrt{g}d^d x - \frac{1}{8\pi}\int_{\partial M}(K-K_0)\sqrt{\gamma}d^{d-1}x \\ &-\frac{(d-3)}{4\Omega_{d-2}}\int_M F_{ab}F^{ab}\sqrt{g}d^d x \\ &+\frac{(d-3)}{\Omega_{d-2}}\int_M A_a\nabla_b F^{ab}\sqrt{g}d^d x\,. \end{aligned} \tag{A2}$$

Now, we develop the line element. We want to treat spherically symmetric Euclidean spaces, so that the Euclidean path integral is to be performed along metrics which have spherical symmetry. The space is then given by the warped product $\mathbb{R}^2 \times \mathbb{S}^{d-2}$ with $\mathbb{R}^2$ being the Euclidean two-space, $\mathbb{S}^{d-2}$ being a $(d-2)$-sphere with radius r, and r^2 being the warping function. The line element ds^2 of such a space is given by

$$ds^2 = b^2(y)d\tau^2 + \alpha^2(y)dy^2 + r^2(y)d\Omega^2_{d-2}\,, \tag{A3}$$

where τ is the periodic Euclidean time with range $0 \le \tau < 2\pi$, and is in fact an angular coordinate, y is a spatial radial coordinate with range $0 \le y \le 1$, $b(y)$ and $\alpha(y)$ are functions of y, the radius of the $(d-2)$-sphere is $r(y)$, and $d\Omega^2_{d-2}$ is the line element of the unit $(d-2)$-sphere with total area $\Omega_{d-2} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}$, where Γ is the gamma function. Since $0 \le y \le 1$, it is clear that the boundaries to this space given by the line element of Eq. (A3) are at $y=0$ and $y=1$. The functions $b(y)$, $\alpha(y)$, and $r(y)$ are to be integrated in the path integral.

Given the line element Eq. (A3), we can develop the action of Eq. (A2) with the considered terms involved. The Ricci tensor R_{ab} and its contraction Ricci scalar R which depend on second derivatives of the metric g_{ab}, form the Einstein tensor G_{ab}, with $G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$. Then, the Ricci scalar for the metric in Eq. (A3) is given by

$$-\frac{1}{16\pi}R = \frac{1}{8\pi\alpha b r^{d-2}}\left(\frac{r^{d-2}b'}{\alpha}\right)' + \frac{1}{8\pi}G^{\tau}{}_{\tau}\,, \tag{A4}$$

where $G^{\tau}{}_{\tau}$ is the time-time component of the Einstein tensor and is given by

$$G^{\tau}{}_{\tau} = \frac{(d-2)}{2r' r^{d-2}}\left[r^{d-3}\left(\frac{r'^2}{\alpha^2}-1\right)\right]'\,. \tag{A5}$$

The Gibbons-Hawking-York boundary term is given by

$$\begin{aligned} -\frac{1}{8\pi}(K-K_0)_{y=1} = &\left(\frac{(d-2)}{8\pi r}\left(1-\frac{r'}{\alpha}\right)\right)_{y=1} \\ &-\left(\frac{1}{8\pi b r^{d-2}}\left(\frac{r^{d-2}b'}{\alpha}\right)\right)_{y=1}\,, \end{aligned} \tag{A6}$$

where it was used that the extrinsic curvature of a constant y hypersurface is $\mathbf{K} = \frac{bb'}{\alpha}d\tau + \frac{rr'}{\alpha}d\Omega^2_{d-2}$ and that $\mathbf{K}_0 = r d\Omega^2_{d-2}$ is the extrinsic curvature of the hypersurface embedded in flat space. With respect to the bulk terms depending on the Maxwell field, one has

$$-\frac{(d-3)}{4\Omega_{d-2}}F_{ab}F^{ab} = -\frac{(d-3)}{2\Omega_{d-2}}\frac{A'_\tau{}^2}{\alpha^2 b^2}\,, \tag{A7}$$

where $F_{y\tau} = A'_\tau$ was used, and

$$\frac{(d-3)}{\Omega_{d-2}}\nabla_b F^{ab}A_a = -\frac{(d-3)}{\Omega_{d-2}\alpha b r^{d-2}}\left(\frac{r^{d-2}A'_\tau}{b\alpha}\right)' A_\tau\,, \tag{A8}$$

where $\nabla_a F^{\tau a} = -\frac{1}{\alpha b r^{d-2}}\left(\frac{r^{d-2}A'_\tau}{\alpha b}\right)'$ was used.

We now study the boundary conditions. We study first the boundary conditions for the geometry at $y=0$ and at $y=1$, and afterward the boundary conditions for the Maxwell field at $y=0$ and at $y=1$.

The boundary conditions for the geometry at $y=0$ have several important features. We also comment on the connection of these to the Euler characteristic. We assume that the hypersurface $y=0$ corresponds to the bifurcate two-surface of the event horizon of the electrically charged black hole, so we must impose the conditions

$$b(0) = 0\,, \tag{A9}$$

$$r(0) = r_+\,, \tag{A10}$$

where r_+ is the horizon radius. The conditions given in Eqs. (A9) and (A10) impose that the $y=0$ hypersurface corresponds to $\{y=0\}\times\mathbb{S}^{d-2}$, i.e., a point times a $(d-2)$-sphere. The $y=0$ point in the (τ, y) sector coincides with the central point of the $\mathbb{R}^2$ plane in polar coordinates, since τ is an angular coordinate and y is a radial coordinate. The $y=0$ hypersurface can be seen as the limit $y\to 0$ of $y=$ constant hypersurfaces, with these latter having an $\mathbb{S}^1\times\mathbb{S}^{d-2}$ topology. For the metric to be smooth as y goes to zero, these $y=$ constant hypersurfaces $\mathbb{S}^1\times\mathbb{S}^{d-2}$ must go smoothly to $\{y=0\}\times\mathbb{S}^{d-2}$. Analytically, one can expand the line element given in Eq. (A3) around $y=0$ with the boundary conditions set in Eqs. (A9)-(A10). This yields

$$ds^2 = \left[\left(\frac{b'}{\alpha}\right)^2_{y=0}\varepsilon^2 + \left(\frac{b'}{\alpha^2}\left(\frac{b'}{\alpha}\right)'\right)_{y=0}\varepsilon^3 + \mathcal{O}(\varepsilon^4)\right]d\tau^2 + d\varepsilon^2 + \Big[r_+ + (r'\alpha)_{y=0}\,\varepsilon + \mathcal{O}(\varepsilon^2)\Big]^2 d\Omega^2_{d-2}\,, \qquad \text{(A11)}$$

where b' is defined as $b' = \frac{db}{dy}$, r' is defined as $r' = \frac{dr}{dy}$, $\left(\frac{b'}{\alpha}\right)'$ is defined as $\left(\frac{b'}{\alpha}\right)' = \frac{d}{dy}\left(\frac{1}{\alpha}\frac{db}{dy}\right)$, $\left(\frac{b'}{\alpha}\right)_{y=0}$ means $\left(\frac{b'}{\alpha}\right)$ evaluated at $y=0$, ε is defined as $\varepsilon = \int_0^\delta \alpha dy$ for small δ and small ε, assuming that the integral is well-defined, as it should be if the metric is smooth. The term $\left(\frac{b'}{\alpha}\right)_{y=0}$ may be absorbed into a redefinition of τ with the caveat that the period of τ becomes $2\pi\left(\frac{b'}{\alpha}\right)_{y=0}$. This means there is a deficit angle and so a conical singularity. Therefore, for smoothness of the metric, we impose a third condition

$$\left(\frac{b'}{\alpha}\right)(0) = 1\,, \qquad \text{(A12)}$$

where $\left(\frac{b'}{\alpha}\right)(0) \equiv \left(\frac{b'}{\alpha}\right)_{y=0}$. With Eq. (A12) considered, one can compute the Ricci scalar of the metric in Eq. (A11) and obtain the problematic terms at $y=0$. One finds

$$R = -\frac{2(d-2)}{\varepsilon r_+}\left(\frac{r'}{\alpha}\right)_{y=0} - \frac{2}{\varepsilon}\left(\frac{1}{\alpha}\left(\frac{b'}{\alpha}\right)'\right)_{y=0} + \mathcal{O}(1)\,, \qquad \text{(A13)}$$

where we have used Eq. (A12). For the curvature invariant R to be well-defined and so for the space to be smooth, one must impose a fourth and a fifth condition, namely,

$$\left(\frac{r'}{\alpha}\right)(0) = 0\,, \qquad \text{(A14)}$$

$$\left(\frac{1}{\alpha}\left(\frac{b'}{\alpha}\right)'\right)(0) = 0\,, \qquad \text{(A15)}$$

with $\left(\frac{r'}{\alpha}\right)(0) \equiv \left(\frac{r'}{\alpha}\right)_{y=0}$ and $\left(\frac{1}{\alpha}\left(\frac{b'}{\alpha}\right)'\right)(0) \equiv \left(\frac{1}{\alpha}\left(\frac{b'}{\alpha}\right)'\right)_{y=0}$. In even dimensions, the condition given in Eq. (A14) is equivalent to requiring that the Euclidean space considered has an Euler characteristic $\chi = 2$ by the Chern-Gauss-Bonnet formula. For odd dimensions, the Euler characteristic vanishes and so this requirement is not satisfactory. Nevertheless, the requirement that the Ricci scalar is well-defined suffices. One can also see that this condition is equivalent to requiring that the event horizon of the black hole is a null hypersurface, if one performs a Wick transformation. The condition Eq. (A15) means that b does not have a dependence in ε^3, but, for some coordinate y, it may indicate that if $\left(\frac{b'}{\alpha}\right)'_{y=0}$ is nonzero and finite, then $\alpha|_{y=0}$ must diverge. Indeed, this is satisfied by the Reissner-Nordström line element with coordinate choice $y = r$ found by solving Einstein's equations, as it is the case in this setting. We note that condition Eq. (A15) is not referred in [11, 14, 30]. The boundary conditions for the geometry at $y = 1$ are now given. Here, we impose the condition

$$b(1) = \frac{\beta}{2\pi}\,, \qquad \text{(A16)}$$

$$r(1) = R\,, \qquad \text{(A17)}$$

where, β is the inverse temperature of the cavity. The condition Eq. (A16) is usually written as $\beta = 2\pi b(1)$. This condition, Eq. (A16), comes from the definition of the path integral as stated in Sec. II A, and it imposes that the total Euclidean proper time of the boundary of the cavity is fixed and it is equal to the inverse temperature of the cavity. The condition given in Eq. (A17) states that the hypersurface $y = 1$ corresponds to the boundary of the cavity with radius $r(1) = R$.

The boundary conditions for the Maxwell field are now given. Due to spherical symmetry and admitting the nonexistence of magnetic monopoles, the only nonvanishing components of the Maxwell tensor F_{ab} are $F_{y\tau} = -F_{\tau y}$. Moreover, we choose a gauge where the only nonvanishing component of the vector potential is $A_\tau(y)$. Therefore, the Maxwell tensor F_{ab} is only described by $F_{y\tau} = A'_\tau$. Therefore, the boundary condition for the Maxwell field at $y = 0$ is given by the requirement that

$$A_\tau(0) = 0\,. \qquad \text{(A18)}$$

At $y = 1$, we fix the electric charge by fixing the electric flux given by $\int_{\substack{y=1\\ \tau=c}} F^{ab} dS_{ab} = 2i\Omega_{d-2}Q$, where c is a constant, Q is the charge of the black hole and dS_{ab} is the surface element of the $y = 1$ and $\tau = 0$ surface, i.e.,

$$\int_{\substack{y=1\\ \tau=c}} F^{\tau y} dS_{\tau y} = i\Omega_{d-2}Q\,. \qquad \text{(A19)}$$

Putting together all these conditions with the action Eq. (A1), or Eq. (A2), the partition function is given only in terms of the radius of the cavity R, the inverse temperature β and the charge Q, which are fixed quantities of the system. Given the boundary conditions just found one can use them in Eqs. (A4)-(A8) to find the final form of the action Eq. (A1), or Eq. (A2). We observe that the first term integrated over y of Eq. (A4) yields $\frac{1}{8\pi}\left(\frac{r^{d-2}b'}{\alpha}\right)_{y=1} - \frac{1}{8\pi}\left(\frac{r^{d-2}b'}{\alpha}\right)_{y=0}$, i.e., a boundary term at $y = 0$ and a boundary term at $y = 1$. The boundary term at $y = 1$ cancels with the last term in Eq. (A6), therefore the only surviving boundary term of the Ricci scalar is $-\frac{1}{8\pi}\left(\frac{r^{d-2}b'}{\alpha}\right)_{y=0}$. Moreover, by using the boundary condition Eq. (A12), this term becomes $-\frac{1}{8\pi}\left(\frac{r^{d-2}b'}{\alpha}\right)_{y=0} = -\frac{1}{8\pi}r_+^{d-2}$. One can proceed with the integrations at the cavity, since the integrands do not depend on time or on the angles, and one obtains the full action

$$I[\beta,Q,R;b,\alpha,r,A_\tau] = \frac{\beta R^{d-3}}{\mu}\left(1-\left(\frac{r'}{\alpha}\right)(1)\right) - \frac{\Omega_{d-2}}{4}r_+^{d-2} - \frac{(d-3)}{\Omega_{d-2}}\int_M \left(\frac{r^{d-2}A_\tau'}{b\alpha}\right)' A_\tau d\tau dy d\Omega_{d-2} + \int_M \frac{\alpha b r^{d-2}}{8\pi}\left(G^\tau{}_\tau - \frac{4\pi(d-3)}{\Omega_{d-2}}\frac{{A_\tau'}^2}{\alpha^2 b^2}\right) d\tau dy d\Omega_{d-2}\,, \tag{A20}$$

where it was used that the time length at the cavity is given by $\beta = 2\pi b(1)$ and that $r(1) = R$, see Eqs. (A16) and (A17). We have then the action as a functional of b, α, r and A_τ to be integrated in all paths, in the path integral. This is the action displayed in Eq. (14). When one integrates the action in b, α, r and A_τ in the path integral one indeed obtains a partition function that depends on the radius of the cavity R, the inverse temperature β and the charge Q, the fixed quantities in the ensemble.

Appendix B: Calculation of the radii where the free energies of the electrically charged black hole are zero: Results for different ensembles and the generalized Buchdahl radius in d dimensions

1. The electric uncharged case: Canonical ensemble radius and the generalized Buchdahl radius in d dimensions

We want to analyze a thermodynamic energy or mass to radius ratio for the d-dimensional canonical ensemble, namely, the energy or mass for which the black hole free energy is zero, $F = 0$. We want to compare this mass to the Buchdahl bound mass in d dimensions.

In the canonical ensemble of an uncharged spherically symmetric black hole in d dimensions [26], which is described by the Euclidean Schwarzschild-Tangherlini black hole space, the canonical ensemble is realized with a fixed temperature at the boundary of the cavity. There are two black hole solutions, where the one with the largest mass is stable and the one with the least mass is unstable. Here we are interested in the large stable black hole. The free energy of the ensemble also has a critical point at zero horizon radius, which is a minimum, the hot flat space case. Therefore, one can analyze which are the favorable states in comparing the free energies of the zero horizon radius, i.e., hot flat space, and the stable black hole solution. The free energy of hot flat space is zero. The black hole solution also has zero free energy for a given horizon radius, which is thus an important thermodynamic radius. The larger the temperature of the ensemble, the larger this radius, and the lower the corresponding free energy. Thus, one can argue that a stable black hole is favored to hot flat space when the free energy of the black hole is lower than the zero, which is the free energy of hot flat space. The radius of the black hole horizon that yields zero free energy, i.e., $F = 0$, is $\left(\frac{r_+}{R}\right)_{F=0} = \left(\frac{4(d-2)}{(d-1)^2}\right)^{\frac{1}{d-3}}$. In terms of the spacetime mass m this is

$$\left(\frac{\mu m}{R^{d-3}}\right)_{F=0} = \left(\frac{2(d-2)}{(d-1)^2}\right)\,. \tag{B1}$$

The Buchdahl bound radius marks the maximum compactness of a spherically symmetric star before spacetime turns singular. The Buchdahl bound for a star or matter configuration of gravitational radius r_+ and radius R is [42] $\left(\frac{r_+}{R}\right)_{\text{Buch}} = \left(\frac{4(d-2)}{(d-1)^2}\right)^{\frac{1}{d-3}}$, which in terms of the spacetime mass m and radius R is

$$\left(\frac{\mu m}{R^{d-3}}\right)_{\text{Buch}} = \left(\frac{2(d-2)}{(d-1)^2}\right)\,. \tag{B2}$$

It is a structural bound coming from mechanics. Self-gravitating matter for which the mass, or the energy, content within a radius R is above the bound, in principle collapses to a black hole.

We see that both masses, or radii, although conceptually different, have the same expression, indeed, $\left(\frac{\mu m}{R^{d-3}}\right)_{F=0} = \left(\frac{\mu m}{R^{d-3}}\right)_{\text{Buch}}$. Therefore, one can argue that as soon as the black hole phase is thermodynamically favorable over the hot flat space, it is actually the only phase that exists, the energy within the reservoir collapses to form a black hole. This could indicate that there is a link between black hole thermodynamics and matter mechanics.

2. The electric charged case: Canonical and grand canonical ensembles radii and the generalized Buchdahl radius in d dimensions

We now want to analyze a thermodynamic energy or mass to radius ratio for two ensembles, one is the d-dimensional canonical ensemble with electric charge that is being treated here, and the other is the grand canonical ensemble that we treated before, for which the black hole free energies are zero, i.e., $F = 0$, and $W = 0$, respectively. We want to compare these two energy or mass to radius ratio to the generalized Buchdahl bound, i.e., the Buchdahl bound in the electric charged case in d dimensions, also called the Buchdahl-Andréasson-Wright bound, see [42].

In the canonical ensemble of a charged black hole inside a cavity in d dimensions, the construction has been described throughout the paper. The canonical ensemble in this case is realized with a fixed temperature and fixed electric charge at the boundary of the cavity. One has in this case two stable black hole solutions for a charge below a saddle, or critical, charge Q_s, and one stable black hole solution for a charge larger than Q_s. In this case, it can be shown that the stable solution with the largest mass for every charge can have a negative free energy, if the black hole has a larger mass than the one that solves this equation $a\left(\frac{\mu m}{R^{d-3}}\right)^4+b\left(\frac{\mu m}{R^{d-3}}\right)^3+c\left(\frac{\mu m}{R^{d-3}}\right)^2+d\left(\frac{\mu m}{R^{d-3}}\right)+e=0$, where $a=\left(\left(\frac{d-3}{d-2}\right)^2-4\right)^2$, $b=-4\left(4+8y-\left(\frac{d-3}{d-2}\right)^2(3+2y)\right)$, $c=-2\left(\frac{d-3}{d-2}\right)^4 y-2\left(\frac{d-3}{d-2}\right)^2(y^2-y+2)+4+24(y+6y^2)$, $d=-4y\left((1+y)(1+2y)+\left(\frac{d-3}{d-2}\right)^2(3+2y)\right)$, $e=\left(\frac{d-3}{d-2}\right)^4 y^2+y^2(1+y)^2+2\left(\frac{d-3}{d-2}\right)^2 y(1+y)(2+y)$, with y being the electric charge parameter given by $y\equiv\frac{\mu Q^2}{R^{2d-6}}$, as before. We see that the equation is a quartic equation in $\frac{\mu m}{R^{d-3}}$. The solution can be written formally as

$$\left(\frac{\mu m}{R^{d-3}}\right)_{F=0}=g\left(d,\frac{\mu Q^2}{R^{2d-6}}\right), \tag{B3}$$

for some calculable function $g\left(d,\frac{\mu Q^2}{R^{2d-6}}\right)$. In the case $Q=0$ one gets $\left(\frac{\mu m}{R^{d-3}}\right)_{F=0}=\left(\frac{2(d-2)}{(d-1)^2}\right)^{\frac{1}{d-3}}$ as required, see Eq. (B1). The largest stable black hole with this mass has a zero Helmholtz free energy, $F=0$. Contrasting to the canonical ensemble of the electrically uncharged black hole discussed above, the free energy in the electrically charged case does not include the zero horizon radius case. The minimum possible horizon radius is the extremal black hole point $r_{+e}=(\mu Q^2)^{\frac{1}{2d-6}}$, yielding a free energy $F_{r_{+e}}=\frac{Q}{\sqrt{\mu}}$. To emulate hot flat space, we used an electrically charged nonself-gravitating shell. We have then compared the black hole configuration with the electrically charged shell with no self-gravity at the boundary of the cavity, having then hot flat space inside the cavity with the electric charge near the boundary. This configuration would require one to look into the matter sector which we have not done here. It is unclear if a transition can occur between hot flat space with electric charges near the cavity and the stable black holes. Nevertheless, we still regard the thermodynamic radius of zero free energy in the canonical ensemble as an important quantity.

In the grand canonical ensemble of a charged Reissner-Nordström black hole inside a cavity for d dimensions, the construction and its thermodynamics were described in [30]. The grand canonical ensemble is realized with a fixed temperature and fixed electric potential at the boundary of the cavity. In this ensemble, the partition function in the zero loop approximation is given in terms of the grand potential, or Gibbs free energy, $W=E-TS-Q\phi$, where E is the mean energy, T is the temperature, S is the entropy, Q is the mean charge and ϕ is the electric potential. The grand potential yields $W[r_+,Q]=\frac{R^{d-3}}{\mu}\left(1-\sqrt{f}\right)-Q\phi-T\frac{\Omega_{d-2}r_+^{d-2}}{4}$, with $f=\left(1-\frac{r_+^{d-3}}{R^{d-3}}\right)\left(1-\frac{\mu Q^2}{(r_+R)^{d-3}}\right)$, and the equilibrium equations that yield the black hole solutions are $\frac{1}{T}=\frac{4\pi}{(d-3)}\frac{r_+^{d-2}}{r_+^{2d-6}-\mu Q^2}\sqrt{f}$ and $\phi=\frac{Q}{\sqrt{f}}\left(\frac{1}{R^{d-3}}-\frac{1}{r_+^{d-3}}\right)$, where the convention for the electromagnetic coupling and electric charge was chosen so that $Q\to\sqrt{(d-3)\Omega_{d-2}}Q$ and $\phi\to(\sqrt{(d-3)\Omega_{d-2}})^{-1}\phi$ in the expressions in [30]. One has in this case up to two solutions, depending on the fixed quantities T and ϕ, with only one being stable. The grand canonical free energy of the ensemble also has a critical point at zero horizon radius, which is a minimum, it is the hot flat space case. The stable black hole solution also has zero free energy for a given horizon radius, which is thus an important thermodynamic radius. The larger the temperature of the ensemble, the larger this radius, and the lower the corresponding free energy. Thus, one can argue that a stable black hole is favored to hot flat space when the free energy of the black hole is lower than the zero, which is the free energy of hot flat space. The radius of the black hole horizon that yields zero grand potential energy, i.e., $W=0$ is complicated to find, but the corresponding mass has a simple expression given by

$$\left(\frac{\mu m}{R^{d-3}}\right)_{W=0}=\frac{-4(d-2)^2}{(d-1)^2(d-3)^2}+\frac{2(d-2)((d-2)^2+1)}{(d-1)^2(d-3)^2}\times\sqrt{1+\frac{(d-1)^2(d-3)^2}{4(d-2)^2}\frac{\mu Q^2}{R^{2d-6}}}\,. \tag{B4}$$

Since hot flat space is described here by the grand potential $W[r_+,Q]$, a possible transition can occur from the charged hot flat space to the stable black hole for temperatures corresponding to stable black holes with higher mass than Eq. (B4). In the case $Q=0$, one has that $W=F$, so one gets $\left(\frac{\mu m}{R^{d-3}}\right)_{W=0}=\left(\frac{\mu m}{R^{d-3}}\right)_{F=0}=\left(\frac{2(d-2)}{(d-1)^2}\right)^{\frac{1}{d-3}}$ as required, see Eq. (B1).

The Buchdahl bound was originally given for the electrically uncharged case and in $d=4$. For electrically charged matter in d dimensions one has the generalized Buchdahl bound that is given by [42]

$$\left(\frac{\mu m}{R^{d-3}}\right)_{\rm Buch}=\frac{d-2}{(d-1)^2}+\frac{1}{d-1}\frac{\mu Q^2}{R^{2d-6}}+\frac{d-2}{(d-1)^2}\sqrt{1+(d-1)(d-3)\frac{\mu Q^2}{R^{2d-6}}}\,. \tag{B5}$$

In the no charge case, $Q=0$, one gets $\frac{\mu m}{R^{d-3}}=\left(\frac{2(d-2)}{(d-1)^2}\right)^{\frac{1}{d-3}}$, as required.

We see that the three mass to radius ratios, are conceptually different, and now in the electrically charged case, have generically different expressions, indeed, $\left(\frac{\mu m}{R^{d-3}}\right)_{F=0}$, $\left(\frac{\mu m}{R^{d-3}}\right)_{W=0}$, and $\left(\frac{\mu m}{R^{d-3}}\right)_{\rm Buch}$ are not equal.

One has $\left(\frac{\mu m}{R^{d-3}}\right)_{F=0} \geq \left(\frac{\mu m}{R^{d-3}}\right)_{\rm Buch} \geq \left(\frac{\mu m}{R^{d-3}}\right)_{W=0}$. This is an interesting result. In the canonical ensemble, the thermodynamic energy content within the cavity when the black hole phase starts to be favorable, i.e., when $F = 0$, is higher than the Buchdahl bound, and so even before the black hole is thermodynamically favored, collapse should occur, i.e., as soon as a black hole forms there is no possibility of a thermodynamic phase transition to hot flat space, indeed the black hole has been formed dynamically. In the grand canonical ensemble, the energy content within the cavity when the black hole phase starts to be favorable, i.e., when $W = 0$, is less than the Buchdahl bound, and so there should be no collapse at this stage, indeed, collapse should only occur when the energy content is increased above the bound. In the grand canonical ensemble this occurs only for some negative W. Both thermodynamic mass to radius ratios are equal to the generalized Buchdahl bound bound when the electric charge is put to zero, and all the three are also equal at the extremal point. The plots given in Fig. 7 for $d = 5$ help in the understanding of this behavior. These results present a counter example to the possible link between the black hole thermodynamics and stability of spherically symmetric matter. The uncharged case seems to be a coincidence.

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