Reaction-diffusion approximation of nonlocal interactions in high-dimensional Euclidean space

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Abstract

In various phenomena such as pattern formation, neural firing in the brain and cell migration, interactions that can affect distant objects globally in space can be observed. These interactions are referred to as nonlocal interactions and are often modeled using spatial convolution with an appropriate integral kernel. Many evolution equations incorporating nonlocal interactions have been proposed. In such equations, the behavior of the system and the patterns it generates can be controlled by modifying the shape of the integral kernel. However, the presence of nonlocality poses challenges for mathematical analysis. To address these difficulties, we develop an approximation method that converts nonlocal effects into spatially local dynamics using reaction-diffusion systems. In this paper, we present an approximation method for nonlocal interactions in evolution equations based on a linear sum of solutions to a reaction-diffusion system in high-dimensional Euclidean space up to three dimensions. The key aspect of this approach is identifying a class of integral kernels that can be approximated by a linear combination of specific Green functions in the case of high-dimensional spaces. This enables us to demonstrate that any nonlocal interactions can be approximated by a linear sum of auxiliary diffusive substances. Our results establish a connection between a broad class of nonlocal interactions and diffusive chemical reactions in dynamical systems.

Keywords: Nonlocal interaction; Reaction-diffusion system; Approximation; Nonlocal evolution equation
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1. Introduction

Various interactions plays a crucial role in phenomena such as developmental processes of multicellular organisms, behavior of biological populations, cell migration and neuronal information processing. The evolutionary dynamics of each factor involved in these phenomena depend on their interactions. These interactions give rise to spatiotemporal patterns and complex behaviors in each system. In particular, some interactions affect the distant objects globally in space as observed in the aforementioned phenomena. Such interactions are often referred as to nonlocal interactions or long-range interactions. The presence of the nonlocal interactions has been suggested by biological experiments in various contexts, including neural firing in the brain, pigment cells in skin of zebrafish, cell migration and cell adhesion.

Kuffler experimentally demonstrated that the light response of a ganglion cell in the receptive field of the cat brain exhibits local excitation and lateral inhibition [20]. This experimental result can be represented as a function, as shown in Figure 1, where local excitation and lateral inhibition correspond to positive and negative values, respectively. This function is referred to as the Mexican-hat function due to its shape, or the LALI function (standing for the local activation and lateral inhibition) in context of pattern formation. Nakamasu *et*

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al. experimentally demonstrated that the presence of local inhibition and lateral activation among the pigment cells in the skin of the zebrafish through laser ablation control experiments [25]. Yamanaka and Kondo reported that cell to cell contact via the cellular projections between two types of pigment cells in zebrafish generates the local inhibition, leading to a run-and-chase behavior [33]. Hamada *et al.* further showed that pigment cells in skin of zebrafish extend cellular projections longer than the previously mentioned ones to transmit survival signals across the pigment stripe patterns [15]. These studies on pigment cell interactions and skin patterns formation in zebrafish are comprehensively reviewed by Watanabe and Kondo [32]. Murakawa and Togashi experimentally demonstrated that variations in the strength of cell adhesion molecules in artificially cultured cells lead to changes in adhesion surface patterns [24]. It is also suggested that these cultured cells may sense the surrounding cell density by extending cellular projections longer than their body size.

The nonlocal interaction is often modeled by the spatial convolution with an appropriate integral kernel representing distance-dependent weight and a variable representing the factor or density of individual organism. Let $u = u(t,x)$ be the unknown variable at position x at time $t \geq 0$, and we assume that $K \in L^1(\mathbb{R}^n)$ is a radial integral kernel. In this paper, we mainly treat the nonlocal interactions described by the following form:

$$(K * u)(t,x) := \int_{\mathbb{R}^n} K(x-y)u(t,y)dy. \tag{1.1}$$

Various mathematical models have been proposed and analyzed from the aforementioned biological backgrounds.

Amari proposed a neural field equation of nonlocal type with the convolution of the Mexican-hat kernel and the Heaviside function, and derived the condition for the existence of traveling wave solutions [2]. Coombes *et al.* derived the equation of motion for the interface dynamics of the planar neural field, and demonstrated the numerical simulations and linear stability analysis [8]. The following model has been proposed to describe the behavior including nonlocal dispersion when an individual organism makes a long-range jump:

$$u_t = d\Delta u + K * u + f(u), \tag{1.2}$$

where $d \geq 0$ is the diffusion coefficient and f is the reaction term. The nonlocal model for the plant dispersal has been proposed by Alfaro *et al.* [1]. The existence and wave speed of the traveling wave solutions to this type of model have been analyzed by [3, 9, 10] and [16]. In particular, Ei and Ishii have proposed equations of motion for the interactions between pulse solutions or traveling wave solution to this type of models with the sign changed kernels [11]. Additionally, a continuation method have been proposed that can convert the spatially discretized models into the form of (1.2) while conserving the discreteness information by Ei *et al.* [13]. From this method it has been theoretically shown that the intercellular interactions such as diffusion, lateral inhibition on uniform and nonuniform lattice, and signal transduction through cellular projections can be described in the form of (1.1).

Mathematical models of nonlocal saturation and nonlocal growth rate have been proposed by Berestycki *et*

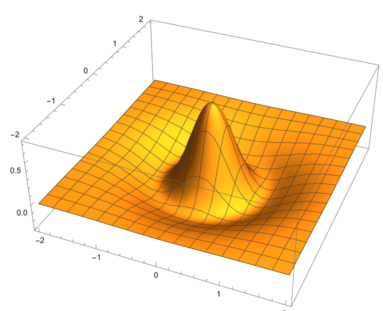

Figure 1: Profile of a Mexican-hat function. $K(x) = \sqrt{3}\exp(-3(x^2+y^2)) - \exp(-(x^2+y^2))$.

al.[5] and Ninomiya *et al*.[26], respectively:

$$u_t = d\Delta u + (1 - K * u)u, \tag{1.3}$$

$$u_t = d\Delta u + (K * u)u + f(u). \tag{1.4}$$

The stationary solution and traveling wave solution of (1.3) have been analyzed in [5]. In [26], it has been rigorously shown that the instability for the stationary constant solution induced by a nonlocal interaction can be regarded as the diffusion-driven instability proposed by Turing[30] in the nonlocal evolution equations including (1.4) through the reaction-diffusion approximation. Kondo proposed a nonlocal evolution equation that combines the nonlocal interaction and the cut function motivated by pattern formations [18]. This model is referred to as the KT model. In particular, this paper reported that the KT model can replicate various patterns such as those induced by the diffusion-driven instability simply by changing the shape of the integral kernel even though the equation has only one variable. Ei *et al*. proposed a methodology to extract the pattern-generating information from an arbitrary dimensional network with spatial interactions by representing it as the shape of an integral kernel in a convolution [12]. According to this approach, it is possible to visually interpret the pattern-forming information within a network of interacting factors as an integral kernel. By applying this reduction technique to the network of pigment cells in zebrafish summarized in [25], it was shown that a LALI-type integral kernel can be theoretically derived. Furthermore, this study was the first to demonstrate that a LALI-type integral kernel can also be theoretically derived from a reaction-diffusion system of an activator and an inhibitor with greater diffusion coefficient. Additionally, it was shown that by incorporating the extracted integral kernel into a nonlocal evolution equation and performing numerical calculations, it is possible to reproduce patterns similar to those generated by the original, unreduced model.

Murakawa and Togashi[24], and Carrillo *et al*.[6] proposed the following mathematical model for the cell adhesion and cell sorting phenomena:

$$u_t = \Delta u^m - \nabla \cdot (u(1-u)\nabla(K * u)), \tag{1.5}$$

where $m \geq 1$ is a constant. It has been reported that the above model can replicate the cell adhesion phenomena qualitatively and almost quantitatively by Carrillo *et al.* [6].

As discussed above, nonlocal interactions can directly model interactions between cells, the potentials between particles, and the intrinsic interactions in density or concentration fields that derive pattern formation by appropriately selecting the shape of the integral kernel. These nonlocal evolution equations can reproduce the various patterns depending on the shape of the integral kernel. However, nonlocal evolution equations present analytical and computational challenges due to their inherent nonlocality. These difficulties include the inapplicability of standard analytical methods designed for spatially local operators directly and the high computational cost of numerical simulations. One approach to overcoming these challenges is to approximate nonlocal interactions using local differential equations. Ninomiya *et al*. demonstrated that in one-dimensional spaces, the nonlocal evolution equations with arbitrary even kernel can be approximated by a reaction-diffusion system with multiple auxiliary factors [26, 27]. Murakawa and Tanaka have shown that in one-dimensional bounded domain, the nonlocal Fokker-Planck equation with an advective nonlocal interaction involving an arbitrary even kernel can be approximated by a Keller-Segel system with multiple chemotactic factors [23]. These theories reveal that various types of nonlocal interactions can be effectively described using multi-component systems introducing diffusive auxiliary factors. Furthermore, in [23] and [26], it has been rigorously established that for nonlocal evolution equations, including (1.4) and the nonlocal Fokker-Planck equation, the instability for the stationary constant solution induced by the shape of the integral kernel closely resembles to the diffusion-driven instability originally proposed by Turing. Despite these theoretical insights, nonlocal interactions are frequently observed in high-dimensional spatial settings in real-world phenomena. Therefore, investigating the relationship between nonlocal interactions and local partial differential equations in high spatial dimensions is a significant problem. Motivated by this, in this study, we examine the connection between nonlocal interactions and reaction-diffusion systems in high spatial dimensions using the singular limit analysis. Specifically, we show that the solutions to nonlocal evolution equations with any radial kernel can be approximated by that to a reaction-diffusion system in dimensions up to three. Moreover, depending on the given integral kernel, we demonstrate that the parameters of the approximating reaction-diffusion system can be explicitly determined, and we provide explicit estimates for the approximation error in terms of both the limit parameter and the number of the auxiliary factors. The core idea of this result is that any radial integral kernel in $L^1(\mathbb{R}^n)$ space can be approximated by an appropriately weighted superposition of the Green functions corresponding to the stationary problem of a reaction-diffusion equation. This Green function is provided by the modified Bessel function that is the fundamental solution to the modified Helmholtz equation. This expansion result can be proved using the Bernstein polynomial or the Lagrange polynomial with the Chebyshev nodes.

This paper is organized as follows. In Section 2, we explain the mathematical settings and state our main results. In Section 3, we prove the singular limit of the reaction diffusion system. In Section 4, we give the proof for the result of the Green function expansion in any dimensions up to three. Thereafter, we perform the numerical simulations of the Green function expansions in Section 5. We give some remarks and conclude this paper in Section 6.

2. Mathematical setting and main results

Based on the motivation introduced in Section 1, we consider the following nonlocal reaction-diffusion equation:

$$\text{(NP)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = D\Delta u + f(u, K*u), & (t>0,\ x\in\mathbb{R}^n), \\ u(0,x) = u_0(x), & (x\in\mathbb{R}^n), \end{cases}$$

where $u = u(t,x) \in \mathbb{R}$ $(t>0,\ x\in\mathbb{R}^n)$, D is a positive constant, the function $f \in \mathrm{Lip}(\mathbb{R}^2;\mathbb{R})$ satisfies $f(0,0)=0$, and $K(x)$ is a radial function in $L^1(\mathbb{R}^n)$. Here, there exists a constant $C_f > 0$ such that

$$|f(u_1,v_1) - f(u_2,v_2)| \le C_f(|u_1-u_2| + |v_1-v_2|)$$

for all $(u_1,v_1),(u_2,v_2) \in \mathbb{R}^2$. For the case that f is the local Lipschitz as introduced in models in Section 1, we provide Remark 2.3 below.

We first consider the existence of the solution to (NP). Denote the set of all bounded continuous functions by $BC(\mathbb{R}^n)$. Introducing the heat kernel as

$$H(t,x;D) := \frac{1}{(4\pi Dt)^{n/2}} e^{-|x|^2/4Dt},$$

we define the following map as

$$\mathscr{P}[\phi](t,x) := H(t;D)*u_0 + \int_0^t H(t-s;D)*f(\phi(s),(K*\phi)(s))ds$$

for $\phi \in C([0,T];BC(\mathbb{R}^n)\cap L^p(\mathbb{R}^n))$ with $1 \le p \le +\infty$. We say that a function $u \in C([0,T];BC(\mathbb{R}^n)\cap L^p(\mathbb{R}^n))$ for $T>0$ is a mild solution to (NP), provided $u = \mathscr{P}[u]$. Then we obtain the following existence result.

Theorem 2.1. *For any $T>0$ and $1\le p\le +\infty$, there exists a unique mild solution $u \in C([0,T];BC(\mathbb{R}^n)\cap L^p(\mathbb{R}^n))$ to* (NP) *with an initial datum $u_0 \in BC(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)$. This mild solution u belongs to $C^{1,2}((0,T]\times\mathbb{R}^n)$, that is, u is the unique classical solution to* (NP). *Moreover, we have*

$$\|u(t)\|_{L^q} \le e^{C_f(1+\|K\|_{L^1})t}\|u_0\|_{L^q}$$

for any $p \le q \le +\infty$ and $t \in [0,T]$.

We construct the mild solution based on the fixed point theorem for integral equations. The proof is a standard argument and is summarized in Appendix A.

Next, we prepare a reaction–diffusion system used for the approximation of the solution to (NP) with any integral kernel. Introducing the auxiliary diffusive substances $v_j = v_j(x,t)$, $(j = 1,\ldots,N)$, $(N \in \mathbb{N})$, we consider the following reaction-diffusion system with $N+1$ components:

$$(\mathrm{RD}_\delta) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = D\Delta u + f\left(u,\ \displaystyle\sum_{j=1}^{N} \alpha_j v_j\right), \\ \dfrac{\partial v_j}{\partial t} = \dfrac{1}{\delta}(d_j \Delta v_j - v_j + u), \quad (j = 1,2,\ldots,N), \end{cases} \qquad (t > 0,\ x \in \mathbb{R}^n),$$

where $\delta > 0$, $\alpha_j \in \mathbb{R}$, $d_j > 0$ for $j = 1,2,\ldots N$. The idea using the reaction-diffusion equations for approximating nonlocal interactions is same as that in [26] and [27]. The initial condition is imposed as

$$(u, v_1, \ldots, v_N)(0,x) = (u_0, k_1 * u_0, \ldots, k_N * u_0)(x), \tag{2.6}$$

where $k_j, (j = 1,\ldots,N)$ are defined as follows:

$$k_j(x) := \left(\frac{1}{d_j}\right)^{n/2} G\left(\frac{|x|}{\sqrt{d_j}}\right), \tag{2.7}$$

$$G(|x|) := \frac{1}{(2\pi)^{n/2}} \left(\frac{1}{|x|}\right)^{n/2-1} M_{n/2-1}(|x|), \tag{2.8}$$

$$M_\nu(r) := \int_0^{+\infty} e^{-r\cosh s} \cosh(\nu s) ds.$$

Here $M_\nu(r)$ is the modified Bessel function of the second kind with the order ν. For $j \in \mathbb{N}$, k_j is represented as

$$k_j(x) = \begin{cases} \dfrac{1}{2\sqrt{d_j}} e^{-|x|/\sqrt{d_j}}, & (n = 1), \\ \dfrac{1}{2\pi d_j} M_0(|x|/\sqrt{d_j}), & (n = 2), \\ \dfrac{1}{4\pi d_j |x|} e^{-|x|/\sqrt{d_j}}, & (n = 3). \end{cases}$$

We note that k_j is the Green function of the differential operator $-d_j\Delta + 1$, that is, $w = k_j * u$ satisfies

$$d_j \Delta w - w + u = 0.$$

Moreover, it is known that $k_j \in L^1(\mathbb{R}^n)$. The property we use are mentioned in Subsection 3.1.

We now show the existence of the solution to (RD_δ) with an initial condition (2.6). Define the two following maps as

$$\Phi[\phi](x,t) := H(t;D) * u_0 + \int_0^t H(t-s;D) * f\left(\phi(s), \sum_{j=1}^{N} \alpha_j \Psi_j[\phi](s)\right) ds,$$

$$\Psi_j[\phi](x,t) := e^{-t/\delta} H(t; d_j/\delta) * (k_j * u_0) + \frac{1}{\delta}\int_0^t e^{-(t-s)/\delta} H(t-s; d_j/\delta) * \phi(s) ds, \quad (j = 1,2,\ldots,N)$$

for $\phi \in C([0,T]; BC(\mathbb{R}^n) \cap L^p(\mathbb{R}^n))$ with $1 \le p \le +\infty$, respectively. We say that a function $(u, v_1, \ldots,\ v_N) \in (C([0,T]; BC(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)))^{N+1}$ for $T > 0$ is a mild solution to (RD_δ), provided $u = \Phi[u]$ and $v_j = \Psi_j[u]$, respectively. By using the Banach fixed point theorem, we obtain the existence result for (RD_δ).

Proposition 2.1. *For any $T>0$, $\delta>0$ and $1\le p\le+\infty$, there exists a unique mild solution $(u^\delta, v_1^\delta,\dots,v_N^\delta)\in\{C([0,T];BC(\mathbb{R}^n)\cap L^p(\mathbb{R}^n))\}^{N+1}$ to* (RD$_\delta$) *with an initial condition* (2.6) *and $u_0\in BC(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)$. This solution belongs to $\{C^{1,2}((0,T]\times\mathbb{R}^n)\}^{N+1}$. Moreover, we have*

$$\begin{aligned}\|u^\delta(t)\|_{L^q} &\le \left(1+\delta C_f\sum_{j=1}^N|\alpha_j|\right)\|u_0\|_{L^q}\exp\left(C_f\left(1+\sum_{j=1}^N|\alpha_j|\right)t\right),\\ \|v_j^\delta(t)\|_{L^q} &\le e^{-t/\delta}\|u_0\|_{L^q}+\sup_{0\le s\le t}\|u^\delta(s)\|_{L^q},\quad (j=1,2,\dots,N)\end{aligned}$$

for any $q\in[p,+\infty]$ and $t\in[0,T]$.

The proof is similar to the proof of Theorem 2.1 and its main part about the boundedness is described in Appendix B.

For constants $\{\alpha_j\}_{1\le j\le N}$, and positive constants $\{d_j\}_{1\le j\le N}$, we set the linear combination of the Green functions as

$$K_N(x):=\sum_{j=1}^N\alpha_jk_j(x). \tag{2.9}$$

For the approximation of nonlocal interactions by the reaction-diffusion system we consider the singular limit as $\delta\to+0$ in (RD$_\delta$) as follows.

Lemma 2.1. *Let $T>0$, $\delta>0$, $1\le p\le+\infty$ and $u_0\in BC(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)$. Suppose that $(u^\delta,v_1^\delta,\dots,v_N^\delta)\in\{C([0,T];BC(\mathbb{R}^n)\cap L^p(\mathbb{R}^n))\}^{N+1}$ is the solution to* (RD$_\delta$) *with an initial condition* (2.6)*. Then, there exists $C_1=C_1(f,D,\{\alpha_j\}_{1\le j\le N},\{d_j\}_{1\le j\le N},T)$ such that*

$$\begin{aligned}\sup_{0\le t\le T}\|u^\delta(t)-u^0(t)\|_{L^p} &\le C_1\|u_0\|_{L^p}\delta,\\ \sup_{0\le t\le T}\|v_j^\delta(t)-k_j*u^0(t)\|_{L^p} &\le C_1\|u_0\|_{L^p}\delta\end{aligned}$$

hold, where u^0 is the solution to (NP) *with $K=K_N$.*

This lemma shows the relationship between the solutions to (RD$_\delta$) and (NP) with K_N. To approximate the solution to (NP) with any integral kernel by that to (RD$_\delta$), we prepare the following lemma.

Lemma 2.2. *For $1\le p\le+\infty$, let u be the solution to* (NP) *with an initial datum $u_0\in BC(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)$. Then, for any $T>0$, there exists $C_2=C_2(f,K,T)>0$ such that*

$$\sup_{0\le t\le T}\|u(t)-u^0(t)\|_{L^p}\le C_2e^{C_f\|K-K_N\|_{L^1}T}\|K-K_N\|_{L^1}\|u_0\|_{L^p}$$

holds, where $u^0\in C([0,T];BC(\mathbb{R}^n)\cap L^1(\mathbb{R}^n))$ is the solution to (NP) *with $K=K_N$.*

This implies that the difference of the solutions to (NP) with K and K_N can be bounded by the difference of K and K_N in $L^1(\mathbb{R}^n)$ space.

Here, we introduce the following assumptions for K.

Assumption 2.1. *Let $K\in L^1(\mathbb{R}^n)$ be a radial function. Let us denote $K(x)=J(|x|)$.*

- *When $n=1$, we assume that $J\in C([0,+\infty))$ and that the limit $\lim_{r\to+\infty}e^rJ(r)$ exists. Then, we set*

  $$h(\lambda):=\frac{J(-\log\lambda)}{\lambda},$$

  where we define $h(0):=\lim_{r\to+\infty}e^rJ(r)$;

- *When $n=2$, we assume that $J \in C^1([0,+\infty))$, and that there exists $\alpha > 1/2$ such that the limit*

$$\lim_{r\to+\infty} r^{\alpha} e^r J'(r)$$

exists. Then, introducing the following function

$$A(r) := -\frac{2r}{\pi}\int_0^{+\infty} J'(r\cosh s)ds,$$

we set

$$h(\lambda) := \frac{A(-\log\lambda)}{\lambda}.$$

Here $h(0) := \lim_{r\to+\infty} e^r A(r) = 0$*;*

- *When $n=3$, we assume that $J \in C((0,+\infty))$ and that both limits $\lim_{r\to+0} rJ(r)$ and $\lim_{r\to+\infty} re^r J(r)$ exist. Then, we set*

$$h(\lambda) := -(\log\lambda)\frac{J(-\log\lambda)}{\lambda},$$

where we define $h(0) := \lim_{r\to+\infty} re^r J(r)$ *and* $h(1) := \lim_{r\to+0} rJ(r)$.

The following error estimate of an expansion by the Green function k_j is one of our main results.

Theorem 2.2. *Let $n \in \{1,2,3\}$ and $d_j = j^{-2}$ for $j \in \mathbb{N}$. Let Assumption 2.1 be enforced. Then, for any $N \in \mathbb{N}$ and constants $\{\alpha_j\}_{1\le j\le N+1}$,*

$$\|K - K_{N+1}\|_{L^1} \le \frac{2\pi^{n/2}}{\Gamma(n/2)} \max_{\lambda\in[0,1]} |h(\lambda) - P_N(\lambda)|$$

holds, where $P_N(\lambda)$ is the polynomial defined as

$$P_N(\lambda) := \sum_{j=0}^{N} \alpha_{j+1} c_{j,n} \lambda^j, \quad c_{j,n} := \begin{cases} \dfrac{j+1}{2}, & (n=1), \\ \dfrac{(j+1)^2}{2\pi}, & (n=2), \\ \dfrac{(j+1)^2}{4\pi}, & (n=3). \end{cases}$$

This theorem shows that the approximation error between K and K_{N+1} is evaluated in terms of the absolute error between the function h determined from the integral kernel and the polynomial P_N. This allows us to examine how to determine $\{\alpha_j\}_{1\le j\le N+1}$ based on polynomial approximation theory. Since the properties of the Green function depend on spatial dimensions, we need Assumption 2.1 for the approximation by K_{N+1}. In the cases that $n=1$ or $n=3$, k_j satisfies this assumption for all $0 < d_j \le 1$. On the other hand, when $n=2$, k_j does not satisfy the assumption for any $d_j > 0$. In this sense, the case that $n=2$ is a technical assumption. The proof is given in Section 4.

We provide examples of how to choose the coefficients $\{\alpha_j\}_{1\le j\le N+1}$. Before stating the result, we prepare some notions. Let $h \in C([0,1])$. We define the modulus of continuity of h on $[0,1]$ as

$$\omega(h,\eta) := \sup\{|h(\lambda_1) - h(\lambda_2)|\ ;\ \lambda_1,\lambda_2 \in [0,1],\ |\lambda_1-\lambda_2| \le \eta\}.$$

For $N \in \mathbb{N}$, let us define constants $\beta_{j,N}[h]$ $(0 \le j \le N)$ as

$$\beta_{j,N}[h] := \sum_{\nu=0}^{j} (-1)^{j-\nu} h\left(\frac{\nu}{N}\right)\binom{N}{j}\binom{j}{\nu}, \quad (j=0,1,\dots,N), \tag{2.10}$$

where $\binom{\cdot}{\cdot}$ is the binomial coefficient. The constant $\beta_{j,N}[h]$ plays the role to determine the coefficients of the Bernstein polynomial below. Furthermore, we set the following constant

$$l_{j,N}[h] := \frac{1}{2^{2N+1}} \sum_{\nu=0}^{N} \zeta_{\nu,N}(h) \tau_{j,\nu}^{(N+1)}, \quad (j = 0, 1, \dots, N),$$

where $\zeta_{\nu,N}(h)$ and $\tau_{j,\nu}^{(N+1)}$ are defined in Subsection 4.3. The constant $l_{j,N}[h]$ plays the role to determine the coefficients of the Lagrange polynomial with the Chebyshev nodes below. Using these coefficients of the polynomials, we obtain the following explicit estimates.

Corollary 2.1. *Let $n \in \{1,2,3\}$ and $d_j = j^{-2}$ for $j \in \mathbb{N}$. Let Assumption 2.1 be enforced.*

- *When $\alpha_j = \beta_{j-1,N}[h]/c_{j-1,n}$, for $m \in \{0,1\}$, there exists a constant $E(m) > 0$ independent of K such that if $h \in C^m([0,1])$ holds, then*

$$\|K - K_{N+1}\|_{L^1} \leq \frac{2E(m)\pi^{n/2}}{\Gamma(n/2)} N^{-m/2} \omega(h^{(m)}, N^{-1/2}).$$

- *When $\alpha_j = l_{j-1,N}[h]/c_{j-1,n}$, for $h \in \mathrm{Lip}([0,1])$, it holds that*

$$\|K - K_{N+1}\|_{L^1} \leq \frac{2\pi^{n/2}}{\Gamma(n/2)} \Big(2 + \frac{2}{\pi} \log N\Big) \omega(h, N^{-1}).$$

Remark 2.1. *As in Lemma 4.7, for $h \in C^m[0,1]$ with $m \in \mathbb{N}$ in the case of $\alpha_j = l_{j-1,N}[h]/c_{j-1,n}$, the convergence order becomes $O(N^{-m})$.*

We have described our results on the approximation of integral kernels for $n \in \{1,2,3\}$. While this result is sufficient to deal with actual phenomena, the case that $n \geq 4$ remains as a mathematical problem. The case is left as a future work. The proof of Corollary 2.1 is in Section 4.

Under the above mentioned settings, we obtain the main approximation result to the nonlocal problems (NP) by using the reaction-diffusion system (RD_δ) as follows.

Theorem 2.3. *Let $n \in \{1,2,3\}$ and $d_j = j^{-2}$ for $j \in \mathbb{N}$. For any integral kernel K satisfying Assumption 2.1, $\varepsilon > 0$, $\delta > 0$ and $T > 0$, there exist $N = N(K,n,\varepsilon) \in \mathbb{N}$, a reaction diffusion system (RD_δ) with $N+1$ components, and positive constants*

$$C_1 = C_1(f, D, \{\alpha_j\}_{1\leq j\leq N}, \{d_j\}_{1\leq j\leq N}, T), \quad C_3 = C_3(f, K, T),$$

such that

$$\begin{aligned} \sup_{0\leq t\leq T} \|u^\delta(t) - u(t)\|_{L^p} &\leq (C_1\delta + C_3\varepsilon)\|u_0\|_{L^p}, \\ \sup_{0\leq t\leq T} \|v_j^\delta(t) - (k_j * u)(t)\|_{L^p} &< (C_1\delta + C_3\varepsilon)\|u_0\|_{L^p}, \quad (j = 1, 2, \dots, N), \end{aligned}$$

where u is the solution to (NP) with $u_0 \in BC(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, and $(u^\delta, v_1^\delta, \dots, v_N^\delta)$ is the solution to (RD_δ) with (2.6).

Remark 2.2. *We can take the limit of $\delta \to +0$ in (RD_δ) since the constants C_1 and C_3 are independent of δ. Furthermore, after this limit, we can take the limit of $\varepsilon \to 0$ in (RD_0) since the constant C_3 is independent of N. If Lemma 4.3 in Subsection 4.2 or Lemma 4.6 in Subsection 4.3 is applied, Theorem 2.3 also shows the convergence rate with respect to δ and N.*

Remark 2.3. *The global Lipschitz condition on $f(u,v)$ can be removed if the boundedness of the solutions can be guaranteed. When $f(u,v)$ is a locally Lipschitz continuous function on $\mathbb{R}^2$ and $u_0 \in BC(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for $1 \le p \le +\infty$, the same assertion as Theorem 2.3 follows by choosing a sufficiently large $C_f > 0$ if*

$$\sup_{0 \le t \le T} (\|u(t)\|_{L^\infty} + \|u^0(t)\|_{L^\infty} + \|u^\delta(t)\|_{L^\infty}) < +\infty$$

is obtained a priori for some $T > 0$ and $\delta > 0$. Here, u^0 is a solution to (NP) *with $K = K_N$.*

Remark 2.4. *When $p = +\infty$ and $u_0 \in BC(\mathbb{R}^n)$ is a periodic function, the solution is also a spatially periodic function with the same period. Therefore, the approximation in Theorem 2.3 can be used to evaluate solutions with periodic boundary conditions.*

The necessary lemmas for proof of Theorem 2.3 is given in Section 3 and the proof of Theorem 2.3 is given in Section 5.

3. Error estimates

3.1. Properties of the Green function

Here we describe some properties of the Green function that we use. Some properties are also described in [17], but we summarize those necessary properties for the ease of the reader.

The Fourier transform of $G(|x|)$ defined by (2.8) is represented by

$$\begin{aligned}
\hat{G}(\xi) = \mathscr{F}_n[G](\xi) &:= \int_{\mathbb{R}^n} e^{-ix\cdot\xi} G(|x|)dx \\
&= (2\pi)^{n/2} \left(\frac{1}{|\xi|}\right)^{n/2-1} \int_0^{+\infty} r^{n/2} J_{n/2-1}(|\xi|r)G(r)dr \\
&= \left(\frac{1}{|\xi|}\right)^{n/2-1} \int_0^{+\infty} rJ_{n/2-1}(|\xi|r)M_{n/2-1}(r)\,dr
\end{aligned}$$

for $|\xi| > 0$, where $J_\nu(r)$ is the Bessel function of the first kind with the order ν. Using formula

$$\int_0^{+\infty} rJ_{n/2-1}(|\xi|r)M_{n/2-1}(r)\,dr = |\xi|^{n/2-1} {}_2F_1\left(\frac{n}{2}, 1; \frac{n}{2}; -|\xi|^2\right) = \frac{|\xi|^{n/2-1}}{1+|\xi|^2}$$

from §13.45 in [31], we obtain $\hat{G}(\xi) = \dfrac{1}{1+|\xi|^2}$. Here, ${}_2F_1(a,b;c;z)$ is the hypergeometric function.

It is obvious that $G(|x|)$ is positive for $|x| > 0$. Moreover, we find that

$$\begin{aligned}
\int_{\mathbb{R}^n} G(|x|)dx &= \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^{+\infty} r^{n-1}G(r)dr \\
&= \frac{1}{2^{n/2-1}\Gamma(n/2)} \int_0^{+\infty} r^{n/2} M_{n/2-1}(r)dr = 1
\end{aligned}$$

by using the integral formula

$$\int_0^{+\infty} r^{\mu-1} M_\nu(r)dr = 2^{\mu-2}\Gamma\left(\frac{\mu-\nu}{2}\right)\Gamma\left(\frac{\mu+\nu}{2}\right)$$

with $|\mathrm{Re}(\nu)| < \mathrm{Re}(\mu)$ from §13.21 in [31].

In summary, k_j defined in (2.7) has the following properties:

Lemma 3.1. *For $j \in \mathbb{N}$, we have* *(i)* $k_j \in C(\mathbb{R}^n \backslash \{0\}) \cap L^1(\mathbb{R}^n)$ *and*

$$\mathscr{F}_n[k_j](s) = \frac{1}{1+d_j|\xi|^2};$$

(ii) $k_j(x) > 0$ $(x \neq 0)$ *and* $\|k_j\|_{L^1} = 1$.

In Section 4, we use the asymptotic properties of M_ν. From [28], it is known the asymptotic properties

$$M_\nu(r) \simeq \sqrt{\frac{\pi}{2r}} e^{-r}, \quad (r \to +\infty) \tag{3.11}$$

and

$$M_\nu(r) \simeq \begin{cases} \dfrac{\Gamma(\nu)}{2}\left(\dfrac{r}{2}\right)^{-\nu}, & (\nu > 0), \\ -\log r, & (\nu = 0), \end{cases} \quad (r \to +0)$$

for any $\nu > 0$, where $f(r) \simeq g(r)$ $(r \to a)$ means

$$\lim_{r\to a} \frac{f(r)}{g(r)} = 1.$$

3.2. Error estimates for reaction-diffusion approximation

In this subsection, we consider the singular limit problem for solutions to (RD$_\delta$) with an initial condition (2.6).

Proof of Lemma 2.1. Let $w(t) = u^\delta(t) - u^0(t)$ and $z_j(t) = v_j^\delta(t) - (k_j * u^0)(t)$. Then, we have

$$\delta \frac{\partial z_j}{\partial t} = d_j \Delta z_j - z_j + w - \delta k_j * \frac{\partial u^0}{\partial t}.$$

This implies that

$$z_j(t) = \frac{1}{\delta}\int_0^t e^{-(t-s)/\delta} H(t-s; d_j/\delta) * \left[w(s) - \delta \left(k_j * \frac{\partial u^0}{\partial t} \right)(s) \right] ds$$

holds for any $j = 1, 2, \ldots, N$. Notice that

$$\begin{aligned} k_j * \frac{\partial u^0}{\partial t}(t) &= [k_j * (D\Delta u^0 + f(u^0, K_N * u^0))](t) \\ &= \frac{D}{d_j} u^0(t) + \frac{D}{d_j}(k_j * u^0)(t) + [k_j * f(u^0, K_N * u^0))](t), \end{aligned}$$

and then, we find that

$$\left\| \left(k_j * \frac{\partial u^0}{\partial t} \right)(t) \right\|_{L^p} \leq \left[\frac{2D}{d_j} + C_f(1 + \|K_N\|_{L^1}) \right] \|u^0(t)\|_{L^p} \tag{3.12}$$

for any $t \in (0, T]$ from the Young inequality and Lemma 3.1.

Let $t \in (0, T]$. Then, we obtain that

$$\|z_j(t)\|_{L^p} \leq \frac{1}{\delta}\int_0^t e^{-(t-s)/\delta} \left[\|w(s)\|_{L^p} + \delta \left\| \left(k_j * \frac{\partial u^0}{\partial t} \right)(s) \right\|_{L^p} \right] ds. \tag{3.13}$$

Moreover, we have

$$
\begin{aligned}
\|w(t)\|_{L^p} &\leq C_f \int_0^t \left[\|w(s)\|_{L^p} + \sum_{j=1}^N |\alpha_j| \|z_j(s)\|_{L^p} \right] ds \\
&\leq C_f \int_0^t \|w(s)\|_{L^p} ds \\
&\quad + \frac{C_f}{\delta} \sum_{j=1}^N |\alpha_j| \int_0^t \int_0^s e^{-(s-\eta)/\delta} \left[\|w(\eta)\|_{L^p} + \delta \left\| \left(k_j * \frac{\partial u^0}{\partial t} \right)(\eta) \right\|_{L^p} \right] d\eta ds \\
&\leq C_f \int_0^t \|w(s)\|_{L^p} ds \\
&\quad + C_f \sum_{j=1}^N |\alpha_j| \int_0^t (1 - e^{-(t-s)/\delta}) \left[\|w(s)\|_{L^p} + \delta \left\| \left(k_j * \frac{\partial u^0}{\partial t} \right)(s) \right\|_{L^p} \right] ds \\
&\leq C_f \left(1 + \sum_{j=1}^N |\alpha_j| \right) \int_0^t \|w(s)\|_{L^p} ds + \delta C_f \sum_{j=1}^N |\alpha_j| \int_0^t \left\| \left(k_j * \frac{\partial u^0}{\partial t} \right)(s) \right\|_{L^p} ds.
\end{aligned}
$$

By using the Gronwall inequality, we deduce

$$
\|w(t)\|_{L^p} \leq \delta C_f \left(\sum_{j=1}^N |\alpha_j| \int_0^t \left\| \left(k_j * \frac{\partial u^0}{\partial t} \right)(s) \right\|_{L^p} ds \right) \exp\left(C_f \left(1 + \sum_{j=1}^N |\alpha_j| \right) t \right).
$$

From (3.12) and Theorem 2.1, there exists

$$
C_{11} = C_{11}(f, D, \{\alpha_j\}_{1\leq j\leq N}, \{d_j\}_{1\leq j\leq N}, T) > 0
$$

such that

$$
\|w(t)\|_{L^p} \leq C_{11} \|u_0\|_{L^p} \delta
$$

holds. From the similar argument for w and (3.13), there exists a positive constant $C_{12} = C_{12}(f, D, \{\alpha_j\}_{1\leq j\leq N}, \{d_j\}_{1\leq j\leq N}, T)$ such that

$$
\|z_j(t)\|_{L^p} \leq C_{12} \|u_0\|_{L^p} \delta
$$

holds. Thus, we obtain the desired assertion. □

Next, we evaluate the continuity of the solution to (NP) with respect to the integral kernel in Lemma 2.2.

Proof of Lemma 2.2. Let $1 \leq p \leq +\infty$ and $t \in (0, T]$. Let $u_{\mathrm{err}}(t) := u(t) - u^0(t)$ and $K_{\mathrm{err}}(x) := K(x) - K_N(x)$. Then, we have

$$
\begin{aligned}
|u_{\mathrm{err}}(t)| &\leq C_f \int_0^t H(t-s; D) * (|u_{\mathrm{err}}(s)| + |(K * u)(s) - (K_N * u^0)(s)| ds \\
&\leq C_f \int_0^t H(t-s; D) * (|u_{\mathrm{err}}(s)| + |(K_N * u_{\mathrm{err}})(s)| + |(K_{\mathrm{err}} * u)(s)|) ds.
\end{aligned}
$$

From the Young inequality, we deduce

$$
\|u_{\mathrm{err}}(t)\|_{L^p} \leq C_f \int_0^t [(1 + \|K_N\|_{L^1}) \|u_{\mathrm{err}}(s)\|_{L^p} + \|K_{\mathrm{err}}\|_{L^1} \|u(s)\|_{L^p}] ds.
$$

Using the Gronwall inequality and Theorem 2.1 yields that

$$\begin{aligned}
\|u_{\mathrm{err}}(t)\|_{L^p} &\leq C_f \left(\int_0^t \|u(s)\|_{L^p} ds\right) e^{C_f(1+\|K_N\|_{L^1})t} \|K_{\mathrm{err}}\|_{L^1} \\
&\leq C_f \left(\int_0^t e^{C_f(1+\|K\|_{L^1})t} ds\right) e^{C_f(1+\|K_N\|_{L^1})t} \|K_{\mathrm{err}}\|_{L^1} \|u_0\|_{L^p} \\
&\leq \frac{1}{1+\|K\|_{L^1}} e^{C_f(2+\|K\|_{L^1}+\|K_N\|_{L^1})t} \|K_{\mathrm{err}}\|_{L^1} \|u_0\|_{L^p} \\
&\leq \frac{1}{1+\|K\|_{L^1}} e^{C_f(2+2\|K\|_{L^1}+\|K_{\mathrm{err}}\|_{L^1})t} \|K_{\mathrm{err}}\|_{L^1} \|u_0\|_{L^p}.
\end{aligned}$$

Thus, we obtain the desired assertion. □

4. Approximation of a kernel by the Green function in $L^1(\mathbb{R}^n)$

Throughout of this section, we set $d_j = j^{-2}$ for $j \in \mathbb{N}$. For simplicity, we use the notation K_N defined in (2.9).

4.1. Error estimate for integral kernels

For the result of $L^1(\mathbb{R})$ convergence stated in Theorem 2.2, we can obtain the one in $W^{1,1}(\mathbb{R})$. Although this result is not used for the reaction-diffusion approximation of nonlocal interactions, we provide the following result on derivative approximation for Theorem 2.2.

Theorem 4.1. *Let $n \in \{1,2,3\}$ and $d_j = j^{-2}$ for $j \in \mathbb{N}$. We assume the following conditions in each dimension:*

- *When $n = 1$, we assume that $J \in C^1([0,+\infty))$ and that the limits $\lim_{r\to+\infty} e^{2r}J(r)$ and $\lim_{r\to+\infty} e^{2r}J'(r)$ exist.*
- *When $n = 2$, we assume that $J \in C^2([0,+\infty))$ and $\lim_{r\to+0} J'(r) + rJ''(r) = 0$, that the limit $\lim_{r\to+0} r^{-1}(J'(r) + rJ''(r))$ exists, and that there exists $\alpha > 1/2$ such that the limits*

  $$\lim_{r\to+\infty} r^{\alpha} e^{2r} J'(r), \quad \lim_{r\to+\infty} r^{\alpha} e^{2r} J''(r)$$

  exist.
- *When $n = 3$, we assume that $J \in C^1((0,+\infty))$ and that the limits $\lim_{r\to+0} rJ(r)$, $\lim_{r\to+0}(J(r)+rJ'(r))$, $\lim_{r\to+\infty} re^{2r}J(r)$, and $\lim_{r\to+\infty} re^{2r}J'(r)$ exist.*

Let the same notation of h in Assumption 2.1 be enforced. Then, for any $N \in \mathbb{N}$ and constants $\{\alpha_j\}_{1\leq j\leq N+1}$,

$$\begin{aligned}
\|K - K_{N+1}\|_{W^{1,1}} \leq{}& \frac{2(n+1)\pi^{n/2}}{\Gamma(n/2)} \max_{\lambda\in[0,1]} |h(\lambda) - P_N(\lambda)| \\
&+ \frac{2n\pi^{n/2}}{2^n\Gamma(n/2)} \max_{\lambda\in[0,1]} |h'(\lambda) - P_N'(\lambda)|
\end{aligned}$$

holds, where P_N is the polynomial defined in Theorem 2.2.

We provide the proofs of Theorem 2.2 and Theorem 4.1 in the successive sub-subsections, respectively.

4.1.1. One-dimensional case

We first give the proof of Theorem 2.2 when $n = 1$. Note that in this case k_j is represented as

$$k_j(x) = \frac{j}{2} e^{-j|x|}.$$

Proof of Theorem 2.2. Let Assumption 2.1 for $n = 1$ be enforced. Then, we obtain

$$\begin{aligned}
\|K - K_{N+1}\|_{L^1} &= 2\int_0^{+\infty} \left| J(r) - \sum_{j=1}^{N+1} \frac{j\alpha_j}{2} e^{-jr} \right| dr \\
&\leq 2\left(\int_0^{+\infty} e^{-r} dr\right) \sup_{r\geq 0} \left| e^r J(r) - \sum_{j=0}^{N} \alpha_{j+1} c_{j,n} e^{-jr} \right| \\
&= 2 \max_{\lambda\in[0,1]} |h(\lambda) - P_N(\lambda)|
\end{aligned}$$

from the definition of $\{\alpha_j\}_{1\leq j\leq N+1}$. □

From this proof, we obtain a point-wise absolute error as follows.

Corollary 4.1. *Let $n = 1$ and $d_j = j^{-2}$ for $j \in \mathbb{N}$. Let Assumption 2.1 be enforced. Then, we have*

$$|K(x) - K_{N+1}(x)| \leq e^{-|x|} \max_{\lambda\in[0,1]} |h(\lambda) - P_N(\lambda)|$$

for any $x \in \mathbb{R}$.

Proof of Theorem 4.1. We note that

$$h'(\lambda) = -\frac{J'(-\log\lambda) + J(-\log\lambda)}{\lambda^2} = -e^{2r}(J'(r) + J(r)),\ (-\log\lambda = r)$$

and we define $h'(0) := \lim_{r\to+\infty} -e^{2r}(J(r) + J'(r))$. Then, we compute that

$$\begin{aligned}
&\left\| \frac{\partial}{\partial x}(K - K_{N+1}) \right\|_{L^1} = 2\int_0^{+\infty} \left| J'(r) + \sum_{j=1}^{N+1} \frac{j^2\alpha_j}{2} e^{-jr} \right| dr \\
&\leq 2\int_0^{+\infty} e^{-2r} \left| e^{2r}(J'(r) + J(r)) + \sum_{j=0}^{N} j\alpha_{j+1} c_{j,n} e^{-(j-1)r} \right| dr + \|K - K_{N+1}\|_{L^1} \\
&\leq 2\Big(\int_0^{+\infty} e^{-2r} dr\Big) \max_{\lambda\in[0,1]} \left| h'(\lambda) - P_N'(\lambda) \right| + \|K - K_{N+1}\|_{L^1} \\
&= \max_{\lambda\in[0,1]} \left| h'(\lambda) - P_N'(\lambda) \right| + 2 \max_{\lambda\in[0,1]} |h(\lambda) - P_N(\lambda)|,
\end{aligned}$$

Thus, the proof in one-dimensional case is complete. □

4.1.2. Two-dimensional case

We show the case that $n = 2$ in Theorem 2.2. Remark that k_j is expressed as

$$k_j(x) = \frac{j^2}{2\pi} M_0(j|x|) = \frac{j^2}{2\pi} \int_0^{+\infty} e^{-j|x|\cosh s} ds.$$

Let Assumption 2.1 for $n = 2$ be enforced. We prepare a lemma for $A(r)$. The integral transformation $A(r)$ of $J(r)$ is defined as inspired by the Abel transformation [4]. Based on the theory, it follows

Lemma 4.1. *A is well-defined on $[0,+\infty)$ and satisfies $A(0)=0$, $A\in C([0,+\infty))$ and*

$$\lim_{r\to+\infty} e^r A(r) = 0.$$

Moreover, J is represented by

$$J(r) = \int_0^{+\infty} A(r\cosh s)ds.$$

Proof. From Assumption 2.1, there exists a constant $C_J > 0$ such that

$$|J'(r)| \le C_J \min\{1, r^{-\alpha}\} e^{-r} \quad (r>0).$$

The continuity of A is obtained by the Lebesgue dominated theorem. For $r>0$, we obtain that

$$\begin{aligned} |A(r)| &\le \frac{2r}{\pi}\int_0^{+\infty} |J'(r\cosh s)|ds \\ &\le \frac{2C_J r}{\pi}\int_0^{+\infty} \min\{1,(r\cosh s)^{-\alpha}\}e^{-r\cosh s}ds \\ &\le \frac{2C_J}{\pi}\min\{r, r^{1-\alpha}\}M_0(r). \end{aligned}$$

This implies that

$$\lim_{r\to+\infty} e^r A(r) = 0$$

from (3.11). Taking a limit as $r\to+0$ yields $A(0)=0$. Finally, we have

$$\begin{aligned} \int_0^{+\infty} A(r\cosh s)ds &= \int_r^{+\infty} \frac{A(s)}{\sqrt{s^2-r^2}}ds \\ &= -\frac{2}{\pi}\int_r^{+\infty}\int_s^{+\infty} \frac{sJ'(\eta)}{\sqrt{s^2-r^2}\sqrt{\eta^2-s^2}}d\eta ds \\ &= -\frac{2}{\pi}\int_r^{+\infty}\int_r^{\eta} \frac{s}{\sqrt{s^2-r^2}\sqrt{\eta^2-s^2}}dsJ'(\eta)d\eta \\ &= -\int_r^{+\infty} J'(\eta)d\eta = J(r). \end{aligned}$$

Thus, we get the desired assertion. □

Proof of Theorem 2.2. From the definitions of h and $\{\alpha_j\}_{1\le j\le N+1}$, we have

$$\begin{aligned} \|K-K_{N+1}\|_{L^1} &= 2\pi\int_0^{+\infty} r\left|J(r) - \sum_{j=1}^{N+1}\frac{j^2\alpha_j}{2\pi}M_0(jr)\right|dr \\ &\le 2\pi\int_0^{+\infty}\int_0^{+\infty} r\left|A(r\cosh s) - \sum_{j=1}^{N+1}\alpha_j c_{j-1,n}e^{-jr\cosh s}\right|dsdr \\ &\le 2\pi\left(\int_0^{+\infty}\int_0^{+\infty} re^{-r\cosh s}dsdr\right)\sup_{r\ge 0}\left|e^rA(r) - \sum_{j=0}^{N}\alpha_{j+1}c_{j,n}e^{-jr}\right| \\ &= 2\pi\max_{\lambda\in[0,1]}|h(\lambda)-P_N(\lambda)|. \end{aligned}$$

□

Similarly, we get a point-wise absolute error.

Corollary 4.2. *Let $n=2$ and $d_j=j^{-2}$ for $j\in\mathbb{N}$. Let Assumption 2.1 be enforced. Then, we have*

$$|K(x)-K_{N+1}(x)|\le M_0(|x|)\max_{\lambda\in[0,1]}|h(\lambda)-P_N(\lambda)|$$

for all $x\in\mathbb{R}^2\backslash\{0\}$.

Proof of Theorem 4.1. First we show the following properties.

Lemma 4.2. *A' is well-defined on $[0,+\infty)$, $A\in C^1([0,+\infty))$, $A'(0)=0$ and $\lim_{r\to+\infty}e^{2r}A'(r)=0$. Moreover, J' is represented by*

$$J'(r)=\int_0^{+\infty}A'(r\cosh s)\cosh s ds.$$

Proof of Lemma 4.2. From the assumptions of Theorem 4.1, there exists a constant C_J such that

$$|J'(r)+rJ''(r)|\le C_J\min\{r,r^{-\alpha}\}e^{-2r}$$

for $r\in[0,\infty)$. Using this boundedness, we obtain that

$$\begin{aligned}|A'(r)|&\le\frac{2C_J}{\pi}\int_0^{\infty}(r\cosh s)e^{-2r\cosh s}ds\\&=\frac{2C_J}{\pi}\Big(\int_r^{\sqrt{r^2+1}}te^{-2t}\frac{dt}{\sqrt{t^2-r^2}}+\int_{\sqrt{r^2+1}}^{\infty}te^{-2t}\frac{dt}{\sqrt{t^2-r^2}}\Big)\\&\le\frac{2C_J}{\pi}\frac{5}{4}\end{aligned}$$

for $r\ge0$. For any $r_n\in[0,+\infty)$, we set $\mathscr{J}_n(s):=J'(r_n\cosh s)+r_n(\cosh s)J''(r_n\cosh s)$ for $s\in[0,+\infty)$. Then, for any $r_n\to r,(n\to+\infty)$, we have $\lim_{n\to+\infty}\mathscr{J}_n(s)=J'(r\cosh s)+r(\cosh s)J''(r\cosh s)$ for a fixed s from the continuity of J' and J''. Thus, using the dominated convergence theorem, we see that A' is continuous for $r\ge0$. Moreover, for any $r_n\to+0,(n\to\infty)$, we have

$$\lim_{n\to+\infty}A'(r_n)=\lim_{n\to+\infty}-\frac{2}{\pi}\int_0^{\infty}\mathscr{J}_n(s)ds=-\frac{2}{\pi}\int_0^{\infty}\lim_{n\to+\infty}\mathscr{J}_n(s)ds=0.$$

Similarly to the proof of Lemma 4.1, we can compute that

$$|A'(r)|\le\frac{2C_Jr}{\pi}\min\{1,r^{-1-\alpha}\}M_1(2r).$$

This yields that $\lim_{r\to+\infty}e^{2r}A'(r)=0$. Finally, we see that

$$\int_0^{+\infty}A'(r\cosh s)\cosh s ds=-\frac{2}{\pi r}\int_r^{\infty}\int_r^{\eta}\frac{s(J'(\eta)+\eta J''(\eta))}{\sqrt{s^2-r^2}\sqrt{\eta^2-s^2}}dsd\eta=J'(r).$$

□

From this lemma, we define $h'(0)=\lim_{r\to+\infty}-e^{2r}(A(r)+A'(r))$. Then, we see that

$$h'(\lambda)=-\frac{A'(-\log\lambda)+A(-\log\lambda)}{\lambda^2}=-e^{2r}(A'(r)+A(r)),\ (-\log\lambda=r).$$

Now, we can estimate that

$$\begin{aligned}
&\sum_{j=1}^{2}\left\|\frac{\partial}{\partial x_j}(K-K_{N+1})\right\|_{L^1}=8\int_0^{+\infty} r\left|J'(r)+\sum_{j=1}^{N+1}\frac{j^3\alpha_j}{2\pi}M_1(jr)\right|dr\\
&\leq 8\int_0^{+\infty}\int_0^{+\infty} r\cosh s e^{-2r\cosh s}\\
&\qquad\times\left|A'(r\cosh s)e^{2r\cosh s}+\sum_{j=1}^{N+1} j\alpha_j c_{j-1,n}e^{-(j-2)r\cosh s}\right|dsdr\\
&\leq 8\Big(\int_0^{+\infty}\int_0^{+\infty} r\cosh s e^{-2r\cosh s}dsdr\Big)\\
&\qquad\times\max_{R\geq 0}\left|e^{2R}(A'(R)+A(R))+\sum_{j=0}^{N} j\alpha_{j+1}c_{j,n}e^{-(j-1)R}\right|\\
&\quad+8\Big(\int_0^{+\infty}\int_0^{+\infty} r\cosh s e^{-r\cosh s}dsdr\Big)\max_{R\geq 0}\left|e^{R}A(R)-\sum_{j=0}^{N}\alpha_{j+1}c_{j,n}e^{-jR}\right|\\
&=\pi\max_{\lambda\in[0,1]}|h'(\lambda)-P_N'(\lambda)|+4\pi\max_{\lambda\in[0,1]}|h(\lambda)-P_N(\lambda)|,\quad (\lambda=e^{-R}).
\end{aligned}$$

Thus, we obtain the desired assertion in two-dimensional case. □

Here, we introduce some examples of A.

Example 4.1. *Let $K(x)=J(|x|)=(a+b|x|)e^{-c|x|}$ with $a,b\in\mathbb{R}$ and $c>1$. Then, K satisfies Assumption 2.1, and A is represented by*

$$\begin{aligned}
A(r) &= -\frac{2r}{\pi}\int_0^{+\infty}\{(b-ac)-bcr\cosh s\}e^{-cr\cosh s}ds\\
&= \frac{2r}{\pi}\{(ac-b)M_0(cr)+bcrM_1(cr)\}.
\end{aligned}$$

Example 4.2. *Let $K(x)=J(|x|)=e^{-a|x|^2}$ with $a>0$. It is easy to see that K satisfies Assumption 2.1. Then, A is computed as*

$$\begin{aligned}
A(r) &= \frac{4ar^2}{\pi}\int_0^{+\infty} e^{-ar^2\cosh^2 s}\cosh s ds\\
&= \frac{2ar^2}{\pi}\int_0^{+\infty}\frac{e^{-ar^2(s+1)}}{\sqrt{s}}ds\\
&= \frac{2ar^2}{\pi}e^{-ar^2}\sqrt{\frac{\pi}{ar^2}}=2\sqrt{\frac{a}{\pi}}re^{-ar^2}.
\end{aligned}$$

4.1.3. Three-dimensional case

Finally, we prove Theorem 2.2 for the case that $n=3$. It should be noted that in this case, k_j is represented as

$$k_j(x)=\frac{j^2}{4\pi|x|}e^{-j|x|}.$$

Proof of Theorem 2.2. Let Assumption 2.1 for $n = 3$ be enforced. Then, we obtain

$$\begin{aligned}
\|K - K_{N+1}\|_{L^1} &= 4\pi \int_0^{+\infty} r^2 \left| J(r) - \sum_{j=1}^{N+1} \frac{j^2 \alpha_j}{4\pi r} e^{-jr} \right| dr \\
&= 4\pi \int_0^{+\infty} r e^{-r} \left| r e^r J(r) - \sum_{j=0}^{N} \alpha_{j+1} c_{j,n} e^{-jr} \right| dr \\
&\leq 4\pi \max_{\lambda \in [0,1]} |h(\lambda) - P_N(\lambda)|
\end{aligned}$$

from the definition of $\{\alpha_j\}_{1 \leq j \leq N+1}$. The desired assertion in three-dimensional case is obtained. □

By a quite similar argument, we have a point-wise absolute error.

Corollary 4.3. *Let $n = 3$ and $d_j = j^{-2}$ for $j \in \mathbb{N}$. Let Assumption 2.1 be enforced. Then, we have*

$$|K(x) - K_{N+1}(x)| \leq \frac{e^{-|x|}}{|x|} \max_{\lambda \in [0,1]} |h(\lambda) - P_N(\lambda)|$$

for all $x \in \mathbb{R}^3 \backslash \{0\}$.

Proof of Theorem 4.1. We note that

$$\begin{aligned}
h'(\lambda) &= -\frac{J(-\log\lambda) - (\log\lambda) J'(-\log\lambda) - (\log\lambda) J(-\log\lambda)}{\lambda^2} \\
&= -e^{2r}(J(r) + rJ'(r) + rJ(r)), \ (-\log\lambda = r).
\end{aligned}$$

We can define $h'(0) = \lim_{r\to+\infty} -e^{2r}(J(r) + rJ'(r) + rJ(r))$ and $h'(1) = \lim_{r\to+0} -e^{2r}(J(r) + rJ'(r) + rJ(r))$. Using Corollary 4.3, we obtain that

$$\begin{aligned}
\|(K - K_{N+1})/|\cdot|\|_{L^1} &\leq 4\pi \left(\int_0^{+\infty} r e^{-r} dr \right) \max_{\lambda \in [0,1]} |h(\lambda) - P_N(\lambda)| \\
&= 4\pi \max_{\lambda \in [0,1]} |h(\lambda) - P_N(\lambda)|.
\end{aligned}$$

Additionally, using the notation $J_N(r) := K_N(|x|) = \sum_{j=1}^{N} a_j k_j(|x|), \ r = |x|$, we see that

$$J'_{N+1}(r) = -\sum_{j=0}^{N} j\alpha_{j+1} k_{j+1}(r) - J_{N+1}(r) - \frac{1}{r} J_{N+1}(r).$$

Then, we can estimate that

$$\begin{aligned}
&\sum_{j=1}^{3} \left\| \frac{\partial}{\partial x_j}(K - K_{N+1}) \right\|_{L^1} = 6\pi \int_0^{+\infty} r^2 \left| J'(r) - J'_{N+1}(r) \right| dr \\
&\leq 6\pi \int_0^{+\infty} r^2 \left| J(r) + J'(r) + \frac{1}{r} J(r) - J_{N+1}(r) - J'_{N+1}(r) - \frac{1}{r} J_{N+1}(r) \right| dr \\
&\quad + \frac{3}{2} \|K - K_{N+1}\|_{L^1} + \frac{3}{2} \|(K - K_{N+1})/|\cdot|\|_{L^1} \\
&\leq 6\pi \left(\int_0^{+\infty} r e^{-2r} dr \right) \max_{\lambda \in [0,1]} |h'(\lambda) - P'_N(\lambda)| + 12\pi \max_{\lambda \in [0,1]} |h(\lambda) - P_N(\lambda)| \\
&= \frac{3\pi}{2} \max_{\lambda \in [0,1]} |h'(\lambda) - P'_N(\lambda)| + 12\pi \max_{\lambda \in [0,1]} |h(\lambda) - P_N(\lambda)|.
\end{aligned}$$

Thus, we obtain the desired assertion in three-dimensional case. □

4.2. Bernstein Polynomial

We discuss the approximation of integral kernels using polynomial approximations. Here, we review a polynomial approximation by the Bernstein polynomial.

Let $h:[0,1]\to\mathbb{R}$ and $N\in\mathbb{N}$. We denote the Bernstein polynomial of degree N to the function h by

$$B_N[h](\lambda):=\sum_{\nu=0}^{N} h\left(\frac{\nu}{N}\right) b_{\nu,N}(\lambda),\ b_{\nu,N}(\lambda):=\binom{N}{\nu}\lambda^{\nu}(1-\lambda)^{N-\nu},\ (\nu=0,1,\ldots,N).$$

We now proceed to polynomial expansion of $B_N[h]$. Since we find

$$\begin{aligned} b_{\nu,N}(\lambda) &= \binom{N}{\nu}\sum_{j=0}^{N-\nu}(-1)^j\binom{N-\nu}{j}\lambda^{j+\nu}=\binom{N}{\nu}\sum_{j=\nu}^{N}(-1)^{j-\nu}\binom{N-\nu}{j-\nu}\lambda^j \\ &= \sum_{j=\nu}^{N}(-1)^{j-\nu}\binom{N}{j}\binom{j}{\nu}\lambda^j, \quad (\nu=0,1,\ldots,N), \end{aligned}$$

we get

$$\begin{aligned} B_N[h](\lambda) &= \sum_{\nu=0}^{N}\sum_{j=\nu}^{N}(-1)^{j-\nu}h\left(\frac{\nu}{N}\right)\binom{N}{j}\binom{j}{\nu}\lambda^j \\ &= \sum_{j=0}^{N}\left(\sum_{\nu=0}^{j}(-1)^{j-\nu}h\left(\frac{\nu}{N}\right)\binom{N}{j}\binom{j}{\nu}\right)\lambda^j=\sum_{j=0}^{N}\beta_{j,N}[h]\lambda^j, \end{aligned}$$

where $\{\beta_{j,N}[h]\}_{0\le j\le N}$ is defined by (2.10). The following theorem of polynomial approximation is known.

Lemma 4.3 ([21])**.** *For $m\in\{0,1\}$, there exists a constant $E(m)>0$ such that if $h\in C^m([0,1])$, then we have*

$$\max_{\lambda\in[0,1]}|h(\lambda)-B_N[h](\lambda)|\le E(m)N^{-m/2}\omega(h^{(m)},N^{-1/2}).$$

Remark 4.1. *It is known that we can choose $E(0)=5/4$ and $E(1)=3/4$ from [21]. Moreover, it is also known that the convergence order becomes faster with the smoothness of h.*

Convergence of derivatives has also been reported in [14] and [21]. Especially, the following result is known with respect to the order of convergence:

Lemma 4.4 ([14])**.** *If $h\in C^{m+2}([0,1])$ for some $m\ge 0$, then*

$$\begin{aligned} &\max_{\lambda\in[0,1]}\left|\frac{d^m}{d\lambda^m}\left[h(\lambda)-B_N[h](\lambda)\right]\right| \\ &\le\frac{1}{2N}\left(m(m+1)\max_{\lambda\in[0,1]}|h^{(m)}(\lambda)|+m\max_{\lambda\in[0,1]}|h^{(m+1)}(\lambda)|+\frac{1}{4}\max_{\lambda\in[0,1]}|h^{(m+2)}(\lambda)|\right). \end{aligned}$$

4.3. Lagrange Polynomial

In this subsection we introduce another candidate for the polynomial P_N by using the Lagrange interpolation polynomial with the Chebyshev nodes. We utilize the result of the coefficient determination of the Lagrange polynomial in the case on $[0,1]$ by [23]. We firstly prepare the notations. We set

$$C_{k,N}:=(-1)^k2^{N-2k-1}\frac{N}{N-k}\binom{N-k}{k},\ (k=0,1,\ldots,\left[\frac{N}{2}\right]),\ N\in\mathbb{N},$$

where $[\cdot]$ is the Gauss symbol. Using this notation to the Chebyshev polynomial, we obtain the expression $T_N(x)=\sum_{k=0}^{[N/2]}C_{k,N}x^{N-2k}$. Next, we prepare the following constants:

$$\mu_{k,j}^{(N)}:=(-1)^j2^{N-2k-j}\binom{N-2k}{j}C_{k,N},N\in\mathbb{N},$$

$$
\xi_{k,N} := \begin{cases} \displaystyle\sum_{\nu=0}^{[N/2]-[(k+1)/2]} \mu^{(N)}_{\nu,N-2\nu-k}, & \text{if } N \text{ is even}, \\ \displaystyle\sum_{\nu=0}^{[N/2]-[k/2]} \mu^{(N)}_{\nu,N-2\nu-k}, & \text{otherwise}. \end{cases}
$$

Utilizing these coefficients, with respect to the shifted Chebyshev polynomial we have $T_N(2x-1) = \sum_{k=0}^{N} \xi_{k,N} x^k$, $x \in [0,1]$. This proof is written in [23].

Let us denote the Chebyshev nodes by

$$
r_{j,N} := \frac{1}{2} + \frac{1}{2}\cos\frac{2j+1}{2N}\pi, \quad (j = 0,1,\ldots,N-1).
$$

For the function h defined in Assumption 2.1, setting

$$
\zeta_{j,N}[h] := \frac{h(r_{j,N+1})}{\prod_{k=0,k\neq j}^{N}(r_{j,N+1} - r_{k,N+1})}, \quad (j = 0,1,\ldots,N),\ N \in \mathbb{N},
$$

we define the Lagrange polynomial as

$$
L_N(\lambda) := \sum_{j=0}^{N} \zeta_{j,N} \prod_{k=0,k\neq j}^{N} (\lambda - r_{k,N+1}).
$$

Regarding the determination of the coefficients of the Lagrange polynomial given interpolation points using Chebyshev nodes, the following lemma holds.

Lemma 4.5 ([23])**.** *For $N \in \mathbb{N}$, we set the coefficients as*

$$
\tau^{(N)}_{j,\nu} := \sum_{k=j}^{N-1} (r_{\nu,N})^{k-j} \xi_{k+1,N}, \quad (j = 0,1,\ldots,N-1),\ (\nu = 0,1,\ldots,N-1),
$$

$$
l_{j,N}[h] := \frac{1}{2^{2N+1}} \sum_{\nu=0}^{N} \zeta_{\nu,N}[h] \tau^{(N+1)}_{j,\nu}, \quad (j = 0,1,\ldots,N).
$$

Then, it holds

$$
L_N(\lambda) = \sum_{j=0}^{N} l_{j,N}[h]\lambda^j, \quad \lambda \in [0,1].
$$

Using the properties of the Lagrange and Chebyshev polynomial as in Section 6.5 of [22] and Section 1.3 of [29], we obtain the following estimates.

Lemma 4.6. *For $h \in \mathrm{Lip}([0,1])$, it holds that*

$$
\max_{\lambda\in[0,1]} |h(\lambda) - L_N(\lambda)| \leq (1+\mu_N)\omega(h, N^{-1}),
$$

where $\mu_N = (2/\pi)\log N + 1$ that comes from the Lebesgue constant.

Furthermore, for the smooth functions, the following estimate is known.

Lemma 4.7 ([7], Section 6)**.** *For $h \in C^m([0,1])$ with $m \in \mathbb{N}$, it holds that*

$$
\max_{\lambda\in[0,1]} |h(\lambda) - L_N(\lambda)| \leq \frac{1}{2}\left(\frac{\pi}{2}\right)^m \|h^{(m)}\|_{C([0,1])} (N-m+2)^{-m}, \quad N \geq m.
$$

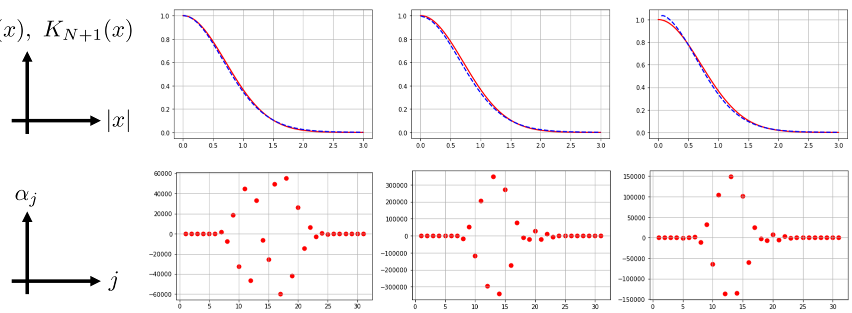

Figure 2: Graphs of K and K_{N+1} with $N = 30$ (top panels), the distributions of $\{\alpha_j\}_{1\leq j\leq N+1}$ (bottom panels) by using the Bernstein polynomial. The graph of K and K_{N+1} are shown by the red line and the blue dashed line, respectively. The numerical examples correspond to the results with $n = 1$ (left), $n = 2$ (middle), and $n = 3$ (right), respectively.

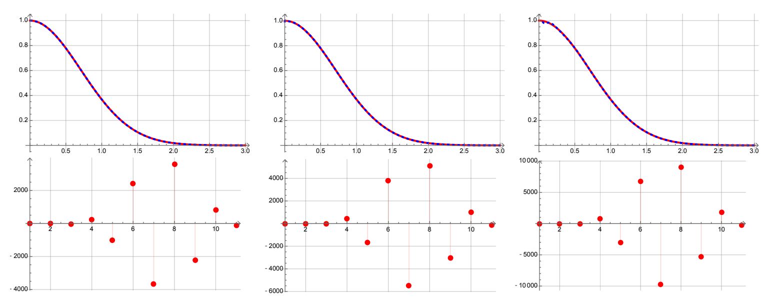

Figure 3: Graphs of K and K_{N+1} with $N = 10$ (top panels), the distributions of $\{\alpha_j\}_{1\leq j\leq N+1}$ (bottom panels) by using the Lagrange polynomial with the Chebyshev nodes. The graph of K and K_{N+1} are shown by the red line and the blue dashed line, respectively. The numerical examples correspond to the results with $n = 1$ (left), $n = 2$ (middle), and $n = 3$ (right), respectively.

4.4. Polynomial approximation and numerical examples

We first provide the proof of Corollary 2.1.

Proof of Corollary 2.1 Setting $\alpha_j = \beta_{j-1,N}[h]/c_{j-1,n}$ for $j = 1,2,\dots,N$ and using Theorem 2.2 and Lemma 4.3, we obtain the first assertion. Similarly, setting $\alpha_j = l_{j-1,N}[h]/c_{j-1,n}$ for $j = 1,2,\dots,N$ and using Theorem 2.2 and Lemma 4.6, we obtain the second assertion. □

Now, we present numerical examples of the approximation of a kernel. We treat the case that the kernel is given by

$$K(x) = e^{-|x|^2}.$$

Note that K satisfies Assumption 2.1 for all $n \in \{1,2,3\}$.

The graphs of K and K_{N+1} using the Bernstein polynomial are shown in Fig. 2. It can be visually confirmed that the graphs of K and K_{N+1} are similar in the case that $N = 30$. When $n = 3$, the graphs are a bit separated near the origin, which is expected due to the singularity of the Green function there. The values in the middle of $\{\alpha_j\}$ are large, on the order of 10^4 or more. Since $\{\alpha_j\}$ are defined by using (2.10), increasing j is thought to cause it by increasing the value of the binomial coefficient.

Figure 3 shows the numerical results by using the Lagrange polynomial with Chebyshev nodes. It is observed that the graphs of K and K_{N+1} are similar in the case that $N = 10$.

5. Reaction-diffusion approximation of nonlocal interactions

Now we explain the proof of the main results.

Proof of Theorem 2.3. Let the assumption of Theorem 2.3 be enforced. As in Corollary 2.1, for any $\varepsilon > 0$, there exist $N \in \mathbb{N}$ and $\{\alpha_j\}_{1\le j\le N}$ such that $\|K - K_N\|_{L^1} \le \varepsilon$. Then, from Lemma 2.2 there exists $C_3 = C_3(f, K, T) > 0$ such that

$$\|u(t) - u^0(t)\|_{L^p} \le C_3\|u_0\|_{L^p}\|K - K_N\|_{L^1} \le C_3\|u_0\|_{L^p}\varepsilon$$

holds for any $t \in (0, T]$. Moreover, there exists a positive constant $C_1 = C_1(f, D, \{\alpha_j\}_{1\le j\le N}, \{d_j\}_{1\le j\le N}, T)$ such that

$$\begin{aligned} \|u^\delta(t) - u^0(t)\|_{L^p} &\le C_1\|u_0\|_{L^p}\delta, \\ \|v_j^\delta(t) - (k_j * u^0)(t)\|_{L^p} &\le C_1\|u_0\|_{L^p}\delta \end{aligned}$$

hold for any $t \in (0, T]$ from Lemma 2.1. Thus, for any $t \in (0, T]$, we have

$$\begin{aligned} \|u(t) - u^\delta(t)\|_{L^p} &\le \|u^\delta(t) - u^0(t)\|_{L^p} + \|u(t) - u^0(t)\|_{L^p} \\ &\le (C_1\delta + C_3\varepsilon)\|u_0\|_{L^p} \end{aligned}$$

and

$$\begin{aligned} \|v^\varepsilon(t) - (k_j * u)(t)\|_{L^p} &\le \|(k_j * u^0_{app})(t) - (k_j * u)(t)\|_{L^p} + \|v^\varepsilon(t) - (k_j * u^0)(t)\|_{L^p} \\ &\le (C_1\delta + C_3\varepsilon)\|u_0\|_{L^p} \end{aligned}$$

from the Young inequality. Therefore, the proof is complete. □

6. Concluding remarks

In this paper, we have demonstrated that in any Euclidean space up to three dimensions, the solution of the nonlocal evolution equation with any radial integral kernel, subject to dimension-depend conditions, can be approximated by that of a reaction-diffusion system with auxiliary factors. We showed that the reaction-diffusion system coupled with auxiliary activators and inhibitors can approximate the time evolution governed by arbitrary nonlocal interactions over any finite time interval. This is achieved by considering the quasi-steady state of the auxiliary factors. In this framework, the parameters of the reaction-diffusion system can be explicitly determined based on the shape of the integral kernel. Consequently, nonlocal problems can be reformulated within the framework of reaction-diffusion systems, and *vice versa*. For instance, the theoretical framework of the n-component reaction-diffusion system can be applied to analyze the nonlocal problems. Conversely, insights from nonlocal problems can also be leveraged to study multi-component reaction-diffusion systems.

By employing this reaction-diffusion approximation, approximate solutions to nonlocal evolution equations can be obtained through numerical simulations of the corresponding reaction-diffusion systems. Since this approach requires solving only the reaction-diffusion system for the auxiliary factors rather than directly handling the nonlocal interactions, it is expected to significantly reduce computational costs and improve the efficiency of solving nonlocal evolution equations.

The essentially important aspect in this reaction-diffusion approximation is the approximation result to any radial kernel by a linear sum of the Green function k_j in $L^1(\mathbb{R}^n)$ space. The error reduces to a form of the polynomial approximation generally. We employed the Bernstein and Lagrange polynomials to obtain the convergence. The other polynomials that have good properties for the approximation can be candidates for L^1 convergence. Regarding this convergence result, we remark that the result of the expansion by k_j in the case of $H^m(\mathbb{R}^n)$ space with arbitrary $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$ is obtained by Ishii and Tanaka [17]. As in Remark 2.1, if $h \in C^m([0, 1])$ for $m \in \mathbb{N}$, the error ε converges to 0 in the order $O(N^{-m})$. Thus, it is possible to obtain numerical solutions at low cost and quickly, especially for nonlocal evolution equations with smooth integral kernels by applying this approximation method to the numerical simulations.

For the equations with advective nonlocal interactions such as cell adhesion model (1.5) and the nonlocal Fokker-Planck equation, the expansion result by the Green function in $W^{1,1}(\mathbb{R}^n)$ is necessary for this type of the PDE approximation. Therefore, Theorem 4.1 can also be useful in the context of such study.

Appendix A. Existence of a global solution to (NP)

We first consider the existence of the mild solution to (NP). The case that $p = +\infty$ is simpler than the case that $1 \le p < +\infty$, thus the discussion is omitted here. For $T > 0$ and $1 \le p < +\infty$, we define the Banach space $X(T,p) := C([0,T]; BC(\mathbb{R}^n) \cap L^p(\mathbb{R}^n))$ and the norm

$$\|\phi\|_{X(T,p)} := \sup_{0\le t\le T} \|\phi(t)\|_{L^\infty} + \sup_{0\le t\le T} \|\phi(t)\|_{L^p}.$$

We first check the property of $\mathscr{P}[\phi]$.

Lemma A.1. *$\mathscr{P} : X(T,p) \to X(T,p)$ is well-defined. Moreover, for any $u_0 \in BC(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, we have*

$$\lim_{t\to+0} \|\mathscr{P}[\phi](t) - u_0\|_{L^p} = 0$$

and

$$\lim_{t\to+0} \mathscr{P}[\phi](t,x) = u_0(x)$$

for any $x \in \mathbb{R}^n$.

Proof. Let $\phi \in X(T,p)$ and $t \in (0,T]$. Then, we obtain that

$$\begin{aligned} \|\mathscr{P}[\phi](t)\|_{L^p} &\le \|u_0\|_{L^p} + C_f(1+\|K\|_{L^1}) \int_0^t \|\phi(s)\|_{L^p} ds \\ &\le \|u_0\|_{L^p} + C_f T(1+\|K\|_{L^1})\|\phi\|_{X(T,p)} \end{aligned}$$

from the Young inequality. A similar argument yields that

$$\|\mathscr{P}[\phi](t)\|_{L^\infty} \le \|u_0\|_{L^\infty} + C_f T(1+\|K\|_{L^1})\|\phi\|_{X(T,p)}.$$

Moreover, since $\phi(t)$, $(K*\phi)(t)$ and $H(t,D)*u_0$ belong to $BC(\mathbb{R}^n)$, we have $\mathscr{P}[\phi](t) \in BC(\mathbb{R}^n)$. Thus, the desired assertion is verified if the convergence of the initial datum is obtained. Since u_0 is a bounded continuous function in $L^p(\mathbb{R}^n)$, it is known that

$$\lim_{t\to+0} \|H(t;D)*u_0 - u_0\|_{L^p} = 0$$

and

$$\lim_{t\to+0} (H(t;D)*u_0)(x) = u_0(x)$$

hold for any $x \in \mathbb{R}^n$. Therefore, we obtain

$$\begin{aligned} \|\mathscr{P}[\phi](t) - u_0\|_{L^p} &\le \|H(t;D)*u_0 - u_0\|_{L^p} + C_f(1+\|K\|_{L^1}) \int_0^t \|\phi(s)\|_{L^p} ds \\ &\le \|H(t;D)*u_0 - u_0\|_{L^p} + C_f t(1+\|K\|_{L^1})\|\phi\|_{X(T,p)} \\ &\to 0 \quad (t\to+0). \end{aligned}$$

Similarly, for any $x \in \mathbb{R}^n$, we have

$$\begin{aligned} |\mathscr{P}[\phi](t,x) - u_0(x)| &\le |(H(t;D)*u_0)(x) - u_0(x)| + C_f t(1+\|K\|_{L^1})\|\phi\|_{X(T,p)} \\ &\to 0 \quad (t\to+0). \end{aligned}$$

The proof is complete. □

Next, we show the existence of the mild solution to (NP).

Proposition A.1. *There exists a unique mild solution $u \in C([0,T]; BC(\mathbb{R}^n) \cap L^p(\mathbb{R}^n))$ to* (NP) *with an initial datum $u_0 \in BC(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$.*

Proof. For $\rho > 0$, we introduce the norm

$$\|\phi\|_\rho := \sup_{0 \le t \le T} e^{-\rho t} \|\phi(t)\|_{L^\infty} + \sup_{0 \le t \le T} e^{-\rho t} \|\phi(t)\|_{L^p}.$$

We note that it is an equivalent norm to $\|\cdot\|_{X(T,p)}$.

Fix $\rho > C_f(1+\|K\|_{L^1})$ and $t \in (0,T]$. For $\phi_1, \phi_2 \in X(T,p)$, we have

$$\begin{aligned} &|\mathscr{P}[\phi_1](t) - \mathscr{P}[\phi_2](t)| \\ &\quad \le C_f \int_0^t H(t-s;D) * (|\phi_1(s) - \phi_2(s)| + |(K*\phi_1)(s) - (K*\phi_2)(s))|)\, ds. \end{aligned}$$

Thus, we deduce

$$\begin{aligned} &e^{-\rho t} \|\mathscr{P}[\phi_1](t) - \mathscr{P}[\phi_2](t)\|_{L^p} \\ &\quad \le C_f \int_0^t e^{-\rho t} \left(\|\phi_1(s) - \phi_2(s)\|_{L^p} + \|(K*\phi_1)(s) - (K*\phi_2)(s))\|_{L^p}\right) ds \\ &\quad \le C_f(1+\|K\|_{L^1}) \int_0^t e^{-\rho(t-s)} e^{-\rho s} \|\phi_1(s) - \phi_2(s)\|_{L^p} ds \\ &\quad = \frac{C_f(1+\|K\|_{L^1})}{\rho} \sup_{0 \le t \le T} e^{-\rho t} \|\phi_1(t) - \phi_2(t)\|_{L^p} \end{aligned}$$

from the Young inequality. A similar argument yields

$$\|\mathscr{P}[\phi_1](t) - \mathscr{P}[\phi_2](t)\|_{L^\infty} \le \frac{C_f(1+\|K\|_{L^1})}{\rho} \sup_{0 \le t \le T} \|\phi_1(t) - \phi_2(t)\|_{L^\infty}.$$

Therefore, we conclude

$$\|\mathscr{P}[\phi_1] - \mathscr{P}[\phi_2]\|_\rho \le \frac{C_f(1+\|K\|_{L^1})}{\rho} \|\phi_1 - \phi_2\|_\rho.$$

Since $\mathscr{P} : (X(T,p), \|\cdot\|_\rho) \to (X(T,p), \|\cdot\|_\rho)$ is a contraction map, there is a unique fixed point $u \in X(T,p)$ from the Banach fixed point theorem. □

Fix $1 \le p \le +\infty$ and $T > 0$. Let $u \in C([0,T]; BC(\mathbb{R}^n) \cap L^p(\mathbb{R}^n))$ be the mild solution to (NP) with an initial datum $u_0 \in BC(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. Next, we estimate the bound of the L^q norm of the solution.

Lemma A.2. *For any $q \in [p, +\infty]$ and $T > 0$,*

$$\|u(t)\|_{L^q} \le e^{C_f(1+\|K\|_{L^1})t} \|u_0\|_{L^q}$$

holds for any $t \in [0,T]$.

Proof. Let $q \in [p, +\infty]$ and $t \in [0,T]$. Then, we have

$$\begin{aligned} \|u(t)\|_{L^q} &\le \|u_0\|_{L^q} + \int_0^t \|f(u(s), (K*u)(s))\|_{L^q} ds \\ &\le \|u_0\|_{L^q} + C_f(1+\|K\|_{L^1}) \int_0^t \|u(s)\|_{L^q} ds \end{aligned}$$

from the Young inequality. The Gronwall inequality yields

$$\|u(t)\|_{L^q} \le e^{C_f(1+\|K\|_{L^1})t} \|u_0\|_{L^q}.$$

Thus, we obtain the desired assertion. □

Let us consider the regularity of the mild solution. From the general theory to the heat equation, we know that

$$H(\cdot;D)*u_0 \in C^{\infty}((0,+\infty)\times\mathbb{R}^n).$$

Thus, we consider the regularity of the Duhamel term

$$\begin{aligned} I(t,x) &:= \int_0^t H(t-s;D)*f(u(s),(K*u)(s))ds \\ &= \int_0^t \int_{\mathbb{R}^n} H(t-s,x-y;D)*f(u(s,y),(K*u)(s,y))dyds. \end{aligned}$$

For sake of simplicity, we set

$$g(t,x) := f(u(t,x),(K*u)(t,x)).$$

It is easy to see that $g \in C([0,T];BC(\mathbb{R}^n))$.

Lemma A.3. *It holds that $I(t) \in BC^1(\mathbb{R}^n)$ for all $t \in (0,T]$. Moreover, $I(\cdot,x) \in C^{\alpha}([\tau,T])$ holds for any $\alpha \in (0,1)$, $\tau \in (0,T)$ and $x \in \mathbb{R}^n$.*

Proof. We first consider the spatial derivative. Fix $j = 1,2,\ldots,n$ arbitrary. Since there is a constant $C_{H1} > 0$ independent of j such that

$$\left\| \frac{\partial H}{\partial x_j}(t;D) \right\|_{L^1} \le \frac{C_{H1}}{\sqrt{t}}$$

holds for any $t > 0$, we have

$$\begin{aligned} \left| \frac{\partial I}{\partial x_j}(t,x) \right| &\le \int_0^t \int_{\mathbb{R}^n} \left| \frac{\partial H}{\partial x_j}(t-s,x-y;D)*g(s,y) \right| dyds \\ &\le \left(\int_0^t \left\| \frac{\partial H}{\partial x_j}(t-s;D) \right\|_{L^1} ds \right) \sup_{0\le t\le T} \|g(t)\|_{L^\infty} \\ &= 2C_{H1}\sqrt{t} \sup_{0\le t\le T} \|g(t)\|_{L^\infty} \end{aligned}$$

for any $t > 0$ and $x \in \mathbb{R}^n$ from differentiating under the integral sign and the Young inequality. Thus, for all $t \in (0,T]$, we obtain $\dfrac{\partial I}{\partial x_j}(t) \in BC(\mathbb{R}^n)$ and thus conclude $I(t) \in BC^1(\mathbb{R}^n)$.

Next, we fix $\alpha \in (0,1)$, $\tau \in (0,T)$ and $x \in \mathbb{R}^n$. Let $t_1,t_2 \in [\tau,T]$ with $t_2 > t_1$. Since we know that there is a constant $C_{H2} > 0$ such that

$$\|\Delta H(t;D)\|_{L^1} \le \frac{C_{H2}}{t}$$

holds for any $t > 0$, we find

$$
\begin{aligned}
|I(t_2,x) - I(t_1,x)| &\leq \int_0^{t_1} |H(t_2-s;D) - H(t_1-s;D)| * |g(s)|ds \\
&\quad + \int_{t_1}^{t_2} H(t_2-s;D) * |g(s)|ds \\
&\leq \left(\int_0^{t_1} \int_{t_1}^{t_2} \left\| \frac{\partial H}{\partial t}(\eta - s;D) \right\|_{L^1} d\eta ds \right) \sup_{0\leq t\leq T} \|g(t)\|_{L^\infty} \\
&\quad + (t_2 - t_1) \sup_{0\leq t\leq T} \|g(t)\|_{L^\infty} \\
&= \left(\int_0^{t_1} \int_{t_1}^{t_2} \|\Delta H(\eta - s;D)\|_{L^1} d\eta ds \right) \sup_{0\leq t\leq T} \|g(t)\|_{L^\infty} \\
&\quad + (t_2 - t_1) \sup_{0\leq t\leq T} \|g(t)\|_{L^\infty} \\
&\leq C_{H2} \left(t_2 \log t_2 - t_1 \log t_1 - (t_2 - t_1) \log(t_2 - t_1) \right) \sup_{0\leq t\leq T} \|g(t)\|_{L^\infty} \\
&\quad + (t_2 - t_1) \sup_{0\leq t\leq T} \|g(t)\|_{L^\infty}.
\end{aligned}
$$

Thus, we conclude $I(\cdot,x) \in C^\alpha([\tau,T])$. □

In addition, the following lemma is provided for Hölder estimate for the solution.

Lemma A.4. *Let $\gamma \in (0,1)$. Then, there exists a positive constant $C = C(\gamma)$ such that*

$$
\left[\frac{\partial H(t)}{\partial x_j} * \phi \right]_\gamma \leq C t^{-(1+\gamma)/2} \|\phi\|_{L^\infty}
$$

for any $t > 0$, $j = 1,2,\dots,n$ and $\phi \in L^\infty(\mathbb{R}^n)$, where $[\phi]_\gamma$ is Hölder seminorm.

Proof. Let $\phi \in L^\infty(\mathbb{R}^n)$. Fix $t > 0$ and $j = 1,2,\dots,n$ arbitrary. For all $x,z \in \mathbb{R}^n$, we find

$$
\begin{aligned}
&\left| \left(\frac{\partial H}{\partial x_j} * \phi \right)(t,x) - \left(\frac{\partial H}{\partial x_j} * \phi \right)(t,z) \right| \\
&\leq \|\phi\|_{L^\infty} \int_{\mathbb{R}^n} \left| \frac{\partial H}{\partial x_j}(t,x-y) - \frac{\partial H}{\partial x_j}(t,z-y) \right| dy \\
&= t^{-1/2} \|\phi\|_{L^\infty} \int_{\mathbb{R}^n} \left| \frac{\partial H}{\partial x_j}(1, t^{-1/2}(x-z) - y) - \frac{\partial H}{\partial x_j}(1,y) \right| dy \\
&=: t^{-1/2} \|\phi\|_{L^\infty} S(t^{-1/2}(x-z)).
\end{aligned}
$$

Since $S(x)$ is a non-negative bounded continuous function on $\mathbb{R}^n$ and satisfies

$$
S(x) \leq |x| \int_{\mathbb{R}^n} \left| \frac{\partial}{\partial y_j} \nabla H(1,y) \right| dy
$$

from the mean value theorem, there exists a constant $C = C(\gamma)$ such that

$$
0 \leq \frac{S(x)}{|x|^\gamma} \leq \sup_{|x|<1} \frac{S(x)}{|x|^\gamma} + \sup_{|x|\geq 1} S(x) \leq C.
$$

Thus, we obtain

$$
\left| \left(\frac{\partial H}{\partial x_j} * \phi \right)(t,x) - \left(\frac{\partial H}{\partial x_j} * \phi \right)(t,z) \right| \leq C t^{-(1+\gamma)/2} \|\phi\|_{L^\infty} |x-z|^\gamma.
$$

The proof is complete. □

Fix $t \in (0,T]$ arbitrary. From Lemma A.3, we obtain $u(t) \in BC^1(\mathbb{R}^n)$. Moreover, since $f(u,v)$ is differentiable almost everywhere for any open set of $\mathbb{R}^2$, we have

$$\|\nabla g(t)\|_{L^\infty} < +\infty.$$

Hence, same argument as in the proof of Lemma A.3 can be applied, leading to $u(t) \in BC^2(\mathbb{R}^n)$. By using Lemma A.4, we conclude $u(t) \in C^{2+\gamma}(\mathbb{R}^n)$ for any $\gamma \in (0,1)$.

We utilize the result of the Schauder estimate, Theorem 9.1.2 by Krylov [19]. Let $\tau \in (0,T)$ and $\gamma \in (0,1)$. Since $u(\tau) \in C^{2+\gamma}(\mathbb{R}^n)$ and $g \in C^{\gamma/2,\gamma}([\tau,T] \times \mathbb{R}^n)$, we obtain that the mild solution satisfies (RD$_\delta$) with $u \in C^{1+\gamma/2,2+\gamma}([\tau,T] \times \mathbb{R}^n)$. As τ is arbitrary, it follows that $u \in C^{1,2}((0,T] \times \mathbb{R}^n)$. Therefore, the mild solution u becomes the unique classical solution to (RD$_\delta$).

Appendix B. Existence of a global solution to (RD$_\delta$)

Here, we describe the proof of Proposition 2.1. Let us consider the existence of a mild solution to (RD$_\delta$) with an initial condition (2.6).

Proposition B.1. *For any $T > 0$, $\delta > 0$ and $1 \le p \le +\infty$, there exists a unique mild solution $(u^\delta, v_1^\delta, \ldots, v_N^\delta) \in \{C([0,T]; BC(\mathbb{R}^n) \cap L^p(\mathbb{R}^n))\}^{N+1}$ to* (RD$_\delta$) *with an initial condition* (2.6)*, where $u_0 \in BC(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. This solution belongs to $\{C^{1,2}((0,T] \times \mathbb{R}^n)\}^{N+1}$.*

This proof is almost same as the argument in Appendix A. It should be noted that the following result for the boundedness of the solution is obtained.

Lemma B.1. *For any $q \in [p,+\infty]$,*

$$\begin{aligned}
\|u^\delta(t)\|_{L^q} &\le \left(1 + \delta C_f \sum_{j=1}^N |\alpha_j|\right) \|u_0\|_{L^q} \exp\left(C_f \left(1 + \sum_{j=1}^N |\alpha_j|\right) t\right), \\
\|v_j^\delta(t)\|_{L^q} &\le e^{-t/\delta} \|u_0\|_{L^q} + \sup_{0 \le s \le t} \|u^\delta(s)\|_{L^q}, \quad (j = 1,2,\ldots,N)
\end{aligned}$$

hold for all $t \in [0,T]$.

Proof. Let $q \in [p,+\infty]$. For any $t \in [0,T]$ and $j = 1,2,\ldots,N$, we obtain that

$$\|v_j^\delta(t)\|_{L^q} \le e^{-t/\delta} \|u_0\|_{L^q} + \frac{1}{\delta} \int_0^t e^{-(t-s)/\delta} \|u^\delta(s)\|_{L^q} ds$$

from the Young inequality. Moreover, we have

$$\begin{aligned}
\|u^\delta(t)\|_{L^q} &\le \|u_0\|_{L^q} + \int_0^t \left\| f\left(u^\delta(s), \sum_{j=1}^N \alpha_j v_j^\delta(s)\right) \right\|_{L^q} ds \\
&\le \|u_0\|_{L^q} + C_f \left(\int_0^t \|u^\delta(s)\|_{L^q} ds + \int_0^t \sum_{j=1}^N |\alpha_j| \|v_j^\delta(s)\|_{L^q} ds \right) \\
&\le \|u_0\|_{L^q} + C_f \int_0^t \|u^\delta(s)\|_{L^q} ds \\
&\quad + C_f \left(\sum_{j=1}^N |\alpha_j|\right) \left(\int_0^t e^{-s/\delta} \|u_0\|_{L^q} ds + \frac{1}{\delta} \int_0^t \int_0^s e^{-(s-\eta)/\delta} \|u^\delta(\eta)\|_{L^q} d\eta ds \right) \\
&\le \left(1 + \delta C_f \sum_{j=1}^N |\alpha_j|\right) \|u_0\|_{L^q} + C_f \left(1 + \sum_{j=1}^N |\alpha_j|\right) \int_0^t \|u^\delta(s)\|_{L^q} ds.
\end{aligned}$$

The Gronwall inequality yields that

$$\|u^{\delta}(t)\|_{L^q} \leq \left(1+\delta C_f \sum_{j=1}^{N} |\alpha_j|\right) \|u_0\|_{L^q} \exp\left(C_f\left(1+\sum_{j=1}^{N}|\alpha_j|\right)t\right)$$

for all $t \in [0,T]$. Finally, we find

$$\begin{aligned}\|v_j^{\delta}(t)\|_{L^q} &\leq e^{-t/\delta}\|u_0\|_{L^q} + \frac{1}{\delta}\sup_{0\leq s\leq t}\|u^{\delta}(s)\|_{L^q}\int_0^t e^{-(t-s)/\delta}ds \\ &\leq e^{-t/\delta}\|u_0\|_{L^q} + \sup_{0\leq s\leq t}\|u^{\delta}(s)\|_{L^q}\end{aligned}$$

for any $t \in [0,T]$. □

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